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POTENTIAL KERNEL FOR TWO-DIMENSIONAL RANDOM WALK

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It is proved that the potential kernel of a recurrent, aperiodic random walk on the integer lattice $\mathbb{Z}^2$ admits an asymptotic expansion of the form

$$\left(2\pi\sqrt{|Q|}\right)^{-1}\ln Q(x_2, -x_1) + \text{const} + |x|^{-1}U_1(\omega^x) + |x|^{-2}U_2(\omega^x) + \cdots,$$

where $|Q|$ and $Q(\theta)$ are, respectively, the determinant and the quadratic form of the covariance matrix of the increment X of the random walk, $\omega^x = x/|x|$ and the $U_k(\omega)$ are smooth functions of ω, $|\omega| = 1$, provided that all the moments of X are finite. Explicit forms of U_1 and U_2 are given in terms of the moments of X.

1. Introduction and statements of results. Let $X^{(1)}, X^{(2)}, \dots$ be a sequence of $\mathbb{Z}^2$-valued i.i.d. mean-zero random variables with finite variance and $\{S_n\}_{n=0}^{\infty}$ the associated random walk on the integer lattice $\mathbb{Z}^2$ starting at the origin; that is, $S_0 = 0$, $S_n = \sum_{i=1}^{n} X^{(i)}$. We write X for $X^{(1)}$ for brevity. We assume that the random walk $\{S_n\}_{n=0}^{\infty}$ is aperiodic (i.e., the smallest additive subgroup containing $\{x \in \mathbb{Z}^2\colon P\{X = x\} > 0\}$ agrees with $\mathbb{Z}^2$). As in [3], we define the potential function (Green function) $a(x)$ by

$$a(x) = \sum_{n=0}^{\infty} \left(P\{S_n = 0\} - P\{S_n = -x\}\right), \qquad x \in \mathbb{Z}^2.$$

Let $Q(\theta)$ be the moment quadratic form of X. That is, $Q(\theta) = E\{(\theta \cdot X)^2\}$, $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$. We sometimes write $Q(\theta_1, \theta_2)$ for $Q(\theta)$. Let Q also denote the covariance matrix of X, and Q^{-1} its inverse matrix, and define

$$\|x\| := \sqrt{x \cdot Q^{-1}x} = \sqrt{Q(x_2, -x_1)/\det Q}$$

[here $x = (x_1, x_2)$ is thought to be a column vector when the matrix is operated from the left]. The square root of Q that is symmetric and positive definite is denoted by $\sqrt{Q}$. We need the moment conditions

$$(\text{MC: } k + \delta) \qquad E\left\{|X|^{k+\delta}\right\} < \infty \quad \text{for some } \delta > 0,$$

where k will take the values $2, 3, \dots$. The following result is due to Spitzer [3].

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THEOREM 1. *Suppose the moment condition* (MC: $2+\delta$) *holds. Then*

$$\lim_{|x|\to\infty}\left[a(x)-(\pi\sigma_1\sigma_2)^{-1}\ln\|x\|-C_0\right]=0, \tag{1.1}$$

where C_0 *is a certain constant that depends on the distribution of* X, *and* σ_1 *and* σ_2 *are the eigenvalues of the matrix* $\sqrt{Q}$.

To state the main result of this paper, we put

$$\psi_{i,1}(\theta) := -2E\{(\theta\cdot X)^3\}/3Q^2(\theta)$$

and

$$\psi_{r,1}(\theta) := \left[\tfrac{1}{2}Q(\theta)E\{(\theta\cdot X)^4\}-\tfrac{2}{3}\left(E\{(\theta\cdot X)^3\}\right)^2\right]\Big/3Q^3(\theta).$$

These functions are the principal parts of the real and imaginary parts of the function $\psi(\theta) := (1-E\{e^{iX\cdot\theta}\})^{-1}-2/Q(\theta)$ [cf. (3.4) and (3.5)]. We also define

$$g^{\sharp}(\theta) := \text{p.v.}\int_{-\infty}^{\infty} g(\theta_1-t\theta_2,\theta_2+t\theta_1)\,dt \tag{1.2}$$

for $\theta\in\mathbb{R}\setminus\{0\}$ and a function g for which the principal value on the right-hand side exists. [The principal value here is, of course, the limit of the integral on the symmetrical interval $(-L,L)$.]

THEOREM 2. *If the moment condition* (MC: $2+m+\delta$) *holds* ($m\geq 1$), *then*

$$a(x)-\frac{1}{\pi\sigma_1\sigma_2}\ln\|x\|-C_0=\frac{U_1(\omega^x)}{|x|}+\cdots+\frac{U_m(\omega^x)}{|x|^m}+o\left(\frac{1}{|x|^m}\right) \tag{1.3}$$

as $|x|\to\infty$ *in* $\mathbb{Z}^2$, *where* $\omega^x=x/|x|$, σ_1, σ_2 *and* C_0 *are the same constants as in* (1.1) *and* U_k, $k=1,2,\dots,$ *are smooth functions of* $\omega=(\omega_1,\omega_2)$, $|\omega|=1$; *moreover, the first and the second of them are given by*

$$U_1(\omega)=\frac{1}{2\pi^2}\psi_{i,1}^{\sharp}(\omega) \quad \text{and} \quad U_2(\omega)=\frac{1}{2\pi^2}(\omega\cdot\nabla\psi_{r,1})^{\sharp}(\omega).$$

REMARK 1. The function $U_1(\omega)$ is identically 0 if and only if all the third moments $E\{(X_1)^k(X_2)^{3-k}\}$, $k=0,1,2,3$, vanish. If X is symmetric, that is, X has the same distribution as $-X$, then U_k vanishes for every odd number k.

REMARK 2. For the simple random walk in particular, Theorem 2 gives the asymptotic expansion

$$a(x)=\frac{2}{\pi}\ln|x|+\frac{\ln 8+2\gamma}{\pi}+\frac{1}{6\pi}\frac{8(\omega_1^x\omega_2^x)^2-1}{|x|^2}+\frac{U_4(\omega^x)}{|x|^4}+\cdots$$

(γ is Euler's constant), which is an improvement of a result of Stöhr [4], where $a(x)$ is computed up to $O(|x|^{-2})$.

REMARK 3. We shall see in Section 5 that $U_k(\theta/|\theta|)/|\theta|^k$ is a rational fraction of the form $\{\theta^{2\nu-k}\}/\|\theta\|^{2\nu}$, where $\{\theta^k\}$ represents a homogeneous polynomial of degree k. Accordingly, (1.3) may be rewritten as

$$(1.4)\quad a(x) - \frac{1}{\pi\sigma_1\sigma_2}\ln\|x\| - C_0 = \frac{\tilde{U}_1(\tilde{\omega}^x)}{\|x\|} + \cdots + \frac{\tilde{U}_m(\tilde{\omega}^x)}{\|x\|^m} + o\left(\frac{1}{|x|^m}\right).$$

Here $\tilde{\omega}^x := \sqrt{Q^{-1}}x/\|x\|$; $\tilde{U}_k(\omega)$ is a polynomial of $\omega = (\omega_1, \omega_2)$ of degree (at most) $3k$ for $k = 1, 2, \dots$; in particular, $\tilde{U}_1(\omega) = (\psi_{i,1} \circ \sqrt{Q^{-1}})^\sharp(\omega)/(2\pi^2\sigma_1\sigma_2)$. [See (1.8) below.]

In the case of a simple random walk, $a(x)$ can be neatly expressed by a contour integral on the complex plane (as given and applied, e.g., in [4] and [5]) and the complex function theory accordingly provides us machinery for computation, though the proof given in [4] is still quite involved.

In our approach, we employ only real analytic arguments as in Spitzer [3] and it is a key step to establish an asymptotic expansion of an integral of the form

$$(1.5)\qquad \int_{[-\pi,\pi]\times[-\pi,\pi]} \frac{p(\theta)}{q(\theta)}\sin(x\cdot\theta)\,d\theta$$

as $x \to \infty$, where $p(\theta)$ and $q(\theta)$ are homogeneous polynomials of degree $2\nu - 1$ and 2ν, $\nu \geq 1$, respectively, and $q(\theta)$ is supposed to be positive for $\theta \neq 0$, so that

$$q(\theta) \geq c|\theta|^{2\nu}, \qquad \theta \in \mathbb{R}^2,$$

for a constant $c > 0$. We formulate the result on the integral (1.5) in the following theorem.

THEOREM 3. *Let p and q be as above. Let D be a two-dimensional bounded domain containing the origin and having piecewise smooth boundary. Let m be a positive integer and $\xi(\theta)$ a function on the closure $\overline{D}$ such that ξ has partial derivatives up to order m that are continuous on $\overline{D}\setminus\{0\}$ and integrable on D. Then for the function*

$$g(\theta) = \frac{p(\theta)}{q(\theta)} + \xi(\theta)$$

it holds that

$$(1.6)\quad \int_D g(\theta)e^{ir\omega\cdot\theta}\,d\theta = -\frac{2}{ir}\left(\frac{p}{q}\right)^\sharp(\omega) + \sum_{l=1}^{m}\frac{1}{(ir)^l}B_l(r,\omega) + o\left(\frac{1}{r^m}\right)$$

as $r \uparrow \infty$ uniformly for ω, $|\omega| = 1$, where $\sharp$ is defined by (1.2) and

$$(1.7)\qquad B_l(r,\omega) := \int_{\partial D}(-\omega\cdot\nabla)^{l-1}g(\theta)e^{ir\omega\cdot\theta}\omega\cdot n\,ds.$$

[Here, $n = n(\theta)$ is the outward unit normal vector to ∂D and $ds = ds_\theta$ is a line element of ∂D.]

REMARK 4. As a consequence of Theorem 3, we obtain a useful property of the transform $p/q \to (p/q)^{\sharp}$. Let A be a regular 2×2 matrix. Then

$$\left(\frac{p}{q}\right)^{\sharp}(\theta) = |\det A| \left(\left(\frac{p}{q}\right) \circ A \right)^{\sharp}(A^t \theta). \tag{1.8}$$

(A^t denotes the transpose of the matrix A.) The verification of (1.8) is immediate from (1.6) [take ξ so that both g and $\nabla \cdot g$ vanish on ∂D, change the variable according to $\theta = A\theta'$ on the left-hand side of (1.6) and let $r \to \infty$], while it is not so simple a matter to establish (1.8) if one only looks at the defining expression (1.2). (See Section 5 for further properties of the transform $\sharp$.)

The following version of Theorem 3 is convenient for application.

REMARK 5. Theorem 3 may be extended to a more complete form. Let D and ξ be as in Theorem 3. Let $\psi(\omega)$ be a smooth function on the unit circle. Then for a function g of the form

$$g(\theta) = h(\theta) + \xi(\theta) \text{ with } h(\theta) = \frac{\psi(\theta/|\theta|)(\theta)}{|\theta|}$$

it holds that

$$\int_D g(\theta) e^{ir\omega\cdot\theta} d\theta = 2\pi h^{\flat}(\theta) + i2h^{\sharp}(\theta) + \sum_{k=1}^{m} \frac{1}{(ir)^k} B_k(r, \omega) + o\left(\frac{1}{r^m}\right),$$

as $r \uparrow \infty$, uniformly in ω, $|\omega| = 1$. Here B_k is the same as in (1.7) and

$$h^{\flat}(\theta) = \tfrac{1}{2}[h(\theta_2, -\theta_1) + h(-\theta_2, \theta_1)].$$

The method developed in this paper can be adapted for deriving the asymptotic expansion of the potential kernel for the higher-dimensional random walk, which will be studied in a separate paper. Theorem 3, in particular, has a d-dimensional version ($d \geq 3$) in which $g^{\sharp}$ takes an analogous or different form according as d is even or odd.

The result of Stöhr [4] mentioned above is used for estimating a certain hitting distribution by Kesten [1] (cf. also [2]). As another example of application of our expansion (1.3), we shall compute in Section 6 the distributions of hitting places of lines $x_2 = N$ up to $O(|x|^{-3})$.

The proof of Theorem 1, which is essentially the same as in [3], prepares that of Theorem 2, and our task for the latter is to get the estimate of the remainder term, which will be reduced to Theorem 3 with not much difficulty.

We shall proceed in logical order, namely, we first prove Theorem 1 in Section 2, secondly Theorem 3 in Section 3 and then Theorem 2 in Section 4.

2. Proof of Theorem 1. Here we outline the proof of Theorem 1. It is identical to that given in Proposition 12.3 of Spitzer [3] [where $Q(\theta)$ is assumed to be a constant multiple of $|\theta|^2$] except for a simple modification by a change of variable, but we need the content of it, since our proof of Theorem 2 being a continuation of it.

Let $\phi(\theta)$ be the characteristic function of X, that is, $\phi(\theta) = E\{e^{iX\cdot\theta}\}$. The function $a(x)$ is expressed as follows:

$$a(x) = \frac{1}{(2\pi)^2}\int_T \frac{1-e^{ix\cdot\theta}}{1-\phi(\theta)}\,d\theta, \qquad x \in \mathbb{Z}^2, \tag{2.1}$$

where $T = [-\pi, \pi] \times [-\pi, \pi]$. Introducing

$$\psi(\theta) := \frac{1}{1-\phi(\theta)} - \frac{2}{Q(\theta)},$$

which is integrable on T since the condition (MC: $2+\delta$) implies $1 - \phi(\theta) = \frac{1}{2}Q(\theta) + O(|\theta|^{2+\delta})$ as $|\theta| \to 0$ (cf. [3], Proposition 12.3). We make the decomposition

$$4\pi^2 a(x) = \int_T \frac{2}{Q(\theta)}(1-\cos x\cdot\theta)\,d\theta + \mathrm{Re}\int_T \psi(\theta)(1-e^{ix\cdot\theta})\,d\theta \tag{2.2}$$

(Re z indicates the real part of a complex number z). In view of the Riemann–Lebesgue lemma, the second term converges, as $|x| \to \infty$, to Re $\int_T \psi(\theta)\,d\theta$, contributing to the constant C_0 and leaving the $o(1)$ term

$$\Psi(x) := -\mathrm{Re}\int_T \psi(\theta)e^{ix\cdot\theta}\,d\theta. \tag{2.3}$$

For the evaluation of the first term, we consider the mapping

$$\theta \mapsto \theta' = \sqrt{Q}\,\theta, \tag{2.4}$$

which, entailing the identity $Q(\theta) = |\theta'|^2$, transforms the ellipse $Q(\theta) = r^2$ into the circle $|\theta'| = r$. Let (r, α) be the polar coordinates of θ', namely,

$$\theta' = \big(r\cos(\alpha - \alpha_0), r\sin(\alpha-\alpha_0)\big), \tag{2.5}$$

where α_0 is a constant chosen arbitrarily. Since $x\cdot\theta = \sqrt{Q^{-1}}x\cdot\theta'$ and $\|x\| = |\sqrt{Q^{-1}}x|$, we can choose the constant $\alpha_0 = \alpha_0(x)$ so as to get $x\cdot\theta = \|x\|r\sin\alpha$.

Now, putting

$$B = \left\{\theta\colon \sqrt{Q(\theta)} \le (\sigma_1 \wedge \sigma_2)\pi\right\}$$

(B is an elliptic region inscribed in T), we decompose the first integral on the right-hand side of (2.2) into two parts, one the integral over B and the other that over $T\setminus B$. The latter converges to $\int_{T\setminus B}[2/Q(\theta)]\,d\theta$, leaving the second $o(1)$ term

$$-\int_{T\setminus B}\frac{2}{Q(\theta)}\cos(x\cdot\theta)\,d\theta. \tag{2.6}$$

The former equals

$$\int_0^{2\pi} d\alpha \int_0^{(\sigma_1 \wedge \sigma_2)\pi} \frac{2}{\sigma_1 \sigma_2 r}[1 - \cos(\|x\| r \sin \alpha)]\, dr$$

$$(2.7)\quad \begin{aligned} &= 4\int_0^{\pi/2} d\alpha \int_0^{c(x)\sin\alpha} \frac{2}{\sigma_1\sigma_2 u}(1 - \cos u)\, du \\ &= \frac{8}{\sigma_1\sigma_2}\left[\int_0^{\pi/2} d\alpha \int_0^1 \frac{1-\cos u}{u}\, du - \int_0^{\pi/2} d\alpha \int_1^\infty \frac{\cos u}{u}\, du \right. \\ &\qquad \left. + \int_0^{\pi/2} d\alpha \int_1^{c(x)\sin\alpha} \frac{1}{u}\, du + \int_0^{\pi/2} d\alpha \int_{c(x)\sin\alpha}^\infty \frac{\cos u}{u}\, du\right], \end{aligned}$$

where $c(x) = (\sigma_1 \wedge \sigma_2)\pi\|x\|$. Within the brackets of the last expression in (2.7), the first and the second terms are constants; the third is equal to

$$\tfrac{1}{2}\pi \log\|x\| + \tfrac{1}{2}\pi \log[(\sigma_1 \wedge \sigma_2)\pi] + \int_0^{\pi/2} \log(\sin\alpha)\, d\alpha,$$

and the fourth vanishes as $|x| \to \infty$ [this gives the last $o(1)$ term]. The proof of Theorem 1 is complete. □

3. Proof of Theorem 3. Put

$$\omega = \frac{x}{|x|} \quad \text{and} \quad R_\omega = \begin{pmatrix} \omega_1 & -\omega_2 \\ \omega_2 & \omega_1 \end{pmatrix}$$

and introduce the new variables

$$(u, v) = R_\omega^{-1}\theta$$

so that $\omega \cdot \theta = u$. Let

$$D^\omega = R_\omega^{-1} D \quad \text{and} \quad g^\omega(u, v) = g(R_\omega(u, v)).$$

Then

$$(3.1)\qquad \int_D g(\theta) e^{ir\omega\cdot\theta}\, d\theta = \int_{D^\omega} g^\omega(u, v) e^{iru}\, du\, dv.$$

We are to carry out the integration by parts for the integral with respect to u on the right-hand side above. This amounts to applying the divergence theorem to the integral on the left-hand side, into which we substitute

$$(3.2)\qquad e^{ir\omega\cdot\theta} = \nabla \cdot \mathbf{A}(\theta), \quad \text{where } \mathbf{A}(\theta) = (ir)^{-1} e^{ir\omega\cdot\theta}\omega.$$

We can apply the divergence theorem repeatedly m times to the integral of $\xi(\theta)e^{ir\omega\cdot\theta} = \nabla\cdot(\xi\mathbf{A}) - \nabla\xi\cdot\mathbf{A}$, which results in the boundary integrals given in (1.7) with ξ in place of g plus the remainder term of the order $o(r^{-m})$. It therefore suffices to prove (1.6) in the case when $\xi = 0$. [Although ξ may be singular at the origin, it can be approximated in Sobolev norm by a smooth function (under the assumption on ξ in Theorem 3) so that the divergence theorem is applicable at least m times.] Let $g = p/q$ with polynomials p and q as described in Theorem 3.

We suppose for simplicity that D^{ω} contains the square

$$K = \{(u, v) \colon |u| \le 1, |v| \le 1\},$$

and we decompose the integral on the right-hand side of (3.1) into that over K and the rest. We formulate the result of the computation as follows.

LEMMA 1. *Let $g = p/q$, where p and q are homogeneous polynomials as given in Theorem 3. Then*

$$\frac{d}{du}\int_{-1}^{1} g^{\omega}(u, v)\, dv = \int_{-1}^{1} \frac{\partial g^{\omega}}{\partial u}(u, v)\, dv = \frac{f(u, \omega)}{q^{\omega}(u, 1) q^{\omega}(u, -1)}, \qquad u \ne 0,$$

where $q^{\omega} = q \circ R_{\omega}$ and $f(u, \omega)$ is a polynomial of (u, ω_1, ω_2), and

$$\begin{aligned}\int_D g(\theta) e^{ir\omega\cdot\theta}\, d\theta = &-\frac{2}{ir} g^{\sharp}(\omega) + \frac{1}{ir}\int_{\partial D} g(\theta) e^{ir\omega\cdot\theta} \omega \cdot n\, ds \\ &- \frac{1}{ir}\int_{D^{\omega}\setminus K} \frac{\partial g^{\omega}}{\partial u}(u, v) e^{iru}\, du\, dv \\ &- \frac{1}{ir}\int_{-1}^{1} e^{iru}\, du \int_{-1}^{1} \frac{\partial g^{\omega}}{\partial u}(u, v)\, dv.\end{aligned}$$

Theorem 3 readily follows from Lemma 1. In fact, if we apply the integration-by-parts formula to the integral relative to u in the last two integrals on the right-hand side above, the contributions of the boundary terms that thereby come up are reduced to

$$\frac{1}{(ir)^2}\int_{\partial D} (-\omega \cdot \nabla) g(\theta) e^{ir\omega\cdot\theta} \omega \cdot n\, ds, \tag{3.3}$$

because of cancellation between those from ∂K [recall the remark made when $\mathbf{A}$ is introduced in (3.2)]. We can repeat the integration by parts in the same way in view of the first half of Lemma 1 to arrive at (1.6). Now it remains to prove Lemma 1.

PROOF OF LEMMA 1. By the divergence theorem

$$\begin{aligned}\int_{D^{\omega}\setminus K} g^{\omega}(u, v) e^{iru}\, du\, dv = &\frac{1}{ir}\int_{\partial(D^{\omega}\setminus K)} g^{\omega}(u, v) e^{iru} n\, ds \\ &- \frac{1}{ir}\int_{D^{\omega}\setminus K} \frac{\partial g^{\omega}}{\partial u}(u, v) e^{iru}\, du\, dv.\end{aligned} \tag{3.4}$$

We cannot apply the divergence theorem to the integral over K directly. We consider the function

$$F(u) := \int_{-1}^{1} g^{\omega}(u, v)\, dv = \int_{-1/u}^{1/u} g^{\omega}(1, t)\, dt.$$

Here, to obtain the second expression, we have applied the assumption that g is in the special form p/q [which the function $g^{\omega} = g \circ R_{\omega}$ clearly inherits, so that $vg^{\omega}(u,v)$ is the ratio of two homogeneous polynomials of degree 2ν]. Clearly, $F(-u) = -F(u)$. Although $g^{\omega}(1,t)$ is not integrable on $\mathbf{R}$, there exists the principal value

$$F(0+) = \text{p.v.}\int_{-\infty}^{\infty} g^{\omega}(1,t)\,dt := \lim_{L\to\infty}\int_{-L}^{L} g^{\omega}(1,t)\,dt.$$

Furthermore, for $u \neq 0$,

$$\begin{aligned} F'(u) &= -u^{-2}\left[g^{\omega}\left(1,\frac{1}{u}\right) + g^{\omega}\left(1,\frac{-1}{u}\right)\right] \\ &= -u^{-1}[g^{\omega}(u,1) + g^{\omega}(u,-1)] \\ &= -\frac{1}{u}\,\frac{p^{\omega}(u,1)q^{\omega}(u,-1) + p^{\omega}(u,-1)q^{\omega}(u,1)}{q^{\omega}(u,1)q^{\omega}(u,-1)}. \end{aligned}$$

Because of cancellation of the constant terms in the numerator of the quotient above, we conclude that, for $u \neq 0$,

$$F'(u) = f(u,\omega)/q^{\omega}(u,1)q^{\omega}(u,-1) \quad \text{with } q^{\omega}(u,\pm 1) \geq c(1+u^2)^{\nu},$$

where $f(u,\omega)$ is a polynomial. This proves the first half of Lemma 1. Now

$$\begin{aligned} \int_K g^{\omega}(u,v)e^{iru}\,du\,dv &= \int_{-1}^{1} F(u)e^{iru}\,du \\ &= \frac{1}{ir}F(u)e^{iru}\Big|_{-1}^{1} - \frac{1}{ir}\int_{-1}^{1} e^{iru}\,dF(u) \\ &= \frac{1}{ir}F(u)e^{iru}\Big|_{-1}^{1} - \frac{2}{ir}F(0+) - \frac{1}{ir}\int_{-1}^{1} e^{iru}F'(u)\,du. \end{aligned}$$

The boundary term appearing above cancels out the contribution from ∂K to the boundary integral in (3.4). The contribution from ∂D^{ω} to the latter integral equals $(ir)^{-1}\int_{\partial D} g(\theta)e^{r\omega\cdot\theta}\omega\cdot n\,ds$. Finally, $F(0+) = g^{\sharp}(\omega)$. Thus we obtain the second relation of Lemma 1. □

4. Proof of Theorem 2. We collect all the error terms that we neglected as $o(1)$ terms in the proof of Theorem 1 and write $\Psi(x) + \Lambda(x)$ for their sum, where

$$\Psi(x) = -\mathrm{Re}\int_T \psi(\theta)e^{ix\cdot\theta}\,d\theta$$

[as already introduced in (2.3)] and

$$\Lambda(x) := \frac{8}{\sigma_1\sigma_2}\int_0^{\pi/2} d\alpha\int_{c(x)\sin\alpha}^{\infty}\frac{\cos u}{u}\,du - \int_{T\setminus B}\frac{2}{Q(\theta)}\cos(x\cdot\theta)\,d\theta.$$

[The first term of $\Lambda(x)$ comes from the last double integral in (2.7) and the second term is (2.6).] As in the previous section, we put $\omega = x/|x|$ and make the change of variables $(u, v) = R_\omega^{-1}\theta$ so that $\omega \cdot \theta = u$. Let

$$T^\omega = R_\omega^{-1} T \quad \text{and} \quad Q^\omega(u, v) = Q(R_\omega(u, v)).$$

LEMMA 2.

$$\Lambda(x) = \int_{-\infty}^{\infty} dv \int_{\{u:\,(u,v)\in\mathbb{R}^2\setminus T^\omega\}} \frac{2}{Q^\omega(u,v)} \cos(|x|u)\, du.$$

PROOF. Recall that we obtained the first term in (2.7), an expression for the integral $\int_B [2/Q(\theta)](1 - \cos x \cdot \theta)\, d\theta$, via a change of variables according to (2.4) and (2.5). By formally reversing the procedure, we see that the first term of the expression defining $\Lambda(x)$ is equal to

$$\int_{\mathbb{R}^2\setminus B} \frac{2}{Q(\theta)} \cos(x \cdot \theta)\, d\theta$$

or, by changing the variables of integration, to the iterated integral

$$\int_{-\infty}^{\infty} dv \int_{\{u:\,(u,v)\in\mathbb{R}^2\setminus B^\omega\}} \frac{2}{Q^\omega(u,v)} \cos(|x|u)\, du, \tag{4.1}$$

where $B^\omega = R_\omega^{-1}B$. Hence we obtain an expression for $\Lambda(x)$ in Lemma 2. This argument, however, must be justified because the function $[1/Q(\theta)]\cos(x \cdot \theta)$, not being Lebesgue integrable on $\mathbb{R}^2 \setminus B$, does not admit the application of Fubini's theorem.

For justification we consider the integral

$$I(L) := \frac{8}{\sigma_1\sigma_2} \int_0^{\pi/2} d\alpha \int_{c(x)\sin\alpha}^{Lc(x)\sin\alpha} \frac{\cos u}{u}\, du. \tag{4.2}$$

Since the inner integral is bounded by $1 + \log^+(1/(c(x)\sin\alpha))$, $I(L)$ converges to the first term of the expression defining Λ as $L \to \infty$. Since the function $u^{-1}\cos u$ is integrable on $\{(\alpha, u): c(x)\sin\alpha < u < Lc(x)\sin\alpha,\ 0 < \alpha \le \pi/2\}$, we may follow the recipe discussed at the beginning of this proof to get $I(L) = \int_{-\infty}^{\infty} f_L(v)\, dv$, where

$$f_L(v) := \int_{\{u:\ \lambda < Q^\omega(u,v) < L\lambda\}} \frac{2}{Q^\omega(u,v)} \cos(|x|u)\, du, \qquad \lambda := \big((\sigma_1 \wedge \sigma_2)\pi\big)^2.$$

We have only to show that $f_L(v)$ is dominated by an integrable function that is independent of L since we can then apply Lebesgue's convergence theorem to see that $I(L)$ converges to (4.1) as $L \to \infty$. Clearly, f_L is bounded uniformly for $L \ge 1$. It therefore suffices to show that, for $a < b$,

$$\left| \int_a^b \frac{1}{Q^\omega(u,v)} \cos(|x|u)\, du \right| \le \frac{M}{|x|v^2}, \tag{4.3}$$

where M is a constant depending on $(\sigma_1 \wedge \sigma_2)$ only. However, since the function $1/Q^{\omega}(u,v)$ with v fixed does not fluctuate at all, that is, it has only one peak for $u \in \mathbb{R}$, and is bounded above by $[(\sigma_1 \wedge \sigma_2)v^2]^{-1}$, the integral of $\cos(|x|u)/Q^{\omega}(u,v)$ over $u \in (a,b)$ is dominated by $2[(\sigma_1 \wedge \sigma_2)|x|v^2]^{-1}$ in absolute value; hence (4.3). The proof of Lemma 2 is complete. □

We decompose

$$\Psi(x) = -\int_T \psi_r(\theta)\cos(x\cdot\theta)\,d\theta + \int_T \psi_i(\theta)\sin(x\cdot\theta)\,d\theta,$$
$$:= \Psi_c(x) + \Psi_s(x) \quad \text{(say)},$$

where $\psi_r(\theta)$ is the real part of $\psi(\theta)$ and $\psi_i(\theta)$ the imaginary part. Put

$$c(\theta) = E\{1-\cos(\theta\cdot X)\} \quad \text{and} \quad s(\theta) = E\{\sin(\theta\cdot X)\}.$$

Then

$$\psi_r(\theta) = \frac{c(\theta)}{c^2(\theta)+s^2(\theta)} - \frac{2}{Q(\theta)} \quad \text{and} \quad \psi_i(\theta) = \frac{s(\theta)}{c^2(\theta)+s^2(\theta)}.$$

Since the random walk is aperiodic, $c^2(\theta)+s^2(\theta) > 0$ for $\theta \in T\setminus\{0\}$. If the moment condition (MC: $4+\delta$) holds, then, putting

$$c_o(\theta) = \tfrac{1}{2}Q(\theta) - \tfrac{1}{24}E\{(\theta\cdot X)^4\} \quad \text{and} \quad s_o(\theta) = -\tfrac{1}{6}E\{(\theta\cdot X)^3\}, \tag{4.4}$$

we obtain

$$\frac{\partial^l}{\partial\theta_1^k\,\partial\theta_2^j}\big(c(\theta)-c_o(\theta)\big) = O\big(|\theta|^{4-l+\delta}\big) \quad \text{for } l := k+j = 0,1,2 \tag{4.5}$$

(as $\theta \to 0$) and the same estimate with $s(\theta)-s_o(\theta)$ in place of $c(\theta)-c_o(\theta)$. Applying these estimates together with the identity $(1+z)^{-1} = (1-z)/(1-z^2)$, we readily deduce

$$\psi_r(\theta) = \frac{1}{3Q^3(\theta)}\left[\frac{1}{2}Q(\theta)E\{(\theta\cdot X)^4\} - \frac{2}{3}\Big[E\{(\theta\cdot X)^3\}\Big]^2\right] + \xi_r(\theta),$$

where

$$\frac{\partial^l}{\partial\theta_1^k\,\partial\theta_2^j}\xi_r(\theta) = O\big(|\theta|^{-l+\delta}\big) \quad \text{for } l = k+j = 0,1,2; \tag{4.6}$$

similarly,

$$\psi_i(\theta) = -\frac{2E\{(\theta\cdot X)^3\}}{3Q^2(\theta)} + \xi_i(\theta), \tag{4.7}$$

where $\xi_i(\theta)$ and its derivatives satisfy (4.6) with ξ_i in place of ξ_r.

Recalling what we noticed just before (3.2) [here $\cos(x\cdot\theta) = \nabla\cdot\mathbf{b}(\theta)$ with $\mathbf{b}(\theta) := |x|^{-1}\sin(x\cdot\theta)\omega$], we apply the divergence theorem to see that, in view of Lemma 2,

$$\begin{aligned}\Psi_c(x) + \Lambda(x) = &-\frac{1}{|x|}\int_{\mathbb{R}^2\setminus T}\omega\cdot\nabla[2/Q](\theta)\sin(x\cdot\theta)\,d\theta \\ &+\frac{1}{|x|}\int_T\omega\cdot\nabla\psi_r(\theta)\sin(x\cdot\theta)\,d\theta,\end{aligned}\tag{4.8}$$

which is valid if (MC: $3+\delta$) holds. Here the boundary terms cancel out each other since $\psi_r(\theta) + 2/Q(\theta)$, as well as $\sin(x\cdot\theta)$, is a doubly periodic function of period $(2\pi, 2\pi)$. Noticing $(\partial/\partial\theta_1)^n(1/Q(\theta)) = O(1/|\theta|^{2+n})$, we see that the first term on the right-hand side of (4.8) is $O(1/|x|^2)$. On the other hand, $|\nabla\psi_r(\theta)|$ is integrable on T, so that the second term is $o(1/|x|)$. Consequently, $\Psi_c(x) + \Lambda(x) = o(1/|x|)$ under (MC: $3+\delta$).

If (MC: $4+\delta$) holds, we can apply Theorem 3 with $m = 2$ to the second integral on the right-hand side of (4.8) in view of (4.6). We can always apply the divergence theorem for the first integral. Again the boundary terms cancel out, resulting in

$$\Psi_c(x) + \Lambda(x) = 2|x|^{-2}(\omega\cdot\nabla\psi_{r,1})^{\sharp}(\omega) + o(|x|^{-2}).$$

As for $\Psi_s(x)$, we have only to apply Theorem 3 with the help of (4.7) to have

$$\Psi_s(x) = \int_T\psi_i(\theta)\sin x\cdot\theta\,d\theta = 4\pi^2\frac{U_1(\omega)}{|x|} + o\left(\frac{1}{|x|^m}\right),$$

where $m = 1$ or 2 according to which moment condition we are assuming. These prove (1.3) for $m = 1$ and 2.

In the case when (MC: $2+m+\delta$) is assumed to hold for $m \geq 3$, we can perform the Taylor expansion of $1-\cos(\theta\cdot X)$ and $\sin(\theta\cdot X)$ up to the mth-order terms for defining c_o and s_o in (4.4). We accordingly obtain the estimates $O(|\theta|^{m+2-l+\delta})$ for $l = 0, 1, \dots, m$ in (4.5), which in turn yields the following expansion for the real and imaginary parts of ψ:

$$\psi_r(\theta) = \frac{\{\theta^6\}}{Q^3(\theta)} + \frac{\{\theta^{12}\}}{Q^5(\theta)} + \cdots + \frac{\{\theta^{3m'+6}\}}{Q^{m'+3}(\theta)} + \xi_r(\theta),\tag{4.9}$$

where $m' = m$ or $m-1$ according to whether m is even or odd and

$$\frac{\partial^l}{\partial\theta_1^k\,\partial\theta_2^j}\xi_r(\theta) = O(|\theta|^{m-2-l+\delta})\quad\text{for } l = k+j = 0, 1, \dots, m;$$

and

$$\psi_i(\theta) = \frac{\{\theta^3\}}{Q^2(\theta)} + \frac{\{\theta^9\}}{Q^4(\theta)} + \cdots + \frac{\{\theta^{3m''+6}\}}{Q^{m''+3}(\theta)} + \xi_i(\theta),\tag{4.10}$$

where $m'' = m - 1$ or m according to whether m is even or odd and

$$\frac{\partial^l}{\partial\theta_1^k\,\partial\theta_2^j}\xi_i(\theta) = O(|\theta|^{m-2-l+\delta}) \quad \text{for } l = k + j = 0, 1, \dots, m.$$

Here $\{\theta^k\}$ denotes a certain homogeneous polynomial of degree k. For evaluating the integral of $\psi_i(\theta)\sin\omega\cdot\theta$, we apply Theorem 3 to the right-hand side of (4.10). All the boundary integrals vanish due to the periodicity of ψ_i. The resultant is

$$\frac{1}{4\pi^2}\int_T \psi_i(\theta)\sin x\cdot\theta\,d\theta = \frac{U_1(\omega)}{|x|} + \frac{U_3(\omega)}{|x|^3} + \cdots + \frac{U_{m''}(\omega)}{|x|^{m''}} + o\left(\frac{1}{|x|^m}\right).$$

Similarly, we obtain the analogous expansion for $\Lambda(x) + \Psi_c(x)$.

5. Self-reciprocity of $\sharp$. Let g be a quotient p/q of two homogeneous polynomials p, q of degrees $2\nu - 1$ and 2ν, respectively ($\nu = 1, 2, \dots$). Suppose $q > 0$, $\theta \neq 0$, and p is relatively prime to q. We prove that

$$g^\sharp(\theta) := \text{p.v.}\int_{-\infty}^{\infty} g(\theta_1 - t\theta_2, \theta_2 + t\theta_1)\,dt$$

is then a function of the same type as g with the same ν and the transform $g \to g^\sharp$ is self-reciprocal, that is, $\pi^2 g = (g^\sharp)^\sharp$. The proof is given in (i)–(v) below.

Let R^α denote rotation by an angle α (counterclockwise). Then:

(i) $$g^\sharp(\theta) = \text{p.v.}\int_{-\pi/2}^{\pi/2} g(R^\alpha\theta)\frac{d\alpha}{\cos\alpha}.$$

This equality is obtained by changing the variable according to $t = \tan\alpha(-\frac{1}{2}\pi < \alpha < \frac{1}{2}\pi)$ so that

$$g(\omega_1 - t\omega_2, \omega_2 + t\omega_1) = g\left(\sqrt{1+t^2}\,R^\alpha\omega\right) = g(R^\alpha\omega)/\sqrt{1+t^2}.$$

From (i) it follows that $(g\circ R^\alpha)^\sharp = g^\sharp\circ R^\alpha$.

(ii) $\quad g^\sharp = |\det A|(g\circ A)^\sharp\circ A^t \quad$ if A is a regular 2×2 matrix.

We have noticed in Remark 4 that (ii) is an easy consequence of Theorem 3. Here we give a direct proof. Once (ii) is proved for diagonal matrices, the general case follows from $(g\circ R^\alpha)^\sharp = g^\sharp\circ R^\alpha$ together with the polar decomposition: $A = SO$ (S, symmetric and O, orthogonal). Let A be a diagonal matrix with diagonal elements λ and μ. Then if $\theta_1 \neq 0$ the right-hand side of (ii) equals

$$\begin{aligned}
&\text{p.v.}\int_{-\infty}^{\infty} g\left(\lambda^2\theta_1 - t\lambda\mu\theta_2, \mu^2\theta_2 + t\lambda\mu\theta_1\right)dt|\lambda\mu| \\
&\quad = \text{p.v.}\int_{-\infty}^{\infty} g\left(\lambda^2\theta_1^2 + \mu^2\theta_2^2 - t\theta_2, t\theta_1\right)dt \\
&\quad = \text{p.v.}\int_{-\infty}^{\infty} g(1 - t\theta_2, t\theta_1)\,dt.
\end{aligned}$$

Thus (ii) follows. Since q is factored into quadratic forms, we can decompose p/q into proper fractions, each of which has for the denominator some power of one of the quadratic forms. Applying (ii) to each fraction of the decomposition, we see that:

(iii) $g^{\sharp}$ is a quotient of two homogeneous polynomials of degrees $2\nu - 1$ and 2ν; the denominator is the product of quadratic forms reciprocal to those making q. [We say here that $(x \cdot A^{-1}x)$ is reciprocal to $(x \cdot Ax)$.]

(iv) $g^{\sharp}(\cos\alpha, \sin\alpha)$ equals the limit value of the allied series for the function $g(-\sin\alpha, \cos\alpha)$; in particular, $\sharp$ is self-reciprocal: $g = (g^{\sharp})^{\sharp}$.

If we put, for a function $\phi(\alpha)$,

$$\phi^{\flat}(\beta) = \frac{1}{\pi}\mathrm{p.v.}\int_{-\pi/2}^{\pi/2} \phi(\beta - \alpha)\frac{d\alpha}{\sin\alpha}. \tag{5.1}$$

Then, making the change of variable $\alpha \to \pi/2 - \alpha$ for the integral in (i) and using the skew symmetry $g(-\theta) = -g(\theta)$, we have $g^{\sharp}(\omega) = \pi(g \circ R^{\pi/2})^{\flat}(\omega)$ (where $\flat$ acts on a restriction of $g \circ R^{\pi/2}$ to the unit circle). Assertion (iv) follows from the next one:

(v) If ϕ is smooth and $\phi(\alpha \pm \pi) = -\phi(\alpha)$, then $\phi^{\flat}$ agrees with the limit value of the series allied with the Fourier series of the function ϕ. [Namely, $\phi^{\flat}(\alpha) = \sum_{n=1}^{\infty} a_n \sin n\alpha - b_n \cos n\alpha$ if $\phi(\alpha) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\alpha + b_n \sin n\alpha$.] In particular, $(\phi^{\flat})^{\flat} = -\phi$.

To prove (v), substitute the identity $1/\sin\alpha = \frac{1}{2}(\cot\frac{1}{2}\alpha + \tan\frac{1}{2}\alpha)$ into the right-hand side of (5.1) and make the change of variable $\alpha \to \pi - \alpha$ in the integral involving $\tan\frac{1}{2}\alpha$. W then deduce from the assumption $\phi(\alpha \pm \pi) = -\phi(\alpha)$ that

$$\begin{aligned}\phi^{\flat}(\beta) &= \frac{1}{\pi}\mathrm{p.v.}\int_0^{\pi} \phi(\beta - \alpha)\frac{d\alpha}{\sin\alpha} \\ &= \frac{1}{2\pi}\int_0^{\pi}[\phi(\beta - \alpha) - \phi(\beta + \alpha)]\cot\frac{1}{2}\alpha\, d\alpha.\end{aligned}$$

This shows the result of (v) since the right-hand side above gives the limit value of the allied series for ϕ.

6. Hitting distribution of lines. We compute the asymptotic form of the hitting distributions of lines $x_2 = N$ for large N. Suppose that the distribution of X is symmetric with respect to the first coordinate axis $x_2 = 0$ and the random walk S_n takes jumps of size at most 1 in the vertical direction, that is, $P\{X_2 = 0, 1 \text{ or } -1\} = 1$. Then the probability that S_n enters the line $x_2 = N$ at a point x, $x_2 = N$, can be expressed by means of the potential function a as follows:

$$H_N(x) = \sum_{j=-\infty}^{\infty}\left[a(-(x_1 + j, N + 1)) - a(-(x_1 + j, N - 1))\right]P(X = (j, 1))$$

(see [1], page 155). We have $Q(\theta) = \sigma_1^2\theta_1^2 + \sigma_2^2\theta_2^2$ and, writing $\tilde{X}_k = X_k/\sigma_k$ and $\tilde{x}_k = x_k/\sigma_k$, $k = 1, 2$, $\psi_{i,1} \circ \sqrt{Q^{-1}}\ (-\sin\alpha, \cos\alpha) = b_1 \sin\alpha + b_3 \cos\alpha$, where $b_1 = \frac{1}{2}(E\{\tilde{X}_1^3\} + E\{\tilde{X}_1, \tilde{X}_2^2\})$ and $b_3 = \frac{1}{6}(-E\{\tilde{X}_1^3\} + 3E\{\tilde{X}_1\tilde{X}_2^2\})$. With the help of (ii) and (iv) of Section 5, it is obvious that

$$U_1(\omega^x)/|x| = -\left[(b_1 - 3b_3)\tilde{x}_1\|x\|^{-2} + 4b_3\tilde{x}_1^3\|x\|^{-4}\right]\Big/(2\pi\sigma_1\sigma_2)$$

($\|x\|^2 = \tilde{x}_1^2 + \tilde{x}_2^2$). Now, applying (1.3) and making elementary computation, we get

$$H_N(x) = \frac{N}{\pi\sigma_1\sigma_2\|x\|^2}\left(1 - 2b_1\frac{\tilde{x}_1}{\|x\|^2} + 8b_3\frac{\tilde{x}_1^3}{\|x\|^4}\right) + O(|x|^{-3}), \tag{6.1}$$

provided that $E\{|X|^{5+\delta}\} < \infty$. (Here we have applied the smoothness of U_2.) Relation (6.1) yields, for example, that $\sum_{-m(N) \le x_1 \le m(N)} x_1 H_N(x)$ converges to $\sigma_1(3b_3 - b_1) = \sigma_1 E\{\tilde{X}_1(\tilde{X}_1^2 - \tilde{X}_2^2)\}$ as $N \to \infty$ whenever $m(N) \to \infty$.

REFERENCES

[1] KESTEN, H. (1987). Hitting probabilities of random walks on Z^d. *Stochastic Process. Appl.* **25** 165–184.

[2] LAWLER, G. F. (1991). *Intersections of Random Walks*. Birkhäuser, Boston.

[3] SPITZER, F. (1976). *Principles of Random Walk*, 2nd ed. Springer, New York.

[4] STÖHR, A. (1950). Über einige lineare partielle Differenzengleichungen mit konstanten Koeffizienten. III. *Math. Nachr.* **3** 330–357.

[5] VAN DER POL, B. (1959). The finite-difference analogy of the periodic wave equation and the potential equation. Appendix IV. In *Probability and Related Topics in Physical Sciences.* Interscience, New York.

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