The Annals of Probability
1996, Vol. 24, No. 4, 1675–1710

MULTIPLE TRANSITION POINTS FOR THE CONTACT PROCESS ON THE BINARY TREE[1]

BY THOMAS M. LIGGETT

University of California, Los Angeles

The contact process on Z^d is known to have only two fundamental types of behavior: survival and extinction. Recently Pemantle discovered that the phase structure for the contact process on a tree can be more complex. There are three possible types of behavior: strong survival, weak survival and extinction. He proved that all three occur on homogeneous trees in which each vertex has $d + 1$ neighbors, provided that $d \geq 3$, but he left open the case $d = 2$. Since $d = 1$ corresponds to Z^1, in which weak survival does not occur, $d = 2$ is the boundary case. In this paper, we complete this picture, by showing that weak survival does occur on the binary tree for appropriate parameter values. In doing so, we extend and develop techniques for obtaining upper and lower bounds for the critical values associated with strong and weak survival of the contact process on more general graphs.

1. Introduction. Let T^d be the homogeneous (connected) tree in which each vertex has $d + 1$ neighbors, and let A_t be the finite contact process on T^d. This is the continuous-time Markov chain on the space of finite subsets of T^d which has the following transitions:

$$A \to A \setminus \{x\} \quad \text{for } x \in A \text{ at rate } 1,$$

$$A \to A \cup \{x\} \quad \text{for } x \notin A \text{ at rate } \lambda\#\{y \in A: |y - x| = 1\}.$$

Here $|y - x|$ denotes the distance between x and y in T^d. Several papers have been written in the past few years which study the behavior of the contact process on a tree. They are listed in the reference section.

We will say that A_t survives strongly if

$$P^{\{x\}}(x \in A_t \text{ for arbitrarily large } t) > 0,$$

and that it survives if

$$P^{\{x\}}(A_t \neq \varnothing \ \forall\ t) > 0.$$

(Using the self-duality of the contact process [see Liggett (1985), Chapter 6], it is easy to check that survival is equivalent to survival of the infinite process starting from all sites occupied.) We will say that A_t dies out if it does not survive, and that it survives weakly if it survives, but does not survive strongly. Then it is natural to define critical values $\lambda_1 \leq \lambda_2$ by the

Received March 1995; revised October 1995.

[1]Preparation of this paper was supported in part by NSF Grants DMS-91-00725 and DMS-94-00644.

AMS 1991 *subject classification*. 60K35.

Key words and phrases. Contact process, critical values, trees, survival, extinction.

requirement that A_t survives strongly for $\lambda > \lambda_2$, survives weakly for $\lambda_1 < \lambda < \lambda_2$ and dies out for $\lambda < \lambda_1$. (*Warning:* our definition of λ_1 is the same as Pemantle's, but our λ_2 is his λ_a.)

Pemantle proved that weak survival can occur by obtaining upper bounds on λ_1 and lower bounds on λ_2 which are good enough to show that $\lambda_1 < \lambda_2$ if $d \geq 3$. Below we list his bounds for $2 \leq d \leq 5$. For comparison purposes, we also include the best bounds for $d = 1$ (in which case $\lambda_1 = \lambda_2$); see Grillenberger and Ziezold (1988) for the lower bound, and Liggett (1995) for the upper bound.

$$
\begin{aligned}
&d = 1, \qquad \lambda_1 \leq 1.942, \qquad \lambda_2 \geq 1.539,\\
&d = 2, \qquad \lambda_1 \leq 0.667, \qquad \lambda_2 \geq 0.561,\\
&d = 3, \qquad \lambda_1 \leq 0.391, \qquad \lambda_2 \geq 0.425,\\
&d = 4, \qquad \lambda_1 \leq 0.279, \qquad \lambda_2 \geq 0.354,\\
&d = 5, \qquad \lambda_1 \leq 0.218, \qquad \lambda_2 \geq 0.309.
\end{aligned} \tag{1.1}
$$

In this paper, we will prove the following result, which settles the remaining case (for homogeneous trees).

THEOREM 1.2. *Suppose that* $d = 2$. *Then*

(a) $\lambda_2 \geq 0.609$;
(b) $\lambda_1 \leq 0.605$.

Therefore, $\lambda_1 < \lambda_2$.

REMARK ADDED IN REVISION. Following the submission of this paper, and largely motivated by it, Stacey (1996) found a very elegant proof that $\lambda_1 < \lambda_2$ for the contact process on T_d, $d \geq 2$ (and on some inhomogeneous trees), which does not rely on finding bounds on the critical values.

We conclude the Introduction by making some remarks about the proof of Theorem 1.2. It is useful to begin by proving the easy bounds $\lambda_2 \geq 0.35$ and $\lambda_1 \leq 1$. For the first of these, order T_2 so that each vertex x has one ancestor and two descendants. (These are the three neighbors of x.) Then one can in a natural way assign a "generation number" $\gamma(x) \in Z$ to each $x \in T_2$, so that the ancestor of x has generation number $\gamma(x) - 1$, and its two descendants have generation number $\gamma(x) + 1$. Define a function on the finite subsets of A by

$$
f(A) = \sum_{x \in A} \rho^{\gamma(x)}, \tag{1.3}
$$

where ρ is a positive parameter to be determined. Then

$$
\begin{aligned}
h(A) &= \frac{d}{dt} E^A f(A_t)\bigg|_{t=0}\\
&= \sum_{x \in A} \left[\lambda \sum_{|y-x|=1,\ y \notin A} \rho^{\gamma(y)} - \rho^{\gamma(x)} \right]\\
&\leq \left[\lambda(2\rho + \rho^{-1}) - 1 \right] f(A).
\end{aligned}
$$

Choosing $\rho = 1/\sqrt{2}$, $\lambda = 1/(2\sqrt{2})$ gives $h(A) \le 0$. This implies that $f(A_t)$ is a positive supermartingale, which must converge a.s. It follows that A_t does not survive strongly, since, if it did, $f(A_t)$ would have to change by significant amounts at arbitrarily large times. We conclude that $\lambda_2 \ge 1/(2\sqrt{2})$ as desired.

This was essentially Pemantle's starting point as well. He then improved on this bound by modifying the definition of f to get the bound quoted in (1.1). What is not clear from his work is how one should "mechanize" this procedure to obtain improved lower bounds. We show how to do this in Section 2. The idea is to express f as the "Fourier transform" of a function g (which turns out to be greater than or equal to 0) which is 0 except for sets of small diameter. We then show how to choose the nonzero values of $g(B)$ in order to make $f(A_t)$ be a positive supermartingale. It turns out that the right choice is obtained by forcing $h(B)$ to be 0 for those B's for which $g(B) \ne 0$. This analysis is done in the context of a general graph. The computations on T_2 which lead to part (a) of Theorem 1.2 are carried out in Section 3.

In order to prove $\lambda_1 \le 1$, let f and h be defined by

$$\begin{aligned} f(A) &= \nu\{\eta\colon \eta(x) = 1 \text{ for some } x \in A\}, \\ h(A) &= \frac{d}{dt} E^A f(A_t)\bigg|_{t=0}, \end{aligned} \tag{1.4}$$

with ν taken to be the simplest probability measure possible—the product measure with $\nu\{\eta(x) = 1\} = \varepsilon$ for each x. Now we wish to choose ε (depending on λ) so that f is subharmonic (i.e., h is nonnegative). This implies survival, since otherwise $f(A_t) \to f(\varnothing) = 0$.

The form of f in (1.4) is motivated by the proof that the critical value of the contact process on Z^1 is at most 2. [See Liggett (1985), Chapter 6.] In that case, ν is taken to be a renewal measure. On Z^1, it turns out that for no λ is there an $\varepsilon > 0$ which makes $h \ge 0$, and it is for that reason that renewal measures were used instead of product measures in that context. On T_2, however, one can compute h as

$$h(A) = (1-\varepsilon)^{|A|-1}\varepsilon[\lambda(1-\varepsilon)N(A) - |A|],$$

where

$$N(A) = \#(x, y)\colon |x - y| = 1, \qquad x \in A,\ y \notin A.$$

It is easy to check that for a connected set A, $|N(A)| = |A| + 2$, and therefore that $|N(A)| \ge |A| + 2$ for all A. Hence $h(A) \ge 0$ for all A provided that $\lambda(1-\varepsilon) \ge 1$. It follows that the process survives for all $\lambda > 1$, and hence $\lambda_1 \le 1$. In Section 4, we will first carry out this argument with $\nu =$ the distribution of a two-state Markov chain (with T_2 viewed as "time"). This gives $\lambda_1 \le 0.637$. To prove part (b) of Theorem 1.2, it is then necessary to use the distribution of a Markov-type chain with a "memory" of length 2. The proof that h is nonnegative becomes increasingly difficult with increasing memory length.

When applied to the tree T_d, the above simple arguments yield

$$\lambda_2 \geq \frac{1}{2\sqrt{d}} \quad \text{and} \quad \lambda_1 \leq \frac{1}{d-1},$$

which imply that $\lambda_1 < \lambda_2$ for $d \geq 6$, as was shown by Pemantle. An interesting consequence is that λ_1 and λ_2 are of different orders of magnitude as $d \to \infty$.

One somewhat unusual feature of the proofs in Section 4 is what we will refer to as the "pyramid scheme" argument. Most readers are probably familiar with pyramid schemes which promise every player a profit. In the simplest version of the game, each player gets \$1 from each of two friends. After paying a dollar to his predecessor, he makes a net profit of \$1. This scheme works well provided there are an infinite number of players who are willing and able to play. The way in which this idea arises in Section 4 is the following. The hard part of the argument there is to show that $h(A) \geq 0$ for all finite $A \subset T_2$. This is not too hard to prove for connected A. For disconnected A, one might try to write

$$h(A) = \sum_B h_B(A) \tag{1.5}$$

over all connected components B of A, where $h_B(A)$ is the contribution to $h(A)$ due to B, and then try to prove $h_B(A) \geq 0$ for each B. Unfortunately, this last inequality is not in general true, so one must rely on some cancellation in (1.5). To organize this cancellation (see Figure 1—here and in other figures, vertices in A are shown as solid disks; vertices not in A are shown as points), let $\{z_1, z_2, \dots, z_m\}$ be the points not in B with a neighbor in B, and suppose that one could prove an inequality of the form

$$h_B(A) \geq \alpha\sigma(z_i) - \beta \sum_{j \neq i} \sigma(z_j) \tag{1.6}$$

for any i, where $\sigma(z_i)$ is some measure of how much of A lies away from B in the z_i direction, and α and β are positive constants. [In particular, $\sigma(z_i) = 0$ if there is no part of A in that direction.] The bound (1.6) would not appear to be useful, since there are many more negative terms than positive ones. However, suppose we now focus on a component C of A^c, and take $\{w_1, w_2, \dots, w_n\}$ to be the points in C with a neighbor in A (listed with multiplicity). Suppose that the σ's satisfy

$$\beta\sigma(w_i) \leq \alpha \sum_{j \neq i} \sigma(w_j) \tag{1.7}$$

for any i (and the same α and β), which again seems to be overly crude, since we are replacing one σ by the sum of many σ's. However, if one orders the tree as we did in the argument following the statement of Theorem 1.2, and chooses the distinguished i in both (1.6) and (1.7) to be the site with the smallest generation number, then one can use (1.5)–(1.7) to show that $h(A) \geq 0$. This involves a change in the order of summation, but there are

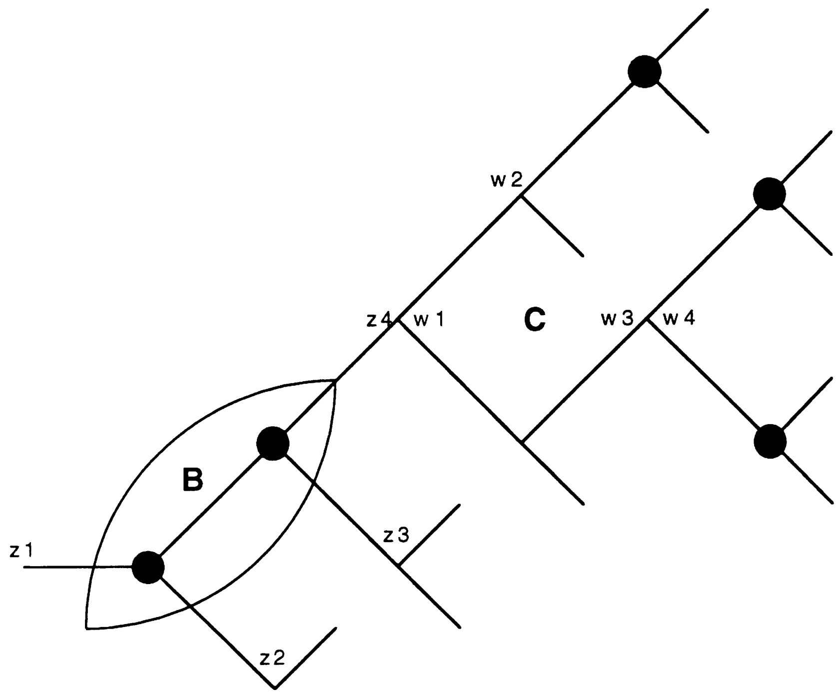

FIG. 1.

only finitely many nonzero summands, so no technical problems arise. Such is the magic of the tree.

2. Extinction results for general graphs. In this section, we prove some results for the contact process A_t on a general graph G which will be used in Section 3 to prove part (a) of Theorem 1.8. It should be clear how the process on G is defined—deaths occur at each site in G at rate 1, and births occur at a rate which is λ times the number of occupied neighbors. Take g to be any function defined for finite subsets of G such that $g(\emptyset) = 0$, and $g(B) \neq 0$ for only finitely many B's containing x for each $x \in G$. (The function g need not be nonnegative at this point, but in our applications, it always turns out to be.) Put

$$f(A) = \sum_{B \subset A} (-1)^{|B|+1} g(B), \tag{2.1}$$

which is a type of "Fourier" representation for f. There is a probabilistic interpretation of this. We will be using f as a sort of approximation to a harmonic function for A_t. If f is of the form

$$f(A) = P^A(x \in A_t \text{ for some } t)$$

for some x, for example, which is harmonic for A away from x, then the corresponding g can be described as follows. Recall that the contact process is additive, so that copies A_t^u of it can be constructed on the same probability space so that $A_0^u = \{u\}$ and $A_t = \bigcup_{u \in A} A_t^u$ is a version of the contact process starting at A. Then

$$f(A) = P(\exists\ u \in A,\ x \in A_t^u \text{ for some } t)$$

and the corresponding

$$g(A) = P(\forall\ u \in A,\ x \in A_t^u \text{ for some } t).$$

Define the function h as in (1.4):

$$h(A) = \frac{d}{dt} E^A f(A_t)\Big|_{t=0}. \tag{2.2}$$

Writing f in terms of g using (2.1), one easily computes

$$\begin{aligned} h(A) &= \sum_{x \in A} [f(A \setminus \{x\}) - f(A)] \\ &\quad + \lambda \sum_{x \notin A,\ y \in A,\ |x-y|=1} [f(A \cup \{x\}) - f(A)] \\ &= \sum_{B \subset A} (-1)^{|B|} |B| g(B) + \lambda \sum_{\substack{|x-y|=1 \\ x \notin A,\ y \in A}} \sum_{B \subset A} (-1)^{|B|} g(B \cup \{x\}). \end{aligned} \tag{2.3}$$

We wish to develop conditions under which one can conclude that $h(A) \le 0$ for all A [i.e., that $f(A_t)$ is a supermartingale]. Looking at (2.3), we see that one difficulty in determining the sign of $h(A)$ is the fact that some terms are multiplied by λ, and some are not. The first proposition rectifies this situation and introduces the key condition which makes it possible to determine the values of $g(B)$ to be used later: h should be 0 whenever g is not. Let S be a collection of finite subsets of G.

PROPOSITION 2.4. *Suppose that*

$$\begin{gathered} B \subset A,\ A \in S \quad \Rightarrow \quad B \in S, \\ h(A) = 0 \qquad \forall\ A \in S \quad \textit{and} \quad g(A) = 0 \qquad \forall\ A \notin S. \end{gathered} \tag{2.5}$$

Then

$$\begin{aligned} h(A) = \lambda \sum_{\substack{B \subset A,\ |x-y|=1 \\ y \in A \setminus B,\ B \cup \{y\} \notin S}} (-1)^{|B|} \big[& g(B) 1_{\{x \in B,\ B \cup \{y\} \setminus \{x\} \in S\}} \\ & + g(B \cup \{x\}) 1_{\{x \notin A\}} \big] \end{aligned} \tag{2.6}$$

for $A \notin S$.

PROOF. For $A \in S$, $h(A) = 0$ by assumption, so that (2.3) becomes

$$\sum_{B \subset A} (-1)^{|B|} |B| g(B) + \lambda \sum_{\substack{|x-y|=1 \\ x \notin A,\, y \in A}} \sum_{B \subset A} (-1)^{|B|} g(B \cup \{x\}) = 0. \tag{2.7}$$

We wish to solve these equations for the g's without a factor of λ in terms of those with this factor. To do so, fix a $C \in S$, multiply (2.7) by $(-1)^{|A|}$ and sum for $A \subset C$, which we can do since all such A are in S. In the computation, we will use the following orthogonality property: for any $B \subset A$,

$$\sum_{C:\, B \subset C \subset A} (-1)^{|C|} = \begin{cases} (-1)^{|A|}, & \text{if } A = B, \\ 0, & \text{otherwise,} \end{cases} \tag{2.8}$$

which is a consequence of the binomial theorem:

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0, \qquad n \geq 1.$$

The result of summing (2.7) with the appropriate factors is

$$\begin{aligned} 0 = &\sum_{B \subset A \subset C} (-1)^{|B|+|A|} |B| g(B) \\ &+ \lambda \sum_{B \subset C,\, |x-y|=1} (-1)^{|B|} g(B \cup \{x\}) \sum_{\substack{A:\, B \subset A \subset C \\ x \notin A,\, y \in A}} (-1)^{|A|}. \end{aligned} \tag{2.9}$$

Using (2.8), we see that

$$\begin{aligned} \sum_{\substack{A:\, B \subset A \subset C \\ x \notin A,\, y \in A}} (-1)^{|A|} &= \sum_{B \cup \{y\} \subset A \subset C \setminus \{x\}} (-1)^{|A|} \\ &= \begin{cases} (-1)^{|B \cup \{y\}|}, & \text{if } B \cup \{y\} = C \setminus \{x\}, \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

for fixed $B \subset C$ and $x, y \in G$. Therefore, (2.9) becomes

$$0 = |C| g(C) + \lambda \sum_{|x-y|=1} \left[g(C \cup \{x\}) - g(C \cup \{x\} \setminus \{y\}) \right] \tag{2.10}$$

for $C \in S$. To check that the factors involving λ are correct, for example, note that there is no contribution to either (2.9) or (2.10) unless $y \in C$, and in that case, if $B \cup \{y\} = C \setminus \{x\}$, then either $B = C \setminus \{x\}$ or $B = C \setminus \{x, y\}$, depending on whether or not $y \in B$. These two cases give rise to the terms $g(C \cup \{x\})$ and $g(C \cup \{x\} \setminus \{y\})$ in (2.10), respectively. Now take $A \notin S$ and use (2.10) in (2.3) to get

$$\begin{aligned} h(A) = &-\lambda \sum_{B \subset A,\, B \in S} (-1)^{|B|} \sum_{|x-y|=1} \left[g(B \cup \{x\}) - g(B \cup \{x\} \setminus \{y\}) \right] \\ &+ \lambda \sum_{\substack{|x-y|=1 \\ x \notin A,\, y \in A}} \sum_{B \subset A} (-1)^{|B|} g(B \cup \{x\}). \end{aligned}$$

In the first sum above, we can add the constraint $y \in A$, since otherwise the summand is 0. In the second sum, we can add the constraint $B \in S$, since otherwise the summand is 0 by (2.5). Therefore,

$$h(A) = \lambda \sum_{\substack{B \subset A,\, B \in S \\ |x-y|=1,\, y \in A}} (-1)^{|B|}\big[g(B \cup \{x\} \setminus \{y\}) - g(B \cup \{x\}) 1_A(x)\big]. \tag{2.11}$$

There is a lot of cancellation which is hidden in (2.11), and which we must remove. First, note that by writing the sum according to whether or not B contains x and then making the change of variables $C = B$ or $C = B \cup \{x\}$, respectively,

$$\begin{aligned} &\sum_{\substack{B \subset A,\, B \in S \\ |x-y|=1;\, x, y \in A}} (-1)^{|B|} g(B \cup \{x\}) \\ &\quad = \sum_{\substack{x \in C \subset A \\ |x-y|=1,\, y \in A}} (-1)^{|C|} g(C)\big[1_{\{C \in S\}} - 1_{\{C \setminus \{x\} \in S\}}\big] \\ &\quad = 0, \end{aligned}$$

since either $C \in S$, in which case $C \setminus \{x\} \in S$ by (2.5), or $C \notin S$, in which case $g(C) = 0$, again by (2.5). To handle the terms in (2.11) which involve $g(B \cup \{x\} \setminus \{y\})$, make the changes of variables

$$C = B, \qquad C = B \cup \{x\}, \qquad C = B \setminus \{y\} \quad \text{or} \quad C = B \cup \{x\} \setminus \{y\},$$

depending on which of x and y is in B (or both or neither). The result is that

$$\begin{aligned} h(A) &= \lambda \sum_{\substack{|x-y|=1,\, y \in A \\ x \in C,\, y \notin C}} (-1)^{|C|} g(C)\big[-1_{\{C \subset A,\, C \cup \{y\} \in S\}} + 1_{\{C \subset A,\, C \in S\}} \\ &\qquad\qquad + 1_{\{C \setminus \{x\} \subset A,\, C \cup \{y\} \setminus \{x\} \in S\}} - 1_{\{C \setminus \{x\} \subset A,\, C \setminus \{x\} \in S\}}\big] \\ &= \lambda \sum_{\substack{|x-y|=1,\, y \in A \\ x \in C,\, y \notin C}} (-1)^{|C|} g(C)\big[1_{\{C \subset A,\, C \cup \{y\} \notin S\}} - 1_{\{C \setminus \{x\} \subset A,\, C \cup \{y\} \setminus \{x\} \notin S\}}\big] \\ &= \lambda \sum_{\substack{|x-y|=1,\, y \in A \\ x \in C,\, y \notin C}} (-1)^{|C|} g(C)\big[1_{\{C \subset A,\, C \cup \{y\} \setminus \{x\} \in S,\, C \cup \{y\} \notin S\}} \\ &\qquad\qquad - 1_{\{C \setminus \{x\} \subset A,\, x \notin A,\, C \cup \{y\} \setminus \{x\} \notin S\}}\big]. \end{aligned}$$

In checking the second two equalities, repeated use of (2.5) is made. For example, $C \cup \{y\} \in S$ implies $C \in S$ by (2.5), and hence

$$1_{\{C \subset A,\, C \in S\}} - 1_{\{C \subset A,\, C \cup \{y\} \in S\}} = 1_{\{C \subset A,\, C \in S,\, C \cup \{y\} \notin S\}}.$$

Since $g(C) = 0$ for $C \notin S$ by (2.5), the constraint $C \in S$ can be omitted in going from the first line to the second. To go from the second line to the third, write

$$1_{\{C \subset A,\, C \cup \{y\} \notin S\}} = 1_{\{C \subset A,\, C \cup \{y\} \notin S,\, C \cup \{y\} \setminus \{x\} \in S\}} + 1_{\{C \subset A,\, C \cup \{y\} \notin S,\, C \cup \{y\} \setminus \{x\} \notin S\}}.$$

Since $C \cup \{y\} \setminus \{x\} \notin S$ implies $C \cup \{y\} \notin S$ by (2.5),

$$1_{\{C \subset A,\, C \cup \{y\} \notin S,\, C \cup \{y\} \setminus \{x\} \notin S\}} = 1_{\{C \subset A,\, C \cup \{y\} \setminus \{x\} \notin S\}}.$$

Then note that if $x \in A$, $C \subset A$ is equivalent to $C \setminus \{x\} \subset A$, while if $x \notin A$ (and $x \in C$), then C is automatically not a subset of A.

Finally, make the change of variables $B = C$ in the positive terms of the final expression for $h(A)$ above, and $B = C \setminus \{x\}$ in the negative terms, to get (2.6). □

Next we will impose some additional conditions on S and g.

COROLLARY 2.12. *Suppose that* (2.5) *and the following conditions hold*:

(2.13) *S contains all singletons and all nearest-neighbor pairs*,

$$\{x, y\} \in S \quad \forall\, x, y \in B \quad \Rightarrow \quad B \in S \tag{2.14}$$

and

$$\sum_{B \subset C,\, B \cup \{y\} \notin S} (-1)^{|B|} g(B \cup \{x\}) \le 0, \tag{2.15}$$

whenever x, y, C satisfy $x, y \notin C$, $|x - y| = 1$.

Then

$$h(A) \le 0 \quad \forall\, A. \tag{2.16}$$

PROOF. Since $h(A) = 0$ for $A \in S$, we may assume $A \notin S$. Use expression (2.6) for $h(A)$ given in Proposition 2.4. Fix x, y and A satisfying $|x - y| = 1$ and $y \in A$. If $x \in A$, the terms in (2.6) which must be considered correspond to B's which satisfy

$$x \in B \subset A \setminus \{y\}, \qquad B \cup \{y\} \setminus \{x\} \in S, \qquad B \cup \{y\} \notin S.$$

By (2.14), there is a $z \in B \cup \{y\}$ so that $\{x, z\} \notin S$. By (2.13), $z \ne y$, so $z \in B$, and hence $\{x, z\} \subset B$. Therefore, $B \notin S$ and $g(B) = 0$ by (2.5). Hence there is no contribution to (2.6) if $x \in A$. The sum of terms in (2.6) corresponding to $x \notin A$ is nonpositive by (2.15). (Let $C = A \setminus \{y\}$.) □

We now put these results together in the following form.

THEOREM 2.17. *Assume that* (2.5) *and* (2.13)–(2.15) *all hold*, *that $f(A) \ge 0$ for all A and that $g(A) \ne 0$ for $|A| = 1$. Then A_t does not survive strongly.*

PROOF. By Corollary 2.12, $f(A_t)$ is a nonnegative supermartingale, and hence converges with probability 1. Each time a transition occurs at site x, the value of this supermartingale changes by

$$\pm \sum_{x \in B \subset A_t \cup \{x\}} (-1)^{|B|} g(B). \tag{2.18}$$

Therefore, the expression in (2.18) tends to 0 a.s. for every $x \in G$. If A_t survives strongly, then, with positive probability, there will be infinitely many times at which $x \in A_t$ and $A_t \cap B = \{x\}$ for every $B \in S$ containing x. Since $g(\{x\}) \neq 0$, we reach a contradiction. □

3. On T_2, there is no strong survival for $\lambda \leq 0.609$. We will verify that the assumptions of Theorem 2.17 are satisfied in this case for an appropriate choice of S. Take $g(B) = 0$ if $\operatorname{diam}(B) > 2$. Define the generation number $\gamma(x)$ as we did following the statement of Theorem 1.2 in the Introduction. We will choose g to have the following form:

$$g(\{x\}) = \rho^{\gamma(x)}, \tag{3.1}$$

$$g(\{x,y\}) = \begin{cases} C_1\rho^{\gamma(x)}, & \text{if } |x-y| = 1,\ \gamma(x) < \gamma(y), \\ C_2\rho^{\gamma(x)+1}, & \text{if } |x-y| = 2,\ \gamma(x) < \gamma(y), \\ C_3\rho^{\gamma(x)-1}, & \text{if } |x-y| = 2,\ \gamma(x) = \gamma(y), \end{cases} \tag{3.2}$$

$$\begin{aligned} &g(\{x,y,z\}) \\ &\quad = \begin{cases} C_4\rho^{\gamma(x)+1}, & \text{if } |x-y| = |y-z| = 1,\ \gamma(x) < \gamma(y) < \gamma(z), \\ C_5\rho^{\gamma(x)}, & \text{if } |x-y| = |x-z| = 1,\ \gamma(x) < \gamma(y) = \gamma(z), \\ C_6\rho^{\gamma(x)+1}, & \text{if } |x-y| = |x-z| \\ & \qquad = |y-z| = 2,\ \gamma(x) < \gamma(y) = \gamma(z), \end{cases} \end{aligned} \tag{3.3}$$

$$g(\{x,y,z,w\}) = C_7\rho^{\gamma(w)} \quad \text{if } |x-w| = |y-w| = |z-w| = 1. \tag{3.4}$$

Using (2.3), it is not hard to write down explicitly the equations giving $h(A) = 0$ for $\operatorname{diam}(A) \leq 2$. They are given below, in the order in which the sets appear in (3.1)–(3.4), and are labeled in a manner consistent with the indexes on the constant C_1–C_7.

$$0 = -\rho + 2\lambda\rho^2 + \lambda - C_1\lambda(2\rho+1), \tag{3.5.0}$$

$$\begin{aligned} 0 = {} & (\rho+1)(-\rho+\lambda-\lambda\rho+2\lambda\rho^2) \\ & -C_1(\lambda+\lambda\rho+2\lambda\rho^2-2\rho) \\ & +(C_4-C_2)\lambda\rho(2\rho+1)+(C_5-C_3)\lambda\rho, \end{aligned} \tag{3.5.1}$$

$$\begin{aligned} 0 = {} & (\rho^2+1)(-\rho+2\lambda\rho^2+\lambda) - C_1\lambda(1+3\rho+2\rho^2+2\rho^3) \\ & +2C_2\rho^2+2C_4\lambda\rho^2, \end{aligned} \tag{3.5.2}$$

$$0 = -\rho+\lambda+2\lambda\rho^2-2C_1\lambda(1+\rho)+C_3+C_5\lambda, \tag{3.5.3}$$

$$\begin{aligned} 0 = {} & \rho^3-\rho^2-\rho+\lambda \\ & +(C_4-C_2-C_1+\rho)\rho(2\lambda\rho^2+\lambda\rho+\lambda-2\rho)+C_1(2\rho-\lambda) \\ & -C_4\rho^2+(C_6-C_7)\lambda\rho^2+(C_5-C_3)\lambda\rho(1+\rho), \end{aligned} \tag{3.5.4}$$

$$0 = -\rho - 2\rho^2 + \lambda + 4\lambda\rho^3 + C_1(4\rho - \lambda - 4\lambda\rho^2) + 2(C_4 - C_2)\lambda\rho(2\rho + 1) + 2C_3\rho - 3C_5\rho + (C_6 - C_7)\lambda\rho, \tag{3.5.5}$$

$$0 = (1 + 2\rho^2)(\lambda + 2\lambda\rho^2 - \rho) - C_1\lambda(1 + 2\rho)(1 + 2\rho + 2\rho^2) + (4C_2 + 2C_3 - 3C_6)\rho^2 + 3(2C_4 + C_5 - C_7)\lambda\rho^2, \tag{3.5.6}$$

$$\begin{aligned} 0 = {} & -\rho^2 - \rho - 2\rho^3 + \lambda + \lambda\rho^2 + 4\lambda\rho^4 \\ & + C_1(1 + 2\rho)(2\rho - \lambda + \lambda\rho - 2\lambda\rho^2) \\ & + C_2\rho(4\rho - \lambda - 4\lambda\rho^2) \\ & + C_3\rho(2\rho - \lambda) + C_4\rho(\lambda + 4\lambda\rho^2 - 6\rho) \\ & - C_5\rho(3\rho - \lambda) - 3C_6\rho^2 + 4C_7\rho^2. \end{aligned} \tag{3.5.7}$$

Before considering (3.5.0)–(3.5.7) in full generality, let us see what happens if we choose S to be a subcollection of the sets of diameter less than or equal to 2. The computations below were carried out with the help of Mathematica. In each case, we use the given equations to eliminate the C_i's (which appear in the equations linearly), leaving a polynomial equation in λ and ρ:

$$P(\lambda, \rho) = 0. \tag{3.6}$$

One then finds that there are solutions $\rho \in (0, 1)$ to (3.6) for small λ, but not for larger λ. The largest $\lambda \in (0, 1)$ for which there is a solution is identified by the fact that for the largest λ the corresponding ρ is a double root of (3.6); that is, it satisfies

$$P(\lambda, \rho) = 0 \quad \text{and} \quad \frac{\partial}{\partial\rho} P(\lambda, \rho) = 0.$$

The ρ can be eliminated from these two equations, leaving a polynomial equation for λ.

Here are the natural choices which satisfy (2.5), (2.13) and (2.14):

(a) $S = \{A\colon \operatorname{diam}(A) \le 1\}$. This corresponds to solving (3.5.0) and (3.5.1) for ρ and C_1 with $C_2 = C_3 = C_4 = C_5 = C_6 = C_7 = 0$. Eliminating C_1 leads to

$$0 = \lambda(\lambda + 2) - (2 + 2\lambda + 3\lambda^2)\rho + 2\lambda(\lambda + 2)\rho^2.$$

For $\lambda \in (0, 1)$, this has real roots if and only if $\lambda \le 0.561722\ldots$ (a root of $\lambda^4 - 20\lambda^3 - 16\lambda^2 + 8\lambda + 4$). At this value of λ, one gets $\rho = 1/\sqrt{2}$ and $C_1 = 0.307007\ldots$. Thus we will be able to conclude from Theorem 2.17 that $\lambda_2 \ge 0.561722\ldots$, which is Pemantle's result.

(b) Solve (3.5.0), (3.5.1), (3.5.3) and (3.5.5) for ρ, C_1, C_3 and C_5. There is a solution to these equations up to $\lambda = 0.571514\ldots$, a root of

$$\lambda^8 - 8\lambda^7 - 62\lambda^6 - 154\lambda^5 - 180\lambda^4 - 76\lambda^3 + 37\lambda^2 + 42\lambda + 9.$$

At this value of λ, the solution is given by

$$\rho = \frac{1}{\sqrt{2}}, \qquad C_1 = 0.315941\dots, \qquad C_3 = 0.125955\dots, \qquad C_5 = 0.095553\dots.$$

(c) Solve (3.5.0), (3.5.1), (3.5.2) and (3.5.4) for ρ, C_1, C_2 and C_4. There is a solution to these equations up to $\lambda = 0.595237\dots$, a root of

$$16\lambda^8 + 128\lambda^7 - 288\lambda^6 - 1760\lambda^5 - 2679\lambda^4 - 1272\lambda^3 + 568\lambda^2 + 672\lambda + 144.$$

At this value of λ, the solution is given by

$$\rho = \frac{1}{\sqrt{2}}, \qquad C_1 = 0.336366\dots, \qquad C_2 = 0.166312\dots, \qquad C_4 = 0.126624\dots.$$

(d) Finally, solve the full system (3.5.0)–(3.5.7) for ρ and C_1–C_7. There is a solution to these equations up to $\lambda = 0.609152\dots$, a root of

$$\begin{aligned}1296\lambda^{14} &+ 14688\lambda^{13} + 56896\lambda^{12} + 12704\lambda^{11} - 664087\lambda^{10}\\ &- 2821858\lambda^9 - 6365933\lambda^8 - 9107526\lambda^7 - 8302491\lambda^6 - 4062131\lambda^5\\ &+ 245628\lambda^4 + 1857600\lambda^3 + 1263600\lambda^2 + 388800\lambda + 46656.\end{aligned}$$

At this value of λ, the solution is given by

$$\begin{aligned}\rho = \frac{1}{\sqrt{2}}, \qquad C_1 = 0.347606\dots, \qquad & C_2 = 0.173973\dots, \qquad C_3 = 0.144034\dots,\\ C_4 = 0.133999\dots, \qquad & C_5 = 0.111157\dots,\\ C_6 = 0.0511674\dots, \qquad & C_7 = 0.42387\dots.\end{aligned}$$

It now remains to check the assumptions of Theorem 2.17. Assumptions (2.5), (2.13) and (2.14) are satisfied by construction. It remains for us to check that f is nonnegative and that g satisfies (2.15). For the nonnegativity of f, note that since all of the C_i's are nonnegative, the only negative terms in (2.1) correspond to B's which have cardinality 2 or 4. If $B \subset A$, $|B| = 4$, contributes a term $-C_7\rho^n$, then B has a unique subset of cardinality 3 which contributes a term $+C_6\rho^n$. Since $C_7 \le C_6$, the overall contribution of these sets is nonnegative. In a similar way, we must associate parts of the contributions of singletons in A with the contributions of the doubletons which contain them. Associate the following amounts of the term corresponding to $B = \{x\}$ to the terms corresponding to $B = \{x, y\}$:

$$\begin{aligned}&C_1\rho^{\gamma(x)-1} && \text{if } |x-y| = 1,\ \gamma(y) < \gamma(x),\\ &C_2\rho^{\gamma(x)-1} && \text{if } |x-y| = 2,\ \gamma(y) < \gamma(x),\\ &C_3\rho^{\gamma(x)-1}/2 && \text{if } |x-y| = 2,\ \gamma(y) = \gamma(x).\end{aligned}$$

This is possible if

$$C_1 + C_2 + \frac{C_3}{2} \le \rho. \tag{3.7}$$

Note that (3.7) holds in all four cases (a)–(d). Thus we conclude that $f \ge 0$ in all four cases.

To check that g satisfies (2.15), take $|x - y| = 1$, and let w_1, w_2 be the neighbors of x other than y, let u_1, u_2 be the neighbors of w_1 other than x and let u_3, u_4 be the neighbors of w_2 other than x. (See Figure 2.) Choose the labels so that $\gamma(w_1) > \gamma(x)$ and $\gamma(u_4) > \gamma(w_2)$. (See Figure 2.) Then the following are the sets B which can occur in the sum in (2.15) in the first three cases (i.e., $B \cup \{x\} \in S$, $B \cup \{y\} \notin S$):

(a) $$B \subset \{w_1, w_2\};$$

(b) $B \subset \{u_4, w_2\}$ if $\gamma(x) < \gamma(y)$; $B \subset \{w_1, w_2\}$ if $\gamma(x) > \gamma(y)$;

(c) $$B \subset \{u_1, u_2, u_3, w_1\} \quad \text{if } \gamma(x) < \gamma(y);$$
$$B \subset \{u_1, u_2, u_3, u_4\} \quad \text{if } \gamma(x) > \gamma(y).$$

In case (a), all terms in (2.15) are less than or equal to 0. In case (b), the only possible positive terms are of the form $C_5 \rho^n$, and they always appear with a term of the form $-C_1 \rho^n$. In case (c), the only possible positive terms are of the form $C_4 \rho^n$, and they always appear with a term of the form $-C_2 \rho^n$. Thus all contributions are nonpositive.

We now consider case (d) in more detail. Use the same labels as before, but without requiring the convention involving the γ's. Any set B which occurs in (2.15) must be a subset of $\{u_1, u_2, u_3, u_4, w_1, w_2\}$. It must contain one of the u_i's, since otherwise adding y would not make the diameter greater than 2; it cannot contain one of $\{u_1, u_2\}$ and also one of $\{u_3, u_4\}$, since otherwise its diameter would be greater than 2. Also, $w_1 \in B$ implies $u_3, u_4, w_2 \notin B$ and $w_2 \in B$ implies $u_1, u_2, w_1 \notin B$ for the same reason. Therefore, the following

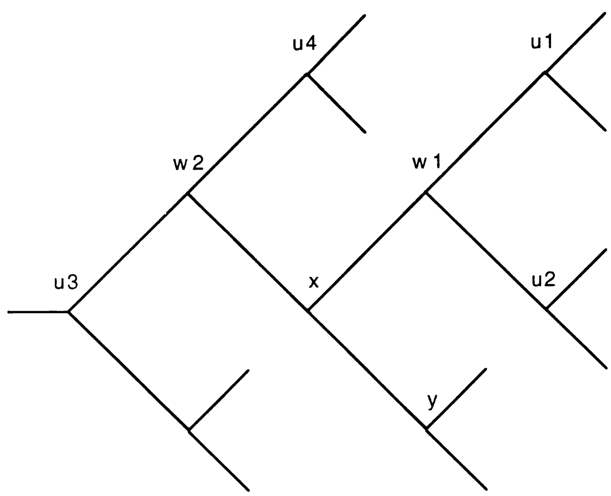

FIG. 2.

is a complete list of the B's which can occur in (2.15):

$$\{u_1\},\{u_2\},\{u_3\},\{u_4\},\{u_1,u_2\},\{u_3,u_4\},$$
$$\{u_1,w_1\},\{u_2,w_1\},\{u_1,u_2,w_1\},\{u_3,w_2\},\{u_4,w_2\},\{u_3,u_4,w_2\}.$$

We can consider separately the contributions to the sum in (2.15) from those B's involving $\{u_1, u_2, w_1\}$ and $\{u_3, u_4, w_2\}$, since there is no B which involves some of each. (We omitted the convention involving the γ's in order to be able to treat these two cases symmetrically.)

So, we need to consider only the following B's:

$$\{u_1\},\{u_2\},\{u_1,u_2\},\{u_1,w_1\},\{u_2,w_1\},\{u_1,u_2,w_1\}.$$

Each of these contributions to (2.15) has a factor of $\rho^{\gamma(w_1)}$, which we ignore below. What remains is the following:

$$\begin{aligned} &C_6 1_{\{u_1,u_2\in C\}} - C_7 1_{\{u_1,u_2,w_1\in C\}} + C_4\sum_{i=1}^{2} 1_{\{u_i,w_1\in C,\,\gamma(u_i)\neq\gamma(x)\}}\\ &\quad + C_5\sum_{i=1}^{2} 1_{\{u_i,w_1\in C,\,\gamma(u_i)=\gamma(x)\}} - C_2\sum_{i=1}^{2} 1_{\{u_i\in C,\,\gamma(u_i)\neq\gamma(x)\}}\\ &\quad - C_3\sum_{i=1}^{2} 1_{\{u_i\in C,\,\gamma(u_i)=\gamma(x)\}}. \end{aligned}$$

All that is relevant about C is its intersection with $\{u_1, u_2, w_1\}$, so we can assume that $C \subset \{u_1, u_2, w_1\}$. If $|C| \le 1$, all the contributions to (2.15) are of the form $-C_2$ or $-C_3$. If $C = \{u_1, u_2\}$, one gets $C_6 - C_2 - C_3$ or $C_6 - 2C_2$. If $C = \{u_i, w_1\}$, one gets $C_4 - C_2$, $C_4 - C_3$, $C_5 - C_2$ or $C_5 - C_3$. The only other case is $C = \{u_1, u_2, w_1\}$. Then one gets $C_6 - C_7 + C_4 - C_2 + C_5 - C_3$ or $C_6 - C_7 + 2C_4 - 2C_2$. In all these cases, the contribution is nonpositive.

Therefore, (2.15) is satisfied in all these cases, and the proof is complete. Applying this argument in each of the four cases gives the successive bounds

$$\lambda_2 \ge 0.561722\ldots,\ \lambda_2 \ge 0.571514\ldots,$$
$$\lambda_2 \ge 0.595237\ldots \text{ and } \lambda_2 \ge 0.609152\ldots.$$

REMARK. The observant reader will have noticed that in each of the four cases handled above, the optimal ρ turns out to be $1/\sqrt{2}$. This is presumably not an accident. The author is currently investigating the implications of this observation for contact processes (and other growth models) on trees [see Liggett (1997)].

4. On T_2, there is survival for $\lambda \ge 0.605$. In this section, we will develop a technique for showing that contact processes on trees survive for appropriate values of λ. This technique can be viewed as an adaptation to trees of the Holley–Liggett (1978) approach. In order to simplify matters, we will restrict our attention to T_2, which, in view of Pemantle's results, is the tree of greatest interest in this context. The idea is now to find nontrivial semiharmonic functions f for A_t. Unlike the f's used in the previous section, these will be homogeneous on the tree. Earlier work suggests two forms which one might expect good choices of f to take:

(a) f is of the form (2.1), where $g(B) = 0$ except for small sets B. This is the form which Pemantle used. If $g(B) = 0$ for all sets except singletons [i.e., $f(A) = |A|$], one gets the bound $\lambda_1 \le 1$. Pemantle took $g(B) = 0$ for all sets except singletons and nearest-neighbor pairs to get $\lambda_1 \le \frac{2}{3}$.

(b) f is of the form

$$f(A) = 1 - \nu\{\eta: \eta(x) = 0 \;\forall\; x \in A\}, \tag{4.1}$$

where ν is an appropriately chosen homogeneous probability measure on $\{0, 1\}^{T_2}$. Support for this choice comes from two (related) directions. First, this is the one (with ν a renewal measure) used by Holley and Liggett [see Liggett (1985)] to prove that $\lambda_c \le 2$ for the contact process on Z^1. Second, duality implies that there is a nontrivial subharmonic (in fact, harmonic) function of the form (4.1) (with ν the upper invariant measure of the infinite system) if $\lambda > \lambda_1$. So, one can imagine that choosing f of the form (4.1) for an appropriately chosen substitute for the upper invariant measure might yield a subharmonic f.

In the Introduction, we used f of the form (4.1), with ν taken as a product measure of density ε, and found that the corresponding h is nonnegative for $\lambda(1 - \varepsilon) \ge 1$. Note that we could divide f (and hence h) by ε and let $\varepsilon \to 0$ in that computation, and conclude that survival occurs for $\lambda = 1$ as well. Effectively, this would mean that $f(A) = |A|$, and we would be back to case (a) above. The main point of this observation is not so much to prove survival for $\lambda = 1$ (as opposed to $\lambda > 1$), but to suggest a simplification which will be important in the sequel when we consider more complex ν in order to get better bounds on λ_1.

Next, we formalize these ideas a bit. In order to do so, we need some notation. For $y \ne x$, let $S_y(x)$ be the component of $T_2 \setminus \{x\}$ which contains y. (See Figure 3.) We will consider only measures ν which have the following two properties:

(a) *Renewal property:* Conditional on $\eta(x) = 1$, the collections of random variables $\{\eta(z),\ z \in S_{y_1}(x)\}$, $\{\eta(z),\ z \in S_{y_2}(x)\}$ and $\{\eta(z),\ z \in S_{y_3}(x)\}$ are independent, where y_1, y_2, y_3 are the three neighbors of x.

(b) *Dependence of range n:* If $A, B \subset T_2$ are separated in the sense that $x \in A$, $y \in B$ implies $|x - y| > n$, then the collections $\{\eta(x),\ x \in A\}$ and $\{\eta(x),\ x \in B\}$ are conditionally independent, given $\{\eta(x),\ x \notin A \cup B\}$.

For a fixed n, the idea is to take a natural family of homogeneous measures ν_ε which have the renewal and dependence of range n properties and which tend to the point mass on $\eta \equiv 0$ as $\varepsilon \to 0$. Then we will let

$$f(A) = \lim_{\varepsilon \to 0} \frac{1 - \nu_\varepsilon\{\eta = 0 \text{ on } A\}}{\nu_\varepsilon\{\eta(x) = 1\}},$$

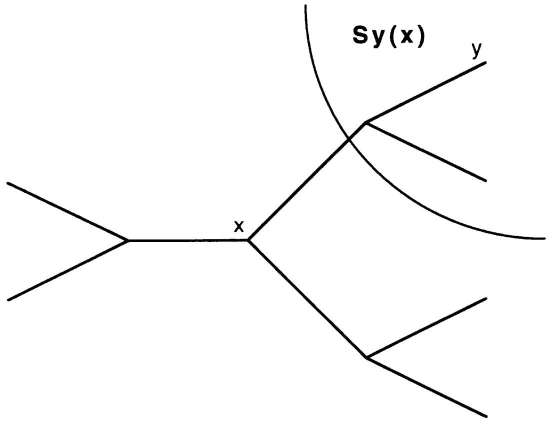

FIG. 3.

so that $h(A)$ will be (the limiting value of)

$$\lambda \sum_{x \in A,\, y \notin A,\, |x-y|=1} \nu\{\eta = 0 \text{ on } A \mid \eta(y) = 1\} - \sum_{x \in A} \nu\{\eta = 0 \text{ on } A \setminus \{x\} \mid \eta(x) = 1\}. \tag{4.2}$$

[See the first line of (2.3).] In each case we consider, it will be clear how to compute the conditional probabilities which arise, even though ν itself is the point mass on $\eta \equiv 0$. For small values of n, we will show that $h(A) \geq 0$ for all finite $A \subset T_2$ for a particular value λ and a particular choice of conditional probabilities. In each case, the conditional probabilities will satisfy

$$\nu\{\eta = 0 \text{ on } A \mid \eta(x) = 1\} \geq C > 0, \qquad x \notin A,$$

independently of A so that

$$C \leq f(A \cup \{x\}) - f(A) \leq 1 \quad \text{for } x \notin A, \qquad C|A| \leq f(A) \leq |A|.$$

Once we prove that $h(A) \geq 0$ for all finite $A \subset T_2$, it will follow that A_t survives, at least for a slightly larger value of λ than the one for which we will have proved these inequalities.

There are at least two ways of arriving at this conclusion. The first goes back to first principles, while the second uses a recent result of Morrow, Schinazi and Zhang (1994). Here is the general fact that we will use for the first approach [which is a more formal version of an argument used by Pemantle (1992)].

FACT. Suppose X_t is a pure jump process on a countable set with transition rates $q(x, y)$, $x \neq y$, and set $q(x) = \sum_{y:\, y \neq x} q(x, y)$. Suppose f satisfies the following properties for some $\varepsilon > 0$, $M < \infty$:

(i) $\sum_y q(x, y)[f(y) - f(x)] \geq \varepsilon f(x) \ \forall\ x$;
(ii) $|f(y) - f(x)| \leq M$ whenever $q(x, y) > 0$;
(iii) $q(x) \leq M f(x) \ \forall\ x$.

Let τ be the hitting time of the set $\{x: f(x) \le c\}$ for some large fixed c. Then $P^x(\tau < \infty) < 1$ provided that $f(x)$ is sufficiently large.

PROOF. To prove this fact, note that, by (i),

$$\sum_y q(x,y)\left[\frac{1}{f(y)} - \frac{1}{f(x)}\right] \le 0,$$

provided that

$$(*) \qquad \sum_{y,z \ne x} q(x,y)q(x,z)\left[\frac{f(z)}{f(y)} + \frac{f(y)}{f(z)} - 2\right] \le 2\varepsilon q(x).$$

Rewriting the left-hand side of $(*)$ and using (ii) and the Schwarz inequality, we get the following bound for it:

$$\sum_{y,z\ne x} \frac{q(x,y)q(x,z)}{f(y)f(z)}[f(z)-f(y)]^2 \le 4M^2\left[\sum_{y\ne x}\frac{q(x,y)}{f(y)}\right]^2 \le 4M^2 q(x)\sum_{y\ne x}\frac{q(x,y)}{f^2(y)}.$$

Using (iii), we see that $(*)$ is satisfied when f is sufficiently large. Therefore, $[f(X_t)]^{-1}$ is a supermartingale when f is sufficiently large. Applying the stopping time theorem, it follows that $P^x(\tau < \infty) < 1$ provided that $f(x)$ is sufficiently large. □

To apply this to the contact process on the tree, we use the f defined above (4.2). Once we have proved the inequalities following (4.2) for a given λ, it follows that (i), (ii) and (iii) hold for any strictly larger value of λ, and hence $|A_t|$ survives. The alternative argument for survival uses (i) only, but also uses the fact proved by Morrow, Schinazi and Zhang (1994) that, at λ_1, $E^{\{x\}}|A_t|$ is bounded in t. Since (i) above implies that $E^{\{x\}}|A_t|$ grows exponentially rapidly, it follows that any λ for which (i) holds is an upper bound for λ_1.

We now turn to the analysis of the various cases. We have already considered the easy case $n = 0$ in the Introduction. The measures with dependence of range 0 are just the product measures, so that, if ν_ε = product measure with density ε,

$$f(A) = \lim_{\varepsilon\to 0}\frac{1-(1-\varepsilon)^{|A|}}{\varepsilon} = |A|$$

and

$$h(A) = \lambda(|A| + 2(\#\text{ components of } A)) - |A|,$$

which is nonnegative for all A provided that $\lambda \ge 1$.

The case $n = 1$ is somewhat harder. The measures with dependence of range 1 have the property that $\{\eta(x)\}$, for x in a subset of T_2 which is isomorphic to the integers, is a stationary two-state Markov chain. One can then construct the $\eta(x)$ for other x's by using the same Markov mechanism. More explicitly, fix a parameter $q \in (0,1)$, and let ν_ε have the conditional probabilities

$$\nu_\varepsilon\{\eta(y) = 0 | \eta(x) = 0\} = 1 - \varepsilon, \qquad \nu_\varepsilon\{\eta(y) = 0 | \eta(x) = 1\} = q$$

for $|x - y| = 1$, and then let $\varepsilon \to 0$ as explained earlier. The resulting conditional distributions can be described in the following way if $q > 0.5$ (which will be true in the case of interest): given $\eta(x) = 1$, the set $\{y: \eta(y) = 1\}$ is a finite connected set containing x, and any such set B occurs with probability $(1-q)^{|B|-1} q^{|B|+2}$. In this case, it is not hard to check that f is of the form (2.1) with $g(B) = (1-q)^{m-1}$, $B \neq \emptyset$, where m is the cardinality of the smallest connected set containing B. Note that $g(B)$ does not have the property that it is 0 except for small sets.

Fix a finite set $A \subset T_2$, and let B be a component of A. In order to show $h \geq 0$, we need to compute the contributions to (4.2) attributable to B (i.e., the terms corresponding to $x \in B$). For $x \in B$, $y \notin B$, $|x - y| = 1$, let

$$\delta(y) = 1 - \nu\big\{\eta = 0 \text{ on } A \cap S_y(x) | \eta(y) = 1\big\}.$$

This is a measure of how much A there is in $S_y(x)$. As explained in the Introduction, we will not be able to show that the contributions for each B are nonnegative, but rather will have to rely on tradeoffs between the contributions from different components. In order to keep track of these tradeoffs, we will use the pyramid scheme described in the Introduction: the contributions from each B will be bounded below by a sum of constant multiples of the $\delta(y)$'s, where the constant is positive for one y and negative for the others.

First, consider a singleton component $B = \{x\}$, and let x_1, x_2, x_3 be the three neighbors of x. (See Figure 4.) Abbreviate $\delta(x_i) = \delta_i$. Then the contributions to (4.2) attributable to B are

$$\begin{aligned} &\lambda \sum_{i=1}^{3} \nu\{\eta = 0 \text{ on } A | \eta(x_i) = 1\} - \nu\{\eta = 0 \text{ on } A \setminus \{x\} | \eta(x) = 1\} \\ &\qquad = \lambda q \sum_{i=1}^{3} \nu\big\{\eta = 0 \text{ on } A \cap S_{x_i}(x) | \eta(x_i) = 1\big\} \\ &\qquad\quad - \prod_{i=1}^{3} \nu\big\{\eta = 0 \text{ on } A \cap S_{x_i}(x) | \eta(x) = 1\big\} \\ &\qquad = \lambda q \sum_{i=1}^{3} [1 - \delta_i] - \prod_{i=1}^{3} [1 - (1-q)\delta_i]. \end{aligned} \tag{4.3}$$

Note that this is not nonnegative if the δ's are close to 1. We want to bound this below by an expression of the form $c(\delta_1 - \delta_2 - \delta_3)$. To do so, note that, when expanded, (4.3) is an expression in which each δ_i occurs to at most the first power. Therefore, in order to show the needed inequality for $0 \leq \delta_i \leq 1$ for each i, it is sufficient to check it at each corner of the unit cube $[0,1]^3$. The

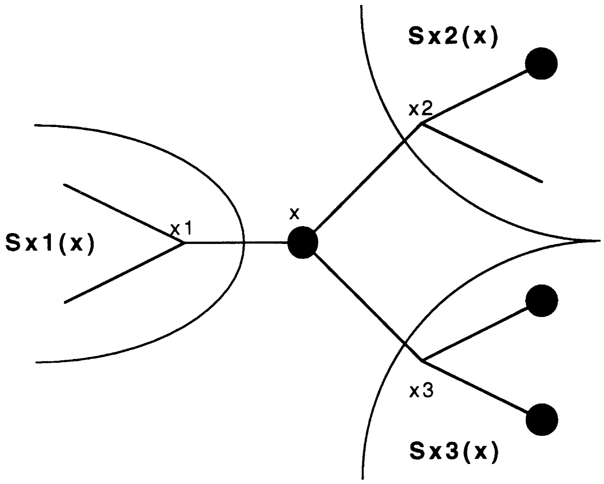

FIG. 4.

eight inequalities which must be satisfied reduce easily to the following four:

$$3\lambda q \geq 1, \qquad q(2\lambda - 1) \geq c, \qquad \lambda \geq q, \qquad q^3 \leq c.$$

Combining the second and fourth gives $q^2 \leq 2\lambda - 1$. Thus the best choice is obtained by setting

$$3\lambda q = 1 \quad \text{and} \quad 2\lambda = 1 + q^2, \tag{4.4}$$

which we do from now on. Mathematica gives the solution as $\lambda = 0.6369\ldots$ and $q = 0.5233\ldots$. Then $c = q(2\lambda - 1) = q^3$. It follows that (4.3) is bounded below by

$$q^3(\delta_1 - \delta_2 - \delta_3). \tag{4.5}$$

This is the bound we needed. (The other two which are needed to use the pyramid scheme follow by symmetry.)

Next, consider the case $|B| > 1$. We need to classify the points in B according to the number of neighbors which are in B. Let $k \geq 0$ be the number of points x in B such that all three neighbors of x are in B, and let $l \geq 0$ be the number of points with exactly two neighbors in B. Since B is connected, there are $k + 2$ points with exactly one neighbor in B. This is easily proved by induction on $|B|$. Let $y_1, y_2, \ldots, y_l$ be the points $\notin B$ with a neighbor in B whose other two neighbors are in B, and let $x_1, x_1^*, x_2, x_2^*, \ldots, x_{k+2}, x_{k+2}^*$ be the points not in B with a neighbor in B which has only one neighbor in B. (The points x_i and x_i^* have a common neighbor in B.) Then the contributions to (4.2) which are attributable to B are

$$\begin{aligned}\lambda q \sum_{i=1}^{l} [1-\delta(y_i)] + \lambda q \sum_{i=1}^{k+2} [2-\delta(x_i)-\delta(x_i^*)] - kq^3 \\ - q^2 \sum_{i=1}^{l} [1-(1-q)\delta(y_i)] \\ - q \sum_{i=1}^{k+2} [1-(1-q)\delta(x_i)][1-(1-q)\delta(x_i^*)].\end{aligned} \tag{4.6}$$

Algebraic manipulations give the following equivalent form for (4.6):

$$\begin{aligned}2q^3 - q^3 \sum_{i=1}^{l} \delta(y_i) - q^3 \sum_{i=1}^{k+2} [\delta(x_i)+\delta(x_i^*)] + q(\lambda-q) \sum_{i=1}^{l} [1-\delta(y_i)] \\ + q \sum_{i=1}^{k+2} \big[(2\lambda-1-q^2)[1-\delta(x_i)\delta(x_i^*)] \\ +(q^2+1-\lambda-q)[\delta(x_i)+\delta(x_i^*)-2\delta(x_i)\delta(x_i^*)]\big].\end{aligned} \tag{4.7}$$

Using (4.4), it follows that the contributions to (4.2) which are attributable to B are greater than or equal to

$$2q^3 - q^3 \sum_{i=1}^{l} \delta(y_i) - q^3 \sum_{i=1}^{k+2} [\delta(x_i)+\delta(x_i^*)].$$

Combining this with (4.5), we see that, for any component B of A, the contributions to (4.2) which are attributable to B are greater than or equal to

$$q^3 \left[\delta(z_i) - \sum_{\substack{1 \le j \le m \\ j \ne i}} \delta(z_j) \right] \tag{4.8}$$

for any $1 \le i \le m$, where $\{z_1, \dots, z_m\}$ is the set of all points not in B with a neighbor in B.

So far, we have concentrated on a given component of A. Now, we will focus on a given component C of its complement, A^c. (See Figure 5.) Let $(x_1, y_1), (x_2, y_2), \dots, (x_m, y_m)$ be all the pairs for which $|x-y| = 1$, $x \in A$, $y \in C$. Note that the y_i's need not be distinct, but that the x_i's are. Our earlier definition of $\delta(y)$ depended on which component of A it was viewed as being a neighbor of (in case it was a neighbor of more than one component). When we use $\delta(y_i)$ below, the relevant component is the one to which the corresponding x_i belongs. Using the translation invariance of the conditional probabilities and the fact that, conditional on $\{\eta(x) = 1\}$, the set of sites y

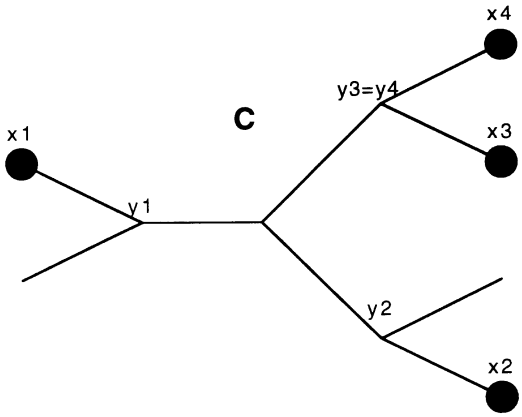

FIG. 5.

with $\eta(y) = 1$ is connected, we can write, for any $1 \le i \le m$,

$$\begin{aligned}\delta(y_i) &= \nu\{\eta\colon \eta(x_j) = 1 \text{ for some } 1 \le j \le m,\, j \ne i \mid \eta(y_i) = 1\}\\ &\le \sum_{\substack{1 \le j \le m\\ j \ne i}} \nu\{\eta\colon \eta(x_j) = 1 \mid \eta(y_i) = 1\}\\ &= \sum_{\substack{1 \le j \le m\\ j \ne i}} \nu\{\eta\colon \eta(x_i) = 1 \mid \eta(y_j) = 1\} \le \sum_{\substack{1 \le j \le m\\ j \ne i}} \delta(y_j).\end{aligned}$$

Rewrite this as

$$\sum_{\substack{1 \le j \le m\\ j \ne i}} \delta(y_j) - \delta(y_i) \ge 0. \tag{4.9}$$

It now remains to combine (4.8) and (4.9) to show that (4.2) is nonnegative for any A. To do so, order T_2 so that each point has one ancestor and two descendants. Let $B_1, \dots, B_m$ be the components of A, and let $z_{i,0}, \dots, z_{i,l_i}$ be the points in A^c with a neighbor in B_i. There is exactly one of these which is the ancestor of its neighbor in B_i. Call it $z_{i,0}$. By (4.8),

$$h(A) \ge q^3 \sum_{i=1}^{m} \left[\delta(z_{i,0}) - \sum_{j=1}^{l_i} \delta(z_{i,j})\right].$$

Rearranging this as a sum over components of A^c and using (4.9) (with the i which appears there being the one with smallest generation number), we see that $h(A) \ge 0$.

Thus we have proved that $\lambda_1 \le 0.637$. This is a significant improvement on Pemantle's bound of $\frac{2}{3}$, but it is still not good enough to combine with the results of Section 3 to conclude that $\lambda_1 < \lambda_2$. Since this is the main point of this paper, we must go on to the next step.

From now on, we take $n = 2$. We will pass immediately to describing the limiting conditional probabilities of a family of measures with the renewal property and dependence of range 2. There are three parameters, q_0, q_1, q_2, satisfying

$$0 \le q_0 \le q_1, \qquad q_2 \le 1, \qquad 2q_1 \le q_2 + q_0. \tag{4.10}$$

If $x \in T_2$ and x_1, x_2, x_3 are its three neighbors, let

$$\nu\{\eta\colon \eta(x) = 1 | \eta(x_1) = 1\} = 1 - q_2,$$
$$\nu\{\eta\colon \eta(x) = 0,\ \eta(x_2) = \eta(x_3) = 1 | \eta(x_1) = 1\} = q_2 - 2q_1 + q_0,$$
$$\nu\{\eta\colon \eta(x) = 0,\ \eta(x_2) = 1,\ \eta(x_3) = 0 | \eta(x_1) = 1\} = q_1 - q_0,$$
$$\nu\{\eta\colon \eta(x) = \eta(x_2) = \eta(x_3) = 0 | \eta(x_1) = 1\} = q_0.$$

Using the renewal and dependence of range 2 properties, all other relevant conditional probabilities can be generated easily. Given $\eta(x) = 1$, as one constructs $\eta(y)$ successively, if one encounters two consecutive 0's, then all sites beyond them are assigned the value 0 automatically. Note that no longer is it the case that $\{y\colon \eta(y) = 1\}$ is connected. We wish to choose q_0, q_1, q_2 and λ so that (4.2) is nonnegative for all finite A.

Fix a finite $A \subset T_2$. For a component B of A and $x \in B$, $y \notin B$, $|x - y| = 1$, define

$$\delta_B(y) = 1 - \nu\big\{\eta = 0 \text{ on } A \cap S_y(x) | \eta(y) = 1\big\},$$
$$\sigma_B(y) = 1 - \nu\big\{\eta = 0 \text{ on } A \cap S_y(x) | \eta(x) = 1\big\}.$$

We will often omit the subscript B from this notation when it is clear which component is relevant.

Before we work on a lower bound for (4.2), we will prove two inequalities. The first will allow us to replace σ's by δ's, and the second is the analog of (4.9). Here is the first of these inequalities: if

$$2\big(q_1 - q_2^2\big) \ge q_0 - q_2^3 \ge 0, \tag{4.11}$$

then

$$(1 - q_1)\delta(y) \le \sigma(y) \le \delta(y). \tag{4.12}$$

[The right-hand inequality is an analog of the monotonicity statements used on Z^1 to show that the last three sums of (1.2) are nonnegative.] We will first prove the right-hand inequality by coupling, and we will then use it to prove the left-hand inequality. For the right-hand inequality, it suffices to construct $\{(\zeta_1(z), \zeta_2(z))\colon z \in S_y(x)\}$ in such a way that $\{\zeta_1(z)\colon z \in S_y(x)\}$ is distributed according to $\nu(\cdot | \eta(x) = 1)$, $\{\zeta_2(z)\colon z \in S_y(x)\}$ is distributed according to $\nu(\cdot | \eta(y) = 1)$ and $\zeta_1(z) \le \zeta_2(z)$ for all $z \in S_y(x)$ with probability 1. Start with $\zeta_1(x) = \zeta_2(y) = 1$, and then let $\zeta_1(y) = 1$ with probability $1 - q_2$. If $\zeta_1(y) = 1$, then the renewal property permits the construction of the rest of the variables so that $\zeta_1(z) = \zeta_2(z)$ for all $z \in S_y(x)$. Let y_1, y_2 be the neighbors of y other than x. If $\zeta_1(y) = 0$, construct $\{\zeta_i(y_j);\ i, j = 1, 2\}$ so that the required inequalities and (conditional) distributions are satisfied. This requires that the distribution of $(\zeta_2(y_1), \zeta_2(y_2))$ be stochastically larger than the conditional distribution of $(\zeta_1(y_1), \zeta_1(y_2))$ given $\zeta_1(y) = 0$. This means that the following must be satisfied:

$$(1 - q_2)^2 \geq \frac{q_2 - 2q_1 + q_0}{q_2} \quad \text{and} \quad q_2^2 \leq \frac{q_0}{q_2}.$$

These inequalities follow from (4.11). For each j, if $\zeta_1(y_j) = 1$, then use the renewal property to construct the variables corresponding to points z "beyond" y_j so that $\zeta_1(z) = \zeta_2(z)$, while if $\zeta_1(y_j) = 0$, it follows that $\zeta_1(z) = 0$ automatically for all such z.

To prove the left-hand inequality in (4.12) (which is the one we will use later), proceed as follows. Use the renewal property to write

$$1 - \delta(y) = [1 - \sigma(y_1)][1 - \sigma(y_2)] \tag{4.13}$$

and

$$\begin{aligned}1 - \sigma(y) = {} & (1 - q_2)[1 - \delta(y)] + q_0 + (q_1 - q_0)[1 - \delta(y_1)]1_{A^c}(y_1) \\ & + (q_1 - q_0)[1 - \delta(y_2)]1_{A^c}(y_2) \\ & + (q_2 - 2q_1 + q_0)[1 - \delta(y_1)]1_{A^c}(y_1)[1 - \delta(y_2)]1_{A^c}(y_2).\end{aligned} \tag{4.14}$$

In (4.13) and (4.14), $\delta(y_i)$ and $\sigma(y_i)$ are defined as before, but relative to the set $A' = A \cup \{y\}$ and its component B' which contains y. Use the right-hand side of (4.12) applied to y_i to replace $\delta(y_i)$ in (4.14) by $\sigma(y_i)$, and replace the indicators by 1. This gives

$$\begin{aligned}1 - \sigma(y) \leq {} & (1 - q_2)[1 - \delta(y)] + q_0 + (q_1 - q_0)[2 - \sigma(y_1) - \sigma(y_2)] \\ & + (q_2 - 2q_1 + q_0)[1 - \sigma(y_1)][1 - \sigma(y_2)].\end{aligned}$$

Then replace $2 - \sigma(y_1) - \sigma(y_2)$ by $2 - \delta(y)$, and replace $[1 - \sigma(y_1)][1 - \sigma(y_2)]$ by $1 - \delta(y)$, which can be done by (4.13). The resulting inequality is the left-hand side of (4.12).

Next, we will prove the analog of (4.9). As in the discussion of that inequality, let C be a component of A^c. Let $(x_1, y_1), (x_2, y_2), \dots, (x_m, y_m)$ be an enumeration of all the pairs (x, y) for which $|x - y| = 1$, $x \in A$, $y \in C$. Then let B_j be the component of A which contains x_j. It is with respect to B_j that $\delta(y_j)$ is defined. Fix a $1 \leq i \leq m$. Then, since $S_{y_i}(x_i) = \bigcup_{1 \leq j \leq m,\, j \neq i} S_{x_j}(y_j)$ (Figure 5 may be helpful here),

$$\begin{aligned}\delta(y_i) &= \nu\left\{\eta\colon \eta(x) = 1 \text{ for some } x \in A \cap \bigcup_{\substack{1 \leq j \leq m \\ j \neq i}} S_{x_j}(y_j) \,\middle|\, \eta(y_i) = 1\right\} \\ &\leq \sum_{\substack{1 \leq j \leq m \\ j \neq i}} \nu\left\{\eta\colon \eta(x) = 1 \text{ for some } x \in A \cap S_{x_j}(y_j) \,\middle|\, \eta(y_i) = 1\right\}.\end{aligned}$$

To bound the summand, note that [conditional on $\eta(y_i) = 1$] if there is a 1 in $A \cap S_{x_j}(y_j)$, then either $\eta(x_j) = 1$, or $\eta(y_j) = 1$, $\eta(x_j) = 0$, and there is a 1 at one of the neighbors of x_j other than y_j. Therefore, for $j \neq i$,

$$\nu\Big\{\eta: \eta(x) = 1 \text{ for some } x \in A \cap S_{x_j}(y_j) \big| \eta(y_i) = 1\Big\}$$
$$\leq \nu\{\eta: \eta(x_j) = 1|\eta(y_i) = 1\} + (q_2 - q_0)\nu\{\eta: \eta(y_j) = 1|\eta(y_i) = 1\}.$$

Next, use the inequalities

$$\nu\{\eta: \eta(x_j) = 1|\eta(y_i) = 1\} \geq \nu\{\eta: \eta(y_j) = 1|\eta(y_i) = 1\}(1 - q_2)$$

and

$$\delta(y_j) \geq \nu\{\eta: \eta(x_i) = 1|\eta(y_j) = 1\} = \nu\{\eta: \eta(x_j) = 1|\eta(y_i) = 1\}.$$

Combining these relations, we can conclude that

$$(1 - q_2)\delta(y_i) \leq (1 - q_0) \sum_{\substack{1 \leq j \leq m \\ j \neq i}} \delta(y_j). \tag{4.15}$$

This is the analog of (4.9). (Note that it reduces to it if $q_0 = q_1 = q_2$.)

We need to lower-bound the contributions to (4.2) attributable to a component B of A. We will assume that (4.11) holds, so that we may use (4.12). (This will be checked later.) Suppose $B = \{u\}$, and u_1, u_2, u_3 are the three neighbors of u. Using the renewal property and (4.12), the contributions to (4.2) attributable to this B are

$$\begin{aligned}
&\lambda \sum_{i=1}^{3} \nu\{\eta = 0 \text{ on } A|\eta(u_i) = 1\} - \nu\{\eta = 0 \text{ on } A\setminus\{u\}|\eta(u) = 1\} \\
&\quad = \lambda \sum_{i=1}^{3} [1 - \delta(u_i)]\Bigg[q_0 + (q_1 - q_0)\sum_{j \neq i}[1 - \delta(u_j)] \\
&\qquad\qquad + (q_2 - 2q_1 + q_0)\prod_{j \neq i}[1 - \delta(u_j)]\Bigg] \\
&\qquad - \prod_{i=1}^{3}[1 - \sigma(u_i)] \qquad\qquad (4.16)\\
&\quad \geq 3\lambda q_2 - \lambda(3q_2 - 2q_1)[\delta(u_1) + \delta(u_2) + \delta(u_3)] \\
&\qquad + \lambda(q_0 - 4q_1 + 3q_2)[\delta(u_1)\delta(u_2) \\
&\qquad\qquad + \delta(u_1)\delta(u_3) + \delta(u_2)\delta(u_3)] \\
&\qquad - 3\lambda(q_2 - 2q_1 + q_0)\delta(u_1)\delta(u_2)\delta(u_3) \\
&\qquad - \prod_{i=1}^{3}[1 - (1 - q_1)\delta(u_i)].
\end{aligned}$$

We wish to bound this expression below by

$$c(1 - q_0)\delta(u_1) - c(1 - q_2)[\delta(u_2) + \delta(u_3)], \tag{4.17}$$

where c is a nonnegative constant to be determined later. The factors which appear in (4.17) and in similar expressions below are used because of the way they appear in (4.15), since we will take advantage of cancellation in the same way we did for $n = 1$ (using the pyramid scheme argument). In each case, all the neighbors of B except one make a negative contribution with a factor $c(1-q_2)$, while the distinguished neighbor makes a positive contribution with a factor $c(1-q_0)$. By the symmetry of (4.16) in the u_i's, it does not matter which neighbor is taken to play the distinguished role.

To check bound (4.17), note that both sides [i.e., the right-hand sides of (4.16) and (4.17)] are linear in each of the δ's and that $0 \le \delta(u_i) \le 1-q_0^2$ for each i. So, it is enough that the inequality holds when $(\delta(u_1), \delta(u_2), \delta(u_3))$ is any corner of the cube $[0, 1-q_0^2]^3$. This verification will be discussed after we choose particular values of the parameters $\lambda, q_0, q_1, q_2, c$.

Suppose next that $B=\{u,v\}$ with $|u-v|=1$, and suppose that u_1, u_2 and v_1, v_2 are the neighbors of u and v, respectively, which are not in B. (See Figure 6.) The contributions from this B to (4.2) are

$$
\begin{aligned}
&\lambda \sum_{i=1}^{2} \nu\{\eta=0 \text{ on } A \mid \eta(u_i)=1\} + \lambda \sum_{i=1}^{2} \nu\{\eta=0 \text{ on } A \mid \eta(v_i)=1\} \\
&\quad -\nu\{\eta=0 \text{ on } A\setminus\{u\} \mid \eta(u)=1\} \\
&\quad -\nu\{\eta=0 \text{ on } A\setminus\{v\} \mid \eta(v)=1\} \\
&\qquad = \lambda[1-\delta(u_1)][q_0+(q_1-q_0)[1-\delta(u_2)]] \\
&\qquad\quad +\lambda[1-\delta(u_2)][q_0+(q_1-q_0)[1-\delta(u_1)]] \\
&\qquad\quad +\lambda[1-\delta(v_1)][q_0+(q_1-q_0)[1-\delta(v_2)]] \\
&\qquad\quad +\lambda[1-\delta(v_2)][q_0+(q_1-q_0)[1-\delta(v_1)]] \\
&\qquad\quad -[q_0+(q_1-q_0)[1-\delta(v_1)]+(q_1-q_0)[1-\delta(v_2)] \\
&\qquad\qquad +(q_2-2q_1+q_0)[1-\delta(v_1)][1-\delta(v_2)]] \\
&\qquad\qquad \times[1-\sigma(u_1)][1-\sigma(u_2)] \\
&\qquad\quad -[q_0+(q_1-q_0)[1-\delta(u_1)]+(q_1-q_0)[1-\delta(u_2)] \\
&\qquad\quad +(q_2-2q_1+q_0)[1-\delta(u_1)][1-\delta(u_2)]] \\
&\qquad\qquad \times[1-\sigma(v_1)][1-\sigma(v_2)] \\
&\qquad \ge 4\lambda q_1 - \lambda(2q_1-q_0)[\delta(u_1)+\delta(u_2)+\delta(v_1)+\delta(v_2)] \\
&\qquad\quad +2\lambda(q_1-q_0)[\delta(u_1)\delta(u_2)+\delta(v_1)\delta(v_2)] \\
&\qquad\quad -[q_2-(q_2-q_1)[\delta(v_1)+\delta(v_2)] \\
&\qquad\qquad\quad +(q_2-2q_1+q_0)\delta(v_1)\delta(v_2)] \prod_{i=1}^{2}[1-(1-q_1)\delta(u_i)] \\
&\qquad\quad -[q_2-(q_2-q_1)[\delta(u_1)+\delta(u_2)] \\
&\qquad\qquad\quad +(q_2-2q_1+q_0)\delta(u_1)\delta(u_2)] \prod_{i=1}^{2}[1-(1-q_1)\delta(v_i)].
\end{aligned}
\tag{4.18}
$$

The renewal property is used in the first equality, and (4.12) is used in the inequality. Arguing as before, we want to bound the right-hand side of this

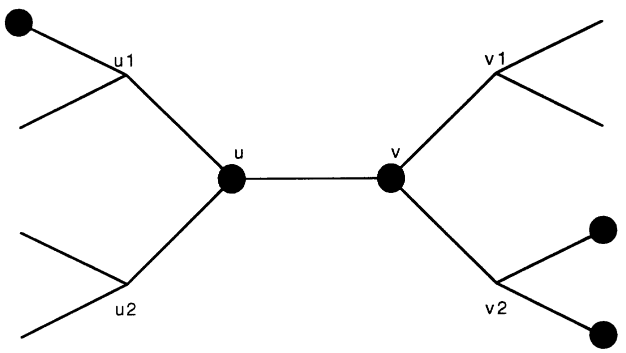

FIG. 6.

expression below by

$$c(1-q_0)\delta(u_1) - c(1-q_2)[\delta(u_2)+\delta(v_1)+\delta(v_2)] \tag{4.19}$$

for any $(\delta(u_1), \delta(u_2), \delta(v_1), \delta(v_2))$ in $[0, 1-q_0^2]^4$.

To decide how to choose the parameters

$$q_0, q_1, q_2, \lambda, c$$

so that the lower bounds (4.17) and (4.19) hold, note that (4.16) and (4.18) involve the δ's in a reasonably symmetric way, while (4.17) and (4.19) are generally larger if $\delta(u_1)$ is large than if the other δ's are. This suggests that we require that the lower bounds (4.17) and (4.19) hold with equality for all choices of $\delta(u_1)$ whenever $\delta(u_2) = \delta(u_3) = 0$ in the case of (4.17) and whenever $\delta(u_2) = \delta(v_1) = \delta(v_2) = 0$ in the case of (4.19). This requirement leads (after some simplification) to the following equations, which we assume to hold from now on:

$$3\lambda q_2 = 1, \qquad 2\lambda q_1 = q_2, \qquad \lambda q_0 = q_1 q_2, \qquad c(1-q_0) = q_2 - q_1. \tag{4.20}$$

Later we will discuss the verification that bounds (4.17) and (4.19) hold in the full relevant cube of δ values.

Next, take $B = \{u_1, u_2, v\}$, where $|u_1 - v| = |u_2 - v| = 1$, $u_1 \neq u_2$. Let u_i', u_i'' be the neighbors of u_i other than v, and let v' be the neighbor of v other than u_1, u_2. Its contributions to (4.2) are

$$\begin{aligned}
&\lambda\nu\{\eta = 0 \text{ on } A \mid \eta(v') = 1\} \\
&\quad + \lambda \sum_{i=1}^{2} [\nu\{\eta = 0 \text{ on } A \mid \eta(u_i') = 1\} + \nu\{\eta = 0 \text{ on } A \mid \eta(u_i'') = 1\}] \\
&\quad - \nu\{\eta = 0 \text{ on } A\setminus\{v\} \mid \eta(v) = 1\} - \sum_{i=1}^{2} \nu\{\eta = 0 \text{ on } A\setminus\{u_i\} \mid \eta(u_i) = 1\} \\
&\qquad = \lambda q_0[1-\delta(v')] + \lambda \sum_{i=1}^{2} [[1-\delta(u_i')][q_0 + (q_1 - q_0)[1-\delta(u_i'')]] \\
&\qquad\qquad + [1-\delta(u_i'')][q_0 + (q_1-q_0)[1-\delta(u_i')]]]
\end{aligned} \tag{4.21}$$

$$
\begin{aligned}
&-[1-\sigma(v')]\prod_{i=1}^{2}\big[q_0+(q_1-q_0)[1-\delta(u_i')]\\
&\qquad\qquad +(q_1-q_0)[1-\delta(u_i'')]\\
&\qquad\qquad +(q_2-2q_1+q_0)[1-\delta(u_i')][1-\delta(u_i'')]\big]\\
&-\sum_{i=1}^{2}\big[q_0+(q_1-q_0)[1-\delta(v')]\big][1-\delta(u_i')][1-\delta(u_i'')]\\
&\geq \lambda q_0[1-\delta(v')]+\lambda\sum_{i=1}^{2}\big[2q_1-(2q_1-q_0)[\delta(u_i')+\delta(u_i'')]\\
&\qquad\qquad +2(q_1-q_0)\delta(u_i')\delta(u_i'')\big]\\
&-[1-(1-q_1)\delta(v')]\prod_{i=1}^{2}\big[q_2-(q_2-q_1)[\delta(u_i')+\delta(u_i'')]\\
&\qquad\qquad +(q_2-2q_1+q_0)\delta(u_i')\delta(u_i'')\big]\\
&-[q_1-(q_1-q_0)\delta(v')]\sum_{i=1}^{2}[1-(1-q_1)\delta(u_i')]\\
&\qquad\qquad \times[1-(1-q_1)\delta(u_i'')].
\end{aligned}
$$

To get the inequality, use the left-hand side of (4.12) to replace the σ's by δ's, and then expand the parts that originally had only δ's. We need to bound the right-hand side of (4.21) below by

$$c(1-q_0)\delta(u_1')-c(1-q_2)[\delta(u_1'')+\delta(u_2')+\delta(u_2'')+\delta(v')] \tag{4.22}$$

and by

$$c(1-q_0)\delta(v')-c(1-q_2)[\delta(u_1')+\delta(u_1'')+\delta(u_2')+\delta(u_2'')], \tag{4.23}$$

depending on which type of boundary point plays the distinguished role. Again, this is verified by checking each corner of the cube $[0,1-q_0^2]^5$ after choosing values for the parameters at the end of this section.

Our final special case is $B=\{u_1,u_2,u_3,v\}$, where $|u_i-v|=1$ for each i (u_i distinct). Let u_i', u_i'' be the neighbors of u_i other than v. (See Figure 7.) Then the contributions to (4.2) corresponding to this B are greater than or equal to

$$
\begin{aligned}
&\lambda\sum_{i=1}^{3}\big[2q_1-(2q_1-q_0)[\delta(u_i')+\delta(u_i'')]\\
&\qquad\qquad +2(q_1-q_0)\delta(u_i')\delta(u_i'')\big]\\
&-\prod_{i=1}^{3}\big[q_2-(q_2-q_1)[\delta(u_i')+\delta(u_i'')]\\
&\qquad +(q_2-2q_1+q_0)\delta(u_i')\delta(u_i'')\big]\\
&-q_0\sum_{i=1}^{3}[1-(1-q_1)\delta(u_i')][1-(1-q_1)\delta(u_i'')].
\end{aligned}
\tag{4.24}
$$

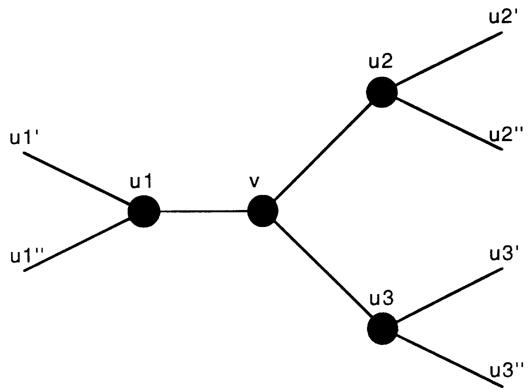

FIG. 7.

Arguing as in the previous cases, we need to have this bounded below by

$$c(1-q_0)\delta(u_1') \\ -c(1-q_2)\big[\delta(u_1'')+\delta(u_2')+\delta(u_2'')+\delta(u_3')+\delta(u_3'')\big]. \tag{4.25}$$

Now, suppose the component B of A is not one of the four cases we have considered so far; that is, $|B| \geq 5$ or $|B| = 4$ and B has diameter 3. For $x \in B$, $|x-y| = 1$, say that y is of type 3 if $y \notin B$, y is of type 2 if $y \in B$ but neither of the two neighbors (call them y', y'') of y other than x is in B, y is of type 1 if $y \in B$ and exactly one of the two neighbors (call it y') of y other than x is not in B and y is of type 0 if $y \in B$ and both of the two neighbors of y other than x is in B. (See Figure 8.) Let $B_{i,j,k,l}$ be the set of $x \in B$ with i neighbors of type 0, j neighbors of type 1, k neighbors of type 2 and l

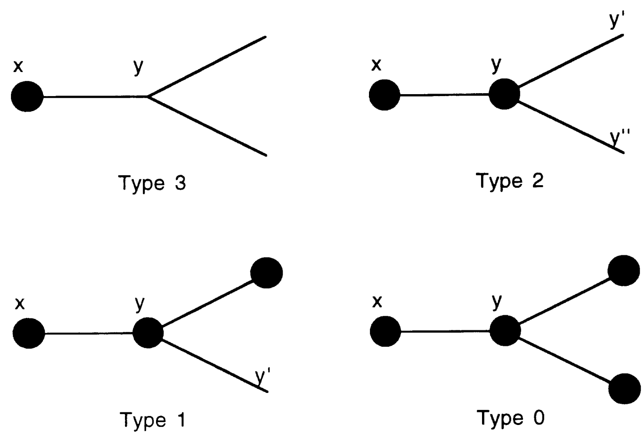

FIG. 8.

neighbors of type 3. The subscripts must satisfy $i + j + k + l = 3$. Also, note that $B_{0,0,k,l} = \varnothing$ for $k + l = 3$, since

$$\begin{aligned} B_{0,0,0,3} \neq \varnothing &\quad\Rightarrow\quad |B| = 1,\\ B_{0,0,1,2} \neq \varnothing &\quad\Rightarrow\quad |B| = 2,\\ B_{0,0,2,1} \neq \varnothing &\quad\Rightarrow\quad |B| = 3,\\ B_{0,0,3,0} \neq \varnothing &\quad\Rightarrow\quad |B| = 4 \text{ and } B \text{ has diameter } 2. \end{aligned}$$

In the expressions below, the neighbors y_1, y_2, y_3 of x are numbered in such a way that the type number of y_i is nondecreasing in i. The contributions to (4.2) attributable to B are then

$$\begin{aligned} h_B(A) \equiv \lambda q_0 &\sum_{x \in \bigcup_{i+j+k=2} B_{i,j,k,1}} [1 - \delta(y_3)] - \sum_{x\in B} \prod_{i=1}^{3} \gamma(y_i)\\ &+ \lambda \sum_{x \in \bigcup_{i+j+k=1} B_{i,j,k,2}} \big[2q_1 - (2q_1 - q_0)[\delta(y_2) + \delta(y_3)]\\ &\qquad\qquad + 2(q_1 - q_0)\delta(y_2)\delta(y_3)\big], \end{aligned} \tag{4.26}$$

where

$$\begin{aligned} &\gamma(y) = 1 - \sigma(y) \quad \text{if } y \text{ is of type 3},\\ &\gamma(y) = q_2 - (q_2 - q_1)[\delta(y') + \delta(y'')] + (q_2 - 2q_1 + q_0)\delta(y')\delta(y'') \\ &\hspace{22em} \text{if } y \text{ is of type 2},\\ &\gamma(y) = q_1 - (q_1 - q_0)\delta(y') \quad \text{if } y \text{ is of type 1},\\ &\gamma(y) = q_0 \quad \text{if } y \text{ is of type 0}. \end{aligned}$$

In finding a lower bound for (4.26), the most awkward term to deal with is the product of the γ's. Note that the other parts of (4.26) are at most quadratic in the δ's, while the product of the γ's can have products of more than two δ's. To rectify this situation, we will obtain and use quadratic upper bounds for the products of γ's. The particular bound depends on which of the $B_{i,j,k,l}$ contains the x in question. In each case, the σ's are first replaced by δ's using (4.12). Then the following inequality is used: suppose that $0 \le t_i \le a_i \le 1$. Then

$$\prod_{i=1}^{m} (1 - t_i) \le 1 - \sum_{i=1}^{m} \left[\prod_{j=1}^{i-1} (1 - a_i) \right] t_i.$$

The proof of this inequality is not hard, and is left to the reader. In any case, we will use it only for $m \le 3$, in which cases it is a simple verification. In applying this inequality below, we will take

$$a_i = \begin{cases} \dfrac{q_1 - q_0,}{q_1} & \text{if } y \text{ is of type 1,} \\ \dfrac{q_2 - q_0}{q_2}, & \text{if } y \text{ is of type 2,} \\ 1 - q_1, & \text{if } y \text{ is of type 3.} \end{cases}$$

We also arrange the factors in order of increasing type number. In cases involving y's of type 2, one needs to note that

$$0 \le (q_2 - q_1)[\delta_1 + \delta_2] - (q_2 - 2q_1 + q_0)\delta_1\delta_2 \le q_2 - q_0,$$

whenever $0 \le \delta_1, \delta_2 \le 1$, in order to check $0 \le t_i \le a_i \le 1$. The upper bound for $\prod_{i=1}^{3}\gamma(y_i)$ which one obtains is

$$\begin{aligned}
x \in B_{3,0,0,0}:&\quad q_0^3,\\
x \in B_{2,1,0,0}:&\quad q_0^2[q_1 - (q_1 - q_0)\delta(y_3')],\\
x \in B_{2,0,1,0}:&\quad q_0^2[q_2 - (q_2 - q_1)[\delta(y_3') + \delta(y_3'')]\\
&\qquad + (q_2 - 2q_1 + q_0)\delta(y_3')\delta(y_3'')],\\
x \in B_{2,0,0,1}:&\quad q_0^2[1 - (1 - q_1)\delta(y_3)],\\
x \in B_{1,2,0,0}:&\quad q_0q_1^2 - q_0q_1(q_1 - q_0)\delta(y_2') - q_0^2(q_1 - q_0)\delta(y_3'),\\
x \in B_{1,1,1,0}:&\quad q_0q_1q_2 - q_0q_2(q_1 - q_0)\delta(y_2')\\
&\quad - q_0^2(q_2 - q_1)[\delta(y_3') + \delta(y_3'')]\\
&\quad + q_0^2(q_2 - 2q_1 + q_0)\delta(y_3')\delta(y_3''),\\
x \in B_{1,0,2,0}:&\quad q_0q_2^2 - q_0q_2(q_2 - q_1)[\delta(y_2') + \delta(y_2'')]\\
&\quad - q_0^2(q_2 - q_1)[\delta(y_3') + \delta(y_3'')]\\
&\quad + q_0q_2(q_2 - 2q_1 + q_0)\delta(y_2')\delta(y_2'')\\
&\quad + q_0^2(q_2 - 2q_1 + q_0)\delta(y_3')\delta(y_3''),\\
x \in B_{1,1,0,1}:&\quad q_0q_1 - q_0(q_1 - q_0)\delta(y_2') - q_0^2(1 - q_1)\delta(y_3),\\
x \in B_{1,0,1,1}:&\quad q_0q_2 - q_0(q_2 - q_1)[\delta(y_2') + \delta(y_2'')]\\
&\quad - q_0^2(1 - q_1)\delta(y_3)\\
&\quad + q_0(q_2 - 2q_1 + q_0)\delta(y_2')\delta(y_2''),\\
x \in B_{1,0,0,2}:&\quad q_0 - q_0(1 - q_1)[\delta(y_2) + \delta(y_3)]\\
&\quad + q_0(1 - q_1)^2\delta(y_2)\delta(y_3),
\end{aligned} \tag{4.27}$$

$$
\begin{aligned}
x \in B_{0,3,0,0}: \quad & q_1^3 - q_1^2(q_1 - q_0)\delta(y_1') - q_1 q_0(q_1 - q_0)\delta(y_2') \\
& -q_0^2(q_1 - q_0)\delta(y_3'), \\
x \in B_{0,2,1,0}: \quad & q_1^2 q_2 - q_1 q_2(q_1 - q_0)\delta(y_1') - q_0 q_2(q_1 - q_0)\delta(y_2') \\
& -q_0^2(q_2 - q_1)[\delta(y_3') + \delta(y_3'')] \\
& +q_0^2(q_2 - 2q_1 + q_0)\delta(y_3')\delta(y_3''), \\
x \in B_{0,2,0,1}: \quad & q_1^2 - q_1(q_1 - q_0)\delta(y_1') - q_0(q_1 - q_0)\delta(y_2') \\
& -q_0^2(1 - q_1)\delta(y_3), \\
x \in B_{0,1,2,0}: \quad & q_1 q_2^2 - q_2^2(q_1 - q_0)\delta(y_1') \\
& -q_0 q_2(q_2 - q_1)[\delta(y_2') + \delta(y_2'')] \\
& -q_0^2(q_2 - q_1)[\delta(y_3') + \delta(y_3'')] \\
& +q_0 q_2(q_2 - 2q_1 + q_0)\delta(y_2')\delta(y_2'') \\
& +q_0^2(q_2 - 2q_1 + q_0)\delta(y_3')\delta(y_3''), \\
x \in B_{0,1,1,1}: \quad & q_1 q_2 - q_2(q_1 - q_0)\delta(y_1') - q_0^2(1 - q_1)\delta(y_3) \\
& -q_0(q_2 - q_1)[\delta(y_2') + \delta(y_2'')] \\
& +q_0(q_2 - 2q_1 + q_0)\delta(y_2')\delta(y_2''), \\
x \in B_{0,1,0,2}: \quad & q_1 - (q_1 - q_0)\delta(y_1') - q_0(1 - q_1)[\delta(y_2) + \delta(y_3)] \\
& +q_0(1 - q_1)^2\delta(y_2)\delta(y_3).
\end{aligned}
$$

Now we replace the products of γ's in (4.26) by the expressions in (4.27). Then the sums are rearranged, so that each summand containing a δ is included in the sum corresponding to the (unique) x which is a nearest neighbor of the argument of the δ. (Note that the x to which the summand is attached before this rearrangement can be a second nearest neighbor of the argument of the δ.) For example, suppose we consider $x \in B_{2,0,0,1}$, and let y_1, y_2, y_3 be its neighbors, with $y_3 \notin B$. Since $x \in B_{2,0,0,1}$, we know that, for each $m = 1, 2$, $y_m \in B_{i,j,k,0}$, where $j \geq 1$. Looking at all the possible cases in (4.27), we see that the multiple of $\delta(y_3)$ which each y_m contributes is one of the following:

$$
\begin{aligned}
& q_0^2(q_1 - q_0),\ q_1^2(q_1 - q_0),\ q_2^2(q_1 - q_0),\ q_0 q_1(q_1 - q_0), \\
& \quad q_0 q_2(q_1 - q_0),\ q_1 q_2(q_1 - q_0).
\end{aligned}
$$

Taking the worst case, we see that the contributions from y_1 and y_2 are bounded below by $2q_0^2(q_1 - q_0)$. The other contribution comes from x itself, and is $q_0^2(1 - q_1)$. This argument is used in each of the sums in (4.28) below. Since we are aiming at a lower bound of the form (4.25), we will add appropriate multiples of the δ's corresponding to the points $\{z_i,\ 1 \leq i \leq m\}$ in

B^c which have a neighbor in B. The result is

$$\begin{aligned}
&h_B(A) + c(1-q_2)\sum_{i=1}^{m}\delta(z_i)\\
&\quad\geq -\sum_{i+j+k=3}|B_{i,j,k,0}|q_0^i q_1^j q_2^k\\
&\qquad+\sum_{i+j+k=1}\sum_{x\in B_{i,j,k,2}}\Big[2\lambda q_1 - q_0^i q_1^j q_2^k + b_{i,j,k,2}[\delta(y_2)+\delta(y_3)]\\
&\qquad\qquad\qquad -c_{i,j,k,2}\delta(y_2)\delta(y_3)\Big]\\
&\qquad+\sum_{i+j+k=2}\sum_{x\in B_{i,j,k,1}}\big[\lambda q_0 - q_0^i q_1^j q_2^k - b_{i,j,k,1}\delta(y_3)\big],
\end{aligned}\tag{4.28}$$

where

$$\begin{aligned}
b_{2,0,0,1} &= \lambda q_0 - c(1-q_2) - q_0^2(1+q_1-2q_0),\\
b_{1,1,0,1} &= \lambda q_0 - c(1-q_2) - q_0(q_1-q_0^2),\\
b_{0,2,0,1} &= \lambda q_0 - c(1-q_2) - 2q_0q_1 + q_0^2 + q_0^2q_1,\\
b_{1,0,1,1} &= \lambda q_0 - c(1-q_2) + q_0 + q_0^3 - q_0^2 - q_1,\\
b_{0,1,1,1} &= \lambda q_0 - c(1-q_2) + q_0 + q_0^2q_1 - q_1 - q_0q_1
\end{aligned}$$

and

$$\begin{aligned}
b_{1,0,0,2} &= -\lambda(2q_1-q_0) + c(1-q_2) + q_0(1-q_1) + q_0^2(q_2-q_1),\\
c_{1,0,0,2} &= -2\lambda(q_1-q_0) + q_0(1-q_1)^2 + q_0^2(q_2-2q_1+q_0),\\
b_{0,1,0,2} &= -\lambda(2q_1-q_0) + c(1-q_2) + q_0(1+q_2-2q_1),\\
c_{0,1,0,2} &= -2\lambda(q_1-q_0) + q_0(1-q_1)^2 + q_0(q_2-2q_1+q_0).
\end{aligned}$$

We need to find a good lower bound for the right-hand side of (4.28). When we choose particular values for the parameters, we will find that $b_{i,j,k,l}\geq 0$ and $c_{i,j,k,l}\geq 0$ for all choices of the subscripts which appear above, and

$$2(1-q_0^2)b_{i,j,k,2} - (1-q_0^2)^2 c_{i,j,k,2} \leq 0$$

for $i=1$ or $j=1$. Thus we may replace all the δ's on the right-hand side of (4.28) by $1-q_0^2$. The result is

$$\begin{aligned}
&h_B(A) + c(1-q_2)\sum_{i=1}^{m}\delta(z_i)\\
&\quad\geq -\sum_{i+j+k=3}|B_{i,j,k,0}|q_0^i q_1^j q_2^k\\
&\qquad+\sum_{i+j+k=1}|B_{i,j,k,2}|\Big[2\lambda q_1 - q_0^i q_1^j q_2^k + 2b_{i,j,k,2}(1-q_0^2)\\
&\qquad\qquad\qquad -c_{i,j,k,2}(1-q_0^2)^2\Big]\\
&\qquad+\sum_{i+j+k=2}|B_{i,j,k,1}|\big[\lambda q_0 - q_0^i q_1^j q_2^k - b_{i,j,k,1}(1-q_0^2)\big].
\end{aligned}\tag{4.29}$$

In order to continue bounding (4.29) from below, we need to find some relations among the cardinalities which appear in it. First, note that every $x \in B_{1,0,0,2}$ has a unique neighbor in $\bigcup_{i+j+k=3} B_{i,j,k,0}$, and each point in $B_{i,j,k,0}$ occurs as the neighbor of k points in $B_{1,0,0,2}$. Hence

$$|B_{1,0,0,2}| = \sum_{i+j+k=3} k|B_{i,j,k,0}|. \tag{4.30}$$

A similar counting argument gives

$$|B_{0,1,0,2}| = |B_{1,0,1,1}| + |B_{0,1,1,1}|. \tag{4.31}$$

For the next identity, let

$$W = \left\{(u,v)\colon |u-v| = 1,\, u \in \bigcup_{i+j+k=2} B_{i,j,k,1},\, v \in \bigcup_{i+j+k=3} B_{i,j,k,0}\right\}.$$

Then one can compute $|W|$ in two different ways (summing first on u and then on v, and vice versa), yielding

$$\sum_{i+j+k=2} i|B_{i,j,k,1}| = \sum_{i+j+k=3} j|B_{i,j,k,0}|. \tag{4.32}$$

It is easy to check that because B is connected, the number of points in B with exactly one neighbor in B is two more than the number of points in B with all three neighbors in B. This gives

$$|B_{1,0,0,2}| + |B_{0,1,0,2}| = \sum_{i+j+k=3} |B_{i,j,k,0}| + 2. \tag{4.33}$$

Finally, using (4.30) and (4.31) in (4.33) yields

$$\begin{aligned} &|B_{1,0,2,0}| + |B_{0,1,2,0}| + |B_{1,0,1,1}| + |B_{0,1,1,1}| \\ &\quad = |B_{3,0,0,0}| + |B_{2,1,0,0}| + |B_{1,2,0,0}| + |B_{0,3,0,0}| + 2. \end{aligned} \tag{4.34}$$

Add 0 to the right-hand side of (4.29) in the form

$$\begin{aligned} &+ a[\text{r.h.s. of } (4.32) - \text{l.h.s. of } (4.32)] \\ &+ b[\text{r.h.s. of } (4.34) - \text{l.h.s. of } (4.34)] \\ &+ d[\text{r.h.s. of } (4.30) - \text{l.h.s. of } (4.30)] \\ &+ e[\text{r.h.s. of } (4.31) - \text{l.h.s. of } (4.31)], \end{aligned} \tag{4.35}$$

where a, b, d, e are constants to be determined. We wish to choose the unknown parameters so that the coefficients of all of the cardinalities $|B_{i,j,k,l}|$ in the resulting expression are nonnegative. Once done, we will have

$$h_B(A) + c(1-q_2)\sum_{i=1}^{m} \delta(z_i) \ge 2b. \tag{4.36}$$

[The $2b$ on the right-hand side comes from the 2 on the right-hand of (4.34).] This will imply the desired bound

$$h_B(A) \ge c(1-q_0)\delta(z_i) - c(1-q_2)\sum_{j \ne i} \delta(z_j), \tag{4.37}$$

provided that

$$2b \geq c(1 - q_0^2)(2 - q_0 - q_2). \tag{4.38}$$

[Recall that $0 \leq \delta(z_i) \leq 1 - q_0^2$.] Once this has been done, the bounds (4.15), (4.17), (4.19), (4.22), (4.23), (4.25) and (4.37) combine as in the case $n = 1$ by using the pyramid scheme argument to imply that $h(A) \geq 0$.

Some trial and error using Mathematica suggests that we choose $\lambda, q_0, q_1, q_2, a, b, c, d, e$ so that the coefficients of $|B_{3,0,0,0}|$, $|B_{2,0,0,1}|$, $|B_{1,0,0,2}|$, $|B_{0,1,1,1}|$ and $|B_{0,1,0,2}|$ are 0. This gives the following equations:

$$\begin{aligned}
0 &= b - q_0^3,\\
0 &= \lambda q_0 - q_0^2 - b_{2,0,0,1}(1 - q_0^2) - 2a,\\
0 &= 2\lambda q_1 - q_0 + 2b_{1,0,0,2}(1 - q_0^2) - c_{1,0,0,2}(1 - q_0^2)^2 - d,\\
0 &= \lambda q_0 - q_1 q_2 - b_{0,1,1,1}(1 - q_0^2) - b + e,\\
0 &= 2\lambda q_1 - q_1 + 2b_{0,1,0,2}(1 - q_0^2) - c_{0,1,0,2}(1 - q_0^2)^2 - e.
\end{aligned}$$

Solving these equations simultaneously with (4.20) gives

$$\begin{aligned}
&\lambda = 0.60485\ldots, && q_0 = 0.41507\ldots, && q_1 = 0.45556\ldots,\\
&q_2 = 0.55109\ldots, && a = 0.01042\ldots, && b = 0.07151\ldots,\\
&c = 0.16332\ldots, && d = 0.10481\ldots, && e = 0.09356\ldots.
\end{aligned} \tag{4.39}$$

With these values of the parameters, it is straightforward, though somewhat tedious, to check that the coefficients of the other $|B_{i,j,k,l}|$ are nonnegative, and that all the inequalities we have assumed in this argument actually hold. We give a few examples:

1. Equation (4.11) is satisfied since $2(q_1 - q_2^2) = 0.3037\ldots$, $q_0 - q_2^3 = 0.2477\ldots$.
2. Bound (4.17) holds, since the right-hand side of (4.16) minus (4.17) can be written as

$$\begin{aligned}
&0.1688\ldots[\delta(u_2) + \delta(u_3)]\\
&\quad - 0.1475\ldots[\delta(u_1)\delta(u_2) + \delta(u_1)\delta(u_3) + \delta(u_2)\delta(u_3)]\\
&\quad + 0.0614\ldots\delta(u_1)\delta(u_2)\delta(u_3).
\end{aligned}$$

It is obvious that this is nonnegative when any one of the δ's is 0. The value of this expression is $0.0111\ldots$ when $\delta(u_1) = \delta(u_2) = \delta(u_3) = 1 - q_0^2 = 0.8277\ldots$, and therefore it is nonnegative in the entire cube $[0, 1 - q_0^2]^3$.

3. Bound (4.19) holds, since the right-hand side of (4.18) minus (4.19) can be written as

$$\begin{aligned}
&0.1688\ldots[\delta(u_2)+\delta(v_1)+\delta(v_2)]\\
&\quad-0.1694\ldots[\delta(u_1)\delta(u_2)+\delta(v_1)\delta(v_2)]\\
&\quad-0.1040\ldots[\delta(u_1)+\delta(u_2)][\delta(v_1)+\delta(v_2)]\\
&\quad+0.0582\ldots[\delta(u_1)\delta(u_2)\delta(v_1)+\delta(u_1)\delta(u_2)\delta(v_2)\\
&\qquad\qquad+\delta(u_1)\delta(v_1)\delta(v_2)+\delta(u_2)\delta(v_1)\delta(v_2)]\\
&\quad-0.0326\ldots\delta(u_1)\delta(u_2)\delta(v_1)\delta(v_2).
\end{aligned}$$

Again, it is easy to check that this is nonnegative whenever two or more of the δ's are 0. Up to symmetries, that leaves three corners of the cube $[0, 1-q_0^2]^4$ to check explicitly, and the above expression is nonnegative at those three corners. Bounds (4.22), (4.23) and (4.25) are similar, except that there are more corners to check. Using Mathematica, this is not difficult.

4. Here are the values of the $b_{i,j,k,l}$ and $c_{i,j,k,l}$ which are needed in passing from (4.28) to (4.29), rounded to four decimals:

$$\begin{aligned}
b_{2,0,0,1}&=0.0700, & b_{1,1,0,1}&=0.0602, & b_{0,2,0,1}&=0.0503,\\
b_{1,0,1,1}&=0.0365, & b_{0,1,1,1}&=0.0266, & b_{1,0,0,2}&=0.0157,\\
c_{1,0,0,2}&=0.0835, & b_{0,1,0,2}&=0.0389, & c_{0,1,0,2}&=0.0969.
\end{aligned}$$

Also, with these values, one finds that

$$\begin{aligned}
2(1-q_0^2)b_{1,0,0,2}-(1-q_0^2)^2c_{1,0,0,2}&=-0.0312\ldots,\\
2(1-q_0^2)b_{0,1,0,2}-(1-q_0^2)^2c_{0,1,0,2}&=-0.0019\ldots.
\end{aligned}$$

5. Here are the coefficients of $|B_{i,j,k,l}|$ on the right-hand side of (4.29) + (4.35), rounded to four decimal places:

(3,0,0,0)	0.0000,	(2,1,0,0)	0.0034,
(2,0,1,0)	0.0099,	(2,0,0,1)	0.0000,
(1,2,0,0)	0.0062,	(1,1,1,0)	0.0110,
(1,0,2,0)	0.0121,	(1,1,0,1)	0.0018,
(1,0,1,1)	0.0038,	(1,0,0,2)	0.0000,
(0,3,0,0)	0.0082,	(0,2,1,0)	0.0113,
(0,2,0,1)	0.0019,	(0,1,2,0)	0.0102,
(0,1,1,1)	0.0000,	(0,1,0,2)	0.0000.

Inequality (4.38) is also easy to check. Thus we have proved that $\lambda_1 \le 0.605$ as required.

We conclude this section with a remark. Using part of the above argument (it corresponds to the case in which all δ's are 0), it is not hard to show that

q_0, q_1, q_2, λ can be chosen so that $h(A) \geq 0$ for all connected sets A if λ lies above the one obtained by solving

$$\begin{gathered} 3\lambda q_2 = 1, \qquad 2\lambda q_1 = q_2, \\ 2(q_2 - q_1) = q_0(q_0^2 + q_2^2), \\ 2(\lambda q_0 + q_2) = q_0(q_0^2 + q_1^2) + 2(q_1 + q_0 q_2). \end{gathered} \tag{4.40}$$

This condition is also necessary, as can be seen by considering the four connected sets: $A_1 = \{x\}$, $A_2 = \{x, y\}$ with $|x - y| = 1$, $A_3 = \{y: |x - y| \leq N\}$ for a fixed x and large N, and A_4, which is obtained from A_3 by adding one neighbor to each boundary point of A_3. The solution of (4.40) is given by

$$q_0 = 0.4485\ldots, \qquad q_1 = 0.4795\ldots, \qquad q_2 = 0.5654\ldots, \qquad \lambda = 0.5895\ldots.$$

Thus we lose about 0.015 in our bound by having to prove $h(A) \geq 0$ for disconnected sets also. It is certainly possible that one could improve our proof to get $\lambda_1 \leq 0.5895\ldots$, but there is little reason to do so.

REFERENCES

DURRETT, R. and SCHINAZI, R. (1995). Intermediate phase for the contact process on a tree. *Ann. Probab.* **23** 668–673.

GRILLENBERGER, C. and ZIEZOLD, H. (1988). On the critical infection rate of the one dimensional basic contact process: numerical results. *J. Appl. Probab.* **25** 1–8.

HOLLEY, R. and LIGGETT, T. M. (1978). The survival of contact processes. *Ann. Probab.* **6** 198–206.

LIGGETT, T. M. (1985). *Interacting Particle Systems*. Springer, New York.

LIGGETT, T. M. (1995). Improved upper bounds for the contact process critical value. *Ann. Probab.* **23** 697–723.

LIGGETT, T. M. (1997). Branching random walks and contact processes on homogeneous trees. *Probab. Theory Related Fields*. To appear.

MADRAS, N. and SCHINAZI, R. (1992). Branching random walks on trees. *Stochastic Process. Appl.* **42** 255–267.

MORROW, G., SCHINAZI, R. and ZHANG, Y. (1994). The critical contact process on a homogeneous tree. *J. Appl. Probab.* **31** 250–255.

PEMANTLE, R. (1992). The contact process on trees. *Ann. Probab.* **20** 2089–2116.

STACEY, A. M. (1996). The existence of an intermediate phase for the contact process on trees. *Ann. Probab.* **24** 1491–1506.

WU, C. C. (1995). The contact process on a tree: behavior near the first phase transition. *Stochastic Process. Appl.* **57** 99–112.

ZHANG, Y. (1996). The complete convergence theorem of the contact process on trees. Unpublished manuscript.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
LOS ANGELES, CALIFORNIA 90024
E-MAIL: tml@math.ucla.edu