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BOUNDEDNESS OF LEVEL LINES FOR TWO-DIMENSIONAL RANDOM FIELDS

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Every two-dimensional incompressible flow follows the level lines of some scalar function ψ on $\mathbb{R}^2$; transport properties of the flow depend in part on whether all level lines are bounded. We study the structure of the level lines when ψ is a stationary random field. We show that under mild hypotheses there is only one possible alternative to bounded level lines: the "treelike" random fields, which, for some interval of values of a, have a unique unbounded level line at each level a, with this line "winding through every region of the plane." If the random field has the FKG property, then only bounded level lines are possible. For stationary C^2 Gaussian random fields with covariance function decaying to 0 at ∞, the treelike property is the only alternative to bounded level lines provided the density of the absolutely continuous part of the spectral measure decays at ∞ "slower than exponentially," and only bounded level lines are possible if the covariance function is nonnegative.

1. Introduction. Every two-dimensional incompressible flow, described by its velocity field, can be represented as the curl of some scalar potential $\psi(x)$, $x \in \mathbb{R}^2$. This means that each Lagrangian trajectory of the flow is an appropriately parametrized level line of ψ (at least provided there are no critical points of ψ on the level line). The qualitative structure of the level lines is thus of considerable physical importance, as it determines transport properties of the flow, such as heat propagation. Of particular interest is whether there are unbounded trajectories, which contribute to transport, or bounded trajectories only, which in the rescaled limit only contribute to diffusion. In some contexts it is natural to think of the potential as random but fixed in time. In plasma physics, for example, velocity fields can be generated by magnetic fields which change at most very slowly with time. This motivates our study here of the qualitative structure of the level lines of stationary random fields.

There is an extensive physics literature on this subject, generally based on intuition and numerical simulations. The central working hypothesis among physicists, sometimes called the *Sagdeev hypothesis*, is that when the mean of the velocity field is 0, for "typical" two-dimensional velocity fields the Lagrangian trajectory containing a given point is a bounded loop a.s., so that

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transport of passive particles is not possible; see the review [13] and the references therein. The diameter of this loop should also have finite moments of all orders. This contrasts with the assumed situation in three or more dimensions, where unbounded trajectories "typically" exist.

There is a more limited rigorous mathematical literature on the subject. We call a connected component of $\{x \in \mathbb{R}^2: \psi(x) = a\}$ a *level line* of ψ. Molchanov and Stepanov [18] showed that, under mild hypotheses, for sufficiently large a the level lines at level a of a stationary Gaussian random field are all bounded. Avellaneda, Elliot and Apelian [6] considered a particular type of random field built from a randomized lattice, and established boundedness of all level lines and moment bounds on the diameters of trajectories. Alexander and Molchanov [5] studied level line boundedness and gave moment bounds for level line diameters for shot-noise random fields with lattice symmetry, and gave examples in which some properties considered generic in the physics literature were violated.

Alexander and Molchanov [5] also considered random fields ψ associated with stationary random infinite trees in the plane, with $\{x \in \mathbb{R}^2: \psi(x) > a\}$ being a neighborhood of the tree, at least for an interval of values of a. One of our main results will be that, under mild hypotheses, such "treelike" random fields represent the only possible type of violation of the Sagdeev hypothesis. Attention in [5] was restricted to trees in the integer lattice, but the main example is readily modified to remove the lattice, as in the next example. We abbreviate $\{x \in \mathbb{R}^2: \psi(x) = a\}$ henceforth as $\{\psi = a\}$, and similarly for $\{\psi > a\}$ and $\{\psi < a\}$.

EXAMPLE 1.1. It was shown in [2] and [3] that the usual definition (for finite point sets) of the minimal spanning tree can be extended in a natural way to define the minimal spanning tree of the set V of sites of a Poisson process of intensity 1 in the plane. We denote this tree, viewed as a union of line segments in the plane, by T. In [3] it is shown that the tree T a.s. has one topological end; that is, there is a unique path to ∞ in the tree from each site in V. The tree T has a dual tree T^*, defined as follows, the derivation being "valid with probability 1." For each $v \in V$ let $Q(v)$ denote the polygon $\{x \in \mathbb{R}^2: d(x, v) = d(x, V)\}$, where $d(\cdot, \cdot)$ denotes Euclidean distance and $d(x, A) := \inf\{d(x, y): y \in A\}$ for a point x and set A. The vertices and edges of these polygons form a graph G^* called the *Voronoi diagram*; we let W^* denote the set of all sites of G^*, that is, the vertices of the polygons. The *Delaunay triangulation* of the plane is the graph G formed by placing a bond $\langle u, v\rangle$ between sites $u, v \in V$ if and only if $Q(u)$ and $Q(v)$ have an edge in common. Each $w \in W^*$ is a vertex of exactly three Voronoi polygons, say $Q(x)$, $Q(y)$ and $Q(z)$, and then x, y and z are the vertices of a triangle, denoted $R(w)$, of the Delaunay triangulation. There is a bond in G^* between $w, x \in W^*$ if and only if $R(w)$ and $R(x)$ have an edge in common. The graphs G and G^* are dual: for each bond b of G there is a unique bond b^* of G^* to which it is perpendicular, and vice versa. Specifically, for $b = \langle u, v\rangle$, b^* is the common edge of $Q(u)$ and $Q(v)$. Exactly as in the well-known case of finite V, T is a subtree of G. Further,

$$\text{(1.1)} \qquad \begin{array}{l} \text{if } b = \langle u, v\rangle \text{ meets } Q(w) \text{ for some } w \text{ other than } u \text{ and } v, \\ \text{then } b \notin T. \end{array}$$

(See [21] for these and additional facts about the Voronoi diagram and Delaunay triangulation.) We now define T^* to be the subgraph of G^* with bonds $\{b^* \in G^*: b \in G \setminus T\}$; T^* is also viewed as a union of line segments in the plane; by (1.1), T and T^* are disjoint. We claim that, like T, T^* is a one-ended tree. To see that T^* is acyclic, note that any cycle C^* in G^* encloses at least one site of V. Since T spans V, some bond b of T must thus cross C^*, and then $b^* \in C^*$ but $b^* \notin T^*$, so $C^* \not\subset T^*$. To see that T^* is connected, suppose S^* is a component of T^* which spans only a proper subset Y^* of the site set W^*. Let $A := \bigcup_{w \in Y^*} R(w)$. Since S^* is connected, so is A, so ∂A is a closed loop if Y^* is finite, and includes a doubly infinite path if Y^* is infinite. But $\partial A \subset T$, and T is a one-ended tree and so contains no closed loop or doubly infinite path. Thus there can be no such S^*; that is, T^* is connected. Thus T^* is a tree. To see that T^* is one-ended, note that if T^* contained a doubly infinite path γ and $x, y \in V$ were on opposites sides of γ, then there would be no path in T from x to y. This is impossible since T spans V.

We now define a random field ψ on $\mathbb{R}^2$ by

$$\psi(x) := (d(x,T^*) - d(x,T))/(d(x,T^*) + d(x,T)), \qquad x \in \mathbb{R}^2,$$

so $-1 \le \psi(x) \le 1$, $\psi = 1$ on T and $\psi = -1$ on T^*. The following properties are easily seen to hold for each $-1 < a < 1$, a.s. The set $\{\psi > a\}$ is a neighborhood of T; it and its complement are each unbounded and connected. The same holds for the neighborhood $\{\psi < a\}$ of T^*. The set $\{\psi = a\}$ is a single unbounded line and is the boundary of both $\{\psi > a\}$ and $\{\psi < a\}$. In [5] it was shown for the lattice analog that the covariance function $\rho(t)$ for ψ satisfies $\rho(t) \to 0$ as $|t| \to \infty$; essentially the same proof yields the same result here as well.

It is common practice to view a random field $\psi(x)$ as representing the elevation of a landscape at x and some constant a as the water level, so $\{\psi > a\}$ is land and $\{\psi < a\}$ is water. An unbounded component of $\{\psi < a\}$ is an *ocean* at level a; an unbounded component of $\{\psi > a\}$ is a *continent* at level a. An ocean or continent is *k-sided* if its complement has exactly k unbounded components.

Motivated by Example 1.1, we call a random field ψ on $\mathbb{R}^2$ *treelike* if there is a nonempty interval I with endpoints $-\infty \le c \le d \le \infty$ such that, with probability 1, there is exactly one ocean and exactly one continent at each level $a \in I$, there is one continent and no ocean for each $a < c$ and there is one ocean and no continent for each $a > d$. Thus a treelike random field (subject to mild regularity conditions) has no k-sided oceans or continents with $k \ge 2$.

We say ψ is *strongly treelike* if the interval I consists of more than a single point. Thus the random field in Example 1.1 is strongly treelike and the corresponding flow does not satisfy the Sagdeev hypothesis.

We can now formulate the main questions we wish to consider. First, under what hypotheses on a random field are all level lines bounded? Second, under what weaker hypotheses can we conclude that the treelike property is the only possible alternative to all level lines being bounded? We will pay particular attention to the case of Gaussian processes.

2. Main results. We begin with some additional definitions. Throughout, ψ is a continuous stationary random field on $\mathbb{R}^2$, and $\rho(s-t) := \operatorname{cov}(\psi(s), \psi(t))$ is its covariance function. A subset of $\mathbb{R}^n$ is called *increasing* if its indicator is a nondecreasing function of each coordinate separately. A probability distribution in $\mathbb{R}^n$ has the *FKG property* if every pair of increasing subsets has a nonnegative correlation. The random field ψ has the *FKG property* if the distribution of $(\psi(t^{(1)}), \dots, \psi(t^{(n)}))$ has the FKG property for all n and all $t^{(1)}, \dots, t^{(n)} \in \mathbb{R}^2$. A *level line* of ψ is a component of $\{\psi = a\}$ for some a. We say ψ has *bounded level lines* if, with probability 1, every connected component of $\{\psi = a\}$ is bounded for every $a \in \mathbb{R}$. If ψ is C^1, we say $t \in \mathbb{R}^2$ is a *critical point of ψ at level a* if $\psi(t) = a$ and $\nabla\psi(t) = 0$, and we say ψ is *critically regular* if, with probability 1, for each $a \in \mathbb{R}$ there are at most finitely many critical points of ψ at level a.

REMARK 2.1. Critical regularity is slightly stronger than the standard property (see [1]) that

(2.1) for each fixed $a \in \mathbb{R}$, with probability 1, there is no critical point at level a.

More precisely, if for some a there is a positive probability of a critical point of ψ at level a, then by stationarity there is a positive probability that the set of critical points at level a is infinite. But critical regularity of ψ implies critical regularity of almost every ergodic component of ψ; the same does not hold for property (2.1).

We say a random field is *ergodic in each coordinate* if it is ergodic with respect to horizontal and vertical translation separately. Here is our first main result, to be proved later in this section.

THEOREM 2.2. *Suppose ψ is a stationary C^1 random field on $\mathbb{R}^2$, ergodic in each coordinate, with the FKG property and property* (2.1). *Then ψ has bounded level lines.*

Modulo certain regularity conditions, for a given level a, a random field has all level lines bounded provided there is either one ocean and no continent, one continent and no ocean or no continent and no ocean, and the field has the treelike property if there is one continent and one ocean. Thus the level line structure is closely tied to uniqueness of oceans and continents. For lattice percolation models, the concept of finite energy, originated in [20], has been central in establishing uniqueness of the infinite cluster. Here we need an analogous concept for continuous random fields. Let $\psi|_A$ denote the restriction of ψ to the set A, and let $\mathscr{F}_A$ denote the σ-algebra $\sigma(\psi(t), t \in A)$. For a continuous random field ψ and an open rectangle $R \subset \mathbb{R}^2$, conditionally on $\mathscr{F}_{R^c}$, $\psi|_{\overline{R}}$ is a.s. a random element of $C(\overline{R})$. For g continuous on R^c and F a finite subset of ∂R, let $H_{a+}(R, g, F) := \{f \in C(\overline{R})$: $f = g$ on ∂R, and a single component of $\{x \in \overline{R}$: $f(x) > a\}$ contains $F\}$, that is, those functions f, agreeing with g on ∂R, for which every pair of points in F is connected together by a path in $\overline{R}$ on which $f > a$. [Of course, $H_{a+}(R, g, F)$ is empty unless $g > a$ on F.] Define $H_{a-}(R, g, F)$ similarly for $f < a$. We say ψ has *signed finite energy* if, for each $a \in \mathbb{R}$, each open square R and each finite $F \subset \partial R$, there is a σ-field $\mathscr{G}_{R^c} \supset \mathscr{F}_{R^c}$ such that

$$
\begin{aligned}
&P\big(\psi|_{\overline{R}} \in H_{a+}(R, \psi|_{R^c}, F)\big|\mathscr{G}_{R^c}\big) > 0\\
&\qquad\qquad\text{a.s. on the event } \big[\psi(s) > a \text{ for every } s \in F\big],\\
&P\big(\psi|_{\overline{R}} \in H_{a-}(R, \psi|_{R^c, F})\big|\mathscr{G}_{R^c}\big) > 0\\
&\qquad\qquad\text{a.s. on the event } \big[\psi(s) < a \text{ for every } s \in F\big].
\end{aligned}
\tag{2.2}
$$

Note that $H_{a+}(R, \psi|_{R^c}, F)$ is an increasing subset of $C(\overline{R})$, and $H_{a-}(R, \psi|_{R^c}, F)$ is decreasing. Signed finite energy is an interpolation property; a sufficient condition for it is that

(2.3) with probability 1, given $\mathscr{F}_{R^c}$, the support of the distribution of $\psi|_{\overline{R}}$ consists of all functions in $C(\overline{R})$ which agree with $\psi|_{R^c}$ on ∂R.

Sufficient conditions for signed finite energy in the case of Gaussian processes will be given in Section 3.

For $a, b \in \mathbb{R}^2$ let $[a, b]$ denote the closed line segment from a to b. Let $e_1 := (1,0)$ and $e_2 := (0,1)$ be the unit coordinate vectors. Given the C^1 random field ψ, let $N_{\mathrm{cr}}([a, b])$ denote the number of critical points of the function $f(t) = \psi(ta + (1-t)b)$ in the interval $[0,1]$. Here is our second main result, also to be proved later in this section.

THEOREM 2.3. *Suppose ψ is a stationary, critically regular C^1 random field on $\mathbb{R}^2$ with signed finite energy satisfying $EN_{\mathrm{cr}}([0, e_i]) < \infty$, $i = 1, 2$. Then either ψ has bounded level lines or ψ is treelike.*

The main step in proving Theorem 2.3 is establishing uniqueness of oceans and continents. Note this uniqueness must a.s. hold simultaneously for all a, not just hold a.s. for each fixed a. An analogous simultaneous uniqueness property for infinite clusters in coupled percolation models was considered in [4]. Example 1.7 of that paper showed that even a.s. uniqueness for each fixed a and finite energy together are not enough to guarantee a.s. simultaneous uniqueness. By analogy, we expect that some additional condition, like the requirement $EN_{\mathrm{cr}}([0, e_i]) < \infty$ in Theorem 2.3, is needed here as well, to restrict how tightly packed the distinct components of $\{\psi > a\}$ can be.

The *fill* of a set B, denoted $\mathrm{fill}(B)$, is the union of B and all bounded components of B^c.

REMARK 2.4. In some of our proofs it is desirable to know that a connected set in $\mathbb{R}^2$ with a connected complement must have a connected boundary. For general sets this is a difficult question involving the properties of local connectedness and simple connectedness—see [22], Chapter 6. But when the set in question is, for example, $\mathrm{fill}(\Theta_a)$ for some component Θ_a of $\{\psi > a\}$ and ψ is critically regular, there is no problem, as the boundary is the image of a continuous curve consisting of finitely many C^1 segments.

To prove Theorem 2.3, we will need some preliminary results. We begin with some observations about the level lines of treelike stationary random fields. We say that a subset A of $\mathbb{R}^2$ *winds through every region of* $\mathbb{R}^2$ if

$$\lim_{M\to\infty} \liminf_{T\to\infty} T^{-2} \int_{[-T,T]^2} 1_{[d(x,A)\le M]}\, dx = 1.$$

That is, for sufficiently large M, the fraction of $\mathbb{R}^2$ that is within distance M of A is close to 1. Let $N(A)$ denote the number of components of a set A, let $N_\infty(A)$ denote the number of unbounded components and let $N_\infty(A; B)$ denote the number of unbounded components of A which intersect another set B. Part (i) of the next lemma is included to support the idea that treelike fields are an oddity rather than "typical"; it says roughly that in a treelike random field the coexisting ocean and continent at each level must be extremely intertwined, as in Example 1.1, because nearly all of $\mathbb{R}^2$ is reasonably close to the "beach" where the ocean and continent meet.

LEMMA 2.5. *Suppose ψ is a stationary random field on $\mathbb{R}^2$.*

(i) *For each $a \in \mathbb{R}$, with probability 1, if there is a unique unbounded level line at level a, then that level line winds through every region of $\mathbb{R}^2$.*

If ψ is C^1 and critically regular and $N_{\mathrm{cr}}([0, e_i]) < \infty$ a.s., $i = 1, 2$, then, with probability 1, the following conclusions hold for each $a \in \mathbb{R}$.

(ii) *If Γ is an unbounded level line at level a and R is an open rectangle, then $N_\infty(\Gamma \cap R^c) \ge 1$.*

(iii) *If G is a neighborhood of an unbounded level line at level a, then G contains an unbounded connected set where $\psi > a$ and an unbounded connected set where $\psi < a$.*

(iv) *The following are equivalent*: (a) *There is a unique unbounded level line Γ_a at level a, and for every open rectangle R, $N_\infty(\Gamma_a \cap R^c) = 1$ or* 2. (b) *There are a unique ocean and a unique continent at level a.*

PROOF. For part (i), we may assume ψ is ergodic. The result is then an immediate consequence of the multidimensional ergodic theorem (see [15], Section 6.2) and the fact that, if A is the unique unbounded component of $\{\psi = a\}$ at some level a, then $\lim_{M\to\infty} P(d(x, A) \le M) = 1$.

For (ii), we may assume R is large enough to contain all critical points of ψ at level a. Let S be a closed rectangle containing $\overline{R}$ in its interior, let C be a component of $(S \setminus R) \cap \{\psi = a\}$ and fix $x \in C$. By the inverse function theorem, x has a neighborhood D such that $D \cap (S \setminus R) \cap \{\psi = a\} \subset C$. Let A be a set containing one point from each component of $\{\psi = a\} \cap (S \setminus R)$. Then D contains at most one point of A, so x is not an accumulation point of A; since C and x are arbitrary, it follows that A has no accumulation points, so A is finite. Therefore $N(\Gamma \cap R^c)$ is finite, and (ii) follows.

Turning to (iii), let Γ be an unbounded level line at level a and R an open rectangle containing all critical points of ψ at level a. By (ii), $\Gamma \cap R^c$ has an unbounded connected component Π. The result now follows easily from the inverse function theorem applied at each point of Π, together with local compactness of Π.

To prove (iv), let R be an open rectangle containing all critical points of ψ at level a. Suppose (a) holds. A point in the intersection of the boundaries of two components A and B of $\{\psi > a\}$ is a critical point of ψ, so there are no such points outside R. If A and B are both unbounded and R is large enough to intersect both A and B, then $N_\infty(\partial A \cap R^c) \ge 2$, $N_\infty(\partial B \cap R^c) \ge 2$, so $N_\infty(\Gamma_a \cap R^c) \ge 4$, a contradiction, so there is at most one continent at level a; by (iii), there is exactly one. Similarly, there is exactly one ocean; that is, (b) holds.

Conversely suppose (b) holds, and let Θ_a be the unique continent at level a. Note that the components of $\mathrm{fill}(\Theta_a)^c$ are the unbounded components of Θ_a^c. Let A and B be components of $\mathrm{fill}(\Theta_a)^c$. Let C be the component of Θ_a in A^c, and let $D \ne A$ be a component of Θ_a^c. Then $\partial D \subset \partial\Theta_a \cap D$, so the latter is nonempty and we must have $D \subset C$. Since D is arbitrary this means $C = A^c$, so A^c is connected. Since A and A^c are unbounded and connected (cf. Remark 2.4) so is ∂A. By (iii), A contains an ocean; similarly so does B, so $A = B$ is unique, and $\partial A = \partial\,\mathrm{fill}(\Theta_a)$. Let Γ_a be the component of $\partial\,\mathrm{fill}(\Theta_a)$ in $\{\psi = a\}$. If Γ_a' is any unbounded component of $\{\psi = a\}$, then $\Gamma_a' \subset A$ by uniqueness of A, but, by (iii), every neighborhood of Γ_a' meets an ocean, so $\Gamma_a' \subset \partial A$ and hence $\Gamma_a' = \Gamma_a$. This proves uniqueness of the unbounded level line. If $N_\infty(\Gamma_a \cap R^c) > 2$, then, after enlarging R if necessary, $\Gamma_a \cap R^c$ includes three or more disjoint C^1 paths from ∂R to ∞, and, by the proof of (iii), each of these paths has an ocean on one side and a continent on the other. This means $\{\psi > a\} \cap R^c$ has two unbounded components A_1 and A_2 intersecting ∂R, and $\{\psi < a\} \cap R^c$ has two unbounded components B_1 and B_2 intersecting ∂R, arranged so that, traversing ∂R clockwise, we enter A_1, then B_1, then A_2 and then B_2. Since A_1 and A_2 are part of the unique ocean Θ_a, they are connected together inside R; similarly so are B_1 and B_2. But this is impossible, so $N_\infty(\Gamma_a \cap R^c) = 2$. □

We call a square $R \subset \mathbb{R}^2$ a *branch node* of a set $A \subset \mathbb{R}^2$ if there are three or more unbounded components of $A \cap R^c$, each containing a path from ∂R to ∞, which are all part of a single component of A. [Here by a path from ∂R to ∞ we mean the image of a continuous mapping $\gamma\colon [0,\infty) \to \mathbb{R}^2$ with $\gamma(0) \in \partial R$ and $|\gamma(u)| \to \infty$ as $u \to \infty$. Such a path need not exist even if A is connected, unbounded and open, and A meets ∂R.] Recall that an unbounded connected open set is called *k-sided* if its complement has exactly k unbounded components. Let $\Lambda_M(x)$ denote the translate $x + (-M, M)^2$ [or $x + (-M, M)^d$ if we are in general dimension d]. We call a function A on a probability space for which the values are open subsets of $\mathbb{R}^2$ a *random open set* if the event $[B \subset A]$ is measurable for each open ball B in $\mathbb{R}^2$. Since $\mathbb{R}^2$ is separable, this is sufficient to make all events in the next lemma and its proof measurable. The proof is an adaptation of the proof of the main theorem of [8].

LEMMA 2.6. *Suppose A is a stationary random open set in $\mathbb{R}^2$ with $EN([0, e_i] \cap A) < \infty$, $i = 1, 2$. Then, with probability 1, there are no branch nodes of A in $\mathbb{R}^2$.*

PROOF. We may assume A is ergodic. If a given square is a branch node, then so is any square containing it. Therefore is it enough to show that $P(\Lambda_M(0)$ is a branch node$) = 0$ for all $M > 0$. Fix $M > 0$ and let B_k be the number of branch nodes of A of the form $\Lambda_M(2Mx)$, $x \in \mathbb{Z}^2$, contained in $\Lambda_{(2k+1)M}(0)$. Then, as in the lattice analog in [8],

$$N\big(\partial\Lambda_{(2k+1)M}(0) \cap A\big) \geq N_\infty\Big(\Lambda_{(2k+1)M}(0)^c \cap A;\, \partial\Lambda_{(2k+1)M}(0)\Big) \geq B_k + 2.$$

Now

$$\begin{aligned} EN\big(\partial\Lambda_{(2k+1)M}(0) \cap A\big) &\leq 2M(2k+1)EN([0, e_1] \cap A) \\ &\quad + 2M(2k+1)EN([0, e_2] \cap A), \end{aligned}$$

so letting $k \to \infty$,

$$E \liminf B_k/k \leq \limsup EB_k/k \leq \limsup EN\big(\partial\Lambda_{(2k+1)M}(0) \cap A\big)/k < \infty,$$

and therefore,

$$\liminf B_k/k < \infty \quad \text{a.s.} \tag{2.4}$$

But, by the multidimensional ergodic theorem (see [15], Section 6.2),

$$\lim_{k\to\infty} B_k/(2k+1)^2 = P(\Lambda_M(0) \text{ is a branch node}) \quad \text{a.s.} \tag{2.5}$$

Together, (2.4) and (2.5) yield $P(\Lambda_M(0)$ is a branch node$) = 0$. $\square$

The next proposition will be used here only to avoid certain technicalities, but we include it because it may be of independent interest. It is related to [4], Lemma 2.2, which says roughly that in stationary random graphs, any property of a vertex that occurs for only finitely many vertices in each infinite component actually never occurs. It is also similar in spirit to [11], Lemma 8. Let card(B) denote the cardinality of a set B.

PROPOSITION 2.7. *Suppose* Λ *is a stationary random open set in* $\mathbb{R}^d$, *satisfying*

$$EN_\infty(\Lambda;\Omega) < \infty \quad \textit{for some bounded rectangle } \Omega \subset \mathbb{R}^d. \tag{2.6}$$

Then, *with probability* 1,

$$\textit{every unbounded component of } \Lambda \textit{ has infinite volume}, \tag{2.7}$$

and for every locally finite stationary point process X *in* $\mathbb{R}^d$ (*viewed as a countable subset of* $\mathbb{R}^d$), *with probability* 1,

$$\mathrm{card}(X \cap \overline{C}) = 0 \textit{ or } \infty \quad \textit{for every unbounded component } C \textit{ of } \Lambda. \tag{2.8}$$

PROOF. Observe that (2.7) follows from (2.8) when we take X to be a Poisson process independent of Λ. Therefore we will prove (2.8). Also, since $N_\infty(\Lambda;\cdot)$ is monotone and subadditive as a set function, we may replace "some" with "every" in (2.6). Further, we may assume the pair (Λ, X) is ergodic.

Let us call a box $\Lambda_M(x)$ a *containment box* if there exists an unbounded component C of Λ for which $X \cap \overline{C} \subset \Lambda_M(x)$. To prove (2.8), it is sufficient to show that $P(\Lambda_M(0)$ is a containment box$) = 0$ for all $M > 0$. This can be done similarly to the proof of Lemma 2.6. Fix $M > 0$ and let B_k be the number of containment boxes of the form $\Lambda_M(2Mx)$, $x \in \mathbb{Z}^d$, contained in $\Lambda_{(2k+1)M}(0)$. Let $V_k := \{x \in \mathbb{Z}^d\colon \max(|x_1|,\dots,|x_d|) = k+1\}$. Then

$$\begin{aligned}
\big((2k+3)^d - (2k+1)^d\big)&EN(\Lambda;\Lambda_M(0))\\
&= E\sum_{x\in V_k} N(\Lambda;\Lambda_M(x))\\
&\geq EN\big(\Lambda;\Lambda_{(2k+3)M}(0)\setminus\Lambda_{(2k+1)M}(0)\big)\\
&\geq EB_k.
\end{aligned}$$

It follows that

$$P\big(\Lambda_M(0) \text{ is a containment box}\big) = \lim_k EB_k/(2k+1)^d = 0,$$

as desired. □

In (2.8) one can replace $X \cap \overline{C}$ with any subset X_C of X, so long as the subset X_C are associated with the components C in a translation-invariant way.

Being stationary, the random field ψ is a mixture of its ergodic components. To facilitate reduction to the ergodic case, we need the next result, an immediate consequence of [4], Lemma 2.1, once it is adapted (straightforwardly) to the present situation. The result [4], Lemma 2.1, in turn, is an analog of [11], Lemma 1.

LEMMA 2.8. *If ψ is a stationary random field with signed finite energy, then every ergodic component of ψ has signed finite energy.*

Let A° denote the interior of A.

REMARK 2.9. For a critically regular stationary random field ψ and a fixed level a, by Remark 2.1 there are a.s. no critical points of ψ at level a. For $A = \{\psi > a\}$, this means that, for each component C of A or A^c and each component Γ of ∂C, the following holds: (i) Γ is C^1-diffeomorphic to $\mathbb{R}$ if Γ is unbounded, and to S^1 if Γ is bounded; (ii) every neighborhood of Γ contains a smaller neighborhood which is the union of an open connected subset of A, an open connected subset of A^c and Γ itself; and (iii) C° is connected. Further, only finitely many components of A or A^c can intersect a given bounded region. We call any set A with these properties *structurally regular*. Note A is structurally regular if and only if A^c is, and in this case both $\overline{A}$ and A° are structurally regular.

REMARK 2.10. If A is structurally regular and some component C of A is k-sided with $k \geq 3$, then any open square which intersects at least three unbounded components of C^c is a branch node of A.

We next prove a.s. uniqueness of the unbounded component of $\{\psi > a\}$ for each fixed a. Later this will be used to help establish a.s. simultaneous uniqueness for all a. In comparing hypotheses of this and other lemmas, it is useful to observe that

$$N([0, e_i] \cap \{\psi > a\}) \leq N_{\mathrm{cr}}([0, e_i]) + 2 \quad \text{for all } a \in \mathbb{R}. \tag{2.9}$$

The following proof is adapted from analogous results in [19] and [8].

LEMMA 2.11. *Suppose ψ is a stationary, critically regular C^1 random field with signed finite energy satisfying $EN_{\mathrm{cr}}([0, e_i]) < \infty$, $i = 1, 2$. For each $a \in \mathbb{R}$, with probability 1, $\{\psi > a\}$ has at most one unbounded component.*

PROOF. By Lemma 2.8 we may assume ψ is ergodic. Fix $a \in \mathbb{R}$. Then there is an n such that $N_\infty(\{\psi > a\}) = n$ a.s. Suppose $2 \leq n < \infty$. If R is a sufficiently large open square, then there is a positive probability that R intersects all n unbounded components, and hence a positive probability of the event that there are n or more unbounded components of $\{\psi > a\} \cap R^c$, and all of them intersect ∂R. Because of signed finite energy, given the latter event, there is a positive probability that the n or more unbounded components of $\{\psi > a\} \cap R^c$ are all part of a single component of $\{\psi > a\}$, in which case $N_\infty(\{\psi > a\}) = 1$, a contradiction. Thus $n = 0$, 1 or ∞.

If $n = \infty$, then for a sufficiently large open square R there is a positive probability that R intersects at least three distinct components of $\{\psi > a\}$, and hence a positive probability of the event D that there are at least three unbounded components of $\{\psi > a\} \cap R^c$ which intersect ∂R. Since $\{\psi > a\}$ is a.s. structurally regular in the sense of Remark 2.9, each unbounded component of $\{\psi > a\} \cap R^c$ then contains a path from ∂R to ∞. Because of signed finite energy, given the latter event, there is a positive probability that three or more unbounded components of $\{\psi > a\} \cap R^c$ are all part of a single component of $\{\psi > a\}$, in which case R is a branch node of $\{\psi > a\}$. Thus, by Lemma 2.6 and (2.9), we do not have $n = \infty$, and the lemma follows. □

We will need the following topological fact. To keep things simple, we make much stronger regularity assumptions than are really necessary.

LEMMA 2.12. *Suppose $A \subset \mathbb{R}^2$ is open and structurally regular and every component of A is unbounded and one-sided. Then* $\mathrm{fill}(A)^c$ *is unbounded and connected.*

PROOF. Let $\mathscr{C}$ be the set of all components of A, and fix $C_0 \in \mathscr{C}$. Let D be the component of $\partial\,\mathrm{fill}(C_0)$ in $\mathrm{fill}(A)^c$, and let $E := D \cup \bigcup\{\mathrm{fill}(C)\colon C \in \mathscr{C}, \partial\,\mathrm{fill}(C) \cap D \neq \varnothing\}$. Since $\partial\,\mathrm{fill}(C)$ is connected for $C \in \mathscr{C}$, if $\partial\,\mathrm{fill}(C) \cap D \neq \varnothing$, then $\partial\,\mathrm{fill}(C) \subset D$. Since A is open, so is $\mathrm{fill}(A)$, so D is closed.

We claim that E is closed. Since each bounded region intersects at most finitely many $C \in \mathscr{C}$, a limit point x of E is either a limit point of D or a limit point of some $C \in \mathscr{C}$ with $\partial\,\mathrm{fill}(C) \cap D \neq \varnothing$. In the first case we have $x \in D$. In the second case we have either $x \in \mathrm{fill}(C)$ or $x \in \partial\,\mathrm{fill}(C) \subset D$. In both cases $x \in E$, so E is closed.

We claim that E is open. Since $\bigcup\{\mathrm{fill}(C)\colon C \in \mathscr{C}, \partial\,\mathrm{fill}(C) \cap D \neq \varnothing\}$ is open, it is sufficient to show that a fixed $y \in D$ is not a limit point of E^c. Let R be a square centered at y. There are at most finitely many components of $\mathrm{fill}(A)^c \cap R$, and these are separated by strictly positive distances, so y is not a limit point of $\mathrm{fill}(A)^c \cap D^c$. If y is a limit point of $\bigcup\{\mathrm{fill}(C)\colon C \in \mathscr{C}, \partial\,\mathrm{fill}(C) \cap D = \varnothing\}$, then, since each bounded region intersects only finitely many sets $\mathrm{fill}(C)$ with $C \in \mathscr{C}$, y must be a limit point of $\mathrm{fill}(C)$ for a single $C \in \mathscr{C}$ with $\partial\,\mathrm{fill}(C) \cap D = \varnothing$. But then $y \notin \partial\,\mathrm{fill}(C)$ and $y \notin \mathrm{fill}(C)$, a contradiction. Thus y is not a limit point of E^c.

Therefore $E = \mathbb{R}^2$, meaning $\mathrm{fill}(A)^c = D$ which is connected. Since $\partial\,\mathrm{fill}(C_0)$ is unbounded, so is D. □

Let $a_c := \sup\{a\colon N_\infty(\{\psi > a\}) \geq 1 \text{ a.s.}\}$ and $a_c' := \inf\{a \in \mathbb{R}\colon N_\infty(\{\psi < a\}) \geq 1$ a.s.$\}$. The following simultaneous-uniqueness result is related to [4], Theorem 1.8, but the proof is simpler here because we work in two dimensions only.

PROPOSITION 2.13. *Suppose ψ is a stationary, critically regular C^1 random field on $\mathbb{R}^2$ with signed finite energy satisfying*

$$EN_{\mathrm{cr}}([0, e_i]) < \infty, \qquad i = 1, 2. \tag{2.10}$$

Then

$$P[\text{for each } a \in \mathbb{R}, \{\psi > a\} \text{ has at most one unbounded component}] = 1.$$

PROOF. By Lemma 2.8 we may assume ψ is ergodic. By Lemma 2.5(iii), Remark 2.1 and Lemma 2.11 (applied to both ψ and $-\psi$), with probability 1, for each $q \in \mathbb{Q}$ there exist no critical points at level q, so $\{\psi > q\}$ is structurally regular, and $\{\psi > q\}$ has at most one unbounded component, with that component, if any, being zero-sided or one-sided. Since $\{\psi > a\}$ decreases as a increases, the conclusion of the proposition a.s. holds simultaneously for all $a > a_c$. Therefore we assume $a_c > -\infty$.

The following is valid with probability 1. Since $\{\{\psi > q\}: q \in \mathbb{Q}, q < a_c\}$ is a nested collection of connected sets, for each real $a < a_c$ there exists a unique unbounded component M_a of $\{\psi > a\}$ containing every unbounded component of $\{\psi > b\}$ for every real $b > a$; we call M_a the *mainland continent* at level a, and call any other unbounded component of $\{\psi > a\}$ an *Atlantis continent* at level a. Our goal is to show that Atlantis continents do not exist. Note that an Atlantis continent at any level a cannot contain an unbounded component of $\{\psi > b\}$ for any $b > a$. (An Atlantis continent at level a may be pictured as follows: as the water level drops, at precisely level a some bounded islands coalesce to form a new continent outside the mainland continent. But as soon as the water level drops below a, this new continent merges with the mainland continent.) Therefore, if $x \in A$ for some Atlantis continent A at level a, then $a = \sup\{b: x \text{ is in an unbounded component of } \{\psi > b\}\}$. In particular, there exists at most one such a for each x; thus any two Atlantis continents, even at different levels, are disjoint, and hence separated. If A is an Atlantis continent at level a, then $\psi(x) = a$ for every $x \in \partial A$.

Let H denote the union of all Atlantis continents at all levels. By critical regularity there are at most finitely many critical points of ψ on the boundary of each Atlantis continent. Now $N([0, e_i] \cap H) \le N_{\mathrm{cr}}([0, e_i]) + 2$, so, by (2.10) and stationarity, we have $EN(\partial R \cap H) < \infty$ for every square R a.s., so (2.6) holds for H. By Proposition 2.7, with probability 1, there are therefore no critical points of ψ on the boundary of any Atlantis continent. Hence H is structurally regular. Therefore, by Lemma 2.6 and Remark 2.10, every Atlantis continent is at most two-sided.

Let H_1 denote the union of the fills of all one-sided Atlantis continents at all levels. As with H, H_1 is structurally regular and hence so are H_1^c and $(H_1^c)^\circ$. Let us show that one-sided Atlantis continents do not exist. By Lemma 2.12, $H_1^c = \mathrm{fill}(H_1)^c$ is unbounded and connected, hence so is $(H_1^c)^\circ$. Since $N([0, e_i] \cap (H_1^c)^\circ) \le N_{\mathrm{cr}}([0, e_i]) + 2$, (2.10), Lemma 2.6 and Remark 2.10 show that $(H_1^c)^\circ$ is at most two-sided, so there are at most two one-sided Atlantis continents. If there are only one or two one-sided Atlantis continents, then, by ergodicity, there is a constant c such that the level of the lowest one-sided Atlantis continent is c a.s. But, by Lemma 2.11, there are a.s. no Atlantis continents at level c, so there are a.s. no one-sided Atlantis continents.

It remains to rule out two-sided Atlantis continents. Fix $q < a_c$ and let $\mathscr{Q}_q$ be the set of all two-sided Atlantis continents at levels less than q. Suppose $\mathscr{Q}_q$ is nonempty, so that M_q is one-sided. Let $\Gamma_q := \partial\, \mathrm{fill}(M_q)$ be the unique unbounded component of ∂M_q. Fix $B \in \mathscr{Q}_q$, and let γ be a path from Γ_q to ∂B in $(M_q \cup B)^c$ consisting of finitely many horizontal and vertical line segments

$[c, d]$, each necessarily with $N_{\mathrm{cr}}([c, d]) < \infty$. The latter condition means that γ can intersect only finitely many Atlantis continents, so there exists $A \in \mathscr{Q}_q$ for which there is a path from ∂A to Γ_q which intersects no two-sided Atlantis continent. Let us call such an Atlantis continent *q-adjacent*.

Let J_q be the union of M_q and the fills of all q-adjacent Atlantis continents, and let C be the component of J_q^c which contains Γ_q. As with H, J_q^c is structurally regular. If A is the fill of a q-adjacent Atlantis continent, then there is a path from a (necessarily unbounded) component Γ_A of ∂A to Γ_q which intersects no two-sided Atlantis continent, so this path is contained in C. Therefore $\Gamma_A \subset \partial C$. Since A is arbitrary, if there are $n \le \infty$ q-adjacent Atlantis continents, this means C, and hence C°, are at least $(n+1)$-sided. But $N([0, e_i] \cap C^\circ) \le N_{\mathrm{cr}}([0, e_i]) + 1$, so, by (2.10), Lemma 2.6 and Remark 2.10, we must have $n \le 1$. But, as with one-sided Atlantis continents, there cannot be a.s. only one q-adjacent Atlantis continent, so $\mathscr{Q}_q$ must be empty a.s. Since $q < a_c$ is arbitrary, there are a.s. no two-sided Atlantis continents having level below a_c. By Lemma 2.11 there are a.s. no Atlantis continents at level a_c, so there are a.s. no two-sided Atlantis continents at any level. □

PROOF OF THEOREM 2.3. By Lemma 2.8 we may assume ψ is ergodic.

If $a_c' < a_c$, then from Proposition 2.13, applied to ψ and $-\psi$, we see that, with probability 1, for each $a \in (a_c', a_c)$ there is a unique continent and a unique ocean, while for $a > a_c$ there is one ocean and no continent and for $a < a_c'$ there is a continent and no ocean. Thus ψ is strongly treelike.

If $a_c' = a_c$, then at level a_c either there is one continent and one ocean, in which case ψ is treelike, or one of the two does not exist, in which case Lemma 2.5(iii) shows that ψ has bounded level lines.

If $a_c' > a_c$, then for each $a \in \mathbb{R}$ oceans and continents do not coexist, so, by Lemma 2.5(iii) ψ has bounded level lines. □

PROOF OF THEOREM 2.2. Fix $a \in \mathbb{R}$, and suppose $N_\infty(\{\psi > a\}) \ge 1$ a.s. Since $\{\psi > a\}$ is open, we have $P([0, he_i] \subset \{\psi > a\}) > 0$, $i = 1, 2$, for some $h > 0$. By the FKG property, the same is true for every $h > 0$. With this observation, the proof of the main theorem of [12] goes through essentially unchanged to show that, with probability 1, every square $R \subset \mathbb{R}^2$ is encircled by $\{\psi > a\}$ [i.e., there is a continuous mapping γ of S^1 into $\{\psi > a\} \cap R^c$ such that R is in a bounded component of $\mathbb{R}^2 \setminus \gamma(S^1)$]. Therefore, with probability 1, at level a there is exactly one continent and no ocean. Thus $a < a_c$ implies $a \le a_c'$, so $a_c \le a_c'$. If $b < a < a_c'$, then an unbounded level line at level b would be part of an ocean at level a, so, since $a < a_c'$ is arbitrary,

$$P(\text{there exists an unbounded level line at some level } b < a_c') = 0,$$

and similarly

$$P(\text{there exists an unbounded level line at some level } b > a_c) = 0.$$

It remains to consider the level a_c when $a_c = a_c'$. Since continents and oceans a.s. do not coexist at level a_c, and, by (2.1), $\{\psi > a_c\}$ is a.s. structurally regular, as in Lemma 2.5(iii), we have a.s. only bounded level lines at level a_c as well. □

3. Gaussian random fields. We consider now conditions under which a Gaussian random field satisfies the hypotheses of Theorem 2.2 or 2.3. Sufficient conditions for the C^1 hypothesis are summarized in the next lemma; these are standard (see [1] and [23]) so we include no proof. We denote first- and second-order partial derivatives of ψ (when existing at least in the L^2 sense) and of ρ using subscripts; for example, $\psi_i(t) = \partial\psi/\partial t_i$ and $\psi_{ij}(t) = \partial^2\psi/\partial t_i\,\partial t_j$. We denote the spectral measure of ψ by Δ and write $\log^+ x$ for $\max(\log x, 0)$.

LEMMA 3.1. *For a stationary Gaussian random field ψ on $\mathbb{R}^2$, the following are equivalent*:

(i) $\rho_{ii}(0)$ *exists and is finite for all i*;
(ii) $\rho_{ij}(t)$ *exists and is finite for all t, i and j*;
(iii) $\psi(t)$ *is differentiable at every t in the L^2 sense*;
(iv) $\rho(t) = \rho(0) - t^T\Sigma t/2 + o(|t|^2)$ *as $t \to 0$ for some nonnegative definite Σ*;
(v) $\int_{\mathbb{R}^2}|x|^2\, d\Delta(x) < \infty$.

Under (i)–(v) *the L^2 partial derivatives have covariances*

$$\operatorname{cov}\big(\psi_i(t), \psi(s)\big) = \rho_i(t-s) \tag{3.1}$$

and

$$\operatorname{cov}\big(\psi_i(t), \psi_j(s)\big) = -\rho_{ij}(t-s) \tag{3.2}$$

and ψ_i has spectral measure

$$d\Delta_{(i)}(x) = x_i^2\, d\Delta(x).$$

If also

$$\sup_{|h|\le u} \big|\, \rho_{ii}(0) - \rho_{ii}(he_i)\big| = O\big((\log 1/u)^{-(1+\varepsilon)}\big) \quad \text{as } u \to 0, \quad i = 1,2, \tag{3.3}$$

for some $\varepsilon > 0$, then ψ has a version which is C^1. A sufficient condition for (3.3) *is*

$$\int_{\mathbb{R}^2}|x|^2(\log^+|x|)^{(1+\varepsilon)}\, d\Delta(x) < \infty.$$

COROLLARY 3.2. *For a stationary Gaussian random field ψ on $\mathbb{R}^2$, if*

$$\sup_{|h|\le u} \big|\, \rho_{iijj}(0) - \rho_{iijj}(he_j)\big| = O\big((\log 1/u)^{-(1+\varepsilon)}\big) \quad \text{as } u \to 0, \qquad i,j = 1,2, \tag{3.4}$$

for some $\varepsilon > 0$, *then* ψ *has a version which is* C^2. *A sufficient condition for* (3.4) *is*

$$\int_{\mathbb{R}^2} |x|^4 (\log^+ |x|)^{(1+\varepsilon)} \, d\Delta(x) < \infty.$$

LEMMA 3.3. *For a stationary* C^1 *Gaussian random field* ψ *on* $\mathbb{R}^2$, *if property* (2.1) *does not hold*, *then* ψ *is a.s. constant on* $\mathbb{R}^2$.

PROOF. Suppose (2.1) does not hold. By [1], Theorem 3.2.1, the covariance matrix of $(\psi(0), \psi_1(0), \psi_2(0))$ is singular. From (3.1), this means that either $\rho(0) = 0$ [in which case $\psi(t) = E\psi(0)$ for all t a.s.] or $\rho(0) > 0$ and there is a nonzero $v \in \mathbb{R}^2$ such that $(\psi_1(0), \psi_2(0)) \cdot v = 0$ a.s. In the latter case, we may assume $v = e_1$, and then ψ is a.s. constant on every horizontal line; that is, $\psi((t_1, t_2)) = Z(t_2)$ a.s. for some C^1 stationary Gaussian process Z on $\mathbb{R}$. Now (2.1) must also fail for Z, so, by [1], Theorem 3.2.1, the covariance matrix of $(Z(0), Z'(0))$ is singular, which, by (3.1), means we have $Z'(0) = 0$ a.s. Therefore Z, and hence ψ, are a.s. constant. □

It is now a simple matter to give conditions for the application of Theorem 2.2—the main requirement is nonnegative correlations.

THEOREM 3.4. *Suppose* ψ *is a stationary* C^1 *Gaussian random field on* $\mathbb{R}^2$ *with* $\rho(0) > 0$, $\rho(t) \geq 0$ *for all* t *and* $\rho(t) \to 0$ *as* $|t| \to \infty$. *Then* ψ *has bounded level lines*.

PROOF. It is proved in [9] that, in the one-dimensional case, $\rho(t) \to 0$ as $|t| \to \infty$ implies ergodicity. As pointed out by Adler [1], Theorem 6.5.4, the same proof applies in all dimensions, and in fact it yields ergodicity in each coordinate, since all that is needed is that an arbitrary event can be approximated by a finite-dimensional event, and any finite-dimensional event is asymptotically independent of its translates in a fixed direction when $\rho(t) \to 0$ as $|t| \to \infty$.

Since $\rho \geq 0$, by a result of Pitt [20], ψ has the FKG property. By the preceding remarks ψ is ergodic in each coordinate. By Lemma 3.3, (2.1) holds. The conclusion now follows from Theorem 2.2. □

LEMMA 3.5. *If* ψ *is a* C^1 *Gaussian random field on* $\mathbb{R}^2$ *with* $\rho(0) > 0$ *and* $\rho(t) \to 0$ *as* $|t| \to \infty$, *then* $\rho_i(t) \to 0$ *and* $\rho_{ij}(t) \to 0$ *as* $|t| \to \infty$, $i, j = 1, 2$. *If* ψ *is also* C^2, *then the distribution of* $(\psi_{11}(t), \psi_{12}(t), \psi_{22}(t))$ *is nondegenerate*.

PROOF. The first statement follows easily from the fact that, by (3.1) and (3.2), ρ_i and ρ_{ij} are both uniformly continuous functions. For the second statement, observe that, by (3.2), the determinant of the covariance matrix $\Sigma^{(2)}$ of $(\psi_{11}(t), \psi_{12}(t), \psi_{22}(t))$ is $\rho_{1122}(0)[\rho_{1111}(0)\rho_{2222}(0) - \rho_{1122}(0)^2]$. If

$$0 = \rho_{1122}(0) = \int_{\mathbb{R}^2} x_1^2 x_2^2 \, d\Delta(x),$$

then Δ is concentrated on the axes, so ψ can be expressed as the sum of two orthogonal stationary processes, each corresponding to the portion of Δ on one axis: $\psi((t_1, t_2)) = \alpha(t_1) + \beta(t_2)$. But then $\rho((r,0)) + \rho((0,r)) - \rho((r,r)) = \mathrm{var}(\alpha(0)) + \mathrm{var}(\beta(0)) = \rho((0,0))$ for all $r \in \mathbb{R}$, so $\rho(t) \nrightarrow 0$ as $|t| \to \infty$. If

$$0 = \rho_{1111}(0)\rho_{2222}(0) - \rho_{1122}(0)^2$$
$$= \left(\int_{\mathbb{R}^2} x_1^4 \, d\Delta(x)\right)\left(\int_{\mathbb{R}^2} x_2^4 \, d\Delta(x)\right) - \left(\int_{\mathbb{R}^2} x_1^2 x_2^2 \, d\Delta(x)\right)^2,$$

then there exists c such that $x_1^2 = c x_2^2$ a.e. (Δ) or $c x_1^2 = x_2^2$ a.e. (Δ); we may assume the former. If $c < 0$, then $x_1 = x_2 = 0$ a.e. (Δ) so $\rho(t) = \rho(0)$ for all t. If $c = 0$, then $x_1 = 0$ a.e. (Δ) so $\rho((r,0)) = \rho((0,0))$ for all r so $\rho(t) \nrightarrow 0$ as $|t| \to \infty$. If $c > 0$, then Δ is concentrated on the lines $x_1 = c^{1/2} x_2$ and $x_1 = -c^{1/2} x_2$, and, analogously to the above case of Δ concentrated on the axes, we obtain $\rho(t) \nrightarrow 0$ as $|t| \to \infty$. Thus, when $\rho(t) \to 0$ as $|t| \to \infty$, we have $\det(\Sigma^{(2)}) \neq 0$. $\square$

Let $\mathrm{diag}(\lambda_1, \dots, \lambda_n)$ denote the diagonal matrix with diagonal entries $\lambda_1, \dots, \lambda_n$.

LEMMA 3.6. *If ψ is a C^2 Gaussian random field on $\mathbb{R}^2$ with $\rho(0) > 0$ and $\rho(t) \to 0$ as $|t| \to \infty$, then ψ is critically regular.*

PROOF. If the matrix Σ of Lemma 3.1(iv) is singular, then the directional derivative of ψ at 0 in some direction v is 0 a.s. We may assume $v = e_1$; then, by (3.2), $\psi_1(t) = 0$ for all t, so ψ is constant on horizontal lines, so $\rho(t) \nrightarrow 0$ as $|t| \to \infty$. Thus Σ is nonsingular. By rotating axes if necessary, we may assume $0 = \Sigma_{12} = -\rho_{12}(0) = \mathrm{cov}(\psi_1(t), \psi_2(t))$.

By Lemma 3.5 the distribution of $(\psi_{11}(t), \psi_{12}(t), \psi_{22}(t))$ is nondegenerate, so, by [7], Corollary 3.1, there are a.s. at most finitely many critical points of ψ in each bounded region. Define $\varphi\colon \mathbb{R}^4 \to \mathbb{R}^5$ by

$$\varphi(s,t) := \big(\psi(t) - \psi(s), \psi_1(t), \psi_1(s), \psi_2(t), \psi_2(s)\big), \qquad s,t \in \mathbb{R}^2.$$

By (3.1), (3.2) and Lemma 3.5, as $|t - s| \to \infty$ the covariance matrix of $\varphi(s,t)$ converges to $\mathrm{diag}(2\rho(0), \Sigma_{11}, \Sigma_{11}, \Sigma_{22}, \Sigma_{22})$ which has strictly positive determinant. Therefore, for sufficiently large M and all $N > M$, the density of $\varphi(s,t)$ is bounded, uniformly for (s,t) in the compact set $K_{MN} := \{(s,t) \in \mathbb{R}^2 \times \mathbb{R}^2\colon\ s \in [-N,N]^2,\ t - s \in [-N,N]^2 \setminus (-M,M)^2\}$. Although the process φ is not stationary, the proof of [1], Theorem 3.2.1, still applies, so there are a.s. no points $(s,t) \in K_{MN}$ where $\varphi(s,t) = 0$. Since N is arbitrary, this means there are a.s. no two critical points s and t of ψ at the same level with $t - s \notin (-M,M)^2$. As there are a.s. at most finitely many critical points of ψ in each bounded region, the lemma follows. $\square$

We continue with a fact from harmonic analysis. Let μ_n denote Lebesgue measure on $\mathbb{R}^n$, let $\operatorname{supp}(f)$ denote the support of a function f and let

$$\check{f}(x) := (2\pi)^{-1} \int_{\mathbb{R}^n} \exp(-it \cdot x) f(t)\, dt$$

denote the inverse Fourier transform of f for $f \in L^2(\mu_n)$.

LEMMA 3.7 ([16]; see [10], page 138). *Suppose $\varepsilon > 0$ and σ is a positive nonincreasing function on $[0, \infty)$ satisfying*

$$\int_0^\infty \frac{\log \sigma(x)}{x^2 + 1}\, dx > -\infty. \tag{3.5}$$

Then there exists a continuous function f: $\mathbb{R} \to \mathbb{C}$, not identically 0, such that $\operatorname{supp}(f) \subset [-\varepsilon, \varepsilon]$ and $|\check{f}(x)| \le \sigma(|x|)$ for all x.

Given nonnegative functions f on $\mathbb{R}^2$ and σ on $[0, \infty)$, we say σ is a *radial nonincreasing minorant* of f if σ is nonincreasing and $f(x) \ge \sigma(|x|)$ for all $x \in \mathbb{R}^2$. Let Δ' denote the density of the absolutely continuous part of Δ. The next result says roughly that the main condition for signed finite energy is that Δ' not decay too fast at ∞.

PROPOSITION 3.8. *Suppose ψ is a stationary continuous Gaussian random field on $\mathbb{R}^2$ such that Δ' has a radial nonincreasing minorant σ satisfying* (3.5). *Then ψ has signed finite energy.*

Note, in particular, that the required radial nonincreasing minorant exists provided Δ' is bounded away from 0 on compact sets and decays only polynomially fast at ∞, for example, if Δ' is nonzero and rational.

Before proving this result let us put it in the context of interpolation theory. We say that ψ has *perfect interpolation* if, for every open square R, ψ is determined on R by its values on R^c; more precisely, if $\operatorname{var}(\psi(t) | \mathscr{F}_{R^c}) = 0$ for all $t \in R$. In general, one would not expect ψ to have signed finite energy if ψ has perfect interpolation. From [10], a sufficient condition for perfect interpolation in one dimension is that

$$\int_0^\infty \frac{\log \Delta'(x)}{x^2 + 1}\, dx = -\infty.$$

Thus it seems unlikely that (3.5) can be improved much; an optimal result might be that the absence of perfect interpolation implies signed finite energy.

For a function f on $\mathbb{R}^n$, define $\tilde{f}(x) := f(-x)$, let $\|f\|_p$ denote the $L^p(\mu_n)$-norm of f and recall that $\operatorname{supp}(f)$ denotes the support of f. For a $C(\mathbb{R}^n)$-valued random process Z, $\operatorname{supp}_C(Z)$ denotes the support of the distribution of Z in $C(\mathbb{R}^n)$.

PROOF OF PROPOSITION 3.8. We begin with a sketch of the proof. Given an open square R and a σ-field $\mathscr{G}_{R^c} \supset \mathscr{F}_{R^c}$, we can decompose ψ into a sum of two independent Gaussian processes:

$$\psi = E(\psi | \mathscr{G}_{R^c}) + \psi_R,$$

with $\psi_R := \psi - E(\psi|\mathscr{G}_{R^c})$. The easiest case is when $\text{supp}_C(\psi_R)$ (which is nonrandom) includes a function γ which is strictly positive on R. Then, given $\mathscr{G}_{R^c}$, the support of the distribution of ψ includes $E(\psi|\mathscr{G}_{R^c}) + M\gamma$ for arbitrarily large M, since $\text{supp}_C(\psi_R)$ is a vector space. This, in turn, yields the finite energy property fairly directly. Unfortunately, there is no reason in general why $\text{supp}_C(\psi_R)$ should include such a function γ. But we can instead look for a Gaussian process X which does have the desired property—the existence of a function γ in $\text{supp}_C(X_R)$ which is strictly positive on R, where X_R is the analog of ψ_R—and which "lies underneath" ψ in the sense that ψ can be decomposed as $\psi = X + Y$ for some process Y independent of X. Under (3.5) we will construct such an X.

Turning to the details, first fix $T > 0$. Since (3.5) also holds for $\alpha(x) := \sigma^{1/4}(4x)$ in place of $\sigma(x)$, by Lemma 3.7 there exists a continuous function $f: \mathbb{R} \to \mathbb{C}$ such that $\varnothing \neq \text{supp}(f) \subset [-T/4, T/4]$ and $|\check{f}(x)| \leq \alpha(|x|)$ for all x. Let $h := |f|^2 * |\tilde{f}|^2$ and let $\theta := \sup(\text{supp}(h))$, so h is nonnegative and even, $\check{h}$ is real and $0 < \theta \leq T/2$. Specifically,

$$\check{h} = \left(\check{f} * \overline{(\check{f})^{\tilde{}}}\right)\left((\check{f})^{\tilde{}} * \overline{\check{f}}\right),$$

while

$$\begin{aligned}
\left|\left(\check{f} * \overline{(\check{f})^{\tilde{}}}\right)(x)\right| &\leq \int_{-\infty}^{\infty} \alpha(|x-u|)\,\alpha(|u|)\,du \\
&= 2\int_0^{\infty} \alpha(|x/2 + v|)\,\alpha(|x/2 - v|)\,dv \\
&\leq 2\|\alpha(|\cdot|)\|_1\,\alpha(|x|/2),
\end{aligned}$$

and similarly for $(\check{f})^{\tilde{}} * \overline{\check{f}}$, so, for some constant c,

$$|\check{h}(x)| \leq c\sigma^{1/2}(2|x|) \quad \text{for all } x.$$

Now define $g((t_1, t_2)) := h(t_1)h(t_2)$. Let W be a white-noise process on the open subsets of $\mathbb{R}^2$, that is, a mean-zero Gaussian process with

$$\text{cov}(W(A), W(B)) = \mu_2(A \cap B),$$

and define the stationary Gaussian process

$$X(t) := \delta \int_{\mathbb{R}^2} g(s-t)W(ds),$$

where $\delta > 0$ satisfies $2\pi\delta^2 c^4 \sigma(0) \leq 1$. The covariance function of X is

$$\rho^X(t) := \delta^2 (g * g)(t) \geq 0,$$

and the corresponding spectral measure Δ_X has a spectral density given by

$$\begin{aligned}\Delta'_X(x) &:= 2\pi\delta^2 h^{\vee}(x_1)^2 h^{\vee}(x_2)^2\\ &\le 2\pi\delta^2 c^4\sigma(2|x_1|)\sigma(2|x_2|)\\ &\le 2\pi\delta^2 c^4\sigma(0)\sigma(|x|)\\ &\le \Delta'(x).\end{aligned}$$

Therefore there exists a stationary Gaussian process Y having spectral measure

$$\Delta_Y := \Delta - \Delta_X.$$

We may take X and Y independent, and construct ψ as

$$\psi = X + Y.$$

Since $\operatorname{var}(X(t) - X(s)) \le \operatorname{var}(\psi(t) - \psi(s))$ for all t and s, by a result from [17], we may assume X, and similarly Y, are continuous.

Let $R := (-T,T)^2$ and $Q := (-(T-\theta),(T-\theta))^2$. Define σ-fields

$$\begin{aligned}\mathscr{W}_{Q^c} &:= \sigma(W(A)\colon A \subset Q^c,\ A \text{ open}),\\ \mathscr{Y}_{R^c} &:= \sigma(Y(t)\colon t \in R^c),\\ \mathscr{G}_{R^c} &:= \mathscr{W}_{Q^c} \vee \mathscr{Y}_{R^c},\end{aligned}$$

so that $\sigma(X(t)\colon t \in R^c) \subset \mathscr{W}_{Q^c}$ and therefore $\mathscr{F}_{R^c} \subset \mathscr{G}_{R^c}$. Define Gaussian processes

$$m_R(t) := E(\psi(t)|\mathscr{G}_{R^c}) = E(X(t)|\mathscr{W}_{Q^c}) + E(Y(t)|\mathscr{Y}_{R^c}),$$

$$X_R(t) := X(t) - E(X(t)|\mathscr{G}_{R^c}) = X(t) - E(X(t)|\mathscr{W}_{Q^c}) = \delta\int_Q g(s-t)W(ds),$$

$$Y_R(t) := Y(t) - E(Y(t)|\mathscr{G}_{R^c}) = Y(t) - E(Y(t)|\mathscr{Y}_{R^c}).$$

Then $X_R = Y_R = 0$ on R^c, and

$$\psi = m_R + X_R + Y_R.$$

Note m_R is $\mathscr{G}_{R^c}$-measurable, while X_R and Y_R are independent of each other and of $\mathscr{G}_{R^c}$. In particular, the distributions of X_R and Y_R, given $\mathscr{G}_{R^c}$, are nonrandom. As with X and Y, the processes m_R, X_R and Y_R are continuous. The covariance function of X_R is

$$\rho^{X_R}(s,t) := \delta^2\int_Q g(u-t)g(u-s)\,du.$$

We claim that $\rho^{X_R}(t,t) > 0$ for all $t \in R$. It is sufficient to show this for $t_1, t_2 \ge 0$. Then since $a \le T/2$ we have

$$\begin{aligned}\rho^{X_R}(t,t) &= \delta^2\int_{t_1-a}^{T-a} h^2(u_1-t_1)\,du_1\int_{t_2-a}^{T-a} h^2(u_2-t_2)\,du_2\\ &= \delta^2\int_{-a}^{T-t_1-a} h^2(u)\,du\int_{-a}^{T-t_2-a} h^2(u)\,du,\end{aligned}$$

and it follows from the definition of a and the symmetry and continuity of h that $\rho^{X_R}(t,t) > 0$. Since ρ^{X_R} is continuous, it follows that $\rho^{X_R}(t,s) > 0$ for all t in a neighborhood of each fixed $s \in R$, and therefore

$$\gamma(s) := \int_R \rho^{X_R}(t,s)\,dt > 0 \quad \text{for all } s \in R.$$

The support of the distribution of X_R [as a random element of $C(\mathbb{R}^2)$] is its reproducing kernel Hilbert space, that is, the uniform closure of the linear span of the functions $\{\rho^{X_R}(t,\cdot),\ t \in \overline{R}\}$. (This standard fact is an easy consequence of the uniform convergence of the orthogonal expansion of X_R—see [14], Theorem 3.3.5.) Let $R_n := (n^{-1}\mathbb{Z}^2) \cap R$; it follows easily from the uniform continuity of ρ^{X_R} that γ is the uniform limit of the functions

$$\gamma_n := n^{-2} \sum_{t \in R_n} \rho^{X_R}(t,\cdot).$$

Therefore $\gamma \in \operatorname{supp}_C(X_R)$, and hence also $M\gamma \in \operatorname{supp}_C(X_R)$ for all $M \in \mathbb{R}$. Since $0 \in \operatorname{supp}_C(Y_R)$ it follows that

$$\text{(3.6)}\quad P\big(\|\psi - (m_R + M\gamma)\|_R < \delta | \mathscr{G}_{R^c}\big) > 0 \quad \text{for all } \delta > 0 \text{ and } M \in \mathbb{R},$$

where $\|\cdot\|_R$ denotes the sup norm over R.

Now let F be a finite subset of ∂R and suppose $\psi(s) > a$ for each $s \in F$. Let Γ be the union of a collection of straight lines connecting each $s \in F$ to a fixed $t \in R$. Let $\varepsilon := \min\{\psi(s) - a\colon s \in F\} > 0$. Since $\psi(s) = m_R(s)$ for $s \in F$ and since m_R and γ are continuous and γ is strictly positive on R, if M is sufficiently large, then $m_R + M\gamma > a + \varepsilon/2$ everywhere on Γ. If also $\|\psi - (m_R + M\gamma)\|_R < \varepsilon/2$, then $\psi|_{\overline{R}} \in H_{a+}(R, \psi|_{R^c}, F)$. The first half of (2.2) now follows from (3.6); the second half is symmetric. □

LEMMA 3.9 ([9], Chapter 10). *Suppose ψ is a stationary C^2 Gaussian random field on $\mathbb{R}^2$. For each $i = 1,2$, the following are equivalent*:

(i) $EN_{\mathrm{cr}}([0,e_i]) = \infty$;
(ii) *ψ is constant on every line parallel to e_i*;
(iii) $\rho_{ii}(0) = 0$;
(iv) $\rho(ue_i) = \rho(0)$ *for all* $u \in \mathbb{R}$.

In particular, if $\rho(0) > 0$ and $\rho(t) \to 0$ as $|t| \to \infty$, then $EN_{\mathrm{cr}}([0,e_i]) < \infty$, $i = 1,2$.

Combining the preceding results immediately yields the following sufficient conditions for the application of Theorem 2.3.

THEOREM 3.10. *Suppose ψ is a stationary C^2 Gaussian random field on $\mathbb{R}^2$ with $\rho(0) > 0$ and $\rho(t) \to 0$ as $|t| \to \infty$. If Δ' has a radial nonincreasing minorant σ satisfying* (3.5), *then either ψ has bounded level lines or ψ is treelike.*

Heuristically, the treelike property of a random field seems to require a kind of long-range structure that is not generally reflected in the covariance. Based on this, we make the following conjecture; a proof, together with Theorem 3.10, would establish a form of the Sagdeev hypothesis for a wide class of Gaussian fields.

CONJECTURE 3.11. *No stationary Gaussian random field on* $\mathbb{R}^2$ *is treelike*.

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REFERENCES

[1] ADLER, R. A. (1981). *The Geometry of Random Fields*. Wiley, New York.

[2] ALDOUS, D. and STEELE, J. M. (1992). Asymptotics for Euclidean minimal spanning trees on random points. *Probab. Theory Related Fields* **92** 247–258.

[3] ALEXANDER, K. S. (1995). Percolation and minimal spanning forests in infinite graphs. *Ann. Probab.* **23** 87–104.

[4] ALEXANDER, K. S. (1995). Simultaneous uniqueness of infinite clusters in stationary random labeled graphs. *Comm. Math. Phys.* **168** 39–55.

[5] ALEXANDER, K. S. and MOLCHANOV, S. A. (1994). Percolation of level sets for two-dimensional random fields with lattice symmetry. *J. Statist. Phys.* **77** 627–643.

[6] AVELLANEDA, M., ELLIOT, F., JR. and APELIAN, C. (1993). Trapping, percolation and anomalous diffusion of particles in a two-dimensional flow. *J. Statist. Phys.* **72** 1227–1304.

[7] BELYAEV, YU. K. (1972). Point processes and first passage problems. In *Proc. Sixth Berkeley Symp. Math. Statist. Probab.* **3** 1–17. Univ. California Press, Berkeley.

[8] BURTON, R. and KEANE, M. (1989). Density and uniqueness in percolation. *Comm. Math. Phys.* **121** 501–505.

[9] CRAMÉR, H. and LEADBETTER, M. R. (1967). *Stationary and Related Stochastic Processes*. Wiley, New York.

[10] DYM, H. and MCKEAN, H. P. (1976). *Gaussian Processes, Function Theory, and the Inverse Spectral Problem*. Academic Press, New York.

[11] GANDOLFI, A., KEANE, M. and NEWMAN, C. M. (1992). Uniqueness of the infinite component in a random graph with applications to percolation and spin glasses. *Probab. Theory Related Fields* **92** 511–527.

[12] GANDOLFI, A., KEANE, M. and RUSSO, L. (1988). On the uniqueness of the infinite occupied cluster in dependent two-dimensional site percolation. *Ann. Probab.* **16** 1147–1157.

[13] ISICHENKO, M. B. (1992). Percolation, statistical topography, and transport in random media. *Rev. Modern Phys.* **64** 961–1043.

[14] JAIN, N. C. and MARCUS, M. B. (1978). Continuity of subgaussian processes. In *Probability on Banach Spaces. Advances in Probability and Related Topics* (J. Kuelbs, ed.) **4** 81–196. Dekker, New York.

[15] KRENGEL, U. (1985). *Ergodic Theorems*. de Gruyter, Berlin.

[16] LEVINSON, N. (1940). *Gap and Density Theorems*. Amer. Math. Soc., Providence, RI.

[17] MARCUS, M. B. and SHEPP, L. A. (1970). Continuity of Gaussian processes. *Trans. Amer. Math. Soc.* **151** 377–392.

[18] MOLCHANOV, S. A. and STEPANOV, A. K. (1983). Percolation in random fields. I, II, III. *Teoret. Mat. Fiz.* **55** 246–256, 419–430; **67** 177–185 [*Theoret. and Math. Phys.* **55** 478–484, 592–599; **67** 434–439 (1983)].

[19] NEWMAN, C. M. and SCHULMAN, L. S. (1981). Infinite clusters in percolation models. *J. Statist. Phys.* **26** 613–628.

[20] PITT, L. D. (1982). Positively correlated normal variables are associated. *Ann. Probab.* **10** 496–499.

[21] PREPARATA, F. P. and SHAMOS, M. I. (1985). *Computational Geometry: An Introduction*. Springer, New York.

[22] WHYBURN, G. T. (1942). *Analytic Topology*. Coll. Publ. **28**. Amer. Math. Soc., Providence, RI.

[23] YADRENKO, M. I. (1983). *Spectral Theory of Random Fields*. Optimization Software Inc., New York.

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