The Annals of Probability
1996, Vol. 24, No. 2, 961–970

INCREASE OF LÉVY PROCESSES

BY R. A. DONEY

University of Manchester

A rather complicated condition is shown to be necessary and sufficient for a Lévy process to have points of increase. A much simpler condition is then shown to be sufficient in the general case, and necessary under certain regularity conditions. The approach used here also gives a unified proof of results for certain special classes of Lévy processes, which have previously been obtained by Bertoin.

1. Introduction and results. If X is a real-valued Lévy process we say that $t > 0$ is an increase time, and write $t \in \mathscr{I}$, if there exist $t' \in [0, t)$ and $t'' \in (t, \infty)$ with

$$(1) \qquad X_s \le X_t \quad \text{for } s \in [t', t], \qquad X_s \ge X_t \quad \text{for } s \in [t, t''].$$

Then it is clear that $P\{\mathscr{I} \ne \varnothing\}$ is either 1 or 0, and we say that X has increase points (has no increase points) in the corresponding cases. This idea was introduced by Dvoretzky, Erdös and Kakutani [12], who showed that Brownian motion has no increase points. Originally, this was considered a remarkable result, but simpler proofs have been given by a number of authors, including Knight [17], Adelman [1] and Aldous [2], and the proof in Burdzy [11] shows that it really follows from the solution to the two-sided exit problem for Brownian motion. From this point of view it is not surprising that the first investigation of this question for Lévy processes, which is Bertoin [3], focussed on the special case of spectrally negative processes. For in this case it is known that there is a function W (the scale function) such that the probability that X exits the interval $[-a, b]$ (where $a > 0$, $b > 0$) at b is given by $W(a)/W(a+b)$. Indeed, although a different approach involving covering with random intervals was used in [3], it is not difficult to see that Burdzy's proof for Brownian motion can be adapted to give Bertoin's result, which is that such a process has increase times if and only if

$$(2) \qquad \int_{0+} \frac{dx}{W(x)} < \infty.$$

Again, using different methods, Bertoin has shown in [5] that a strictly stable process X has points of increase if and only if $P\{X_1 > 0\} > \frac{1}{2}$, and in [6] that a Lévy process which can "creep" downwards (see Millar [18] and Rogers [19]) has no increase points. These results for special cases are all compatible with the following conjecture, which is also due to Jean Bertoin (private communication). Let $\overline{X}$ and $\underline{X}$ denote the supremum and infimum

Received June 27, 1994; revised March 1995.
AMS 1991 *subject classifications*. 60J30, 60G17.
Key words and phrases. Stable processes, spectrally negative, subordinators, increase time.

processes, let T be a random variable which is independent of X and has an $\exp(\eta)$ distribution ($\eta > 0$ fixed) and let F and F^* denote the distribution functions of $\overline{X}(T)$ and $-\underline{X}(T)$, respectively. The conjecture is that X has points of increase if and only if

$$\int_{0+} \frac{F(dx)}{F^*(x)} < \infty. \tag{3}$$

The purpose of this paper is to show that the method used by Burdzy [11] (the original idea seems to stem from Adelman [1]) can be further adapted to throw considerable light on this conjecture and at the same time give a unified method of proof of most of Bertoin's results. Specifically, if the first passage processes σ and σ^* are defined by $\sigma(x) = \inf\{t\colon \overline{X}_t > x\}$ and $\sigma^*(x) = \inf\{t\colon -\underline{X}_t > x\}$, $x \geq 0$, we introduce a quantity R_ε defined for $\varepsilon > 0$ by

$$R_\varepsilon = \begin{cases} \overline{X}(\sigma_\varepsilon^*) - X(\sigma_\varepsilon^*), & \text{if } \sigma_\varepsilon^* \leq T, \\ \infty, & \text{if } \sigma_\varepsilon^* > T. \end{cases}$$

The case when 0 is not regular for $(-\infty, 0)$ being easily dealt with, our main result is the following theorem.

THEOREM. *Any Lévy process X such that* 0 *is regular for* $(-\infty, 0)$ *for* X *has points of increase if and only if*

$$\lim_{\varepsilon \downarrow 0} \left\{ F(\varepsilon) + \int_\varepsilon^\infty P\{y < R_\varepsilon < \infty\} F(dy) \right\} \Big/ F^*(\varepsilon) < \infty. \tag{4}$$

From this, we deduce the sufficiency of (3).

COROLLARY 1. *If X is as in the theorem and* (3) *holds, then X has points of increase.*

In the opposite direction, an easy consequence of the theorem is one of Bertoin's results. [If X has a Brownian component, then it creeps downwards [18], and Bertoin [6] shows that it has no points of increase. Unfortunately, this does not seem to follow easily from (4).]

COROLLARY 2. *If X has no Brownian component and can creep downwards, then X has no increase points and* (3) *does not hold.*

Another of Bertoin's results is the following corollary.

COROLLARY 3. *If X is spectrally negative, then it has points of increase if and only if* (3) *holds or, equivalently, if and only if* (2) *holds.*

However, to state our main result in the opposite direction, we need some notation. We write $\hat{X}$ for the process we get by killing X at time T, so that $\hat{X}_t = X_t$ for $t \leq T$, $\hat{X}_t =$ cemetery point for $t > T$. The subordinators τ and H

are the inverse local time at the supremum of $\hat{X}$ and $\hat{X}(\tau)$, respectively, and μ and ν stand for the drift and Lévy measure of H. The potential measure U of H is defined by

$$U(dx) = \int_{t=0}^{\infty} P\{H_t \in dx\}\, dt,$$

and we write $U(x) = U\{(0,x]\}$. Note that, since $U(\infty) < \infty$ and $\overline{X}(T)$ is the final value of H, we have $F(x) = U(x)/U(\infty)$. Furthermore, if $V(x) = \nu\{(x,\infty]\}$, then in the case $\mu = 0$ the link between U (and hence F) and V is given by the relation

$$\int_0^x V(x-y)U(dy) = 1, \qquad x > 0, \tag{5}$$

which was first proved by Kesten [16]. (See in particular Proposition 8 of [16] for the case of a killed subordinator, which is what we need here.) Finally, write τ^*, H^*, μ^*, ν^*, U^* and V^* for the corresponding quantities evaluated for $-\hat{X}$, and recall that X (and hence $\hat{X}$) creeps downwards if and only if $\mu^* > 0$.

COROLLARY 4. *Assume that X is as in the theorem, that $\mu^* = 0$ and*

$$\int_{0+} V^*(x)F(dx) = \infty. \tag{6}$$

Then X has no points of increase.

It follows easily from (5) that, if either V^* or U^* is regularly varying at 0, then (see Section 3.5) $\lim_{x\downarrow 0} U^*(x)V^*(x) > 0$; clearly whenever

$$\liminf_{x\downarrow 0} U^*(x)V^*(x) > 0, \tag{7}$$

then (3) is again necessary for X to have points of increase. However, it is not difficult to show that (7) is not valid for all subordinators; whether the fact that H^* is a "ladder-height" subordinator can be exploited to establish (7) in our situation is an open question. Finally, we mention that if H_0 denotes the analogue of H for the unkilled process X and U_0 is the potential measure of H_0, then it is not difficult to show that the ratio $F(x)/U_0(x)$ is bounded away from 0 and ∞ for all small enough x. This and the corresponding result for U_0^* suggest that (3) is equivalent to what is in some ways a more natural condition, namely,

$$\int_{0+} \frac{U_0(dx)}{U_0^*(x)} < \infty, \tag{8}$$

and this is indeed true as will be shown.

PROPOSITION. *Conditions* (3) *and* (8) *are equivalent.*

2. Proofs. The cases when X or $-X$ is a subordinator or X is a compound Poisson process (i.e., a Lévy process which has zero drift, no Brownian component and whose Lévy measure is finite) are of no interest and will be implicitly excluded in what follows. Also if 0 is not regular for $(-\infty, 0)$ for X, it is easy to see that X has points of increase, so we will also assume that 0 is regular for $(-\infty, 0)$.

We will say that t is a global increase point for $\hat{X}$ if (1) holds with $t' = 0$, $t'' = T$, and write $\tilde{\mathscr{I}}$ for the set of all global increase points. A simple, but important, observation is that $P\{\mathscr{I} \neq \varnothing\} = 1 \Leftrightarrow P\{\tilde{\mathscr{I}} \neq \varnothing\} > 0$. We will also write $\hat{\sigma}$ and $\hat{\sigma}^*$ for the first passage processes of $\hat{X}$, so that, for example, $\hat{\sigma}(x) = \inf\{t \leq T\colon\ X_t > x\}$, with the usual convention that $\inf(\varnothing) = \infty$.

PROOF OF THE THEOREM. For each fixed $\varepsilon > 0$ we define sequences of random variables $\{W_n,\ n \geq 0\}$ and $\{Z_n,\ n \geq 1\}$ as follows. We set $W_0 \equiv 0$, $Z_1 = \hat{\sigma}^*(\varepsilon)$ and $W_1 = \inf\{t > Z_1\colon\ \hat{X}(t) > \overline{X}(Z_1)\}$ if $Z_1 < \infty$, $W_1 = \infty$ otherwise. If, for $n \geq 1$, $W_n < \infty$ we write $X^{(n)}(\cdot) = X(W_n + \cdot) - X(W_n)$, $Z_{n+1} = W_n + Z_1^{(n)}$ and $W_{n+1} = W_n + W_1^{(n)}$, where $Z_1^{(n)}$ and $W_1^{(n)}$ are Z_1 and W_1 evaluated for $X^{(n)}$, and if $W_n = \infty$, then we write $W_{n+1} = Z_{n+1} = \infty$. Next, for $n \geq 1$ we write $A_n^{(\varepsilon)} = \{W_{n-1} < \infty,\ Z_n = \infty\}$ and put $A^{(\varepsilon)} = \bigcup_{n=1}^{\infty} A_n^{(\varepsilon)}$ so that

$$A^{(\varepsilon)} = \{\exists\ 0 < t \leq T\colon\ X_s \leq X_t \text{ for } s \in [0, t] \text{ and } X_s \geq X_t - \varepsilon \text{ for } s \in [t, T]\}.$$

Clearly, as $\varepsilon \downarrow 0$, $A^{(\varepsilon)}$ decreases to some limit, A say. Since we are assuming that 0 is regular for $(-\infty, 0)$, it holds that, with probability 1, X does not jump upwards at any time t with $X_t = \overline{X}_t$ (see Corollary 1 of Rogers [19]), that $\underline{X}(T) \neq 0$ and, by time reversal, that $\overline{X}(T) \neq X(T)$; it then follows that $P(A) = P\{\tilde{\mathscr{I}} \neq \varnothing\}$. Now clearly the lack of memory property of the exponential distribution and the strong Markov property give $P(A_n^{(\varepsilon)}) = P\{W_{n-1} < \infty\}P\{Z_1 = \infty\} = [P\{W_1 < \infty\}]^{n-1}P\{Z_1 = \infty\}$, so that $P(A^{(\varepsilon)}) = P\{Z_1 = \infty\}/P\{W_1 = \infty\}$. Of course, $P\{Z_1 = \infty\} = P\{-\underline{X}(T) \leq \varepsilon\} = F^*(\varepsilon)$. Therefore, it remains only to evaluate $P\{W_1 = \infty\}$. But another appeal to the strong Markov property at time Z_1 gives

$$\begin{aligned} P\{W_1 = \infty\} &= P\{Z_1 = \infty\} + P\{Z_1 < \infty,\ W_1 = \infty\} \\ &= F^*(\varepsilon) + P\Big\{Z_1 \leq T,\ \sup_{0 \leq s \leq T - Z_1} (X(Z_1 + s) - X(Z_1)) \leq R_\varepsilon\Big\} \\ &= F^*(\varepsilon) + E\{F(R_\varepsilon);\ R_\varepsilon < \infty\} \\ &= F^*(\varepsilon) + F(\varepsilon)\{1 - F^*(\varepsilon)\} + \int_\varepsilon^\infty P\{y < R_\varepsilon < \infty\}F(dy), \end{aligned}$$

which establishes (4). □

PROOF OF COROLLARY 1. If X has a Brownian component, then (see Millar [18]) the drifts μ and μ^* of H and H^* are both positive. If $\kappa(\lambda) =$

$-\log\{E(e^{-\lambda H_1})\}$ is the Laplace exponent of H, we have

$$\kappa(\lambda)=\mu\lambda+\int_{(0,\infty]}(1-e^{-\lambda x})\nu(dx), \tag{9}$$

where $\int_{(0,\infty]}(1-e^{-x})\nu(dx)<\infty$, and also

$$\int_0^\infty e^{-\lambda x}U(dx)=\frac{1}{\kappa(\lambda)}, \qquad \lambda>0. \tag{10}$$

From (9) we have $\kappa(\lambda)\sim\mu\lambda$ as $\lambda\to\infty$, and using this in (10) gives $U(x)\sim cx$ as $x\downarrow 0$, and hence $F(x)\sim cx$ as $x\downarrow 0$. (Here and later c denotes a generic positive constant whose value may change from line to line.) Similarly we get $F^*(x)\sim cx$, and it is easily seen that (3) cannot hold. If X has no Brownian component but $\mu^*>0$, then X creeps downwards and the next proof will show that (3) cannot hold. So, assume $\mu^*=0$, and also that $V^*(0+)=\infty$; for if $V^*(0+)<\infty$, then we would have $\kappa^*(\infty)<\infty$, which by (10) entails $U^*(0+)>0$ and hence $F^*(0+)>0$. But then (3) is automatic; but it is also clear that (4) holds. It then follows (see, e.g., Theorem 3.1 of Horowitz [15]) that, for all $\varepsilon>0$, $x>0$,

$$\begin{aligned}U^*(\varepsilon)V^*(\varepsilon+x)&\le P\{H^* \text{ does not hit } [\varepsilon,\varepsilon+x]\}\\&=\int_0^\varepsilon V^*(\varepsilon+x-s)\,dU^*(s)\le U^*(\varepsilon)V^*(x).\end{aligned} \tag{11}$$

Moreover, U^* is a renewal function (for a simple proof of this, see Bertoin and Doney [8]) and hence for any $c>1$ we have the bounds

$$U^*(x)\le U^*(cx)\le cU^*(x). \tag{12}$$

Thus, when (3) holds, we also have $\int_{0+}\{F^*(\frac{1}{2}x)\}^{-1}F(dx)<\infty$. Let Γ^*_ε denote the overshoot of $\hat{X}$ below $-\varepsilon$, viz.

$$\Gamma^*_\varepsilon=\begin{cases}-\varepsilon-X(\hat{\sigma}^*_\varepsilon), & \text{if } \hat{\sigma}^*_\varepsilon<\infty,\\ \infty, & \text{otherwise,}\end{cases} \tag{13}$$

so that

$$\begin{aligned}P\{\varepsilon+x<R_\varepsilon<\infty\}&=P\{x<\Gamma^*_\varepsilon+\overline{X}(\hat{\sigma}^*_\varepsilon)<\infty\}\\&\le P\{\tfrac{1}{2}x<\Gamma^*_\varepsilon<\infty\}+P\{\tfrac{1}{2}x<\overline{X}(\hat{\sigma}^*_\varepsilon)<\infty\}.\end{aligned} \tag{14}$$

Clearly, $P\{\frac{1}{2}x<\Gamma^*_\varepsilon<\infty\}\le P\{H^*$ does not hit $[\varepsilon,\varepsilon+\frac{1}{2}x]\}$, so we can use (11), together with the inequality $V^*(\frac{1}{2}x)U^*(\frac{1}{2}x)\le 1$, which follows from the starred version of (5), to get $P\{\frac{1}{2}x<\Gamma^*_\varepsilon<\infty\}\le U^*(\varepsilon)/U^*(\frac{1}{2}x)=F^*(\varepsilon)/F^*(\frac{1}{2}x)$. To deal with the other term, note that $P\{\hat{\sigma}^*_\varepsilon<\infty,\ \overline{X}(\hat{\sigma}^*_\varepsilon)>\frac{1}{2}x\}$. $F^*(\varepsilon+\frac{1}{2}x)\le P\{\hat{\sigma}(\frac{1}{2}x)<\infty,\ \underline{X}(\hat{\sigma}(\frac{1}{2}x))>-\varepsilon\}$. $F^*(\varepsilon+\frac{1}{2}x)\le F^*(\varepsilon)$, so that $P\{\frac{1}{2}x<\overline{X}(\hat{\sigma}^*_\varepsilon)<\infty\}\le F^*(\varepsilon)/F^*(\frac{1}{2}x)$. It then follows that $\{F^*(\varepsilon)\}^{-1}\int_\varepsilon^\infty P\{y<R_\varepsilon<\infty\}F(dy)\le 2\int_\varepsilon^\infty\{F^*(\frac{1}{2}(y-\varepsilon))\}^{-1}F(dy)\to 2\int_0^\infty\{F^*(\frac{1}{2}y)\}^{-1}F(dy)<\infty$, and, since $F(\varepsilon)/F^*(\varepsilon)\le\int_0^\varepsilon\{F^*(y)\}^{-1}F(dy)$, the result follows. □

PROOF OF COROLLARY 2. Let $\Psi(s) = \log E(e^{sX_1})$ denote the Lévy exponent of X; then $\hat{X}$ has exponent $\Psi(s) - \eta$ and the Wiener–Hopf factorization can be written as

$$\eta \cdot (\eta - \Psi(s))^{-1} = \psi(s)\psi^*(-s), \qquad \mathrm{Re}(s) = 0, \tag{15}$$

where $\psi(s) = \int_0^\infty e^{sx}\, dF(x)$ and $\psi^*(s) = \int_0^\infty e^{sx}\, dF^*(x)$. Since X creeps downwards, $\mu^* > 0$ and, as we have already remarked, this entails $F^*(x) \sim cx$ as $x \downarrow 0$. Since F is subadditive (being proportional to the potential U), it follows that $f_0 = \lim_{x\downarrow 0} x^{-1}F(x)$ exists. If it were the case that $f_0 < \infty$, then it would follow from (15) that $\lim_{\theta\to\infty} |\theta^2(1 - \Psi(i\theta))^{-1}| < \infty$, and hence that X has a Brownian component. Since this is not the case, $f_0 = \infty$, so that $F(x)/F^*(x) \to \infty$ as $x \downarrow 0$ and the result follows from (4). □

PROOF OF COROLLARY 3. This depends on the result that, if $\hat{W}$ is the scale function for $\hat{X}$, then (see, e.g., Theorem 7 of Bingham [9])

$$P\{\hat{X} \text{ exits } [-a, b] \text{ at } b\} = \hat{W}(a)/\hat{W}(a+b), \qquad a > 0,\ b > 0, \tag{16}$$

together with the observation that

$$\hat{W}(x) \sim W(x) \sim F^*(x) \quad \text{as } x \downarrow 0. \tag{17}$$

The first part of (17) follows easily from the well-known formula which gives the Laplace transform of W and $\hat{W}$ (see, e.g., Proposition 8 of Bingham [9]), the calculation being spelt out in Bertoin [4]; see, in particular, (13) therein, and note that our $\hat{W}$ coincides with Bertoin's W^{a*} when $a = \eta - 1$. The second part of (17) can be derived in a similar manner, since the Laplace transform of F^* can be derived from (15), and the fact that $\overline{X}(T)$ has an exponential distribution. Using this same fact and (17) shows that (2) and (3) are equivalent, and Corollary 1 shows that, when either holds, X has points of increase. In the contrary case, using (16) gives the bound

$$\begin{aligned} P\{\varepsilon + x < R_\varepsilon < \infty\} &\geq P\{x < \overline{X}(\hat{\sigma}_\varepsilon^*) < \infty\} \\ &= P\{\hat{X} \text{ exits } [-\varepsilon, x] \text{ at } x\} P\{\sigma^*(\varepsilon + x) < \infty\} \\ &= \hat{W}(\varepsilon)\{1 - F^*(\varepsilon + x)\}\{\hat{W}(\varepsilon + x)\}^{-1}. \end{aligned}$$

As previously observed we may, without loss of generality, assume that $F^*(0+) = 0$; in this case the fact that X has no points of increase when (3) fails is immediate from (17), the above bound and the theorem. □

PROOF OF COROLLARY 4. Here we use the other obvious lower bound, namely that $P\{\varepsilon + x < R_\varepsilon < \infty\} \geq P\{x < \Gamma_\varepsilon^* < \infty\}$. Indeed, since $F^*(\varepsilon) = P\{\hat{\sigma}_\varepsilon^* = \infty\} = P\{\Gamma_\varepsilon^* = \infty\}$, we need only remark that $F^*(\varepsilon) + P\{x < R_\varepsilon^* < \infty\} = P\{H^* \text{ does not hit } [\varepsilon, \varepsilon + x]\}$ and use the left-hand bound in (11) to see that $P\{y < R_\varepsilon < \infty\} \geq F^*(\varepsilon)\{1 + V^*(y)\}$, and the result follows. □

PROOF OF THE PROPOSITION. If ϕ_0 denotes the exponent of the bivariate subordinator (τ_0, H_0), it follows from Lemma 2.3 of Greenwood and Pitman [14] that

$$\int_0^\infty e^{-\lambda y} F(y)\, dy = \lambda^{-1} E(e^{-\lambda \overline{X}(T)}) = \frac{\phi_0(\eta, 0)}{\{\lambda \phi_0(\eta, \lambda)\}}.$$

On the other hand, if $V_0(dx, dy)$ denotes the potential measure of (τ_0, H_0), then

$$\begin{aligned}\{\lambda\phi_0(\lambda,\eta)\}^{-1} &= \lambda^{-1}\int_{t=0}^\infty \int_{x=0}^\infty \int_{y=0}^\infty e^{-(\eta x+\lambda y)} P\{\tau_0(t) \in dx,\ H_0(t) \in dy\}\, dt \\ &= \int_0^\infty e^{-\lambda y} \int_{t=0}^\infty \int_{x=0}^\infty e^{-\eta x} P\{\tau_0(t)\in dx,\ H_0(t) \le y\}\, dt.\end{aligned}$$

Hence, with $\theta = \phi_0(\eta)$,

$$\begin{aligned} F(y) &= \theta E\left\{\int_0^\infty I_{\{H_0(t)\le y\}} e^{-\eta\tau_0(t)}\, dt\right\} \\ &= \theta E\left\{\int_0^\infty I_{\{\overline{X}(s)\le y\}} e^{-\eta s}\, dL_0(s)\right\},\end{aligned}$$

where L_0 is the local time at 0 of $\overline{X} - X$. Since

$$U_0(y) = \int_0^\infty P\{H_0(t) \le y\}\, dt = E\left\{\int_0^\infty I_{\{\overline{X}(s)\le y\}}\, dL_0(s)\right\},$$

we see immediately that $F(y) \le \theta U_0(y)$ and that

$$F(y) \ge e^{-1}\theta E\left\{\int_0^1 I_{\{\overline{X}(s)\le y\}}\, dL_0(s)\right\}.$$

On the other hand, using the Markov property, we see that

$$\begin{aligned} U_0(y) &= E\left\{\int_0^1 I_{\{\overline{X}(s)\le y\}} dL_0(s)\right\} + E\left\{\int_1^\infty I_{\{\overline{X}(s)\le y\}} dL_0(s)\right\} \\ &\le e\theta^{-1} F(y) + P\{\overline{X}(1) \le y\} U_0(y),\end{aligned}$$

which shows that, for all small enough y and some positive c,

$$cU_0(y) \le F(y) \le c^{-1} V_0(y). \tag{18}$$

Using the result corresponding to (18) for F^*, it is clear that (3) is equivalent to $\int_{0+} \{U_0^*(y)\}^{-1}\, dF(y) < \infty$. By Fubini this is equivalent to $\int_{0+} F(y)\, d\{-1/U_0^*(y)\} < \infty$, and by (18) this is equivalent to $\int_{0+} U_0(y)\, d\{-1/U_0^*(y)\} < \infty$; a final use of Fubini shows that this is equivalent to (8). □

3. Special cases.

3.1. *Spectrally positive processes.* If X is spectrally positive, is not a subordinator and has no Brownian component, then it must creep downwards; thus, by Corollary 3, X has no increase points.

3.2. *Spectrally negative processes.* Although it is possible for (2) to fail, "most" spectrally negative processes have increase points; see Bertoin [3], Section 3. Notice that in this case the nontrivial problem of expressing (2) directly in terms of the Lévy measure of X has recently been solved by Bertoin. In [7], Proposition 10, Chapter VIII, Bertoin shows that (2) holds if and only if X has no Brownian component and

$$\int_{(-1,0)} x^2 \log\left|\frac{1}{x}\right| \pi(dx) < \infty. \tag{19}$$

Also it should be remarked that Bertoin [4] has shown that when (2) holds and X has unbounded variation it is possible to define a local time on the set $\hat{\mathscr{I}}$ of global increase points.

3.3. *Stable processes.* Suppose X is a strictly stable process with index $\alpha \in (0,2)$ and positivity parameter $\rho \in (0,1)$; that is, X is a Lévy process with the scaling property

$$X =_d \{c^{-1/\alpha} X_{cs};\ s \geq 0\}, \tag{20}$$

$\rho = P\{X_1 > 0\}$ and neither X nor $-X$ is a subordinator. Then it is known that X has no Brownian component, that $\max\{\alpha\rho, \alpha(1-\rho)\} \leq 1$ and that the potential functions of H_0 and H_0^* are given by

$$U_0(x) = kx^{\alpha\rho}, \qquad U_0^*(x) = k^* x^{\alpha(1-\rho)} \tag{21}$$

for some constants k and k^*. [Perhaps the easiest way to verify (21) is to use Theorem 9.1 of Fristedt [13] to write the exponent of H_0 as

$$\begin{aligned}
\log \kappa_0(\lambda) &= \int_0^\infty t^{-1} E\{e^{-\lambda X_t} - e^{-t};\ X_t > 0\}\, dt \\
&= \int_0^\infty s^{-1} E\{e^{-X_s} - e^{-s\lambda^{-\alpha}};\ X_s > 0\}\, ds \\
&= \int_0^\infty s^{-1} E\{e^{-X_s} - e^{-s};\ X_s > 0\}\, ds + \rho \int_0^\infty s^{-1}\{e^{-s} - e^{-s\lambda^{-\alpha}}\}\, ds \\
&= \log \kappa_0(1) + \rho\alpha \log \lambda,
\end{aligned}$$

where we have used (21) and Frullani's integral.] Thus (8) holds if and only if $\rho > \frac{1}{2}$, and since it is easy to see that (7) holds, it follows that X has points of increase if and only if $\rho > \frac{1}{2}$. [It should be noted that $\alpha\rho = 1$ only occurs when $\alpha \in (1,2)$ and X is spectrally negative and then X has increase points, whereas $\alpha(1-\rho) = 1$ only occurs when $\alpha \in (1,2)$ and X is spectrally positive and then X has no increase points.]

3.4. *Processes which creep.* Although we have seen that all processes which creep downwards have no points of increase, the reverse implication is certainly false, as stable processes only creep downwards if they are spectrally positive. The same remark shows that the information that a process does or does not creep upwards gives no information about whether it has increase points or not. Finally, we remark that the problem of categorizing all Lévy processes which have downwards creep has still not been completely solved; see Millar [18] (where the term "continuous downwards passage" is used) and Rogers [19].

3.5. *Processes for which* (7) *holds.* If $W^*(x) = \int_0^x V^*(y)\,dy$ and u^* and w^* are the Laplace–Stieltjes transforms of U^* and W^*, then (5) says that $\lambda w^*(\lambda)u^*(\lambda) = 1$. By a Tauberian theorem W^* (respectively U^*) is regularly varying (r.v.) at 0 if and only if w^* (respectively u^*) is r.v. at ∞, and, by the monotone density argument, W^* is r.v. if and only if V^* is. We therefore see that (7) holds whenever either U^* or V^* is r.v. at 0. In fact, since a similar Tauberian result holds for O-regularly varying functions (Theorem 2.10.2 of Bingham, Goldie and Teugels [10]) and the monotone density argument can also be extended under weak regularity conditions (Proposition 2.10.3 of [10]), (7) actually holds under weaker, but more complicated conditions on U^* or V^*. We omit the details of such calculations, since it seems that a more pressing question is that of finding reasonable conditions on the Lévy measure of X for the validity of (7).

Acknowledgments. I am extremely grateful to Jean Bertoin, who not only brought this problem to my attention and suggested condition (3), but also helped me in various ways as I struggled to make progress with it. I would also like to thank Chris Rogers for showing me a condition he has found which is equivalent to (4), though apparently less easy to apply.

Note added in proof. S. Fourati has recently established the validity of the conjecture that X has points of increase if and only if (3) holds, in an as yet unpublished paper, "Points de croissance des processus de Lévy et théorie générale des processus."

REFERENCES

[1] ADELMAN, O. (1985). Brownian motion never increases: a new proof to a result of Dvoretzky, Erdös and Kakutani. *Israel J. Math.* **50** 189–192.

[2] ALDOUS, D. (1989). *Probability Approximations via the Poisson Clumping Heuristic*. Springer, New York.

[3] BERTOIN, J. (1991). Increase of a Lévy process with no positive jumps. *Stochastics* **37** 247–251.

[4] BERTOIN, J. (1993). Lévy processes with no positive jumps at an increase time. *Probab. Theory Related Fields* **96** 123–135.

[5] BERTOIN, J. (1994). Increase of stable processs. *J. Theoret. Probab.* **7** 551–563.

[6] BERTOIN, J. (1994). Lévy processes that can creep downwards never increase. *Ann. Inst. H. Poincaré Probab. Statist.* **31** 379–391.

[7] BERTOIN, J. (1996). *An Introduction to Lévy Processes*. Cambridge Univ. Press.
[8] BERTOIN, J. and DONEY, R. A. (1994). Cramér's estimate for Lévy processes. *Statist. Probab. Lett.* **21** 363–365.
[9] BINGHAM, N. H. (1975). Fluctuation theory in continuous time. *Adv. in Appl. Probab.* **7** 705–766.
[10] BINGHAM, N. H., GOLDIE, C. M. and TEUGELS, J. L. (1987). *Regular Variation*. Cambridge Univ. Press.
[11] BURDZY, K. (1990). On nonincrease of Brownian motion. *Ann. Probab.* **18** 978–980.
[12] DVORETZKY, A., ERDÖS, P. and KAKUTANI, S. (1961). Nonincrease everywhere of the Brownian motion process. *Fourth Berkeley Symp. Math. Statist. Probab.* **2** 103–116. Univ. California Press, Berkeley.
[13] FRISTEDT, B. E. (1973). Sample functions of stochastic processes with stationary independent increments. *Adv. Probab.* **3** 241–396.
[14] GREENWOOD, P. E. and PITMAN, J. (1980). Fluctuation identities for Lévy processes and splitting at the maximum. *Adv. in Appl. Probab.* **12** 893–902.
[15] HOROWITZ, J. (1972). Semilinear Markov process, subordinators and renewal theory. *Z. Wahrsch. Verw. Gebiete* **24** 167–193.
[16] KESTEN, H. (1969). Hitting probabilities of single points for processes with stationary independent increments. *Mem. Amer. Math. Soc.* **93** 1–129.
[17] KNIGHT, F. B. (1981). *The Essentials of Brownian Motion and Diffusion*. *Math. Surveys* **18**. Amer. Math. Soc., Providence, RI.
[18] MILLAR, P. W. (1973). Exit properties of stochastic processes with stationary independent increments. *Trans. Amer. Math. Soc.* **178** 459–479.
[19] ROGERS, L. C. J. (1984). A new identity for real Lévy processes. *Ann. Inst. H. Poincaré Probab. Statist.* **20** 21–34.

STATISTICAL LABORATORY
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MANCHESTER
MANCHESTER M13 9PL
UNITED KINGDOM
E-MAIL: MBBGSRD@mh1.mcc.ac.uk