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SPECTRAL CRITERIA, SLLN'S AND A.S. CONVERGENCE OF SERIES OF STATIONARY VARIABLES[1]

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It is shown here how to extend the spectral characterization of the strong law of large numbers for weakly stationary processes to certain singular averages. For instance, letting $\{X_t,\ t \in R^3\}$ be a weakly stationary field, $\{X_t\}$ satisfies the usual SLLN (by averaging over balls) if and only if the averages of $\{X_t\}$ over spheres of increasing radii converge pointwise. The same result in two dimensions is false. This spectral approach also provides a necessary and sufficient condition for the a.s. convergence of some series of stationary variables.

1. Main results. Gaposhkin [4] has provided a striking characterization of the strong law of large numbers for weakly stationary random variables in terms of the behavior of the associated random spectral measure. This result, as well as a companion due to Jajte [9], forms the motivation for the current paper. In this paper we study extensions of these results to a variety of averaging methods which are more singular than the classical method of averaging over balls. For instance, we shall show that for a weakly stationary sequence $\{X_t\}$, indexed by $t \in R^3$, the usual strong law of large numbers, formed by averaging over balls, holds if and only if the same convergence holds for averages of $\{X_t\}$ over increasing spheres. The restriction to dimension 3 or higher is sharp.

The necessary notation is now introduced. We formulate our theorems in the continuous context. All theorems have analogous forms valid in the discrete setting, but they require a bit more effort to formulate and prove.

Let $(\Omega, \mathscr{B}, \mathscr{P})$ be a probability space, let $\mathscr{E}$ denote expectation with respect to $\mathscr{P}$ and let $X = \{X_t\}_{t \in R^d} \subset L^2(\mathscr{P})$ be a mean square continuous, zero-mean, weakly stationary sequence (more precisely, a zero-mean, homogeneous random field indexed by R^d). The role of dimension d will be of interest below. Then X has a spectral representation

$$X_t = \int_{R^d} e^{it\cdot\lambda} Z(d\lambda),$$

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where $t \cdot \lambda = \sum_{j=1}^{d} t_j \lambda_j$, where Z: $\mathscr{B}(R^d) \to L^2(\mathscr{P})$ is a σ-additive and orthogonally scattered measure on the Borel σ-algebra of R^d; that is, if $A, B \in \mathscr{B}(R^d)$ are disjoint, then $\mathscr{E} Z(A)\overline{Z(B)} = 0$. Then weak stationarity is equivalent to $\mathscr{E} X_t \overline{X}_s = \mathscr{E} X_{t-s} \overline{X}_0$ for $t, s \in R^d$, and the covariance sequence $\{R(t) = \mathscr{E} X_t \overline{X}_0 : t \in R^d\}$ itself has a spectral representation

$$R(t) = \int_{R^d} e^{it\cdot\lambda} \mu(d\lambda)$$

for a unique positive finite Borel measure μ on $\mathscr{B}(R^d)$, given via $\mu(A) = \mathscr{E}|Z(A)|^2$, $A \in \mathscr{B}(R^d)$. The measures μ and Z are respectively called the spectrum and the random spectrum of the sequence X.

Gaposhkin's [4] characterization concerns the usual averages

$$B_t = c_d t^{-d} \int_{|s| \le t} X_s \, ds, \qquad t > 0,$$

where c_d is a normalization constant and $|\cdot|$ is the Euclidean norm. In particular,

$$B_t \to Z(\{0\}) \quad \text{a.s. as } t \to \infty$$

if and only if

$$Z(\{\lambda : |\lambda| \le 2^{-k}\}) \to Z(\{0\}) \quad \text{a.s. as } k \to \infty, k \in Z.$$

Notice that the second condition involves only the behavior of the spectral measure along a thin sequence of sets.

If either of these conditions holds for X, we say that the strong law of large numbers holds, and we write $X \in$ SLLN. Gaposhkin includes many related results in his paper; besides discussing the discrete case, he also discusses the case in which one forms the averages over sequences of increasing sets. Also, Jajte [9], which we mentioned above, considers the case of the discrete form of the Hilbert transform. We will consider continuous, multidimensional singular integrals below.

As mentioned above, we are interested in other averages, in particular, averages with respect to singular measures, such as averages over spheres, in dimension 2 or higher. In R^d, let S_d^r denote the $d-1$-dimensional sphere of radius r, and let σ_d^r be the unique rotationally invariant normalized measure on S_d^r (when $r = 1$, we just write S_d and σ_d). Set

$$A_t = \int_{S_d^t} X_s \sigma_d^t(ds) = \int_{S_d} X_{ts} \sigma_d(ds).$$

If the map $t \to X_t$ is a.s. measurable with respect to the Borel σ-field in t, then A_t will be a random variable. This is not obvious, but is a consequence of the study of the spherical means. See the discussion at the beginning of [15].

We prove in high dimensions that the formally weaker notion of convergence of averages over spheres is equivalent to convergence of averages over balls.

THEOREM 1.1. *If $d \geq 3$, then $A_t \to Z(\{0\})$ a.s. if and only if $X \in SLLN$.*

The restriction on dimension is sharp. In particular, in two dimensions, the a.s. convergence of the averages A_t can fail for strictly stationary sequences X. This was pointed out by Stein [14], but also see the elaboration by Jones [10]. On the other hand, the strong law of large numbers does hold for strictly stationary sequences X with finite pth moment, where p must be strictly greater than 2; see [12].

Also, the corresponding result for averages over the surface of cubes is not true. Indeed, in two dimensions, consider the following convolution problem. If C denotes the square and ν the unit arc length measure on C, one sees that the supremum

$$\sup_t \int_C f(x - ty)\,\nu(dy)$$

can be infinite a.e. even if f is taken to be a bounded function. Indeed, just make f infinite on a single vertical line in the plane. Such examples, which hold in all dimensions, can be transferred to strictly stationary sequences. In short, the interesting feature of the spherical means is that positive results are available, due to the curvature of the sphere.

One can obtain a sharp range of results in all dimensions by considering certain generalizations of Cesàro averages, and in doing this we follow the lead of Stein and Wainger [15]. The averages below are defined initially only for $\alpha > 0$:

$$C_{\alpha,t} = c_{\alpha,d}\, t^{-d} \int_{|s| \leq t} X_s \frac{ds}{\left(1 - |s|^2/t^2\right)^{-1+\alpha}},$$

where $c_{\alpha,d}$ is a normalization constant. They are then extended to the complex plane by analytic continuation (see [15]). In that instance, we recover the spherical averages when $\alpha = 0$. The $C_{\alpha,t}$ admit the representation $\int_{R^d} m_\alpha(t\lambda) Z(d\lambda)$, with

$$m_\alpha(\lambda) = 2^{\alpha-1+d/2}\Gamma(\alpha + d/2)|\lambda|^{1-\alpha-d/2} J_{\alpha-1+d/2}(|\lambda|), \tag{1.1}$$

where J_n is the nth-order Bessel function. Moreover, if $\mathrm{Re}(\alpha) > d/2$ and if $C_{\alpha,t}$ converges, then so does $C_{\alpha',t}$, for $\mathrm{Re}(\alpha') > \mathrm{Re}(\alpha)$.

We have the following motivations for considering the above averages: as already mentioned, a sharp range of results in all dimensions can then be obtained. Also, for certain α, the m_α give rise to the fundamental solution of the wave equation.

THEOREM 1.2. *In any dimension d, if $\mathrm{Re}(\alpha) > 1 - d/2$, then*

$$C_{\alpha,t} \to Z(\{0\}) \quad a.s. \text{ as } t \to \infty$$

if and only if $X \in SLLN$. In particular, for $d \geq 3$, we recover the previous theorem.

In one dimension, the condition $\mathrm{Re}(\alpha) > 1/2$ is sharp. Indeed, Gaposhkin [5] has already shown that, for $d = 1$, the theorem above can fail, for $\alpha = 1/2$, in the weakly stationary case. In the strictly stationary case, however, the $C_{1/2,t}$ means converge a.s., yet the maximal function in t is not square integrable, but only weakly square integrable; see [2]. We remark that the techniques employed in the present paper implicitly prove the square integrability of the maximal function, and so they cannot be used in this delicate case.

We also note that the convergence of the spherical averages trivially implies the convergence of the Cesàro averages, for $\alpha > 0$.

Next, we turn to a result suggested by Jajte [9]. He applied Gaposhkin's approach to the discrete Hilbert transform. We treat the continuous multidimensional case as follows. First, let k denote a Calderón–Zygmund kernel on R^d. Such kernels can be defined in many ways. For specificity, we will require that the kernel satisfy the following size and smoothness conditions. Let k be a kernel on R^d, for which the following hold:

$$\int_{S_d} k(ry)\,\sigma(dy) = 0 \quad \text{for all } 0 < r < \infty \tag{1.2}$$

(in one dimension this means that the kernel must be odd);

$$|k(y)| \le \frac{C}{|y|^d}; \tag{1.3}$$

and for some $\delta > 1/2$ and all $2|y| \le |x|$,

$$|k(x-y) - k(x)| \le C\frac{|y|^\delta}{|x-y|^{d+\delta}}. \tag{1.4}$$

Typically, one only requires that $\delta > 0$ in inequalities such as (1.4).

For such kernels we consider the truncations

$$T_t = \int_{1/t \le |y| \le t} k(y) X_y\,dy, \qquad t \ge 1.$$

THEOREM 1.3. *With the notation above*,

$$\lim_{t\to\infty} T_t \quad \textit{exists a.s.}$$

if and only if

$$\lim_{\substack{j\to+\infty\\ j\in Z}} \int_{M_j} \hat{k}(\lambda) Z(d\lambda) = 0 \quad a.s.,$$

where $\hat{k}$ *is the Fourier transform of* k *and* $M_j = \{\lambda\colon 0 < |\lambda| < 2^{-j}$ *or* $|\lambda| > 2^j\}$.

There is also an interesting equivalence between the strong law of large numbers and the pointwise convergence of singular integrals. This was noted by Jajte [9], and we give an extension of his observation here.

For $1 \le i \le d$, let $\{U_i^t\colon t \in R^d\}$ be a continuous group of unitary operators on $L^2(\Omega, \mathscr{B}, \mathscr{P})$, where $(\Omega, \mathscr{B}, \mathscr{P})$ is a probability space. Suppose that the operators in the different groups commute with one another. Set

$$U^t = U_1^{t_1} \cdots U_d^{t_d} \quad \text{for } t = (t_1, \dots, t_d) \in R^d.$$

For $f \in L^2(\Omega)$, we will consider the following limits, which exist in L^2:

$$Af(\omega) = \lim_{t \to \infty} \left(c_d t^d\right)^{-1} \int_{|s| \le t} U^s f(\omega)\, ds,$$

where c_d is the volume of the d-dimensional unit ball. Also consider the following singular integrals:

$$R_i f(\omega) = \lim_{t \to \infty} \int_{|s| \le t} U^s f(\omega) \frac{s_i}{|s|^{d+1}}\, ds, \qquad 1 \le i \le d.$$

In the integral, s_i is the ith coordinate of the d-dimensional vector s. This makes R_i the ith Riesz transform. Note that in this definition we integrate over s in a compact region of R^d, in analogy to the manner in which the averages are formed.

THEOREM 1.4. *The following are equivalent*:

(α) *The limit $Af(\omega)$ exists a.s. for all $f \in L^2(\Omega)$.*

(β) *For some* (*for all*) $1 \le i \le d$, *the limit*(*s*), $R_i f(\omega)$ *exists a.s. for all* $f \in L^2(\Omega)$.

Moreover, *the existence of the limit $Af(\omega)$, for f fixed*, *is equivalent to the variety of conditions in Theorem* 1.2.

Proofs of these theorems occupy the two subsequent sections. The final section contains some additional remarks on the theorems.

2. A lemma. The examples we treat are unified under the notation

$$M_t = \int_{R^d} m_t(\lambda) Z(d\lambda),$$

where the multipliers $m_t(\lambda)$ are appropriately chosen, that is, for the spherical averages, $m_t(\lambda)$ is $\hat{\sigma}(t\lambda)$, and where $d \ge 1$.

Let us impose the following assumptions on the functions $m_t(\lambda)$:

$$|m_t(\lambda)| \le C \quad \text{for all } t \text{ and } \lambda. \tag{2.1}$$

For some $\beta > 1/2$, for all $t/2 < s < t < \infty$ and all λ,

$$|m_t(\lambda) - m_s(\lambda)| \le C\left(\frac{t-s}{t}\right)^{\beta}, \tag{2.2}$$

$$|m_t(\lambda) - m_s(\lambda)| \le C((t-s)|\lambda|)^{\beta} \tag{2.3}$$

and

$$|m_t(\lambda) - m_s(\lambda)| \le \frac{C}{(t|\lambda|)^{\beta}}, \qquad |\lambda| > 0. \tag{2.4}$$

Notice that these inequalities weaken as β decreases; thus the exponents can be different in each of the last three lines, as long as they are strictly larger than $1/2$. Notice also that in the previous theorems condition (2.1) implies L^2-convergence.

With these inequalities, we can reduce the question of convergence of M_t to the convergence along a lacunary set of t. The lemma below is directly inspired by Gaposhkin's approach, and the proof uses the classical binary decomposition technique.

LEMMA 2.1. *Under the assumptions above*,

$$\sum_{l=-\infty}^{\infty} \mathscr{E} \sup_{2^l < t \le 2^{l+1}} |M_t - M_{2^l}|^2 < \infty. \tag{2.5}$$

And, *in particular*, M_t *converges a.s. as* $t \to \infty$ (*or* $t \to 0$) *if and only if* M_{2^l} *converges a.s. as* $l \to \infty$ (*or* $l \to -\infty$).

The most important special case of this lemma occurs with the functions $m_t(\lambda)$ being $m(t\lambda)$ for a fixed function m. In this instance, Lemma 2.1 can be simplified as follows.

LEMMA 2.2. *Let* m: $R^d \to R$ *be a bounded continuous function which*, *for some* $a > 1/2$, *is Lipschitz of order* a *near the origin. Away from the origin*, *suppose that for* $j \ge 1$ *we can write*

$$m(\lambda) = 2^{-\gamma j} n_j(\lambda), \qquad 2^j \le |\lambda| < 2^{j+1}, \qquad \gamma > 0, \tag{2.6}$$

where, *for some* $\delta \in R$,

$$\|n_j\|_{\mathrm{Lip}(a)} \le C 2^{j\delta}. \tag{2.7}$$

If $\gamma > \delta + 1/2$, *then* (2.5) *holds with* $M_t = \int_{R^d} m(t\lambda) Z(d\lambda)$.

PROOF OF LEMMA 2.1. Following Gaposhkin, the classical technique of dyadic decomposition is used. Fix an integer l. We bound

$$\mathscr{E} \sup_{2^l < t \le 2^{l+1}} |M_t - M_{2^l}|^2$$

by an appropriate integral against the spectral measure μ. Write $2^l < t \le 2^{l+1}$ as

$$t = 2^l \left(1 + \sum_{u=0}^{\infty} \varepsilon_u 2^{-u}\right)$$

for $\varepsilon_u \in \{0, 1\}$. Every $2^l < t \le 2^{l+1}$ can be written in this way. Further, given $(\varepsilon_1, \dots, \varepsilon_u) \in \{0, 1\}^u$, set $a_u = 2^l(1 + \sum_{v=0}^{u} \varepsilon_v 2^{-v})$.

To make the next step clear, in the expectation above, replace the supremum by a stopping time $t(\omega)$: $\omega \to [2^l, 2^{l+1})$. Write

$$t(\omega) = 2^l\left(1 + \sum_{u=0}^{\infty} \varepsilon_u(\omega) 2^{-u}\right).$$

Then, for an appropriate stopping time,

$$\begin{aligned}
\mathscr{E} \sup_{2^l < t \le 2^{l+1}} |M_t - M_{2^l}|^2 &\le 2\mathscr{E}|M_{t(\omega)} - M_{2^l}|^2 \\
&\le 2\mathscr{E}\left|\sum_{u=0}^{\infty} \left(M_{t_u(\omega)} - M_{t_{u-1}(\omega)}\right)\right|^2 \\
&\qquad \text{(where } t_u(\omega) = 2^l(1 + \textstyle\sum_{v=0}^{u} \varepsilon_v(\omega) 2^{-v}) \text{ and } t_{-1} = 2^l\text{)} \\
&= 2\mathscr{E}\left|\sum_{u=0}^{\infty} (u+1)(u+1)^{-1}\left(M_{t_u(\omega)} - M_{t_{u-1}(\omega)}\right)\right|^2 \\
&\le C \sum_{u=0}^{\infty} (u+1)^2 \mathscr{E}\left|M_{t_u(\omega)} - M_{t_{u-1}(\omega)}\right|^2 \\
&\qquad \text{(by the Cauchy–Schwarz inequality in } u\text{)}
\end{aligned}$$

$$\le C \sum_{u=0}^{\infty} (u+1)^2 2^u \max_{(\varepsilon_1, \dots, \varepsilon_u) \in \{0,1\}^u} \mathscr{E}|M_{a_u} - M_{a_{u-1}}|^2. \tag{2.8}$$

This last expectation is

$$\mathscr{E}|M_{a_u} - M_{a_{u-1}}|^2 = \int_{R^d} \left|m_{a_u}(\lambda) - m_{a_{u-1}}(\lambda)\right|^2 \mu(d\lambda). \tag{2.9}$$

The integral against the spectral measure μ is estimated in three distinct ways. In the first instance, set $R_{1,l} = \{\lambda: |\lambda| \le 2^{-l}\}$. Then, by (2.3),

$$\int_{R_{1,l}} \left|m_{a_u}(\lambda) - m_{a_{u-1}}(\lambda)\right|^2 \mu(d\lambda) \le C|a_u - a_{u-1}|^{2\beta} \int_{R_{1,l}} |\lambda|^{2\beta} \mu(d\lambda).$$

Now, a_u and a_{u-1} are 2^l times numbers which disagree in the uth place of their dyadic expansions. Hence $|a_u - a_{u-1}| \le 2^{l-u}$, and let us further denote the annuli $A_r = \{\lambda: 2^{-r-1} \le |\lambda| \le 2^{-r}\}$. Then continue the estimate above as

$$\int_{R_{1,l}} \left|m_{a_u}(\lambda) - m_{a_{u-1}}(\lambda)\right|^2 \mu(d\lambda) \le C 2^{2\beta(l-u)} \sum_{r=l}^{\infty} 2^{-2r\beta} \mu(A_r). \tag{2.10}$$

This estimate is independent of the choice of $(\varepsilon_1, \dots, \varepsilon_u) \in \{0,1\}^u$, as the two subsequent estimates will be.

In the second instance, set $R_{2,l} = \{\lambda\colon 2^{-l} < |\lambda| \le 2^{-l+u}\}$. Use the estimate (2.2) to get

$$\begin{aligned}\int_{R_{2,l}} \left|m_{a_u}(\lambda) - m_{a_{u-1}}(\lambda)\right|^2 \mu(d\lambda) &\le C\left(\frac{a_u - a_{u-1}}{a_u}\right)^{2\beta} \int_{R_{2,l}} \mu(d\lambda)\\ &\le C\left(\frac{2^{l-u}}{2^l}\right)^{2\beta} \int_{R_{2,l}} \mu(d\lambda)\\ &\le C 2^{-2\beta u} \sum_{r=l-u}^{l+1} \mu(A_r).\end{aligned} \tag{2.11}$$

In the third and final instance, set $R_{3,l} = \{\lambda\colon |\lambda| > 2^{-l+u}\}$. Use the estimate (2.4) to get

$$\begin{aligned}\int_{R_{3,l}} \left|m_{a_u}(\lambda) - m_{a_{u-1}}(\lambda)\right|^2 \mu(d\lambda) &\le \frac{C}{a_u^{2\beta}} \int_{R_{3,l}} \frac{\mu(d\lambda)}{|\lambda|^{2\beta}}\\ &\le C 2^{-2l\beta} \sum_{r=-\infty}^{l-u+1} 2^{2\beta r} \mu(A_r).\end{aligned} \tag{2.12}$$

We have completed our estimate of the expectation in (2.9). Putting this into (2.8), we get

$$\mathscr{E} \sup_{2^l < t \le 2^{l+1}} |M_t - M_{2^l}|^2 \le C \sum_{u=0}^{\infty} (u+1)^2 2^u (\rho_{1,l} + \rho_{2,l} + \rho_{3,l}),$$

where $\rho_{i,l}$, for $1 \le i \le 3$, is the contribution from the integration of μ over the region $R_{i,l}$. This must be summed over l. Let us consider $i = 1$. From (2.10),

$$\begin{aligned}\sum_{u=0}^{\infty} (u+1)^2 2^u \rho_{1,l} &\le C \sum_{u=0}^{\infty} (u+1)^2 2^{u+2\beta(l-u)} \sum_{r=l}^{\infty} 2^{-2r\beta} \mu(A_r)\\ &= C 2^{2\beta l} \sum_{u=0}^{\infty} \sum_{r=l}^{\infty} (u+1)^2 2^{u(1-2\beta)} 2^{-2\beta r} \mu(A_r)\\ &= C 2^{2\beta l} \sum_{r=l}^{\infty} 2^{-2\beta r} \mu(A_r),\end{aligned}$$

the last line following because $1 - 2\beta < 0$, that is, $1/2 < \beta$. Summing this over l gives

$$\begin{aligned}\sum_{l=-\infty}^{\infty} 2^{2\beta l} \sum_{r=l}^{\infty} 2^{-2\beta r} \mu(A_r) &= \sum_{r=-\infty}^{\infty} 2^{-2\beta r} \mu(A_r) \sum_{l=-\infty}^{r} 2^{2\beta l}\\ &= C \sum_{r=-\infty}^{\infty} \mu(A_r)\\ &< \infty.\end{aligned}$$

This completes the case of $i = 1$.

In the second case, $i = 2$, from (2.11),

$$\sum_{u=0}^{\infty} (u+1)^2 2^u \rho_{2,l} \le C \sum_{u=0}^{\infty} (u+1)^2 2^{u-2\beta u} \sum_{r=l-u}^{l+1} \mu(A_r).$$

This must be summed over l:

$$\begin{aligned}\sum_{l=-\infty}^{\infty} \sum_{u=0}^{\infty} (u+1)^2 2^{u-2\beta u} \sum_{r=l-u}^{l+1} \mu(A_r) &= \sum_{r=-\infty}^{\infty} \mu(A_r) \sum_{u=0}^{\infty} (u+1)^3 2^{u(1-2\beta)} \\ &\le C \sum_{r=-\infty}^{\infty} \mu(A_r) \\ &< \infty,\end{aligned}$$

since $1 - 2\beta < 0$.

And last of all, with $i = 3$, from (2.12),

$$\begin{aligned}\sum_{u=0}^{\infty} (u+1)^2 2^u \rho_{3,l} &\le C 2^{-2l\beta} \sum_{u=0}^{\infty} (u+1)^2 2^u \sum_{r=-\infty}^{l-u+1} 2^{2\beta r} \mu(A_r) \\ &= C 2^{-2l\beta} \sum_{r=-\infty}^{l+1} 2^{2\beta r} \mu(A_r) \sum_{u=0}^{l-r+1} (u+1)^2 2^u \\ &\le C 2^{(1-2\beta)l} \sum_{r=-\infty}^{l+1} (l-r+2)^2 2^{-(1-2\beta)r} \mu(A_r).\end{aligned}$$

Summing this over l gives

$$\begin{aligned}&\sum_{l=-\infty}^{\infty} 2^{(1-2\beta)l} \sum_{r=-\infty}^{l+1} (l-r+2)^2 2^{-(1-2\beta)r} \mu(A_r) \\ &\quad = \sum_{r=-\infty}^{\infty} 2^{-(1-2\beta)r} \mu(A_r) \sum_{l=r-1}^{\infty} (l-r+2)^2 2^{(1-2\beta)l} \\ &\quad = C \sum_{r=-\infty}^{\infty} \mu(A_r) \\ &\quad < \infty.\end{aligned}$$

This completes the proof of the lemma. □

PROOF OF LEMMA 2.2. With the assumptions placed upon the fixed function m, we need to check that Lemma 2.1 applies to the functions $\{m(t\lambda): t > 0\}$. Write

$$m(\lambda) = m_0(\lambda) + \sum_{j=1}^{\infty} m_j(\lambda),$$

where $m_0(\lambda)$ is supported on $\{\lambda: |\lambda| < 2\}$, and, for $j \ge 1$, $m_j(\lambda)$ is supported on $\{\lambda: 2^j \le |\lambda| < 2^{j+1}\}$. Then m_0 is a bounded $\mathrm{Lip}(a)$ function, and

$$m_j(\lambda) = 2^{-j\gamma} n_j(\lambda) \quad \text{with } \|n_j\|_{\mathrm{Lip}(a)} \le C 2^{j\delta}. \tag{2.13}$$

Since the hypotheses weaken as a decreases to $1/2$ and $\gamma > \delta + 1/2$, we can assume that $\gamma > \delta + a$. We check that the functions $\{m_j(t\lambda): t > 0\}$, for $j \geq 0$, satisfy (2.1), (2.2), (2.3) and (2.4) with constants that are summable in j.

Let us treat m_0. For $t/2 < s < t$, if either $m_0(t\lambda)$ or $m_0(s\lambda)$ are nonzero, then $|\lambda| < 4/t$, since m_0 is supported near the origin. Hence, to check (2.2),

$$\begin{aligned} \left| m_0(t\lambda) - m_0(s\lambda) \right| &\leq C|t\lambda - s\lambda|^a \\ &\leq C\left(\frac{|t-s|}{t} \right)^a. \end{aligned}$$

As $a > 1/2$, (2.2) holds. The second equation (2.3) is immediate. And for the third, notice that $|\lambda|t < 4$; hence

$$\begin{aligned} \left| m_0(t\lambda) - m_0(s\lambda) \right| &\leq 2\|m\|_\infty \\ &\leq \frac{C}{|\lambda|t}. \end{aligned}$$

Now, consider $m_j(\lambda)$. Notice that (2.1) trivially holds with constants summable in j. For (2.2), recall (2.13). If either $m_j(t\lambda)$ or $m_j(s\lambda)$ are nonzero, then $2^{j-1} \leq |t\lambda| < 2^{j+2}$. Hence

$$\begin{aligned} \left| m_j(t\lambda) - m_j(s\lambda) \right| &\leq 2^{-j\gamma} \left| n_j(t\lambda) - n_j(s\lambda) \right| \\ &\leq C 2^{-j(\gamma-\delta)} |t\lambda - s\lambda|^a \\ &\leq C 2^{-j(\gamma-\delta-a)} \left| \frac{t-s}{t} \right|^a. \end{aligned}$$

This is summable, as $\gamma > \delta + a$ is assumed. Notice that the second line above proves (2.3) with a coefficient summable in j, under the weaker condition $\gamma > \delta$. Finally, the last condition (2.4) is seen by

$$\begin{aligned} \left| m_j(t\lambda) - m_j(s\lambda) \right| &\leq 2^{-j\gamma+1} \|n_j\|_\infty \\ &\leq C 2^{-j(\gamma-\delta)} \\ &\leq C 2^{-j(\gamma-\delta-a)} (t|\lambda|)^{-a}, \end{aligned}$$

again, as $\gamma > \delta + a$, this is summable in j. □

3. Proofs. Most of the work for the proofs of the theorems has been done in the previous section. Recall that Theorem 1.2 contains the first theorem of the paper, so that we need only prove it. And, to prove Theorem 1.2, we need only apply Lemma 2.2 to the function m_α defined in (1.1). To do this, asymptotics for the Bessel functions are needed. The classical reference for this is [18]. One can also consult [16]. From properties of the Bessel functions, it follows that m_α is a bounded Lipschitz function at the origin. For $|\lambda|$ large, we have

$$\begin{aligned} m_\alpha(\lambda) &= c_\alpha|\lambda|^{-(d-1)/2-\alpha}\{\cos(|\lambda| - \alpha\pi/2 - \pi/4) + \sin(|\lambda| - \alpha\pi/2 - \pi/4)\} \\ &\qquad \times\big(1 + O(|\lambda|^{-1})\big) \\ &= c_\alpha|\lambda|^{-(d-1)/2-\mathrm{Re}(\alpha)}v_\alpha(|\lambda|), \end{aligned}$$

where v_α is a Lip(1) function on R which is bounded for $|\lambda|$ away from the origin. Thus, provided $(d-1)/2 + \mathrm{Re}(\alpha) > 1/2$, that is, $\mathrm{Re}(\alpha) > 1 - d/2$, the hypotheses of Lemma 2.2 are fulfilled.

We conclude that $C_{\alpha,t}$ converges a.s. to $Z\{0\}$ if and only if the same conclusion holds for $C_{\alpha,2^k}$, for $k = 1, 2 \dots$. Then, since

$$m_\alpha(0) = 1, \qquad |m_\alpha(t\lambda) - 1| \le C(t|\lambda|) \quad \text{and} \quad |m_\alpha(t\lambda)| \le C(t|\lambda|)^{-\beta},$$

where $\beta = (d-1)/2 + \mathrm{Re}(\alpha) > 1/2$, a simple square function inequality shows that

$$\begin{aligned} \sum_{k=1}^{\infty} \mathscr{E}\left|C_{\alpha,2^k} - \int_{|\lambda|\le 2^{-k}} Z(d\lambda)\right|^2 &= \sum_{k=1}^{\infty}\int_{R^d}\left|m_\alpha(2^k\lambda) - 1_{\{|\lambda|\le 2^{-k}\}}\right|^2\mu(d\lambda) \\ &\le C\int_{R^d}\mu(d\lambda). \end{aligned}$$

Thus we arrive at Gaposhkin's characterization of the strong law of large numbers.

For the proof of the third theorem, concerning the singular integrals, we need to check the following result.

LEMMA 3.1. *Let k be a Calderón–Zygmund kernel as defined in the Introduction. Then the functions*

$$m_t(\lambda) = \int_{\{|y|<1/t\}\cup\{|y|>t\}} e^{i\lambda\cdot y}k(y)\,dy$$

satisfy the hypotheses of Lemma 2.1. *In particular*, *if*

$$T_t = \int_{\{|y|<1/t\}\cup\{|y|>t\}} k(y)X_y\,dy,$$

we have that T_t converges a.s. if and only if T_{2^k} converges a.s.

PROOF. There are four hypotheses of Lemma 2.1 to check. The first, that $|m_t(\lambda)| \le C$ for all t and λ, is well known. We refer the reader to, for example, [17], Lemma XI.5.3. Let us check the other three conditions for

$$m_t(\lambda) = \int_{\{|y|>t\}} e^{i\lambda\cdot y}k(y)\,dy,$$

the integration over $\{|y| < 1/t\}$ being similar.

Fix $t/2 < s < t < \infty$ and set

$$\psi(\lambda) = m_s(\lambda) - m_t(\lambda)$$
$$= \int_{s<|y|<t} e^{i\lambda\cdot y} k(y)\, dy.$$

Equation (2.2) is trivial. Using only the size condition on $k(y)$, (1.3),

$$|\psi(\lambda)| \le \int_{s<|y|<t} |k(y)|\, dy$$
$$\le C\int_s^t \frac{1}{r}\, dr$$
$$\le C(\log t - \log s)$$
$$\le C\left(\frac{t-s}{t}\right).$$

The second condition is equally simple. As the spherical averages of k are 0, according to (1.2), we have

$$|\psi(\lambda)| = \left|\int_{s<|y|<t} (e^{i\lambda\cdot y} - 1) k(y)\, dy\right|$$
$$\le C|\lambda| \int_{s<|y|<t} |yk(y)|\, dy$$
$$\le C(t-s)|\lambda|.$$

Less trivial estimates are required for the third inequality, (2.4). We comment that by rescaling k to $k_1(y) = r^d k(ry)$, where $r > 0$, the kernel k_1 satisfies the inequalities for Calderón–Zygmund kernels, with the same constant C. Thus we change the integration in the definition of $\psi(\lambda)$ as follows:

$$\psi(\lambda) = \int_{s<|y|<t} e^{i\lambda\cdot y} k(y)\, dy$$
$$= \int_{s/t<|x|<1} e^{it\lambda\cdot x} t^d k(tx)\, dx.$$

Observe that $k_1(x) = t^d k(tx)$ is a bounded function. By virtue of (1.4), k_1 is Lipschitz of order δ on the annulus $\{s/t < |y| < 1\}$. Hence the decay estimate

$$|\psi(\lambda)| \le \frac{C}{(t|\lambda|)^\delta}$$

is a classical fact (see, e.g., [11], Theorem 1.4.6). Since $\delta > 1/2$ was assumed, the proof of the lemma is done. □

To finish the proof of Theorem 1.3, it is easily seen, using the above estimates on m_t, that

$$\sum_{k=1}^{\infty} \mathscr{E}\left|T_{2^k} - \int_{\{|\lambda| \le 2^{-k},\ |\lambda| > 2^k\}} \hat{k}(\lambda) Z(d\lambda)\right|^2$$

$$= \sum_{k=1}^{\infty} \int_{R^d} \left| m_{2^k}(\lambda) - \hat{k}(\lambda) 1_{\{|\lambda| \le 2^{-k},\ |\lambda| > 2^k\}} \right|^2 \mu(d\lambda)$$

$$\le C \int_{R^d} \mu(d\lambda).$$

In fact, we also note here that such simple square function estimates can also be obtained in the general framework of Lemma 2.1. Hence, under the appropriate conditions and with probability 1,

$$\lim_{k\to\infty} \left\{ M_{2^k} - \int_{\{|\lambda| \le 2^{-k},\ |\lambda| > 2^k\}} m_\infty(\lambda) Z(d\lambda) \right\} = 0, \quad \text{where } m_\infty = \lim_{t\to\infty} m_t.$$

From this, a spectral criterion for the a.s. convergence, as $t \to \infty$, of $\int_{R^d} \widehat{m_t}(s) X_s \, ds$ follows.

We come to the final theorem of the Introduction, Theorem 1.4. Recall the notation introduced for that theorem. Note that, for each $f \in L^2(\Omega)$, the process $\{U^t f\colon\ t \in R^d\}$ is weakly stationary. Indeed, from the commuting property of the transformations,

$$\int_\Omega U^t f \overline{U^s f} \, dP = \int_\Omega f \overline{U^{s-t} f} \, dP,$$

establishing weak stationarity. Thus we have the spectral equivalences of Theorems 1.2 and 1.3 at our disposal.

Now the spectral representation is

$$U^t f = \int_{R^d} e^{it\cdot\lambda} Z(d\lambda) f,$$

where $Z(d\lambda)$ is an orthogonal projection-valued measure. In addition, if A and B are disjoint Borel sets in R^d, then the projections $Z(A)$ and $Z(B)$ are orthogonal. Condition (α) of the theorem, with Gaposhkin's characterization, is equivalent to

$$(\alpha') \qquad Z\left([-2^{-k}, 2^{-k}]^d\right) f \to Z(\{0\}) f \quad \text{a.s. for all } f \in L^2(\Omega).$$

As well, we can characterize condition (β). It is well known that the Fourier transform of the Hilbert transform is

$$\int_{-\infty}^{\infty} \frac{e^{i\lambda y}}{y} \, dy = i\pi \operatorname{sign}(\lambda).$$

From this, and a little work, it follows that

$$\int_{R^d} e^{i\lambda\cdot s}\frac{s_i}{|s|^{d+1}}\,ds = c_d\operatorname{sign}(\lambda_i).$$

We apply Theorem 1.3 to the Riesz transforms. Remembering that we only truncated the singular integral at ∞ in the definition of $R_i f$, we see the following equivalence to condition (β):

$$(\beta') \qquad \begin{aligned} &Z\left\{\lambda \in [-2^{-k}, 2^{-k}]^d : \lambda_i > 0\right\} f \\ &\quad - Z\left\{\lambda \in [-2^{-k}, 2^{-k}]^d : \lambda_i < 0\right\} f \to 0 \\ &\qquad \text{a.s. for all } f \in L^2(\Omega) \text{ and for all (for some) } 1 \le i \le d. \end{aligned}$$

To prove Theorem 1.4, we have to prove the equivalence of (α') and (β'). This is done with the aid of some projections. For $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d) \in \{-1, +1\}^d$, set

$$P^{\varepsilon} f = Z\{\lambda : \varepsilon_i\lambda_i > 0,\ 1 \le i \le d\} f.$$

Assume (β') holds for a single j. Apply it to the functions $P^{\varepsilon} f$ to see that

$$(\beta'') \qquad \begin{aligned} &Z\left\{\lambda : 0 < \varepsilon_i\lambda_i < 2^{-k},\ 1 \le i \le d\right\} f \to 0 \\ &\qquad \text{a.s. for all } f \in L^2(\Omega) \text{ and all } \varepsilon \in \{-1, +1\}^d. \end{aligned}$$

But this condition clearly implies (α'), so the proof that (β) (in its restricted form) implies (α) is done.

Assuming (α) holds, and hence (α') holds, it clearly implies (β''). This last condition implies (β'), so that the proof of the theorem is complete.

4. Complements.

CASE 1. All of the results of this paper admit formulations valid in the discrete setting. To illustrate this point, let us show how to formulate an analog of the result for spherical averages (also see [10]). Let $X = \{X_j : j \in Z^d\}$ be a zero-mean, weakly stationary sequence of random variables indexed by the d-dimensional integers. Then X admits the representation

$$X_j = \int_{T^d} e^{2\pi i j\cdot\lambda} Z(d\lambda),$$

where Z is a σ-additive and orthogonally scattered measure on the Borel σ-algebra of $T^d = (-1/2, 1/2]^d$. The covariances are given by

$$R(j) = \mathscr{E} X_{j+k}\overline{X_k} = \int_{T^d} e^{2\pi i j\cdot\lambda}\mu(d\lambda).$$

Gaposhkin characterized the strong law of large numbers

$$\frac{c_d}{n^d}\sum_{|j|\le n} X_j \to Z(\{0\}) \quad \text{a.s.}$$

by the condition

$$\int_{[-2^{-k}, 2^{-k}]^d} Z(d\lambda) \to Z(\{0\}) \quad \text{a.s.}$$

Again, the point in the second condition is that k goes to ∞ in the integers, so that the second condition, in a sense, requires less than the first. Write $X \in$ SLLN if either condition holds.

Our interest in Gaposhkin-style characterization suggests that we should consider averaging over annuli but only of a restricted type. To do so, let us introduce some notation. Let $\alpha_n = \{x \in R^d\colon n \le |x| < n + w_n\}$ be annuli of inner radius n and outer radius $n + w_n$. The interesting averaging case for these annuli is when $w_n = o(n)$. Denote by $\tilde{\alpha}_n = \alpha_n \cap Z^d$, the lattice points in α_n. Let also

$$D_d(r) = \left| \operatorname{vol}(\{x\colon |x| < r\}) - \#(\{x\colon |x| < r\} \cap Z^d) \right|$$

be the absolute error between the volume of ball of radius r and the number of lattice points in this ball (# denotes cardinality). Again, for $X = \{X_j\colon j \in Z^d\}$, weakly stationary, we write $X \in$ SLLN if and only if $Z([-2^{-k}, 2^{-k}]^d) \to Z(\{0\})$ a.s.

Now, forming the averages

$$A_n = \frac{1}{\#\tilde{\alpha}_n} \sum_{j \in \tilde{\alpha}_n} X_j,$$

we have the following result.

THEOREM 4.1. *Let $d \ge 3$ and assume $w_n = o(n)$. If*

$$\sum_{n=1}^{+\infty} \left(\frac{D_d(n) + D_d(n + w_n)}{n^{d-1} w_n} \right)^2 < +\infty,$$

then $A_n \to Z(\{0\})$ a.s. as $n \to +\infty$, $n \in Z$, if and only if $X \in$ SLLN.

According to Jones [10], for $d \ge 5$, $D_d(r) \le cr^{d-2}$. Hence the assumption of the theorem is satisfied if

$$\sum_{n=1}^{+\infty} \left(\frac{n^{d-2}}{n^{d-1} w_n} \right)^2 = \sum_{n=1}^{+\infty} \left(\frac{1}{n w_n} \right)^2 < +\infty.$$

This is in particular the case if $w_n \gg (\log n)^{1/2+\varepsilon}/n^{1/2}$ for $\varepsilon > 0$.

For $d = 4$, $D_d(r) \le cr^{d-2} \log r$, and the assumption is satisfied if

$$\sum_{n=1}^{+\infty} \left(\frac{\log n}{n w_n} \right)^2 < +\infty,$$

or $w_n \gg (\log n)^{3/2+\varepsilon}/\sqrt{n}$ for $\varepsilon > 0$.

Still following [10], we note that $\#\tilde{\alpha}_n/\mathrm{vol}(\alpha_n) \to 1$ as $n \to \infty$. Indeed,

$$\frac{\#\tilde{\alpha}_n}{\mathrm{vol}(\alpha_n)} = \frac{\#(Z^d \cap B(n + w_n)) - \#(Z^d \cap B(n))}{\mathrm{vol}\, B(n + w_n) - \mathrm{vol}\, B(n)} = 1 + \varepsilon_n,$$

where

$$|\varepsilon_n| \le \frac{D_d(n) + D_d(n + w_n)}{\mathrm{vol}(B(n + w_n) - B(n))}.$$

Now $\mathrm{vol}(B(n + w_n) - B(n)) = \mathrm{vol}(\alpha_n) \simeq c_d n^{d-1} w_n$ as $w_n = o(n)$. Hence, under the assumption of the theorem, $\varepsilon_n \to 0$. So the claim is proved, and we can use either $\mathrm{vol}(\alpha_n)$ or $\#\tilde{\alpha}_n$ for normalizations.

Let us now sketch the proof of Theorem 4.1. We need to study the multipliers

$$\tilde{m}_n(\lambda) = (|\alpha_n|)^{-1} \sum_{j \in \tilde{\alpha}_n} e^{2\pi i j\cdot\lambda}, \qquad \lambda \in [-1/2, 1/2]^d$$

(above and below, $|\alpha_n|$ is the volume of α_n). In particular, we want to know that they satisfy the estimates of Lemma 2.1.

Define

$$m_n(\lambda) = \frac{1}{|\alpha_n|} \int_{\alpha_n} e^{2\pi i \lambda\cdot x}\, dx, \qquad \lambda \in R^d.$$

For $d \ge 3$, these functions do satisfy the estimates of Lemma 2.1. This is so, because the Fourier transform of the surface measure of the sphere satisfies the lemma, and the $m_n(\lambda)$ are smoother than that. Next, to compare $\tilde{m}_n(\lambda)$ and $m_n(\lambda)$, define

$$\begin{aligned}\mu_n(\lambda) &= \frac{1}{|\alpha_n|} \sum_{j \in \tilde{\alpha}_n} \int_{j + [-1/2, 1/2]^d} e^{2\pi i \lambda\cdot x}\, dx \\ &= \left\{ \frac{1}{|\alpha_n|} \sum_{j \in \tilde{\alpha}_n} e^{2\pi i j\cdot\lambda} \right\} \int_{[-1/2, 1/2]^d} e^{2\pi i \lambda\cdot x}\, dx = \tilde{m}_n(\lambda) K(\lambda),\end{aligned}$$

where $K(\lambda)$ is a fixed bounded, Lip(1) function which is bounded away from the origin in the complex plane for $\lambda \in [-1/2, 1/2]^d$.

Now, comparing $\mu_n(\lambda)$ to $m_n(\lambda)$, if Δ denotes the symmetric difference, we get

$$\begin{aligned}|\mu_n(\lambda) - m_n(\lambda)| &\le \frac{1}{|\alpha_n|} \left| \alpha_n \Delta \bigcup_{j \in \tilde{\alpha}_n} \left\{ j + \left[-\frac{1}{2}, \frac{1}{2} \right]^d \right\} \right| \\ &\le \frac{D(n) + D(n + w_n)}{|\alpha_n|}.\end{aligned}$$

This last estimate is an $L^\infty(d\lambda)$ estimate. Under the hypothesis (assumed in Theorem 4.1) that this last term is square summable, we can pass from estimating

$$A_n = \int_{[-1/2,1/2]^d} \tilde{m}_n(\lambda) Z(d\lambda) = \int_{[-1/2,1/2]^d} \frac{\mu_n(\lambda)}{K(\lambda)} Z(d\lambda)$$

to estimating

$$\int_{[-1/2,1/2]^d} \frac{m_n(\lambda)}{K(\lambda)} Z(d\lambda).$$

Then one immediately sees that the functions $m_n(\lambda)/K(\lambda)$, $\lambda \in [-1/2, 1/2]^d$, satisfy the hypotheses of Lemma 2.1. From this, we can conclude the sketch.

CASE 2. There are also modifications of the results for the Cesàro averages. In view of the elegant theorems available for these averages in the deterministic continuous case (see, e.g., [15]), this seems interesting, due to the advantage gained from the curvature of the sphere. We, however, do not pursue this topic here.

CASE 3. The methods of this paper, based as they are on spectral techniques, extend to a wide variety of processes which admit such a representation. Using the terminology and the methods of [7], the above results remain valid for some classes of nonstationary sequences, namely, the so-called (p, q)-bounded ones $0 \le p \le 2 \le q \le +\infty$. These classes include harmonizable stable sequences, periodically correlated sequences, L^2-bounded orthogonal sequences and, for example, some mixing sequences. In fact, the spectral approach also works for different averages, for example, the Borel method of summation, in which case $m_t(\theta) = \exp[-t(1 - \exp i\theta)]$, $t > 0$, $\theta \in (-\pi, \pi]$, satisfy conditions (2.1)–(2.4) (with $\beta = 1$). In the special case of harmonizable stable sequences or Gaussian stationary sequences, the random spectrum Z is independently scattered, which is to say that $Z(A)$ and $Z(B)$ are independent provided A and B are disjoint Borel sets. Consequently, Gaposhkin's spectral condition is always satisfied. Alternatively, the SLLN for harmonizable stable variables can be seen directly from the Gaussian result. This is done by using the conditioning argument provided to us by J. Rosinski (see [7], Theorem 3.9). These arguments rely on the representation of harmonizable stable variables by Fourier integrals. Local ergodic theorems can also be obtained in a similar fashion.

In this regard, we ought to mention, too, that random sequences having a Fourier representation with respect to an independently scattered measure Z are, in general, not strictly stationary. Indeed, strict stationarity is characterized by the random measure Z being rotationally invariant. Another class of variables for which the spectral condition of Theorem 1.1 is satisfied is the class of stationary sequences whose spectrum μ is absolutely continuous with Radon–Nikodym derivative in $L^{1+\varepsilon}$ for some $\varepsilon > 0$. See the proof of Corollary

3.4 of [7]. Other types of sufficient conditions (on the covariances) presented there also apply here.

The results presented here also complement a Rademacher–Menchov-type result obtained in [8] and apply when the framework there is violated. If $\{X_n\}$ is a weakly stationary sequence and if $\{\hat{k}(n)\}$ is an odd Calderón–Zygmund sequence, $\sum \hat{k}(n) X_n$ converges a.s. if and only if with probability 1,

$$\lim_{j \to +\infty} \int_{|\theta| < 2^{-j}} k(\theta) Z(d\theta) = 0.$$

This last condition is equivalent to $\lim_{j \to +\infty} \{Z(0, 2^{-j}) - Z(-2^{-j}, 0)\} = 0$, whenever k has a Lipschitz behavior of order $\alpha > 0$, in a neighborhood of the origin. More generally, if $k(\theta) \sim \sum \hat{k}(n) e^{in\theta}$ has finitely many jumps, say, $-\pi < \theta_1 \cdots < \theta_M < \pi$, with a "Calderón–Zygmund and Lipschitz behavior" near each jump, then $\sum \hat{k}(n) X_n$ converges a.s. if and only if

$$\lim_{j \to +\infty} \sum_{m=1}^{M} \left\{ \frac{k(\theta_m^+) - k(\theta_m^-)}{2} Z(\theta_m, \theta_m + 2^{-j}) + \frac{k(\theta_m^-) - k(\theta_m^+)}{2} Z(\theta_m - 2^{-j}, \theta_m) \right\} = 0.$$

CASE 4. Some operators on Hilbert space, and L^p-spaces, admit spectral representation. By the well-known interchangeability between weakly stationary sequences and unitary operators, the above results have versions for unitary operators. In fact, by the methods of [7], they also apply to contractions on Hilbert space and to some classes of operators on L^p-spaces. The techniques presented in [1] can also be adapted to obtain results as above for some other classes of operators on L^p-spaces. Reinterpreting these last operator theoretic results in a stochastic framework provides, in particular, spectral criteria for (generalized) moving average processes (see [3] and [6]) and so for two of the three elements in the decomposition of a stationary stable process recently obtained by Rosinski [13].

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