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WIENER FUNCTIONALS ASSOCIATED WITH JOINT DISTRIBUTIONS OF EXIT TIME AND POSITION FROM SMALL GEODESIC BALLS[1]

BY KEISUKE HARA

University of Tokyo

Consider the first exit time and position from small geodesic balls for Brownian motion on Riemannian manifolds. We establish a smooth Besselization technique and calculate the asymptotic expansion for the joint distributions by a purely probabilistic approach.

1. Introduction. Pinsky (1988) and Liao (1988) computed the asymptotic expansions for the joint distribution of the first exit time and position:

$$H_\varepsilon(\lambda, f) = E\left[\exp\left(-\lambda\frac{T_\varepsilon}{\varepsilon^2}\right) f\left(\frac{X(T_\varepsilon)}{\varepsilon}\right)\right], \qquad \varepsilon \searrow 0, \tag{1.1}$$

where T_ε is the first exit time from a geodesic ball of radius ε for Brownian motion on a Riemannian manifold and $X(t)$ is its pullback on the tangent space. Their method to get the expansion is the perturbation theory of the partial differential equations. Kôzaki and Ogura (1988) also studied the independence of exit time and position by a similar method.

In this paper, we give an expansion of $H_\varepsilon(\lambda, f)$ in the path space level and it turns out very simple. Furthermore we calculate explicitly Wiener functionals associated with the coefficients of this expansion and give a more detailed expansion than the Pinsky–Liao expansion with a purely probabilistic proof.

Our first strategy is the Besselization of the radial part $\|X(t)\|$ by transformation of drift,

$$E\left[e^{-\lambda T} f\left(\frac{X(T)}{\varepsilon}\right)\right] = E\left[e^{-\lambda \tilde{T}} f\left(\frac{Y(\tilde{T})}{\varepsilon}\right) e^{\Phi}\right], \tag{1.2}$$

where $Y(t)$ is the transformed process so that $\|Y\|$ is a Bessel process, $\tilde{T}$ is the exit time for $Y(t)$ and $\exp\Phi$ is the Girsanov–Maruyama density. This technique is used by Takahashi and Watanabe (1980) to study Onsagar–Machlup functions of diffusion processes. Our Besselization drift is different from theirs and is smooth. The smoothness is essential to our computation.

In the next section, we state our results precisely. In Section 3, we establish the smooth Besselization technique and apply the Cameron–Martin–Girsanov–Maruyama formula for transformation of drift. In Section 4, we apply Brownian scaling to get ε-expansions by formal calculation. In Sections 5

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and 6, we compute Wiener functionals in the expansion explicitly. Our main tools for these explicit calculations are the time-reversed integral and the rotational invariance of Brownian motion.

2. Main results. Let (M, g) be an n-dimensional connected C^∞-Riemannian manifold with $n \geq 2$ and let $B_m(\varepsilon)$ be the geodesic ball centered at $m \in M$ with small radius $\varepsilon > 0$. Let also $X = (X_t, P_m)$ be a Brownian motion starting at m on (M, g). We denote by T_ε the first exit time of X_t from the geodesic ball $B_m(\varepsilon)$, that is,

$$T_\varepsilon = \inf\{t > 0: X_t \notin B_m(\varepsilon)\}. \tag{2.1}$$

The symbol E_m stands for the expectation with respect to the probability measure P_m. We fix a normal coordinate system $(x_1, x_2, \ldots, x_n)$ around m and identify the tangent space M_m with R^n by the exponential map $\exp_m$. Note $\|x\| = \{\sum_{i=1}^n (x^i)^2\}^{1/2} = \mathrm{dist}(m, \exp_m x)$. Then $T_\varepsilon = \inf\{t > 0: \|X_t\| > \varepsilon\}$. We denote by S the unit sphere $\{x: \|x\| = 1\}$.

Let g_{ij}, g^{ij} and Γ^i_{jk} be the metric tensor, the inverse and Christoffel symbol, respectively. Let R_{ijkl}, ρ_{ij}, s and $\partial_h R_{ijkl}$, $\partial_h \rho_{ij}$, $\partial_h s$ be the curvature tensor, the Ricci curvature, the scalar curvature and their derivatives, respectively, all evaluated at m. We will adopt the convention of omitting the summation sign over repeated indices.

Our first goal is to give the following expansion on the path space level.

THEOREM 2.1. *For any smooth function f on the unit sphere S, we have the asymptotic expansion*

$$\begin{aligned} H_\varepsilon(\lambda, f) &= E\left[\exp\left(-\lambda \frac{T_\varepsilon}{\varepsilon^2}\right) f\left(\frac{X(T_\varepsilon)}{\varepsilon}\right)\right] \\ &= c_0(\lambda) I(f) + \varepsilon^2 I(uf) + \varepsilon^3 I(vf) + O(\varepsilon^4), \end{aligned} \tag{2.2}$$

where I is the mean of f with respect to the uniform distribution on S, $c_0(\lambda) = E[\exp(-\lambda\tau)]$, u and v are functions on S defined for $\theta \in S$ by

$$u(\theta) = E\left[-\frac{1}{6}\exp(-\lambda\tau)\int_0^\tau \rho_{ij} B_t^i \, dB_t^j \;\middle|\; B_\tau = \theta\right], \tag{2.3}$$

$$v(\theta) = E\left[-\frac{1}{24}\exp(-\lambda\tau)\int_0^\tau \partial_i \rho_{jk} B_t^j B_t^k \, dB_t^i \;\middle|\; B_\tau = \theta\right] \tag{2.4}$$

and $B_t = (B_t^1, B_t^2, \ldots, B_t^n)$ is the standard Brownian motion starting at 0 in R^n and τ is the first hitting time to the unit sphere by B_t.

Next we compute the Wiener functionals in the theorem above and we get the following theorem, which is more detailed than the Pinsky–Liao expansion.

THEOREM 2.2. *We have the following asymptotic expansion*:

$$H_\varepsilon(\lambda, f) = c_0(\lambda)I(f) + \varepsilon^2 I(uf) + \varepsilon^3 I(vf) + O(\varepsilon^4), \tag{2.5}$$

where

$$u(\theta) = c_1(\lambda)\rho_{ij}\theta^i\theta^j + c_2(\lambda)s, \tag{2.6}$$

$$v(\theta) = c_3(\lambda)\,\partial_i\rho_{jk}\,\theta^i\theta^j\theta^k + c_4(\lambda)\,\partial_i s\,\theta^i, \tag{2.7}$$

where $c_i(\lambda)$ are the following constants: *for* $\lambda \geq 0$,

$$\begin{aligned} &c_1(\lambda) = -\frac{1}{12}\varphi(\lambda), \qquad c_2(\lambda) = -\frac{1}{12}\varphi'(\lambda),\\ &c_3(\lambda) = -\frac{1}{24}\varphi(\lambda), \qquad c_4(\lambda) = \begin{cases} \dfrac{1}{24}\dfrac{\phi''(\lambda)}{\phi(\lambda)\phi'(\lambda)}, & \lambda > 0,\\ \dfrac{1}{24(n+2)}, & \lambda = 0, \end{cases} \end{aligned} \tag{2.8}$$

and

$$\varphi(\lambda) = E[e^{-\lambda\tau}], \qquad \phi(\lambda) = 1/\varphi(\lambda). \tag{2.9}$$

3. Stochastic differential equation and smooth Besselization. We start with the stochastic differential equation (S.D.E.) for Brownian motion on Riemannian manifold. In the normal coordinate system $(x^1, x^2, \dots, x^n)$, Brownian motion is the solution of the S.D.E. [see, e.g., Ikeda and Watanabe (1981)]

$$dX_t^i = \sigma_{ik}(X_t)\,dB_t^k + a^i(X_t)\,dt, \qquad X_0 = 0, \tag{3.1}$$

where $\sigma_{ij}(x)$ is the square root of $g^{ij}(x)$. That is, $\sigma_{ij}(x)$ is symmetric and such that

$$\sum_{k=1}^n \sigma_{ik}(x)\sigma_{kj}(x) = g^{ij}(x) \tag{3.2}$$

and $a^i(x)$ is Coriolis drift

$$a^i(x) = -\tfrac{1}{2}g^{jk}(x)\Gamma^i_{jk}(x). \tag{3.3}$$

Our first idea to compute $H_\varepsilon(\lambda, f)$ is the smooth Besselization of $\|X_t\|$ by the Cameron–Martin–Girsanov–Maruyama formula. The following two lemmas are the essential part in our proof. Lemma 3.1 is proved by Takahashi and Watanabe (1980).

LEMMA 3.1 (Takahashi–Watanabe). *There exists a local vector field $b(x)$ such that the radial part $\|\tilde{X}_t\|$ of the solution $\tilde{X}_t$ of the S.D.E.*

$$d\tilde{X}_t^i = \sigma_{ik}(\tilde{X}_t)\,dB_t^k + b^i(\tilde{X}_t)\,dt, \qquad \tilde{X}_0 = 0, \tag{3.4}$$

is a Bessel process with the dimension n.

PROOF. Though the proof of Lemma 3.1 is in Takahashi and Watanabe (1980), we give the proof here for completeness.

By the Itô formula,

$$d\bigl(\|\tilde{X}_t\|^2\bigr) = 2\sum_{j=1}^{n} \tilde{X}_t^j\,dB_t^j + \left\{2\sum_{j=1}^{n} \tilde{X}_t^j b^j(\tilde{X}_t) + \sum_{j=1}^{n} g^{jj}(\tilde{X}_t)\right\} dt. \tag{3.5}$$

We can choose $b^j(x)$ such that the following condition is satisfied:

$$\sum_{j=1}^{n} b^j(x)x^j = \frac{1}{2}\sum_{j=1}^{n}(1 - g^{jj}(x)). \tag{3.6}$$

For example,

$$\tilde{b}^i(x) = -\frac{x^i}{2\|x\|^2}\sum_{j=1}^{n}(g^{jj}(x) - 1) \tag{3.7}$$

trivially satisfies (3.6). Then (3.5) implies

$$d(\|\tilde{X}_t\|^2) = 2\sum_{j=1}^{n} \tilde{X}_t^j\,dB_t^j + n\,dt. \tag{3.8}$$

Therefore,

$$d(\|\tilde{X}_t\|^2) = 2\|\tilde{X}_t\|\,dM_t + n\,dt, \tag{3.9}$$

where

$$M_t = \sum_{j=1}^{n}\int_0^t \frac{\tilde{X}_t^j}{\|\tilde{X}_t\|}\,dB_t^j \tag{3.10}$$

is a one-dimensional standard Brownian motion. Hence, $\|\tilde{X}_t\|$ is a Bessel process with index n. We obtain Lemma 3.1. □

LEMMA 3.2. *We can choose the following smooth drift as the $b(x)$ in Lemma* 3.1:

$$b^i(x) = \frac{1}{2}\frac{\partial}{\partial x^k} g^{ik}(x). \tag{3.11}$$

PROOF. We check that $b^i(x)$ in Lemma 3.2 satisfies the condition (3.6). As we chose the normal coordinate system, we have the basic relation

$$x^i = \sum_{j=1}^{n} x^j g^{ij}(x). \tag{3.12}$$

Differentiating both sides, we obtain

$$1 = g^{ii} + \sum_{j=1}^{n} x^j \frac{\partial}{\partial x^i} g^{ij}. \tag{3.13}$$

Then

$$\sum_{i=1}^{n}(1 - g^{ii}) = 2\sum_{j=1}^{n} b^j x^j; \tag{3.14}$$

hence the lemma follows. □

REMARK. Takahashi and Watanabe (1980) used (3.7) as the Besselization drift $b(x)$, which is singular at the origin. Recently, Takahashi pointed out that our smooth drift can make their proof very simple.

Now we apply the Cameron–Martin–Girsanov–Maruyama formula to $H_\varepsilon(\lambda, f)$. That is, the probability measure induced by X_t is absolutely continuous with respect to the measure induced by $\tilde{X}_t$ and

$$\begin{aligned} H_\varepsilon(\lambda, f) &= E\left[\exp\left(-\lambda \frac{T_\varepsilon}{\varepsilon^2}\right) f\left(\frac{X(T_\varepsilon)}{\varepsilon}\right)\right] \\ &= E\left[\exp\left(-\lambda \frac{\tilde{T}_\varepsilon}{\varepsilon^2}\right) f\left(\frac{X(T_\varepsilon)}{\varepsilon}\right) \exp(\Phi(\tilde{T}_\varepsilon))\right], \end{aligned} \tag{3.15}$$

where $\tilde{T}_\varepsilon$ is the first exit time of $\tilde{X}_t$:

$$\tilde{T}_\varepsilon = \inf\{t > 0 : \|\tilde{X}_t\| > \varepsilon\} \tag{3.16}$$

and

$$\begin{aligned} \Phi(\tilde{T}_\varepsilon) = &\int_0^{\tilde{T}_\varepsilon} \sum_{i,j=1}^{n} \sigma_{ij}(\tilde{X}_t) c^i(\tilde{X}_t)\, dB_t^j \\ &- \frac{1}{2}\int_0^{\tilde{T}_\varepsilon} \sum_{i,j=1}^{n} g^{ij}(\tilde{X}_t) c^i(\tilde{X}_t) c^j(\tilde{X}_t)\, dt, \end{aligned} \tag{3.17}$$

$$c^i(x) = a^i(x) - b^i(x). \tag{3.18}$$

We will calculate the right-hand side of (3.15) in the next section.

4. Brownian scaling and ε-expansion. In this section, we give the stochastic expansion with respect to a small parameter ε by Brownian scaling and the Itô fomula. Since the radial part of our new process $\tilde{X}_t$ is a Bessel process with index n, we can apply Brownian scaling to the exit time of $\tilde{X}_t$ from the ε-ball:

$$\tilde{T}_\varepsilon = \varepsilon^2 \tilde{T}_1, \quad \text{in law}, \tag{4.1}$$

where $\tilde{T}_1$ is the exit time of the standard Brownian motion from the unit ball in the Euclidean space R^n. For simplicity, let us write τ for $\tilde{T}_1$. Hence setting

$$\tilde{X}^\varepsilon_t = \tilde{X}(\varepsilon^2 t), \tag{4.2}$$

we get

$$\begin{aligned} &\exp\left(-\lambda \frac{\tilde{T}_\varepsilon}{\varepsilon^2}\right) f\left(\frac{\tilde{X}(\tilde{T}_\varepsilon)}{\varepsilon}\right) \\ &\quad = \exp(-\lambda\tau) f\left(\frac{\tilde{X}(\varepsilon^2\tau)}{\varepsilon}\right) = \exp(-\lambda\tau) f\left(\frac{\tilde{X}^\varepsilon(\tau)}{\varepsilon}\right), \end{aligned} \tag{4.3}$$

$$\begin{aligned} \Phi &= \Phi(\tilde{T}_\varepsilon) \\ &= \int_0^{\varepsilon^2\tau} \sum_{i,j=1}^n \sigma_{ij}(\tilde{X}_t) c^i(\tilde{X}_t)\, dB^j_t \\ &\quad - \frac{1}{2}\int_0^{\varepsilon^2\tau} \sum_{i,j=1}^n g^{ij}(\tilde{X}_t) c^i(\tilde{X}_t) c^j(\tilde{X}_t)\, dt \\ &= \varepsilon \int_0^{\tau} \sum_{i,j=1}^n \sigma_{ij}(\tilde{X}^\varepsilon_t) c^i(\tilde{X}^\varepsilon_t)\, dB^j_t \\ &\quad - \frac{1}{2}\varepsilon^2 \int_0^{\tau} \sum_{i,j=1}^n g^{ij}(\tilde{X}^\varepsilon_t) c^i(\tilde{X}^\varepsilon_t) c^j(\tilde{X}^\varepsilon_t)\, dt. \end{aligned} \tag{4.4}$$

Now we need the Taylor expansions of the geometric tensors with respect to the normal coordinate. The following lemma for the metric tensor is proved by E. Cartan [see, e.g., Gray (1990)].

LEMMA 4.1. *For small* x,

$$g_{ij}(x) = \delta_{ij} - \tfrac{1}{3} R_{aibj} x^a x^b - \tfrac{1}{6}\partial_a R_{bicj} x^a x^b x^c + O(\|x\|^4). \tag{4.5}$$

We can get the following lemma by easy calculation from Lemma 3.1.

LEMMA 4.2. *For small* x,

$$a^i(x) = -\tfrac{1}{3}\rho_{ia} x^a + \left(\tfrac{1}{24}\partial_i \rho_{ab} - \tfrac{1}{4}\partial_a \rho_{bi}\right) x^a x^b + O(\|x\|^3), \tag{4.6}$$

$$b^i(x) = -\tfrac{1}{6}\rho_{ia}x^a + \big(\tfrac{1}{12}\partial_i\rho_{ab} - \tfrac{1}{4}\partial_a\rho_{bi}\big)x^a x^b + O(\|x\|^3), \tag{4.7}$$

$$c^i(x) = -\tfrac{1}{6}\rho_{ia}x^a - \tfrac{1}{24}\partial_i\rho_{ab}x^a x^b + O(\|x\|^3). \tag{4.8}$$

Next let us get the formal asymptotic expansion of $\tilde{X}_t^\varepsilon$.

LEMMA 4.3.

$$\frac{\tilde{X}_t^{\varepsilon,i}}{\varepsilon} = B_t^i + \frac{\varepsilon^2}{6}N_t^{(2),i} + \frac{\varepsilon^3}{12}N_t^{(3),i} + O(\varepsilon^4), \tag{4.9}$$

where

$$N_t^{(2),i} = R_{aibj}\int_0^t B_s^a B_s^b \, dB_s^j - \rho_{ia}\int_0^t B_s^a \, ds, \tag{4.10}$$

$$N_t^{(3),i} = \partial_a R_{bicj}\int_0^t B_s^a B_s^b B_s^c \, dB_s^j + (\partial_i\rho_{ab} - 3\,\partial_a\rho_{ib})\int_0^t B_t^a B_t^b \, ds. \tag{4.11}$$

In particular,

$$\tilde{X}_t^\varepsilon = \varepsilon B_t + O(\varepsilon^3). \tag{4.12}$$

PROOF. The stochastic differential equation of $\tilde{X}_t^\varepsilon$ is

$$\tilde{X}_t^{\varepsilon,i} = \varepsilon\int_0^t \sigma_{ij}(\tilde{X}_s^\varepsilon)\, dB_s^j + \varepsilon^2\int_0^t b^i(\tilde{X}_s^\varepsilon)\, ds. \tag{4.13}$$

By Lemma 4.1 and 4.2, we know

$$\begin{aligned} \sigma_{ij}(\tilde{X}_t^\varepsilon) = \delta_{ij} &+ \tfrac{1}{6}R_{aibj}\tilde{X}_t^{\varepsilon,a}\tilde{X}_t^{\varepsilon,b} \\ &+ \tfrac{1}{12}\partial_a R_{bicj}\tilde{X}_t^{\varepsilon,a}\tilde{X}_t^{\varepsilon,b}\tilde{X}_t^{\varepsilon,c} + O(\|\tilde{X}_t^\varepsilon\|^4) \end{aligned} \tag{4.14}$$

and the expansion of $b^i(\tilde{X}_t)$. Then we get the lemma by iterated substitutions in (4.13). □

The next proposition is a direct consequence of Lemma 4.3 and the Itô formula.

PROPOSITION 4.4. *We have*

$$f\left(\frac{\tilde{X}_t^\varepsilon}{\varepsilon}\right) = f(B_t) + \varepsilon^2 N_t^{(2),i}\frac{\partial}{\partial x^i}f(B_t) + \varepsilon^3 N_t^{(3),i}\frac{\partial}{\partial x^i}f(B_t) + O(\varepsilon^4), \tag{4.15}$$

$$\Phi = -\frac{1}{6}\varepsilon^2\int_0^\tau \rho_{ij}B_t^i\, dB_t^j - \frac{1}{24}\varepsilon^3\int_0^\tau \partial_i\rho_{jk}\, B_t^j B_t^k\, dB_t^i + O(\varepsilon^4). \tag{4.16}$$

Now using Proposition 4.4, we can get the following expansion of (3.15) by a standard argument:

$$
\begin{aligned}
H_\varepsilon(\lambda, f) &= E\left[\exp(-\lambda\tau) f\left(\frac{\tilde{X}^\varepsilon(\tau)}{\varepsilon}\right)\exp(\Phi)\right] \\
&= E[\exp(-\lambda\tau) f(B_\tau)] \\
&\quad - \frac{1}{6}\varepsilon^2 E\left[\exp(-\lambda\tau) f(B_\tau)\int_0^\tau \rho_{ij} B_t^i\, dB_t^j\right] \\
&\quad - \frac{1}{24}\varepsilon^3 E\left[\exp(-\lambda\tau) f(B_\tau)\int_0^\tau \partial_i \rho_{jk}\, B_t^j B_t^k\, dB_t^i\right] \\
&\quad + \varepsilon^2 E\left[\exp(-\lambda\tau) N_\tau^{(2),i} \frac{\partial}{\partial x^i} f(B_\tau)\right] \\
&\quad + \varepsilon^3 E\left[\exp(-\lambda\tau) N_\tau^{(3),i} \frac{\partial}{\partial x^i} f(B_\tau)\right] + O(\varepsilon^4).
\end{aligned}
\tag{4.17}
$$

5. Calculation of Wiener functionals I: proof of Theorem 2.1. In Sections 5 and 6, we calculate the Wiener functionals in the expansion (4.17). In this section, we prove that $N_t^{(1)}$ and $N_t^{(2)}$ contribute nothing to the expectation, and so we get Theorem 2.1.

PROPOSITION 5.1. *We have*

$$
E[e^{-\lambda\tau} g(B_\tau) N_\tau^{(K),i}] = 0 \tag{5.1}
$$

for any smooth function g *on* S, *where* $K = 1, 2$ *and* $i = 1, \dots, n$.

PROOF. First, we decompose the Brownian motion in polar form:

$$
B_t^i = \|B_t\| \frac{B_t^i}{\|B_t\|} = R_t U_t^i, \tag{5.2}
$$

where R_t is the radial part and U_t is the spherical part. Note that U_t^i are martingales which are orthogonal to the martingale part of R_t.

Then,

$$
\begin{aligned}
N_t^{(1),i} &= R_{aibj}\int_0^t R_s^2 U_s^a U_s^b d(R_s U_s^j) - \rho_{ia}\int_0^t R_s U_s^a\, ds \\
&= R_{aibj}\int_0^t R^2 U^a U^b (R\, dU^j + U^j\, dR + dR\, dU^j) - \rho_{ia}\int_0^t R U^a\, ds \\
&= R_{aibj}\int_0^t R^3 U^a U^b\, dU^j - \rho_{ia}\int_0^t R U^a\, ds.
\end{aligned}
\tag{5.3}
$$

The last equality is by the skew symmetric property of R_{aibj}: $R_{aibj} = -R_{aijb}$.

In the same manner, we get

$$
N_t^{(2),i} = \partial_a R_{bicj}\int_0^t R^4 U^a U^b U^c\, dU^j + (\partial_i \rho_{ab} - 3\,\partial_a \rho_{ib})\int_0^t R^2 U^a U^b\, ds. \tag{5.4}
$$

Let us calculate the following by the Itô formula:

$$
\begin{aligned}
d(R_{aibj}R^3U^aU^b)\,dU^j &= R_{aibj}R^3U^b\,dU^a\,dU^j \\
&= R_{aibj}R^3U^b\frac{\delta_{aj}-U^aU^j}{R^2}\,dt = R_{aibj}RU^b\delta_{aj}\,dt \\
&= -R_{aijb}RU^b\delta_{aj}\,dt = -\rho_{ib}RU^b\,dt.
\end{aligned} \tag{5.5}
$$

Similarly,

$$
\begin{aligned}
&d(\partial_a R_{bicj}\,R^4U^aU^bU^c)\,dU^j \\
&\quad = R^4\,\partial_a R_{bicj}\{U^aU^b\,dU^c\,dU^j + U^bU^c\,dU^a\,dU^j + U^cU^a\,dU^b\,dU^j\} \\
&\quad = R^2\,\partial_a R_{bicj}\,\{U^aU^b(\delta_{cj}-U^cU^j) + U^bU^c(\delta_{aj}-U^aU^j) \\
&\qquad\qquad + U^cU^a(\delta_{bj}-U^aU^j)\}\,dt \\
&\quad = R^2(\partial_a R_{bijj}\,U^aU^b + \partial_j R_{bicj}\,U^bU^c + \partial_a R_{jicj}\,U^cU^a)\,dt \\
&\qquad\qquad\qquad \text{(by the Ricci identity)} \\
&\quad = R^2(-\partial_a\rho_{bi}U^aU^b + \partial_i\rho_{bc}U^bU^c - \partial_b\rho_{ci}U^cU^i - \partial_a\rho_{ci}U^cU^a)\,dt \\
&\quad = (\partial_i\rho_{bc} - 3\,\partial_b\rho_{ci})R^2U^bU^c\,dt.
\end{aligned} \tag{5.6}
$$

Then we get the representation for $N_t^{(K)}$:

$$
N_T^{(K)} = \int_0^T Z_j^{(K)}\,dU^j + \int_0^T dZ_j^{(K)}\,dU^j = \int_0^T Z_j^{(K)}d^+U^j, \tag{5.7}
$$

where $K = 1$, 2 and

$$
Z_j^{(1)} = R_{aibj}R^3U^aU^b, \qquad Z_j^{(2)} = \partial_a R_{bicj}R^4U^aU^bU^c. \tag{5.8}
$$

That is, we can represent the $N^{(K)}$ as anticipating (time-reversed) stochastic integrals [cf. Elworthy (1982)]. Therefore, they are martingales in terms of the reversed-time variable. That is, for the transformation $\check{\phi} = \phi(T-t)$,

$$
\int_0^T \phi\,d^+U = \int_0^T \check{\phi}\,d\check{U}. \tag{5.9}
$$

If we take the conditional expectation for the filtration $\mathscr{F}_R$ with respect to the radial part, U_t is a $P(\cdot|\mathscr{F}_R)$-martingale and

$$
E[e^{-\lambda\tau}g(B_\tau)N_\tau^{(K)}] = E\left[e^{-\lambda\tau}E\left[g(\check{B}(0))\int_0^\tau \check{Z}^{(K)}d\check{U}\;\middle|\;\mathscr{F}_R\right]\right] = 0. \qquad \square \tag{5.10}
$$

Then, we get Proposition 5.1 and so Theorem 2.1.

6. Calculation of Wiener functionals II: proof of Theorem 2.2. Finally, we prove Theorem 2.2. Our task is now to compute the following Wiener functionals:

$$I(uf) = E\left[-\frac{1}{6}e^{-\lambda\tau}f(B_\tau)\int_0^\tau \rho_{ij}B_t^i\,dB_t^j\right], \tag{6.1}$$

$$I(vf) = E\left[-\frac{1}{24}e^{-\lambda\tau}f(B_\tau)\int_0^\tau \partial_i\rho_{jk}B_t^jB_t^k\,dB_t^i\right]. \tag{6.2}$$

By the Itô formula,

$$\begin{aligned}\rho_{ij}B_\tau^iB_\tau^j &= \int_0^\tau 2\rho_{ij}B_t^i\,dB_t^j + \int_0^\tau \rho_{ij}d\langle B^i, B^j\rangle_t \\ &= 2\int_0^\tau \rho_{ij}B_t^i\,dB_t^j + \int_0^\tau s\,dt.\end{aligned} \tag{6.3}$$

Therefore,

$$\begin{aligned}&E\left[e^{-\lambda\tau}f(B_\tau)\int_0^\tau \rho_{ij}B_t^i\,dB_t^j\right] \\ &\quad = E\left[e^{-\lambda\tau}f(B_\tau)\left(\frac{1}{2}\rho_{ij}B_\tau^iB_\tau^j - \frac{1}{2}\tau s\right)\right] \\ &\qquad\qquad \text{(because of the rotational invariance of Brownian motion)} \\ &\quad = \frac{1}{2}\left(E[e^{-\lambda\tau}]\int_S f(\theta)\rho_{ij}\theta^i\theta^j\,d\theta - E[\tau e^{-\lambda\tau}]\int_S f(\theta)s\,d\theta\right).\end{aligned} \tag{6.4}$$

Then we get (2.6) in Theorem 2.

The calculation of (6.2) is similar, but a little more interesting. First,

$$\partial_i\rho_{jk}\,B_\tau^iB_\tau^j\,B_\tau^k = \partial_i\rho_{jk}\int_0^\tau B_t^jB_t^k\,dB_t^i + 2\,\partial_i s\int_0^\tau B_t^i\,dt \tag{6.5}$$

by the Itô formula and tensor calculation.

Therefore,

$$\begin{aligned}&E\left[e^{-\lambda\tau}f(B_\tau)\int_0^\tau \partial_i\rho_{jk}\,B_t^j\,B_t^k\,dB_t^i\right] \\ &\quad = E[e^{-\lambda\tau}f(B_\tau)\,\partial_i\rho_{jk}\,B_\tau^iB_\tau^jB_\tau^k] - 2\,\partial_i s\,E\left[e^{-\lambda\tau}f(B_\tau)\int_0^\tau B_t^i\,dt\right] \\ &\quad = E[e^{-\lambda\tau}]I[f(\theta)\theta^i\theta^j\theta^k\,\partial_i\rho_{jk}] - 2\,\partial_i s\,E\left[e^{-\lambda\tau}f(B_\tau)\int_0^\tau B_t^i\,dt\right].\end{aligned} \tag{6.6}$$

Now what remains to be proved is the following proposition.

PROPOSITION 6.1. *We have*

$$E\left[e^{-\lambda\tau}f(B(\tau))\int_0^\tau B_t^i\,dt\right] = c(\lambda)I[f(\theta)\theta^i], \tag{6.7}$$

$$c(\lambda) = \begin{cases} \dfrac{1}{2}\dfrac{\phi''(\lambda)}{\phi(\lambda)\phi'(\lambda)}, & \lambda > 0, \\ \dfrac{1}{2(n+2)}, & \lambda = 0, \end{cases} \tag{6.8}$$

where $\phi(\lambda) = 1/\varphi(\lambda)$ *and* $\varphi(\lambda) = E[e^{-\lambda\tau}]$.

PROOF. Let us calculate the following conditional expectation such that the exit position is fixed at $\theta \in S$.

Using the decomposition $B_t = R_t U_t$,

$$E\left[e^{-\lambda\tau} f(B_\tau) \int_0^\tau B_t^i \, dt \,\middle|\, B_\tau = \theta\right]$$
$$= f(\theta) E\left[e^t au \int_0^\tau R_t U_t^i \, dt \,\middle|\, U_\tau = \theta\right]$$

(by the rotational invariance of Brownian motion around the axis θ)

$$\begin{aligned} &= f(\theta) E\left[e^{-\lambda\tau} \int_0^\tau R_t \langle U_t, \theta\rangle \theta^i \, dt \,\middle|\, U_\tau = \theta\right] \\ &= f(\theta)\theta^i E\left[e^{-\lambda\tau} \int_0^\tau R_t \langle U_t, \theta\rangle \, dt \,\middle|\, U_\tau = \theta\right], \\ &= f(\theta)\theta^i E\left[e^{-\lambda\tau} \int_0^\tau B_t^1 \, dt \,\middle|\, B_\tau = (1, 0, \dots, 0)\right], \end{aligned} \tag{6.9}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in R^n.

Then we get the identity

$$\begin{aligned} E\left[e^{-\lambda\tau} f(B_\tau) \int_0^\tau B_t^i \, dt\right] &= E\left[E\left[e^{-\lambda\tau} f(B_\tau) \int_0^\tau B_t^i \, dt \,\middle|\, U_\tau = \theta\right]\right] \\ &= c(\lambda) I[f(\theta)\theta^i], \end{aligned} \tag{6.10}$$

where

$$c(\lambda) = E\left[e^{-\lambda\tau} \int_0^\tau B_t^1 \, dt \,\middle|\, B_\tau = (1, 0, \dots, 0)\right] \tag{6.11}$$

is a constant that depends on λ and the dimension n only. Let us determine this constant by the computation of a special case.

First, we assume $\lambda > 0$. We take $\exp(\sqrt{2\lambda} B_\tau^i)$ especially as $f(B_\tau)$. Note that $\exp(-\lambda t + \sqrt{2\lambda} B_t^i)$ is a martingale.

By the Itô formula,

$$\begin{aligned} \Lambda &= E\left[\exp\left(-\lambda\tau + \sqrt{2\lambda} B_\tau^i\right) \int_0^\tau B_t^i \, dt\right] \\ &= E[\exp(-\lambda\tau) h(B_\tau^i)] = E[\exp(-\lambda\tau)] I[h(\theta^i)], \end{aligned} \tag{6.12}$$

where

$$h(x) = \frac{1}{2}\exp(\sqrt{2\lambda}x)\left(\frac{x^2}{\sqrt{2\lambda}} - \frac{x}{2\lambda}\right). \tag{6.13}$$

On the other hand, by (6.10),

$$\Lambda = c(\lambda) I[\exp(\sqrt{2\lambda}\theta^i)\theta^i]. \tag{6.14}$$

Then, we have

$$c(\lambda) = \frac{E[\exp(-\lambda\tau)] I[h(\theta^i)]}{I[\exp(\sqrt{2\lambda}\theta^i)\theta^i]}. \tag{6.15}$$

Now, note the following two elementary relations:

$$\frac{\partial^2}{\partial\lambda^2}\exp(\sqrt{2\lambda}x) = \frac{1}{\sqrt{2\lambda}} h(x) \tag{6.16}$$

and

$$\begin{aligned} 1 \equiv E[\exp(-\lambda\tau + \sqrt{2\lambda}B_\tau)] &= E[\exp(-\lambda\tau)] I[\exp(\sqrt{2\lambda}\theta^i)] \\ &= \varphi(\lambda) I[\exp(\sqrt{2\lambda}\theta^i)]; \end{aligned} \tag{6.17}$$

that is,

$$I[\exp(\sqrt{2\lambda}\theta^i)] = \phi(\lambda). \tag{6.18}$$

By (6.15) and the relations above, we have

$$c(\lambda) = \frac{\phi''}{2\phi\phi'}. \tag{6.19}$$

Finally, if $\lambda = 0$, we take B_τ^i as $f(B_\tau)$. By the Itô formula,

$$E\left[B_\tau^i \int_0^\tau B_t^i\,dt\right] = \frac{1}{6}E[(B_\tau^i)^4] = \frac{1}{6}\int_S (\theta^i)^4\,d\theta = \frac{1}{2n(n+2)}. \tag{6.20}$$

On the other hand,

$$E\left[B_\tau^i \int_0^\tau B_t^i\,dt\right] = c(0) I[(\theta^i)^2] = \frac{c(0)}{n}. \tag{6.21}$$

Therefore,

$$c(0) = \frac{1}{2(n+2)}. \tag{6.22}$$

Thus, we have proved Proposition 6.1 and, consequently, we get Theorem 2.2. □

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DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF TOKYO
KOMABA, MEGURO-KU, TOKYO 153
JAPAN
E-MAIL: hara@poisson.c.u-tokyo.ac.jp