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HYDRODYNAMICAL LIMIT FOR SPACE INHOMOGENEOUS ONE-DIMENSIONAL TOTALLY ASYMMETRIC ZERO-RANGE PROCESSES[1]

BY C. LANDIM

Instituto de Matemática Pura e Aplicada

We consider totally asymmetric attractive zero-range processes with bounded jump rates on **Z**. In order to obtain a lower bound for the large deviations from the hydrodynamical limit of the empirical measure, we perturb the process in two ways. We first choose a finite number of sites and slow down the jump rate at these sites. We prove a hydrodynamical limit for this perturbed process and show the appearance of Dirac measures on the sites where the rates are slowed down. The second type of perturbation consists of choosing a finite number of particles and making them jump at a slower rate. In these cases the hydrodynamical limit is described by *nonentropy* weak solutions of quasilinear first-order hyperbolic equations. These two results prove that the large deviations for asymmetric processes with bounded jump rates are of order at least e^{-CN}. All these results can be translated to the context of totally asymmetric simple exclusion processes where a finite number of particles or a finite number of holes jump at a slower rate.

Introduction. Totally asymmetric one-dimensional zero-range processes are among the simplest interacting particle systems and can be informally described as follows. Consider indistinguishable particles moving on the one-dimensional integers **Z**. Let $g\colon \mathbf{N} \to \mathbf{R}$ be a nonnegative nondecreasing function with $g(0) = 0$. If a site x is occupied by n particles, then at a rate $g(n)$ one of them jumps to $x + 1$.

For each density $\rho \geq 0$ there exists an invariant measure, denoted by ν_ρ^{ti}, which is translation invariant and has mean density ρ:

$$\nu_\rho^{\mathrm{ti}}[\eta(0)] = \rho.$$

Here and in the sequel "ti" stands for translation invariant.

The configuration space $\mathbf{N}^{\mathbf{Z}}$ is denoted by $\mathscr{X}$ and the configurations by Greek letters η, ξ and χ. In this way, for an integer x, $\eta(x)$ denotes the total number of particles at site x for the configuration η.

Fix an integer N. For each configuration η we associate a Radon measure $\pi^N = \pi^N(\eta)$ on **R** rescaling **Z** and assigning mass N^{-1} to each particle of η:

$$\pi^N = N^{-1} \sum_{x \in \mathbf{Z}} \eta(x) \delta_{x/N}.$$

In this last formula, for a real u, δ_u denotes the Dirac measure on u.

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For a bounded continuous function $m^*\colon \mathbf{R} \to \mathbf{R}_+$, denote by $\mu^N_{m^*}$ the product measure on the configuration space $\mathscr{X}$ with marginals given by

$$\mu^N_{m^*}\{\eta;\ \eta(x) = k\} = \nu^{\mathrm{ti}}_{m^*(x/N)}\{\eta;\ \eta(0) = k\}$$

for $x \in \mathbf{Z}$ and $k \in \mathbf{N}$. At the macroscopic point $u \in \mathbf{R}$, the measure $\mu^N_{m^*}$ is close to the invariant state of the process with density $m^*(u)$ [in our setting the measure $\nu^{\mathrm{ti}}_{m^*(u)}$].

Denote by η_t the state of the process at time t and by π^N_t the empirical measure at time tN: $\pi^N_t = \pi^N(\eta_{tN})$. Rezakhanlou [7] proved a law of large numbers for the empirical measure. More precisely, he showed that, starting from the product measure $\mu^N_{m^*}$, the empirical measure π^N_t converges in probability to the absolutely continuous Radon measure $M(t, du) = m(t, u)\, du$ whose density $m(t, u)$ is the entropy solution of the quasilinear hyperbolic equation

$$\begin{aligned} \partial_t m + \partial_u \varphi(m) = 0,& \\ m(0, \cdot) = m^*(\cdot)\,.& \end{aligned}$$

Here φ is a smooth increasing function associated with the microscopic dynamic: $\varphi(\rho) = \nu^{\mathrm{ti}}_\rho[g(\eta(0))]$. This law of large numbers that describes the macroscopic evolution of the process is called the hydrodynamical limit (cf. [6] for a proof of the conservation of local equilibrium). Once a law of large numbers is established, a natural question is to study the large deviations.

A first step was taken in this direction by Kipnis and Léonard [4], who proved, through explicit computations, a large deviation principle for the empirical measure of a superposition of independent asymmetric random walks. They showed that the exponential decay rate of the large deviation probabilities is of order N^2 in dimension 1.

In the interacting case, however, Varadhan pointed out that large deviations should be of order e^{-CN} and that the perturbations which must be introduced in order to observe large deviations should be a deceleration of the jump rates at a finite number of sites. Moreover, with these perturbations one would obtain as the hydrodynamical limit *nonentropy* weak solutions of the hyperbolic equations.

In this article we consider totally asymmetric zero-range processes in one dimension and show that indeed large deviations are at least of order e^{-CN}. In order to do this we study two different types of perturbations. The first one consists in slowing down the jump rates at a finite number of sites. We prove a hydrodynamical limit for this process and show the appearance of Dirac measures on sites where the jumps are decelerated.

The second type of perturbation consists of slowing down the jump rates of a finite number of particles. We show, through an example, that the hydrodynamical limit of this process is given by nonentropy solutions of the hyperbolic equation.

These two types of perturbations enable us to obtain lower bounds for the large deviations for a class of profiles. This in particular shows that the large deviations are at least of order e^{-CN}.

By a well-known correspondence between nearest-neighbor zero-range processes and nearest-neighbor simple exclusion processes, all the previous results can be translated into the context of simple exclusion processes. In this case the perturbation consists of decelerating the jump rates of some particles or the jump rates of some holes. In both cases we obtain nonentropy solutions of Burgers' inviscid equations.

We consider the totally asymmetric case in order to avoid some technical problems. In fact, the method relies on the existence of product invariant measures. This is also the case for the nearest-neighbor processes. The results should therefore extend to this case.

The article is organized as follows. In Section 1 we introduce the notation used throughout this article and state the hydrodynamical limit for the first kind of perturbation considered. In Section 2, based on the first theorem, we study an example of the second type of perturbation. In Section 3 we prove a lower bound for the large deviations for a class of profiles. In Sections 4, 5 and 6 we prove the theorem stated in Section 1, and in the Appendix we fix the terminology of entropy weak solution of hyperbolic equation used in this article.

1. Notation and results. In this section we establish the notation and state the main results of the article.

The state space of the process $\mathbf{N}^{\mathbf{Z}}$ is denoted by $\mathscr{X}$ and the configurations by Greek letters η and ξ. In this way, for an integer x, $\eta(x)$ represents the number of particles at site x for the configuration η.

Fix a nondecreasing bounded function $g\colon \mathbf{N} \to \mathbf{R}_+$ vanishing at 0:

$$0 = g(0) < g(1), \qquad \sup_{k\geq 0} g(k) = g(\infty) < \infty, \qquad g(k) \leq g(k+1) \tag{1.1}$$

for all nonnegative integers k. Fix also a finite sequence of reals $u_1 < u_2 < \cdots < u_n$ and an associated sequence of functions $\alpha_j\colon \mathbf{R}_+ \to (0,1)$ for $1 \leq j \leq n$. For each positive integer N, the totally asymmetric space inhomogeneous zero-range process $(\eta_t)_{t\geq 0}$ associated with the rate function g and the sequence $\{(u_1,\alpha_1),\dots,(u_n,\alpha_n)\}$ is the Markov process on $\mathscr{X}$ whose generator acts on functions that depend only on a finite number of coordinates as

$$(L_{N,t}f)(\eta) = \sum_{x\in\mathbf{Z}} p_N(t,x) g(\eta(x))[f(\eta^{x,x+1}) - f(\eta)]. \tag{1.2}$$

In this formula, for configurations η with at least one particle at x, $\eta^{x,x+1}$ stands for the configuration obtained from η letting one particle jump from site x to site $x+1$:

$$\eta^{x,x+1}(y) = \begin{cases} \eta(y), & \text{if } y \neq x, x+1,\\ \eta(x) - 1, & \text{if } y = x,\\ \eta(x+1)+1, & \text{if } y = x+1, \end{cases} \tag{1.3}$$

and $p_N\colon \mathbf{R}_+ \times \mathbf{Z} \to [0,1]$ is a function equal to 1 on all but a finite number of macroscopic sites $x_i = [u_i N]$. More precisely,

$$p_N(t,x) = \begin{cases} \alpha_i(t), & \text{if } x = [u_i N] \text{ for } 1 \le i \le n, \\ 1, & \text{otherwise.} \end{cases}$$

We will say that $L_{N,t}$ is the generator associated with the sequence $\{(u_1,\alpha_1),\dots,(u_n,\alpha_n)\}$. This terminology is used throughout the article.

To keep the notation simple, we omitted the dependence of the generator on the sequence $\{(u_1,\alpha_1),\dots,(u_n,\alpha_n)\}$. However, to stress this dependence, we sometimes denote $L_{N,t}$ by $L_{N,t}(\{(u_1,\alpha_1),\dots,(u_n,\alpha_n)\})$. We denote by L_h the generator of the space homogeneous process, that is, the one with $p_N(t,x) = 1$ for all (t,x).

The existence of the space inhomogeneous zero-range process is proved in [1].

The monotonicity assumption made on the rate function $g(\cdot)$ is important since it allows the use of coupling techniques. On the other hand, we will see below that the asumption that the jump rate is bounded is crucial in this article.

To fix ideas, the reader may take n to be equal to 1, u_1 to be equal to 0 and α_1 a constant function. For this process particles which are not at site 0 evolve as in the usual zero-range process; that is, particles at site $x \neq 0$ wait a mean $g(\eta(x))^{-1}$ exponential time after which one of them jumps to $x+1$. On the other hand, jumps at site 0 are slower. At this site particles jump to site 1 at rate $\alpha g(\eta(0))$.

Since stating the theorems in their full generality would require much notation, for didactical reasons in this article we concentrate our attention on the case $n = 1$ and $u_1 = 0$. The reader should notice, however, that all the statements and proofs apply to any $n \ge 1$. Furthermore, for the sake of simplicity, we assume that $\alpha_1(\cdot)$ is a step function. The hypothesis that $\alpha_1(\cdot)$ is strictly smaller than 1 is useful because it avoids unbounded profiles.

It can be seen in a simple example that the macroscopic behavior of the process may change with a modification of the jump rate at site 0. Consider the space inhomogeneous zero-range process with rate $g_0(\cdot)$ given by $g_0(k) = 1_{\{k\ge 1\}}$.

For a nonnegative real ρ, let ν_ρ be the product invariant measure with density ρ for the space homogeneous zero-range process with jump rate g_0. Recall that the marginals of ν_ρ are given by

$$\nu_\rho\{\eta;\eta(x) = k\} = (1-\varphi(\rho))\varphi(\rho)^k$$

for every nonnegative integer k. In this formula $\varphi\colon \mathbf{R}_+ \to [0,1)$ is the function defined by

$$\varphi(\rho) = \frac{\rho}{1+\rho}\cdot$$

Consider the space inhomogeneous zero-range process with jump rates at 0 equal to some constant α in $(0,1)$ and starting from the measure ν_ρ. At the left of the origin particles evolve as the space homogeneous zero-range process. In particular, by Burke's theorem, the jumps of particles from site -1 to site 0 are a Poisson point process with intensity $\varphi(\rho)$ (cf. [8] for a simple proof of this result). On the other hand, for the space inhomogeneous process, particles can leave site 0 at rate at most α. Therefore in this case the total number of particles at site 0 is a birth and death Markov process on $\mathbf{N}$. For a positive integer k, the intensity of a jump from k to $k+1$ is $\varphi(\rho)$ and that of a jump from k to $k-1$ is α. On the other hand, jumps from 0 to -1 are not allowed and jumps from 0 to 1 occur with rate $\varphi(\rho)$. It is now easy to show that in the case where the density of particles for the initial measure is greater than $\rho_\alpha = \varphi^{-1}(\alpha)$ the number of particles at site 0 increases linearly in time.

In the interesting case where $\varphi(\rho) > \alpha$, $\rho > \rho_\alpha$, with a little more work using coupling arguments, one can prove a law of large numbers for the empirical measure associated with the space inhomogeneous zero-range process. More precisely, for each positive integer N, let $P^N_{\nu_\rho}$ represent the probability measure on the path space $D([0,\infty), \mathscr{X})$ of the inhomogeneous zero-range process with generator $L_{N,t}$ defined in (1.2), accelerated by N and starting from the product measure ν_ρ. For every $\gamma > 0$, for every $t \geq 0$ and for every continuous function $H\colon \mathbf{R} \to \mathbf{R}$ with compact support,

$$\lim_{N\to\infty} P^N_{\nu_\rho}\left[\left|N^{-1}\sum_x H\left(\frac{x}{N}\right)\eta_t(x) - \int H(u)M(t,du)\right| > \gamma\right] = 0,$$

where the measure $M(t,du) = m(t,u)\,du + \beta(t)\delta_0$ is such that

$$m(t,u) = \rho\,\mathbf{1}_{\{u\notin[0,C(\alpha,\rho)t]\}} + \rho_\alpha\,\mathbf{1}_{\{u\in[0,C(\alpha,\rho)t]\}},$$

with

$$C(\alpha,\rho) = \frac{\varphi(\rho)-\alpha}{\rho-\rho_\alpha}$$

and

$$\beta(t) = [\varphi(\rho)-\alpha]t.$$

In these formulas δ_0 represents the Dirac measure on 0.

Before stating the first main result of this article, we introduce some invariant product measures of the inhomogeneous process.

Recall the definition of $g(\infty)$ given in (1.1). Define the strictly increasing analytic function $Z\colon [0, g(\infty)) \to \mathbf{R}_+$ by

$$Z(\varphi) := \sum_{k\geq 0} \frac{\varphi^k}{g(k)!},$$

where, for $k \geq 1$, $g(k)!$ stands for $\prod_{1\leq j\leq k} g(j)$ and, by convention, $g(0)! = 1$. Let $\rho\colon [0, g(\infty)) \to \mathbf{R}_+$ be the strictly increasing function defined by

$$\rho(\varphi) = \frac{1}{Z(\varphi)}\sum_{k\geq 0} k\,\frac{\varphi^k}{g(k)!} = \varphi\,\frac{Z'(\varphi)}{Z(\varphi)}. \tag{1.4}$$

It is easy to show that ρ is onto $\mathbf{R}_+$. Denote by φ: $\mathbf{R}_+ \to [0, g(\infty))$ the inverse function:

$$\varphi(\cdot) = \rho^{-1}(\cdot). \tag{1.5}$$

For each $\rho \geq 0$, let ν_ρ^{ti} be the translation invariant product measure with marginals given by

$$\nu_\rho^{\mathrm{ti}}\{\eta; \eta(x) = k\} = \frac{1}{Z(\varphi(\rho))} \frac{\varphi(\rho)^k}{g(k)!}, \qquad k \geq 0. \tag{1.6}$$

In this notation ti stands for translation invariant. Notice that the expected number of particles and the expected value of the jump rate under the measure ν_ρ^{ti} are ρ and $\varphi(\rho)$, respectively:

$$\nu_\rho^{\mathrm{ti}}[\eta(0)] = \rho, \qquad \nu_\rho^{\mathrm{ti}}[g(\eta(0))] = \varphi(\rho).$$

For α in $(0, 1)$, denote by ρ_α the value at $\alpha g(\infty)$ of the function ρ defined in (1.4). In this way

$$g(\infty)\alpha = \varphi(\rho_\alpha). \tag{1.7}$$

For $\rho < \rho_\alpha$, the product measure $\nu_{\rho,\alpha}$ defined by

$$\nu_{\rho,\alpha}\{\eta; \eta(x) = k\} = \begin{cases} \dfrac{1}{Z(\varphi(\rho))} \dfrac{\varphi(\rho)^k}{g(k)!}, & x \neq 0, \\[2ex] \dfrac{1}{Z(\varphi(\rho)\alpha^{-1})} \dfrac{[\varphi(\rho)\alpha^{-1}]^k}{g(k)!}, & x = 0, \end{cases} \tag{1.8}$$

is an invariant measure for the space inhomogeneous zero-range process associated with the rate function g and the sequence $\{(0, \alpha)\}$. Here $\alpha(\cdot)$ is the constant function equal to α.

To obtain invariant measures with density of particles at the left of the origin larger than ρ_α, we have to allow an infinite number of particles at site 0. We therefore denote by $\overline{\mathbf{N}}$ the nonnegative integers with the point $+\infty$ included and consider the zero-range process evolving on $\overline{\mathbf{N}}^{\mathbf{Z}}$. If at some site x there are infinitely many particles, one of them jumps to $x + 1$ at rate $g(\infty)$.

For $\rho \geq 0$ the product measure $\nu_{\rho,\alpha}^0$ with marginals given by

$$\nu_{\rho,\alpha}^0\{\eta; \eta(x) = k\} = \begin{cases} \dfrac{1}{Z(\varphi(\rho))} \dfrac{\varphi(\rho)^k}{g(k)!}, & x < 0,\ 0 \leq k < \infty, \\[2ex] \dfrac{1}{Z(\varphi(\rho_\alpha))} \dfrac{\varphi(\rho_\alpha)^k}{g(k)!}, & x > 0,\ 0 \leq k < \infty, \\[2ex] \mathbf{1}_{\{k=\infty\}}, & x = 0, \end{cases} \tag{1.9}$$

is invariant for the space inhomogeneous zero-range process. Here the superscript 0 indicates that there are infinitely many particles at 0.

We now introduce some notation needed in order to state the first main result. We first define the initial measures and then describe the hydrodynamical equation which governs the macroscopic evolution of space inhomogeneous zero-range processes.

Recall from (1.6) the measures $\{\nu_\rho^{\mathrm{ti}}; \rho \geq 0\}$. For a function m^*: $\mathbf{R} \to \mathbf{R}_+$, denote by $\mu_{m^*}^N$ the product measure with marginals given by

$$\mu_{m^*}^N\{\eta; \eta(x) = k\} = \nu_{m^*(x/N)}^{\mathrm{ti}}\{\eta; \eta(x) = k\} \tag{1.10}$$

for $x \in \mathbf{Z}$ and $k \geq 0$. We will say that the sequence $\mu_{m^*}^N$ is associated with the profile m^*.

Since our purpose in this article is not to present a proof of the hydrodynamical behavior of space inhomogeneous zero-range processes in its full generality but rather to present some phenomena that appear when introducing these inhomogeneous rates, we assume throughout this article that m^* is a bounded continuous function.

Throughout this paper, for a measure μ on $\mathscr{X}$, we denote by P_μ^N the probability measure on the path space $D([0, \infty), \mathscr{X})$ corresponding to the Markov process with generator $L_{N,t}$ defined by (1.2) and associated with the sequence $\{(0, \alpha(\cdot))\}$, accelerated by N and starting from the measure μ.

We now pass to the hydrodynamical equation. Since at the left of the origin the space inhomogeneous zero-range processes behave exactly as ordinary zero-range processes and since the macroscopic behavior of the latter is described by entropy solutions of first-order quasilinear hyperbolic equations, we introduce the following functions.

Recall the definition of the smooth strictly increasing function $\varphi(\cdot)$ given in (1.5). Denote by λ: $\mathbf{R} \times \mathbf{R} \to \mathbf{R}_+$ the *entropy* solution of the one-dimensional conservation law $\partial_t\lambda + \partial_u\varphi(\lambda) = 0$ for the Cauchy problem with initial data m^*:

$$\begin{aligned} &\partial_t\lambda + \partial_u\varphi(\lambda) = 0,\\ &\lambda(0, \cdot) = m^*(\cdot). \end{aligned} \tag{1.11}$$

We saw through an example above that for some initial profiles a Dirac mass at the origin may appear. We will denote by $\beta(t)$ the total mass at the origin at time t. To study the behavior of the Dirac measure at the origin, we recall some facts on the entropy solution of this scalar conservation law. Since the initial profile is bounded and since $\varphi'(\cdot)$ is bounded below by a strictly positive constant on each compact subset of $\mathbf{R}_+$, the speed of the shocks of the entropy solution is bounded below by a strictly positive constant. It is also known that entropy solutions are continuous outside at most a countable number of shock lines. For this reason there exists a version λ which is continuous at $(t, 0)$ for all but a countable set of points t.

The following identity, easy to prove, will be used in the analysis of the evolution of the Dirac measure at the origin:

$$\int_0^\infty [\lambda(t, u) - \lambda(0, u)]\, du = \int_0^t \varphi(\lambda(s, 0))\, ds.$$

Mass arrives at the origin at a rate

$$\partial_t \int_{\mathbf{R}_+} \lambda(t,u)\,du = \varphi(\lambda(t,0)). \tag{1.12}$$

On the other hand, it leaves the origin at a rate bounded above by $\alpha(\cdot)g(\infty)$ which is the maximum rate at which a particle jumps from 0 to 1. Moreover, if $\beta(t) > 0$ this is exactly the rate at which particles leave the origin. In the case where $\beta(t) = 0$, the rate at which mass leaves the origin is the minimum of $\alpha(\cdot)g(\infty)$ and the rate at which mass arrives at the origin. Therefore $\beta(\cdot)$ should be the solution of

$$\partial_t\beta(t) = \begin{cases} \varphi(\lambda(t,0)) - \alpha(t)g(\infty), & \text{if } \beta(t) > 0, \\ [\varphi(\lambda(t,0)) - \alpha(t)g(\infty)]^+, & \text{if } \beta(t) = 0, \end{cases} \tag{1.13}$$

with initial data $\beta(0) = 0$.

Finally, at the right of the origin, particles behave as in ordinary zero-range processes. Therefore the macroscopic behavior should be described by the entropy solution of a first-order quasilinear hyperbolic partial differential equation with mass creation at the origin. Denote by $\omega(t)$ the rate at which mass is created at the origin. If there is mass at 0, that is, if $\beta(t) > 0$, the rate at which mass is transferred from the origin to the positive axis is $\alpha(t)g(\infty)$. Otherwise it is the minimum of the rate at which it arrives at 0 and $\alpha(t)g(\infty)$ which is equal to the minimum between (1.12) and $\alpha(t)g(\infty)$. Therefore $\omega(t)$ is given by

$$\omega(t) = \begin{cases} \alpha(t)g(\infty), & \text{if } \beta(t) > 0, \\ \alpha(t)g(\infty) \wedge \varphi(\lambda(t,0)), & \text{otherwise.} \end{cases} \tag{1.14}$$

We are therefore naturally led to consider the following differential equations in the semi-infinite line. Denote by $\lambda^+\colon \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$ the *entropy* solution of the conservation law in the semiinfinite line with mass production rate at 0 given by $\omega(\cdot)$ defined in (1.14):

$$\begin{gathered} \partial_t\lambda^+ + \partial_u\varphi(\lambda^+) = 0, \\ \partial_t \int_{\mathbf{R}_+} \lambda^+(t,u)\,du = \omega(t), \\ \lambda^+(0,\cdot) = m^*|_{\mathbf{R}_+}(\cdot). \end{gathered} \tag{1.15}$$

In the Appendix we fix the terminology of entropy solutions of the above equations and state a uniqueness theorem.

Let $M(t,du)$ be the measure on $\mathbf{R}$ defined by

$$M(t,du) = m(t,u)\,du + \beta(t)\delta_0, \tag{1.16}$$

where

$$m(t,u) = \lambda(t,u)\mathbf{1}_{\{u<0\}} + \lambda^+(t,u)\mathbf{1}_{\{u>0\}}$$

and $\beta(\cdot)$ is the solution of (1.13). We shall say that $M(t, du)$ is the measure associated with the rate $\alpha(\cdot)$ and the initial profile $m^*(\cdot)$.

We are now in a position to state the first main theorem of this article. Denote by $C_K(\mathbf{R})$ the space of all real continuous functions with compact support.

THEOREM 1. *For every H in $C_K(\mathbf{R})$, every $t \geq 0$ and every strictly positive γ,*

$$\lim_{N\to\infty} P^N_{\mu^N_{m^*}}\left[\left|N^{-1}\sum_x H\left(\frac{x}{N}\right)\eta_t(x) - \int H(u)M(t, du)\right| > \gamma\right] = 0.$$

2. Nonentropy weak solutions as hydrodynamical limits of space inhomogeneous processes. In this section we consider zero-range processes where, instead of modifying the jump rates at fixed sites, we modify the jump rates of a fixed number of particles. Thus, for an integer n and a sequence of functions α_j: $\mathbf{R}_+ \to (0, 1)$, we choose n particles and set the jump rate of the jth particle to be equal to $\alpha_j(t)$ at time t. The interesting feature of this perturbation is that the macroscopic behavior of the process is described by a *nonentropy* weak solution of the hyperbolic conservation law

$$\begin{aligned} \partial_t\lambda + \partial_u\varphi(\lambda) = 0, \\ \lambda(0, \cdot) = m^*(\cdot). \end{aligned} \tag{2.1}$$

To define precisely the process, we need some notation. Throughout this section the jump rate $g(\cdot)$ is fixed to be

$$g(k) = \mathbf{1}_{\{k\geq 1\}}. \tag{2.2}$$

To avoid technical details, we will consider only configurations with infinitely many particles to the left and to the right of the origin. Denote by $\mathscr{X}_0$ this subset of configurations:

$$\mathscr{X}_0 = \left\{\eta; \sum_{x\leq 0}\eta(x) = \infty \text{ and } \sum_{x>0}\eta(x) = \infty\right\}.$$

We label particles of each configuration η in $\mathscr{X}_0$. The labeling is done in such a way that:

(a) all particles are labeled.

We assume also that there are no jumps in the labeling. Since we assumed that there are infinitely many particles this is equivalent to requiring that:

(b) for each integer i there is a particle labeled i.

Denote by $X(i, \eta) = X(i) \in \mathbf{Z}$ the position of the particle labeled by i. The labeling is done from left to right:

(c) $X(i) \leq X(i+1)$ for all integers i.

Finally, to fix ideas, we assume that the particle labeled by 0 is the first one at the left of the origin. Thus we label by 0 a particle at the origin. If there is no particle at the origin we label by 0 a particle on the first nonempty site at the left of the origin:

(d) $X(0) \le 0 < X(1)$.

In this way, with each configuration η of $\mathscr{X}_0$ we associated a configuration X of the subset $\mathscr{Y}$ of $\mathbf{Z}^{\mathbf{Z}}$ defined by

$$\mathscr{Y} = \{X \in \mathbf{Z}^{\mathbf{Z}}; X(i) \le X(i+1)\}.$$

It is easy to recover the configuration η from X since $\eta(x)$ is the number of labeled particles at site x:

$$\eta(x) = \sum_i \mathbf{1}_{\{X(i)=x\}}. \tag{2.3}$$

Notice that the space homogeneous zero-range process with rate (2.2) is the process where each particle $X(i)$ jumps at rate 1 to the site on its right if the particle $X(i+1)$ is not at the same site as $X(i)$. Thus the generator of the zero-range process can be rewritten as

$$(L_h^p f)(X) = \sum_i \mathbf{1}_{\{X(i)<X(i+1)\}}[f(X^i) - f(X)],$$

where, for an integer i, X^i is the configuration obtained from X where the particle labeled i jumped to the right:

$$X^i(j) = \begin{cases} X(i)+1, & \text{if } j = i, \\ X(j), & \text{otherwise.} \end{cases}$$

Notice furthermore that this dynamic preserves the order of the labeling just defined and therefore defines a process on $\mathscr{Y}$.

We consider in this section a perturbation of this process setting some particles to jump at a slower rate.

Fix a positive integer n, a finite sequence of reals $u_1 < u_2 < \cdots < u_n$ and an associated sequence of functions $\alpha_j\colon \mathbf{R}_+ \to (0,1)$ for $1 \le j \le n$. For each positive integer N, consider the Markov process on $\mathscr{Y}$ whose generator acts on functions that depend only on a finite number of coordinates as

$$(L_{N,t}^p f)(X) = \sum_{i\in\mathbf{Z}} p_N(t,i)\mathbf{1}_{\{X(i)<X(i+1)\}}[f(X^i) - f(X)].$$

Here $p_N(t,i)$ is a function equal to 1 at all but a finite number of sites:

$$p_N(t,i) = \begin{cases} \alpha_k(t), & \text{if } i = [u_k N] \text{ for } 1 \le k \le n, \\ 1, & \text{otherwise.} \end{cases}$$

In order to use the result stated in the last section, we restrict our attention to the case where just one particle, say $X(0)$, has a jump rate smaller than 1. The purpose of this section is to present an example where the hydrodynamical behavior of this process is described by a nonentropy solution of (2.1).

The result is the following. For a strictly positive constant α_0, denote by $P^N_{p,\rho}$ the measure on $D([0,\infty),\mathscr{X})$ corresponding to the process η_t evolving according to the generator $L^p_{N,t}$ associated with the couple $\{(0,\alpha_0)\}$ accelerated by N and starting from the equilibrium measure ν^{ti}_ρ.

THEOREM 2. *For every smooth function* H: $\mathbf{R}\to\mathbf{R}$ *with compact support, every* $\rho<\rho_{\alpha_0}$, *every* $t\ge 0$ *and every* $\varepsilon>0$,

$$\lim_{N\to\infty} P^N_{p,\rho}\biggl[\biggl|N^{-1}\sum_x H(x/N)\eta_t(x)-\int H(u)\rho(t,u)\,du\biggr|>\varepsilon\biggr]=0,$$

where

$$\rho(t,u)=\begin{cases}\rho_{\alpha_0}, & \text{if } \varphi(\rho_{\alpha_0})-\rho_{\alpha_0}\dfrac{\varphi(\rho)-\varphi(\rho_{\alpha_0})}{\rho-\rho_{\alpha_0}}<\dfrac{u}{t}\le\varphi(\rho_{\alpha_0}),\\[2ex] 0, & \text{if } \varphi(\rho_{\alpha_0})<\dfrac{u}{t}\le\varphi(\rho),\\[2ex] \rho, & \text{otherwise.}\end{cases}\tag{2.4}$$

Notice that $\rho(t,\cdot)$ has a decreasing shock at $\varphi(\rho_{\alpha_0})t$. Therefore it is not an entropy solution. On the other hand, a simple computation shows that $\rho(t,u)$ is a weak solution of (2.1) with initial profile ρ.

PROOF OF THEOREM 2. Since we only use the assumption that the jump rate is a constant equal to α_0 at the end of the proof when we identify the profile obtained, we suppose for the moment that it is given by a positive function $\alpha(\cdot)$ strictly smaller than 1.

To reduce this problem to the one studied in the previous section, we interchange the roles of particles and sites. So, for a configuration X in $\mathscr{Y}$, define a configuration $\chi=\chi(X)$ in $\mathscr{X}_0$ by

$$\chi(x)=X(x+1)-X(x).\tag{2.5}$$

In this way χ counts the number of sites between two adjacent particles for the configuration η.

It is easy to see that the process $\chi_t=\chi(X_t)$ evolves as the space inhomogeneous asymmetric zero-range process with generator defined in (1.2) associated with the couple $\{(0,\alpha(\cdot))\}$ but with particles jumping to the left instead of to the right.

Notice that the number of jumps of particle $X(i)$ in the interval $[0,t]$ is exactly the number of particles that jumped from site i to site $i-1$ for the χ process:

$$X_t(i)-X_0(i)=\sum_{-\infty}^{i-1}[\chi_t(x)-\chi_0(x)].\tag{2.6}$$

To prove the theorem, the first task is to obtain the initial distribution of the χ particles.

Denote by $\mathscr{T}$: $\mathscr{X}_0\to\mathscr{X}_0$ the functional that sends an η configuration to a χ configuration. Extend this functional to the space of probability measures on

$\mathscr{X}_0$. A classical computation, passing through configurations on $\{0,1\}^{\mathbf{Z}}$, shows that the image of the product measure ν_ρ^{ti} under $\mathscr{T}$ is a product measure, denoted by $\tilde{\nu}_\rho$, with marginals given by

$$\tilde{\nu}_\rho\{\chi; \chi(x) = k\} = \nu_{1/\rho}^{\mathrm{ti}}\{\eta; \eta(0) = k\} \quad \text{if } x \neq 0.$$

At site 0, from the way we labeled the particles, there is at least one particle for the configuration χ. In fact, we have

$$\tilde{\nu}_\rho\{\chi; \chi(0) = k\} = \nu_{1/\rho}^{\mathrm{ti}}\{\eta; \eta(0) + \eta(1) = k-1\} \quad \text{for } k \geq 1.$$

Thus χ starts from a product measure associated with the constant profile $1/\rho$ and evolves as a space inhomogeneous process associated with $\{(0, \alpha(\cdot))\}$ and Theorem 1 describes the hydrodynamical behavior of the process χ_t.

Fix a smooth function $H\colon \mathbf{R} \to \mathbf{R}$ in $C_K(\mathbf{R})$. To prove a law of large numbers for the empirical measure, we need to study the behavior of the sum

$$N^{-1} \sum_x H\left(\frac{x}{N}\right) \eta_t(x).$$

In order to use the result proved in the last section, we have to replace the configuration η by a function of χ. From (2.3), we may rewrite the last expression as

$$N^{-1} \sum_{x \in \mathbf{Z}} \sum_{i \in \mathbf{Z}} H\left(\frac{x}{N}\right) \mathbf{1}_{\{X_t(i) = x\}} = N^{-1} \sum_{i \in \mathbf{Z}} H(N^{-1} X_t(i)).$$

Replacing X by χ, we obtain from (2.6) that the last sum is equal to

$$N^{-1} \sum_{i \in \mathbf{Z}} H\left(N^{-1} X_0(i) + N^{-1} \sum_{-\infty}^{i-1} [\chi_t(x) - \chi_0(x)]\right),$$

which, after some simple computations taking advantage of relation (2.5), is equal to

$$N^{-1} \sum_{i \in \mathbf{Z}} H\left(N^{-1} X_0(0) + N^{-1} \sum_{-\infty}^{i-1} \chi_t(x) - N^{-1} \sum_{-\infty}^{-1} \chi_0(x)\right).$$

Notice that, though both sums are infinite, a simple coupling argument shows that the difference is finite almost surely.

Recall that the initial measure is a product measure with density $1/\rho$ and that χ_t evolves as the space inhomogeneous zero-range process associated with the couple $\{(0, \alpha(\cdot))\}$ with particles jumping to the left. Since $N^{-1} X_0(0)$ converges to 0 in L^1, from Theorem 1, the last expression converges in probability to

$$\int H(M(t, (-\infty, u)) - M(0, (-\infty, 0)))\, du,$$

where $M(t, du)$ is the measure associated with the rate $\alpha(\cdot)$ and the initial profile constant equal to $1/\rho$ in the terminology introduced after (1.16).

Let $v(t,u)$ be the function

$$v(t,u) = M(t,(-\infty,u)) - M(0,(-\infty,0)).$$

A simple computation shows that

$$(2.7)\qquad v(t,u) = \begin{cases} \int_0^u m(t,r)\,dr + \int_0^t \varphi(m(s,0+))\,ds, & \text{for } u > 0,\\ \int_0^u m(t,r)\,dr + \int_0^t \omega(s)\,ds, & \text{for } u < 0. \end{cases}$$

Notice that, for each fixed time t, $v(t,\cdot)$ is an increasing function continuous in $\mathbf{R} - \{0\}$ since $m(t,\cdot)$ is positive and

$$\int_0^t \varphi(m(s,0+)) = \int_0^t \omega(s)\,ds + \beta(t) \ge \int_0^t \omega(s)\,ds.$$

The first equality follows immediately from the definition of $\beta(\cdot)$ and $\omega(\cdot)$ given in (1.13) and (1.14).

A change of variables in the last integral shows that it is equal to

$$\int_{-\infty}^{\int_0^t \omega(s)\,ds} H(v)\frac{1}{m(t,u(t,v))}\,dv + \int_{\int_0^t \varphi(m(s,0+))\,ds}^{\infty} H(v)\frac{1}{m(t,u(t,v))}\,dv.$$

Here, for each fixed t, $u(t,v)$ denotes the inverse function of the one defined in (2.7).

In the case where $\alpha(\cdot) = \alpha_0$ and $\rho > \rho_{\alpha_0}$, a simple computation shows that this last sum is equal to

$$\int H(v)\rho(t,v)\,dv,$$

with $\rho(t,v)$ given by (2.4). □

3. Lower-bound large deviations. This section is devoted to the proof that the large deviations from the hydrodynamical limit of attractive asymmetric zero-range processes with bounded jump rates are of order at least e^{-CN}.

More precisely, for a fixed density ρ and a fixed time T_0, denote by $P_h^{N,\rho}$ the probability on the path space $D([0,T_0],\mathscr{X})$ of the space homogeneous zero-range process with generator accelerated by N starting from the invariant measure ν_ρ^{ti}.

For a positive integer N and a configuration η, denote by π^N the Radon measure on $\mathbf{R}$ obtained by assigning to each particle a mass N^{-1}:

$$(3.1)\qquad \pi^N = \pi^N(\eta) = N^{-1}\sum_x \eta(x)\,\delta_{x/N}.$$

Here, for a real u, δ_u denotes the Dirac measure concentrated on u. We use the shorthand π_t^N for $\pi^N(\eta_t)$ and we denote by $\mathscr{M}_+(\mathbf{R})$ the space of positive Radon measures on $\mathbf{R}$ endowed with the vague topology. For a continuous function H with compact support and a Radon measure π in $\mathscr{M}_+(\mathbf{R})$, we denote by $\langle \pi, H\rangle$ the integral of H with respect to π.

We show that, for a class of positive Radon measures $M(t, du)$ on $\mathbf{R}$,

$$\liminf_{N\to\infty} N^{-1} \log P_h^{N,\rho}[\pi^N \in V_M] \geq -I(M) > -\infty$$

for every neighborhood V_M of the path $M(t, du)$.

Two different types of large deviations are observed. The first one comes from the deviations from the initial product measure and the second from the dynamic. Since the analysis of the first kind of deviation is trivial, we concentrate on the second type.

At a formal level, to obtain a lower bound of order e^{-CN} of the large deviations, it is enough to find a perturbation of the original process for which we are able to prove a law of large numbers for the empirical measure and such that the relative entropy of the perturbation with respect to the original process is bounded by CN.

Indeed, let $\tilde{P}^N$ be a probability measure on $D([0, T_0], \mathscr{X})$ and assume the following:

(a) Under $\tilde{P}^N$ the empirical measure converges to a deterministic path $\tilde{M}(t, du)$.

(b) $\limsup_{N\to\infty} N^{-1} H(\tilde{P}^N | P^N) \leq C$.

Recall that the entropy $H(\tilde{P}^N | P^N)$ is given by

$$H(\tilde{P}^N | P^N) = \tilde{E}^N\left[\log \frac{d\tilde{P}^N}{dP^N}\right].$$

Let V denote a neighborhood of $\tilde{M}(t, du)$. We have that

$$N^{-1} \log P^N[\pi^N \in V] = N^{-1} \log \tilde{E}^N\left[\frac{dP^N}{d\tilde{P}^N} \mathbf{1}_{\{\pi^N \in V\}}\right].$$

Since, by assumption (a), π^N converges to $\tilde{M}(t, du)$ under $\tilde{P}^N$, the indicator function on the right-hand side can be removed. Therefore, by Jensen's inequality, the expression is bounded below by

$$N^{-1} \tilde{E}^N\left[\log \frac{dP^N}{d\tilde{P}^N}\right] = -N^{-1} H(\tilde{P}^N | P^N).$$

The lower bound follows then from assumption (b).

In Section 1 we proved the hydrodynamical limit for a class of perturbations of the space homogeneous zero-range process. In view of this formal argument, to get a lower bound, we just have to compute the limit of the entropy divided by N.

Recall from Section 1 the definition of $P^N_{\nu^{\mathrm{ti}}_\rho}$. Denote by $P^N_{\nu^{\mathrm{ti}}_\rho}(T_0)$ the measure $P^N_{\nu^{\mathrm{ti}}_\rho}$ restricted to $D([0, T_0], \mathscr{X})$. A simple computation shows that

$$N^{-1} H\big(P^N_{\nu^{\mathrm{ti}}_\rho}(T_0) \big| P^{N,\rho}_h\big)$$
$$= E^N_{\nu^{\mathrm{ti}}_\rho}\left[\int_0^{T_0} [\alpha(s)\log\alpha(s) - \alpha(s) + 1] g(\eta_s(0))\, ds\right],$$

which is bounded above by $g(\infty)\int_0^{T_0}[\alpha(s)\log\alpha(s) - \alpha(s) + 1]\, ds$. The formal argument presented above can be made rigorous (details are left to the reader) and provides a lower bound for a large deviation of the empirical measure.

To obtain the best possible lower bound for a fixed path $M(t, du)$, we have to choose among all perturbations for which the empirical measure converges to $M(t, du)$ the one with smallest relative entropy with respect to the original process.

We conclude this section by obtaining a lower bound for certain paths $M(t, du)$ that we believe is the correct lower bound of the large deviation principle.

Fix a positive Radon measure $M(t, du)$ on $\mathbf{R}$ and suppose that $M(t, du)$ is absolutely continuous with respect to the Lebesgue measure away from the origin:

$$M(t, du) = m^-(t, u)\mathbf{1}_{\{u<0\}}\, du + \beta(t)\delta_0 + m^+(t, u)\mathbf{1}_{\{u>0\}}\, du$$

and that $M(0, du) = \rho\, du$.

We now enumerate properties that $M(t, du)$ must satisfy in order to be able to prove a lower bound for the probability that the empirical measure belongs to a neighborhood of $M(t, du)$.

Assume that $m^-(t, u) = \rho$ for every (t, u). Assume that $\beta(\cdot)$ is strictly positive on $(0, T_0]$, continuous, piecewise linear and such that

$$\varphi(\rho) - g(\infty) < (\partial_t\beta)(t) \le \varphi(\rho)$$

for every $0 \le t \le T_0$. Define α: $[0, T_0] \to [0, 1)$ by

$$\alpha(t) = g(\infty)^{-1}[\varphi(\rho) - (\partial_t\beta)(t)]. \tag{3.2}$$

Notice that $\alpha(\cdot)$ is a step function. Finally, assume that m^+ is the entropy solution of (1.15) with boundary conditions ρ at $u = 0$ and $\varphi(\rho) - (\partial_t\beta)(t)$ at $t = 0$. For this path $M(t, du)$ the previous arguments give the following lower bound.

THEOREM 3. *Fix a path $M(t, du)$ possessing the properties just described. For every neighborhood V of $M(t, du)$,*

$$\liminf_{N\to\infty} N^{-1}\log P^{N,\rho}_h\big[\pi^N \in V\big] \ge -g(\infty)\int_0^{T_0}\big[\alpha(s)\log\alpha(s) - \alpha(s) + 1\big]\, ds,$$

where $\alpha(\cdot)g(\infty)$, defined in (3.2), *is the rate at which $\int_0^\infty m^+(t, u)\, du$ increases in time.*

4. Proof of Theorem 1. Throughout this section we use the notation established in Section 1.

The proof of Theorem 1 is divided into several lemmas. We first prove the hydrodynamical behavior at the left of the origin. This part is easy and follows from previous results since at the left of the origin particles evolve as particles in usual space homogeneous zero-range processes. The second step consists of studying the behavior of the Dirac measure at the origin. This is done in Proposition 4.4. Finally, the last step consists of proving the hydrodynamical behavior at the right of the origin. This is postponed until Section 6.

First of all, by coupling arguments, it is easy to show that it is enough to prove Theorem 1 for initial profiles m^* with compact support. Therefore, from now on, we assume that m^* is continuous with compact support. In particular, the total number of particles is $\mu^N_{m^*}$-a.s. finite since

$$\mu^N_{m^*}\left[\sum_x \eta(x)\right] = \sum_x m^*\left(\frac{x}{N}\right) < \infty. \tag{4.1}$$

To start, we need some terminology on attractive processes. On the configuration space $\mathscr{X}$ we introduce the natural partial order defined by $\eta \le \xi$ if $\eta(x) \le \xi(x)$ for all $x \in \mathbf{Z}$. A continuous function f is said to be monotone if $f(\eta) \le f(\xi)$ whenever $\eta \le \xi$. We denote by $\mathbf{M}$ the set of monotone functions and we extend the partial order to the measures on $\mathscr{X}$ in the natural way: $\mu \le \nu$ if $\int f\, d\mu \le \int f\, d\nu$ for every monotone function f. A Feller process is said to be attractive if its semigroup S_t preserves the partial order: $\mu \le \nu \Rightarrow \mu S_t \le \nu S_t$ for every $t > 0$. It is proved in [1] that the monotonocity of g implies the attractiveness of inhomogeneous zero-range processes.

Recall the definition given in (3.1) of the Radon measure $\pi^N = \pi^N(\eta)$ associated with each configuration η.

Fix a time $T_0 > 0$ and recall from (1.10) the definition of the product measure $\mu^N_{m^*}$. Throughout this section we denote by Q^N the probability measure on the path space $D([0, T_0], \mathscr{M}_+(\mathbf{R}))$ corresponding to the Markov process π^N_t evolving according to the generator $L_{N,t}$ defined in (1.2) and associated with $\{(0, \alpha(\cdot))\}$, accelerated by N and starting from the product measure $\mu^N_{m^*}$.

We start with the tightness of the sequence Q^N.

LEMMA 4.1. *The sequence Q^N is tight. All its limit points are concentrated on weakly continuous paths $\pi(t, du)$ which are absolutely continuous with respect to the Lebesgue measure outside the origin and with density bounded by a constant which depends only on the initial profile $m^*(\cdot)$ and the jump rates at the origin $\alpha(\cdot)$*:

$$\pi(t, du) = \rho(t, u)\, du + b(t)\delta_0 \quad \textit{and} \quad \rho(t, u) \le A = A(m^*, \alpha).$$

Moreover, since all limit points are concentrated on continuous paths, every converging subsequence converges also in the uniform topology. In particular, its one-dimensional marginals converge.

The proof of this lemma is omitted. It relies on standard arguments on tightness of interacting particle processes and on the following coupling lemma that will be used several times later.

LEMMA 4.2. *Let μ^N be a sequence of measures on $\mathscr{X}$ bounded by some translation invariant product measure $\nu^{\mathrm{ti}}_{\rho_0}$. Denote by S^N_t the semigroup of the Markov process with generator $L_{N,t}(\{(0, \alpha(\cdot))\})$ defined in* (1.2), *accelerated by N. There exists a density $\rho_1 = \rho_1(\rho_0, \alpha)$ such that, for every N and every $t \geq 0$, $\mu^N S^N_t$ is bounded by $\nu^{\mathrm{ti}}_{\rho_1}$ outside from the origin. More precisely, for every bounded monotone cylinder function Ψ which does not depend on the variable $\eta(0)$,'*

$$\mu^N S^N_t[\Psi] \leq \nu^{\mathrm{ti}}_{\rho_1}[\Psi].$$

PROOF. Let $\alpha^* = \sup_{t\geq 0} \alpha(t)$. By assumption $\alpha^* < 1$. For $\rho > 0$ and $\alpha^* < \alpha < 1$, consider the product invariant measure $\nu^0_{\rho,\alpha}$ of the space inhomogeneous process defined in (1.9). Choose ρ_1 and α_1 such that $\nu_{\rho_0} \leq \nu^0_{\rho_1,\alpha_1}$. This is always possible since $\{\nu^0_{\rho,\alpha}, \rho \geq 0, 0 \leq \alpha < 1\}$ is an increasing family and

$$\lim_{\substack{\rho\to\infty\\ \alpha\to 1}} \inf_x \nu^0_{\rho,\alpha}[\eta(x)] = \infty.$$

By the attractiveness and invariance of $\nu^0_{\rho_1,\alpha_1}$,

$$\mu^N S^N_t \leq \nu_{\rho_0} S^N_t \leq \nu^0_{\rho_1,\alpha_1} S^N_t = \nu^0_{\rho_1,\alpha_1}.$$

The lemma is thus proved since $\nu^0_{\rho_1,\alpha_1}$ coincides with some ν^{ti}_{ρ} away from the origin. □

We now prove the hydrodynamical behavior of the process at the left of the origin.

LEMMA 4.3. *For every continuous function H with compact support in $(-\infty, 0]$ and vanishing at 0, for every $0 \leq t \leq T_0$ and for every positive γ,*

$$\lim_{N\to\infty} P^N_{\mu^N_{m^*}}\left[\left|N^{-1}\sum_x H\left(\frac{x}{N}\right)\eta_t(x) - \int H(u)M(t,du)\right| > \gamma\right] = 0.$$

There are two ways to prove Lemma 4.3. The first method consists of proving that the empirical measure on $[0, T_0] \times (-\infty, 0)$ converges to the entropy solution of the conservation law $\partial_t\rho + \partial_u\varphi(\rho) = 0$ with initial data m^* restricted to $(-\infty, 0)$, repeating the arguments of Rezakhanlou [7]. The second method consists of taking advantage of knowledge of the hydrodynamical behavior of space homogeneous asymmetric processes. The idea is to couple the process with a space homogeneous zero-range process with jump rate $g(\cdot)$ in such a way that at the left of the origin the two processes are equal at any time. The result follows then from the hydrodynamical behavior of the latter process. We sketch the second method, which is simpler.

PROOF OF LEMMA 4.3. The proof requires some notation. Let $(\xi_t)_{t\geq 0}$ denote a totally asymmetric space homogeneous zero-range process associated with the rate function $g(\cdot)$ with generator accelerated by N. We couple $(\eta_t)_{t\geq 0}$ and $(\xi_t)_{t\geq 0}$ in order that the η and ξ particles jump together as much as possible. Assume that, at time 0, ξ coincides with η. The generator of the coupled process, denoted by $\overline{L}_N$, is given by

$$\begin{aligned}
(\overline{L}_N f)(\eta, \xi) \\
&= \sum_{x\in\mathbf{Z}} \min\{p_N(t,x)g(\eta(x)), g(\xi(x))\}[f(\eta^{x,x+1}, \xi^{x,x+1}) - f(\eta,\xi)] \\
&\quad + \sum_{x\in\mathbf{Z}} \big(p_N(t,x)g(\eta(x)) - g(\xi(x))\big)^+[f(\eta^{x,x+1}, \xi) - f(\eta,\xi)] \\
&\quad + \sum_{x\in\mathbf{Z}} \big(g(\xi(x)) - p_N(t,x)g(\eta(x))\big)^+[f(\eta, \xi^{x,x+1}) - f(\eta,\xi)].
\end{aligned} \tag{4.2}$$

This coupling possesses two properties since $p_N(t,x) = 1$ for $x < 0$:

$$\eta_t(x) = \xi_t(x) \quad \text{for all } x < 0 \text{ and } t > 0 \tag{4.3}$$

if these equalities hold at time 0 and

$$\sum_{y\geq x} \eta_t(y) \leq \sum_{y\geq x} \xi_t(y) \quad \text{for all } x \in \mathbf{Z} \text{ and } t > 0 \tag{4.4}$$

if the two processes start from the same configuration.

The statement of the lemma follows therefore from the hydrodynamical limit for attractive space homogeneous zero-range processes starting from product measures, which is proved in [7]. □

Recall from Section 1 that $\lambda(\cdot, 0)$ is discontinuous at most at a countable set of points and that

$$\int_0^\infty [\lambda(t,u) - \lambda(0,u)]\,du = \int_0^t \varphi(\lambda(s,0))\,ds. \tag{4.5}$$

This relation will be used in the analysis of the evolution of the Dirac measure at the origin.

PROPOSITION 4.4. *Let Q^* be any limit point of the sequence Q^N. Let t_0 be a continuity point of $\lambda(\cdot,0)$ and $\alpha(\cdot)$. Then Q^*-a.s.*

$$(\partial_t^+ b)(t_0) := \lim_{h\downarrow 0} h^{-1}[b(t_0+h) - b(t_0)]$$

exists and

$$(\partial_t^+ b)(t_0) = \begin{cases} \varphi(\lambda(t_0,0)) - \alpha(t_0)g(\infty), & \textit{if } b(t_0) > 0, \\ (\varphi(\lambda(t_0,0)) - \alpha(t_0)g(\infty))^+, & \textit{otherwise}. \end{cases} \tag{4.6}$$

This proposition is a corollary of the following result.

LEMMA 4.5. *Let R^N be the probability measure on $D([0, T_0], \mathbf{R}_+)$ corresponding to the process $N^{-1}\eta_t(0)$, where η_t is the Markov process evolving with generator $L_{N,t}(\{(0,\alpha)\})$, accelerated by N and starting from $\mu^N_{m^*}$. The sequence R^N converges in the uniform topology to the probability concentrated on the deterministic path solution of* (4.6) *with initial condition $b(0) = 0$.*

We first show that Proposition 4.4 follows from this result.

PROOF OF PROPOSITION 4.4. Denote by $\beta(t)$ the solution of (4.6) with initial condition $\beta(0) = 0$. Fix a time $0 \le t \le T_0$. For each $\varepsilon > 0$, let H_ε be a positive continuous function bounded by 1, equal to 1 at the origin and with support contained in $[-\varepsilon, \varepsilon]$.

Let Q^* be a limit point of the sequence Q^N. To keep the notation as simple as possible, assume that the sequence Q^N converges to Q^*. By Lemma 4.1 the one-dimensional marginals $Q^N\pi_t^{-1}$ converge to $Q^*\pi_t^{-1}$. By the weak convergence, for every $\gamma > 0$,

$$\begin{aligned} &Q^*\big[|\langle \pi_t, H_\varepsilon\rangle - \beta(t)| > \gamma\big] \\ &\quad \le \lim_{N\to\infty} Q^N\big[|\langle \pi_t, H_\varepsilon\rangle - \beta(t)| > \gamma\big] \\ &\quad \le \lim_{N\to\infty} P^N\left[|N^{-1}\eta_t(0) - \beta(t)| > \left(\frac{\gamma}{2}\right)\right] \\ &\qquad + \lim_{N\to\infty} P^N\left[\left|N^{-1}\sum_{x\neq 0} H_\varepsilon\left(\frac{x}{N}\right)\eta_t(x)\right| > \left(\frac{\gamma}{2}\right)\right]. \end{aligned}$$

For each positive γ the next to the last limit is equal to 0 by virtue of Lemma 4.5. On the other hand, by Lemma 4.2 there exists $\varepsilon(\gamma)$ such that the last limit is equal to 0 for $\varepsilon < \varepsilon(\gamma)$. Therefore, for $\varepsilon < \varepsilon(\gamma)$,

$$Q^*[|\langle \pi_t, H_\varepsilon\rangle - \beta(t)| > \gamma] = 0.$$

Since by Lemma 4.1 the density outside the origin is bounded by a constant, letting $\varepsilon \downarrow 0$, we obtain that

$$Q^*[|b(t) - \beta(t)| > \gamma] = 0$$

for every positive γ. This concludes the proof. □

It remains to prove Lemma 4.5. Its proof is divided into several lemmas. We start by proving that the sequence R^N is tight and that all its limit points are concentrated on continuous paths.

LEMMA 4.6. *The sequence R^N is tight. All its limit points are concentrated on continuous trajectories $b(t)$ that satisfy the following inequalities*:

$$\begin{aligned} &\int_t^{t+h} [\varphi(\lambda(s,0)) - \alpha(s)g(\infty)]\, ds \\ &\qquad \le b(t+h) - b(t) \le \int_t^{t+h} \varphi(\lambda(s,0))\, ds \end{aligned}$$

for every t, $h \geq 0$. In particular, every converging subsequence R^{N_k} converges in the uniform topology and

$$\lim_{k\to\infty} R^{N_k}[\eta_t(0) \geq \theta N_k] = R^*[b(t) \geq \theta]$$

for every $0 \leq t \leq T_0$ and θ such that $R^*[b(t) = \theta] = 0$. In this formula R^* stands for the limit point of R^{N_k}.

PROOF. The proof relies on the coupling (4.2) defined in the proof of Lemma 4.3. For any t and $h \geq 0$, the difference $\eta_{t+h}(0) - \eta_t(0)$ is equal to the total number of particles that arrived at the origin between t and $t+h$ minus the total number of particles that left the origin in this interval. By property (4.3) of the coupling, the total number of particles that arrived at the origin is equal to the total number of particles that arrived at the origin in the interval $[t, t+h]$ for the space homogeneous process:

$$\sum_{x\geq 0}[\eta_{t+h}(x) - \eta_t(x)] = \sum_{x\geq 0}[\xi_{t+h}(x) - \xi_t(x)].$$

Notice that these sums are well defined since by (4.1) the total number of particles is μ^N-a.s. finite.

By the hydrodynamical limit for space homogeneous zero-range processes, the right-hand side divided by N converges in probability to

$$\int_0^\infty [\lambda(t+h, u) - \lambda(t, u)]\, du = \int_t^{t+h} \varphi(\lambda(s, 0))\, ds.$$

This last equality follows from identity (4.5).

On the other hand, the rate at which particles leave the origin is bounded by a Poisson point process of rate $\alpha(\cdot)g(\infty)$.

Therefore, for every $\varepsilon > 0$,

$$\lim_{N\to\infty} P^N_{\mu^N_{m^*}}\bigg[-\int_t^{t+h} \alpha(s)g(\infty)\, ds - \varepsilon \\ \leq N^{-1}[\eta_{t+h}(0) - \eta_t(0)] - \int_t^{t+h} \varphi(\lambda(s,0))\, ds \leq \varepsilon\bigg] = 1.$$

Therefore, to conclude the proof of the lemma, we only have to show the tightness of the sequence R^N and that all limit points are concentrated on continuous paths. This follows from the coupling. Indeed, to prove tightness, it is enough to show that

$$(4.7)\qquad \begin{aligned} &\limsup_{C\to\infty} \limsup_{N\to\infty} \sup_{0\leq t\leq T_0} P^N_{\mu^N_{m^*}}[N^{-1}\eta_t(0) \geq C] = 0, \\ &\limsup_{\gamma\to 0} \limsup_{N\to\infty} P^N_{\mu^N_{m^*}}\bigg[\sup_{|t-s|\leq\gamma} N^{-1}|\eta_t(0) - \eta_s(0)| \geq \varepsilon\bigg] = 0 \end{aligned}$$

for every positive ε. The first statement follows from the inequality

$$N^{-1}\eta_t(0) \leq N^{-1}\eta_0(0) + N^{-1}\sum_{x\geq 0}[\xi_t(x) - \xi_0(x)]$$

and the law of large numbers for the homogeneous process. To prove the second statement, we first remark that we have the following inequality

$$N^{-1}|\eta_t(0) - \eta_s(0)| \le N^{-1} \sum_{x \ge 0} [\xi_t(x) - \xi_s(x)] + N^{-1}[U^N(t) - U^N(s)],$$

where $U^N(\cdot)$ represents a Poisson process with rate $\alpha(\cdot)g(\infty)$ accelerated by N. The tightness follows therefore from the law of large numbers for both processes on the right-hand side of the inequality. Moreover, (4.7) implies that all limit points are concentrated on continuous paths. This, in turn, shows that every converging subsequence converges in the uniform topology. □

A slight refinement of the above arguments gives the exact behavior of $b(\cdot)$ at points where $b(\cdot)$ is strictly positive. Denote by $\overline{\alpha}$ the maximum of $\alpha(\cdot)$ in $[0, T_0]$:

$$\overline{\alpha} = \sup_{t \in [0, T_0]} \alpha(t).$$

LEMMA 4.7. *Let R^* be a limit point of the sequence R^N. Fix $t > 0$ and assume that there exists $\theta > 0$ such that*

$$R^*[b(t) \ge \theta] > 0, \qquad R^*[b(t) = \theta] = 0.$$

Then, for all $0 \le h \le \theta[\overline{\alpha} g(\infty)]^{-1}$,

$$R^*\left[b(t+h) - b(t) = \int_t^{t+h} [\varphi(\lambda(s, 0)) - \alpha(s)g(\infty)]\, ds \,\Big|\, b(t) \ge \theta \right] = 1.$$

PROOF. Fix $h < \theta[\overline{\alpha} g(\infty)]^{-1}$ and $\varepsilon > 0$. Assume, to keep the notation as simple as possible, that R^N converges to R^*. From Lemma 4.6 and from its proof,

$$\begin{aligned} &R^*\left[\left| b(t+h) - b(t) - \int_t^{t+h} [\varphi(\lambda(s, 0)) - \alpha(s)g(\infty)]\, ds \right| \le \varepsilon \,\Big|\, b(t) \ge \theta \right] \\ &\qquad \ge \lim_{N \to \infty} P^N \bigg[\bigg| N^{-1}[\eta_{t+h}(0) - \eta_t(0)] \\ &\qquad\qquad - \int_t^{t+h} [\varphi(\lambda(s, 0)) - \alpha(s)g(\infty)]\, ds \bigg| \le \varepsilon \,\Big|\, \eta_t(0) \ge \theta N \bigg]. \end{aligned} \tag{4.8}$$

Therefore, to prove the lemma, it is enough to show that the right-hand side of the last inequality is equal to 1 for every $\varepsilon > 0$.

Notice that on the set $\{\inf\{\eta_s(0);\ s \in [t, t+h]\} \ge 1\}$ particles leave the origin as a Poisson point process of rate $\alpha(\cdot)g(\infty)$. This is the key remark in order to prove this lemma. Thus $\eta_{t+h}(0) - \eta_t(0)$ is equal to the total number of particles that arrive at the origin in the interval $[t, t+h]$ minus a Poisson point process of rate $\alpha(\cdot)g(\infty)$.

By the hydrodynamical limit for space homogeneous zero-range processes the total number of particles that arrive at the origin in the interval $[t, t+h]$ divided by N converges in probability to $\int_t^{t+h} \varphi(\lambda(s,0))\,ds$. To conclude the argument, we have only to show that the conditional probability of the set $\{\inf\{\eta_s(0);\ s \in [t, t+h]\} \ge 1\}$, given that $\eta_t(0) \ge \theta N$, converges to 1. But this is a simple consequence of the law of large numbers for Poisson processes.

More precisely, the probability appearing on the right-hand side of (4.8) is bounded below by

$$P^N\biggl[\biggl|N^{-1}[\eta_{t+h}(0) - \eta_t(0)] - \int_t^{t+h} [\varphi(\lambda(s,0)) - \alpha(s)g(\infty)]\,ds\biggr| \le \varepsilon \\ \text{and } \inf\{\eta_s(0);\ s \in [t, t+h]\} \ge 1 \,\biggl|\, \eta_t(0) \ge \theta N\biggr].$$

Denote by $(U^N(t))_{t\ge 0}$ a Poisson point process with parameter $\alpha(\cdot)g(\infty)$ accelerated by N. With this notation, on the set $\bigl\{\inf\{\eta_s(0);\ s \in [t, t+h]\} \ge 1\bigr\}$, $\eta_{t+h}(0) - \eta_t(0)$ is equal to

$$\sum_{x\ge 0}[\xi_{t+h}(x) - \xi_t(x)] - [U^N(t+h) - U^N(t)].$$

By Lemma 4.3, by conservation of the total number of particles and by the law of large numbers for Poisson processes, this expression divided by N converges in probability as $N \uparrow \infty$ to

$$\int_t^{t+h} [\varphi(\lambda(s,0)) - \alpha(s)g(\infty)]\,ds.$$

Therefore the last probability, which is bounded below by

$$P^N\biggl[\biggl|N^{-1}\sum_{x\ge 0}[\eta_{t+h}(x) - \eta_t(x)] - N^{-1}[U^N(t+h) - U^N(t)] \\ - \int_t^{t+h} [\varphi(\lambda(s,0)) - \alpha(s)g(\infty)]\,ds\biggr| \le \varepsilon \,\bigl|\, \eta_t(0) \ge \theta N\biggr] \\ - P^N\Bigl[\inf\{\eta_s(0);\ s \in [t, t+h]\} = 0 \,\bigl|\, \eta_t(0) \ge \theta N\Bigr],$$

has a limit bounded below by

$$1 - \limsup_{N\to\infty} P^N[\inf\{\eta_s(0);\ s \in [t, t+h]\} = 0 \,|\, \eta_t(0) \ge \theta N].$$

Since $h < \theta[\bar{\alpha} g(\infty)]^{-1}$ and $\eta_s(0)$ decreases at most at rate $\alpha(s)g(\infty)$, this last expression is equal to 1. □

Letting $h \downarrow 0$ and then $\theta \downarrow 0$, we obtain the following result as an immediate consequence of the previous lemma.

COROLLARY 4.8. *For all limit points R^* of the sequence R^N and all continuity points t_0 of $\lambda(\cdot, 0)$ and $\alpha(\cdot)$,*

$$R^*[(\partial_t b)(t_0) = \varphi(\lambda(t_0, 0)) - \alpha(t_0)g(\infty) \,|\, b(t_0) > 0] = 1.$$

The first claim in Lemma 4.5 is therefore proved. We now complete the proof of this lemma.

PROOF OF LEMMA 4.5. Assume first that $\varphi(\lambda(t_0, 0)) < \alpha(t_0)g(\infty)$. There exist ε_0 and h_0 such that $\varphi(\lambda(t_0+h, 0)) < \alpha(t_0+h)g(\infty)-\varepsilon_0$ for all $0 \le h < h_0$. From Lemma 4.6, b is continuous. On the other hand, Lemma 4.7 gives the behavior of $b(\cdot)$ when $b(\cdot)$ is positive. It is easy to see from these two facts that $b(\cdot)$ has to be identically equal to 0 on the interval $[t_0, t_0 + h_0)$.

We now turn to the case $Z(t_0) := \varphi(\lambda(t_0)) - \alpha(t_0)g(\infty) \ge 0$. By the lower bound of Lemma 4.6,

$$\liminf_{h\downarrow 0} h^{-1}\{b(t_0 + h) - b(t_0)\} \ge Z(t_0).$$

Assume first that $Z(t_0) > 0$. In this case $b(\cdot)$ is positive on some interval (t_0, t_1). Corollary 4.8 describes the behavior of $b(\cdot)$ when $b(\cdot)$ is positive: $\partial_t b(t) = Z(t)$. Thus, in the interval (t_0, t_1), $b(t) = \int_{t_0}^{t} Z(s)\,ds$ and $(\partial_t^+ b)(t_0) = Z(t_0)$.

Suppose now that $Z(t_0) = 0$. We want to show that

$$\limsup_{h\downarrow 0} h^{-1}[b(t_0 + h) - b(t_0)] = 0.$$

Assume by contradiction that this limit is greater than a strictly positive constant 2ε. There would exist therefore a sequence $s_k \downarrow t_0$ with $b(s_k) \ge 2\varepsilon(s_k - t_0)$. Since t_0 is a continuity point of $Z(\cdot)$, there exists $\delta > 0$ such that $Z(t) \le \varepsilon/2$ for t in $[t_0, t_0 + \delta)$. Assume, without loss of generality, that s_k belongs to $[t_0, t_0 + \delta)$ for all k. By definition, $b(s_1) > 2\varepsilon(s_1 - t_0)$. Since $b(\cdot)$ is continuous, $\lim_k b(s_k) = b(t_0)$. There exists therefore k_0 such that $b(s_1) - b(s_{k_0}) \ge \varepsilon(s_1 - s_{k_0})$. This is impossible because when b is positive its derivative at t is equal to $Z(t)$ which in this interval is bounded by $\varepsilon/2$. □

5. An entropy inequality at the microscopic level. In this section and the next we follow the approach proposed by Rezakhanlou [7] to prove the hydrodynamical behavior of asymmetric attractive interacting particle systems. For this reason some proofs are only sketched.

Throughout this section, for an integer x, τ_x denotes the translation by x units on the configuration space. These translations are naturally extended to cylinder functions. We will use repeatedly the following notation throughout this section. For an integer x and a positive integer l, $\eta^l(x)$ represents the mean density of particles in a box of length $2l + 1$ centered at x:

$$\eta^l(x) = (2l + 1)^{-1} \sum_{|y-x|\le l} \eta(y). \tag{5.1}$$

The symbol $\overline{L}_N$ denotes the basic coupling of two inhomogeneous zero-range processes with evolution described by the generator (1.2) associated with the couple $\{(0, \alpha)\}$:

$$
\begin{aligned}
(\overline{L}_N f)(\eta, \xi) & \\
&= \sum_{x\in\mathbf{Z}} p_N(t, x)\min\{g(\eta(x)), g(\xi(x))\}[f(\eta^{x,x+1}, \xi^{x,x+1}) - f(\eta, \xi)] \\
&\quad + \sum_{x\in\mathbf{Z}} p_N(t, x)(g(\eta(x)) - g(\xi(x)))^+[f(\eta^{x,x+1}, \xi) - f(\eta, \xi)] \\
&\quad + \sum_{x\in\mathbf{Z}} p_N(t, x)(g(\xi(x)) - g(\eta(x)))^+[f(\eta, \xi^{x,x+1}) - f(\eta, \xi)].
\end{aligned} \tag{5.2}
$$

This section is devoted to the proof of an entropy inequality at the microscopic level. In order to state the first result, for a probability $\overline{\mu}^N$ on the configuration space $\mathbf{N}^{\mathbf{Z}} \times \mathbf{N}^{\mathbf{Z}} = \mathscr{X}^2$, denote by $\overline{P}^N_{\overline{\mu}^N}$ the probability measure on the path space $D([0, \infty), \mathscr{X}^2)$ corresponding to the Markov process (η_t, ξ_t) evolving according to the generator $\overline{L}_N$ defined above, accelerated by N and starting from $\overline{\mu}^N$.

For a measure $\overline{\mu}^N$ on the product space $\mathscr{X}^2$, we denote by $\overline{\mu}^N_i$ its ith marginal.

PROPOSITION 5.1. *Let $\overline{\mu}^N$ be a measure with both marginals bounded by a translation invariant product measure $\nu^{\mathrm{ti}}_{\rho_0}$:*

$$
\overline{\mu}^N_i \le \nu^{\mathrm{ti}}_{\rho_0} \tag{5.3}
$$

for $i = 1, 2$ and some density ρ_0. Recall the definition of φ given in (1.5) *and the definition of $\eta^l(x)$ given in* (5.1). *For every smooth positive function H of $C_K((0, \infty) \times \mathbf{R})$ and every positive ε,*

$$
\begin{aligned}
\lim_{l\to\infty}\lim_{N\to\infty} \overline{P}^N_{\overline{\mu}^N}\bigg[\int_0^\infty dt\, N^{-1} \sum_x \bigg\{\partial_t H\Big(t, \frac{x}{N}\Big)\big|\eta^l_t(x) - \xi^l_t(x)\big| & \\
+ \partial_u H\Big(t, \frac{x}{N}\Big)\big|\varphi\big(\eta^l_t(x)\big) - \varphi\big(\xi^l_t(x)\big)\big|\bigg\} \ge -\varepsilon\bigg] &= 1.
\end{aligned}
$$

Notice that the assumption (5.3) on the initial measure $\overline{\mu}^N$ implies that there is a finite density of particles on each compact set:

$$
\limsup_{N\to\infty} \overline{\mu}^N\bigg[N^{-1} \sum_{|x|\le CN} \eta(x) + \xi(x)\bigg] < \infty \tag{5.4}
$$

for every $C > 0$. The next lemma requires only this weaker assumption on the sequence of initial measures.

The proof of Proposition 5.1 is divided in several lemmas. We first prove that in the limit as $N \uparrow \infty$ the configurations η and ξ are ordered.

LEMMA 5.2. *Assume that the sequence of initial measure $\bar{\mu}^N$ satisfies hypothesis* (5.4). *Then, for every positive smooth function H with compact support, for every time $T > 0$ and every integer y,*

$$\lim_{N\to\infty} \overline{E}^N_{\bar{\mu}^N}\left[\int_0^T dt\, N^{-1}\sum_x H(x/N) G_{x,x+y}(\eta_t, \xi_t)\right] = 0,$$

where, for two sites x and y, $G_{x,y}(\eta, \xi)$ is an indicator function equal to 1 *if the configurations η and ξ are not ordered at sites x and y*:

$$G_{x,y}(\eta, \xi) = \mathbf{1}_{\{\eta(x)<\xi(x),\,\eta(y)>\xi(y)\}} + \mathbf{1}_{\{\eta(x)>\xi(x),\,\eta(y)<\xi(y)\}}. \tag{5.5}$$

PROOF. We sketch the proof of this result to avoid long but simple computations.

The proof is done in three steps. We first show that

$$\lim_{N\to\infty} \overline{E}^N_{\bar{\mu}^N}\left[\int_0^T dt\, N^{-1}\sum_x H\left(\frac{x+1}{N}\right)\left|g(\eta_t(x)) - g(\xi_t(x))\right| G_{x,x+1}(\eta_t, \xi_t)\right] = 0,$$

considering the martingale

$$N^{-1}\sum_x H\left(\frac{x}{N}\right)|\eta_t(x) - \xi_t(x)| - \int_0^t ds\, N^{-1}\sum_x H\left(\frac{x}{N}\right) N\overline{L}_N |\eta_s(x) - \xi_s(x)|$$

and using the assumption on the initial measure $\bar{\mu}^N$. From this result it follows that

$$\lim_{N\to\infty} \overline{E}^N_{\bar{\mu}^N}\left[\int_0^T dt\, N^{-1}\sum_x H\left(\frac{x+1}{N}\right)\left\{\mathbf{1}_{\substack{0=\eta_t(x)<\xi_t(x)\\ \eta_t(x+1)>\xi_t(x+1)}} + \mathbf{1}_{\substack{\eta_t(x)>\xi_t(x)=0\\ \eta_t(x+1)<\xi_t(x+1)}}\right\}\right] = 0.$$

The second step consists of removing the condition $\eta_t(x) \wedge \xi_t(x) = 0$ in the last expression. This is done by induction. For a positive integer m, let

$$I_m(\eta, \xi) = N^{-1}\sum_x H\left(\frac{x+1}{N}\right)\mathbf{1}_{\substack{m=\eta(x)<\xi(x)\\ \eta(x+1)>\xi(x+1)}}.$$

In the first step we proved that

$$\lim_{N\to\infty} \overline{E}^N_{\bar{\mu}^N}\left[\int_0^T dt\, I_0(\eta_t, \xi_t)\right] = 0.$$

A simple computation of the martingale $I_m(\eta_t, \xi_t) - \int_0^t N\overline{L}_N I_m(\eta_s, \xi_s)\, ds$ shows that

$$g(m+1)\overline{E}^N_{\bar{\mu}^N}\left[\int_0^T dt\, I_{m+1}(\eta_t, \xi_t)\right] \le N^{-1}C(H) + 4g(\infty)\overline{E}^N_{\bar{\mu}^N}\left[\int_0^T dt\, I_m(\eta_t, \xi_t)\right].$$

It follows from this inequality and the statement proved in the first step of the proof that

$$\lim_{N\to\infty} \overline{E}^N_{\bar{\mu}^N}\left[\int_0^T dt\, N^{-1}\sum_x H\left(\frac{x+1}{N}\right)\mathbf{1}_{\substack{m=\eta_t(x)<\xi_t(x)\\ \eta_t(x+1)>\xi_t(x+1)}}\right] = 0$$

for every positive integer m. This result, the assumption made on the initial measure $\overline{\mu}^N$ and a simple coupling argument show that

$$\lim_{N\to\infty} \overline{E}^N_{\overline{\mu}^N}\left[\int_0^T dt\, N^{-1}\sum_x H\left(\frac{x+1}{N}\right) G_{x,\,x+1}(\eta_t,\,\xi_t)\right] = 0.$$

This concludes the second step.

The last step in the proof consists of replacing $x+1$ by $x+y$ in the above formula. This can be done since H is uniformly continuous and $G_{x,\,x+y}(\eta,\xi)$ is bounded above by

$$\sum_{z=x\wedge y}^{x\vee y-1} G_{z,\,z+1}(\eta,\,\xi). \qquad \square$$

This ordering of the coordinates made by the process allows us to replace averages of absolute values of differences of monotone functions by absolute values of averages. This statement is made clear in the next lemma.

A cylinder function Ψ is said to be Lipschitz if there exists a finite subset Λ of $\mathbf{Z}$ and a constant $C(\Psi)$ such that

$$\left|\Psi(\eta) - \Psi(\xi)\right| \le C(\Psi)\sum_{x\in\Lambda}\left|\eta(x) - \xi(x)\right|.$$

Notice that for all Lipschitz cylinder functions Ψ there exists a constant $C'(\Psi)$ and a finite subset Λ of $\mathbf{Z}$ such that

$$\left|\Psi(\eta)\right| \le C'(\Psi)\left(1 + \sum_{x\in\Lambda}\eta(x)\right).$$

LEMMA 5.3. *Let $\overline{\mu}^N$ be a sequence of measures satisfying the assumption stated in Proposition* 5.1. *Let Ψ be a monotone Lipschitz function. Then, for every positive continuous function H: $\mathbf{R}_+\times\mathbf{R}\to\mathbf{R}$ with compact support and for every positive integer l,*

$$\lim_{N\to\infty} \overline{E}^N_{\overline{\mu}^N}\left[\int_0^\infty dt\, N^{-1}\sum_x H\left(t,\frac{x}{N}\right)\left\{(2l+1)^{-1}\sum_{|y-x|\le l}\left|\tau_y\Psi(\eta_t) - \tau_y\Psi(\xi_t)\right| - \left|(2l+1)^{-1}\sum_{|y-x|\le l}\left[\tau_y\Psi(\eta_t) - \tau_y\Psi(\xi_t)\right]\right|\right\}\right] = 0.$$

PROOF. In the case where Ψ is a bounded function, this result is an immediate consequence of the preceding lemma. Indeed, let Λ_l be the subset of $\mathbf{Z}$ consisting of all integers at a distance smaller than l from Λ: $\Lambda_l = \{y\in\mathbf{Z};\ \exists\, x\in\Lambda,\ |x-y|\le l\}$. Define $G_{\Lambda,\,l}(\eta,\xi)$ as the indicator function defined to be equal to 1 if η and ξ are not ordered at Λ_l:

$$G_{\Lambda,\,l}(\eta,\xi) = 1 - \prod_{x,\,y\in\Lambda_l}\left(1 - G_{x,\,y}(\eta,\xi)\right).$$

If the configurations η and ξ are ordered on the set Λ_l translated by x, the expression that appears under brackets in the expectation vanishes because Ψ is monotone. Therefore, to prove the lemma for bounded functions Ψ, it is enough to show that

$$\lim_{N\to\infty} \overline{E}^N_{\overline{\mu}^N}\left[\int_0^\infty dt\, N^{-1}\sum_x H\left(t, \frac{x}{N}\right)\tau_x G_{\Lambda, l}(\eta_t, \xi_t)\right] = 0$$

and this follows from the previous lemma since $G_{\Lambda, l}(\eta, \xi)$ is bounded by

$$\sum_{x, y\in\Lambda_l} G_{x, y}(\eta, \xi).$$

We now turn to the general case. Since the negative part of a monotone Lipschitz function is bounded, we assume, without loss of generality, that Ψ is positive. The idea is to reduce the general case to the bounded case. At the origin, however, the cutoff needed may not work since the number of particles at site 0 can be of order N. This statement will be made clear later in the proof. For this reason we have to consider sites around the origin separately. We therefore divide the sum over all sites x in the expected value appearing in the statement of the lemma into two pieces. The first one takes into account sites near the origin, $|x| \le 2l_0$, and the second one sites far from the origin, $|x| > 2l_0$. Here l_0 is chosen in such a way that the support of $\tau_x\Psi$ is contained in $\{-l_0, \dots, l_0\}$ for every $|x| \le l$.

We show that each piece converges to 0 separately. We concentrate first on the second piece. For a real positive A, let Ψ_A be the cutoff of Ψ at level A:

$$\Psi_A(\eta) = \Psi(\eta) \wedge A.$$

The second piece is bounded above by

$$\begin{aligned}
&\overline{E}^N_{\overline{\mu}^N}\bigg[\int_0^\infty dt\, N^{-1}\sum_{|x|>2l_0} H\left(t, \frac{x}{N}\right)\bigg\{(2l+1)^{-1}\sum_{|y-x|\le l}\left|\tau_y\Psi_A(\eta_t) - \tau_y\Psi_A(\xi_t)\right| \\
&\qquad\qquad - \bigg|(2l+1)^{-1}\sum_{|y-x|\le l}[\tau_y\Psi_A(\eta_t) - \tau_y\Psi_A(\xi_t)]\bigg|\bigg\}\bigg] \\
&\quad + \overline{E}^N_{\overline{\mu}^N}\bigg[\int_0^\infty dt\, 2N^{-1}\sum_{|x|>2l_0} H\left(t, \frac{x}{N}\right) \\
&\qquad \times \bigg\{(2l+1)^{-1}\sum_{|y-x|\le l}[\tau_y\Psi(\eta_t) - \tau_y\Psi(\eta_t)\wedge A] \\
&\qquad\quad + (2l+1)^{-1}\sum_{|y-x|\le l}[\tau_y\Psi(\xi_t) - \tau_y\Psi(\xi_t)\wedge A]\bigg\}\bigg].
\end{aligned}$$

For every A the first term converges to 0 as N increases to ∞ by the first part of the proof. On the other hand, since both marginals of the initial measure $\overline{\mu}^N$ are bounded by the translation invariant measure $\nu^{\mathrm{ti}}_{\rho_0}$, by Lemma 4.2, there exist a density ρ_1 and a rate α_1 such that the state of the process at any time is bounded by the invariant measure $\nu^0_{\rho_1,\alpha_1}$ of the inhomogeneous process. Therefore, since $\Psi(\eta) - \Psi(\eta) \wedge A$ is an increasing function, the second term is bounded by

$$4C_0(H)\nu^0_{\rho_1,\alpha_1}\Bigg[N^{-1}\sum_{2l_0<|x|\le C_1(H)N}\bigg\{(2l+1)^{-1}\sum_{|y-x|\le l}\tau_y\Psi(\eta) - \tau_y\Psi(\eta)\wedge A\bigg\}\Bigg].$$

We now see why we needed to separate from the global sum over all sites the part where $\eta(0)$ interferes. Indeed, to show that this expression converges to 0, we need to use the fact that the sum does not take into account the occupation variable $\eta(0)$.

Since outside the origin the measure $\nu^0_{\rho_1,\alpha_1}$ is bounded by a translation invariant product measure $\nu^{\mathrm{ti}}_{\rho_2}$ with an appropriate density ρ_2, the last expression is bounded by

$$C_2(H)\nu^{\mathrm{ti}}_{\rho_2}\big[\Psi(\eta) - \Psi(\eta)\wedge A\big]$$

and this expected value converges to 0 as A increases to ∞ by the dominated convergence theorem.

We now concentrate on the sum over sites near the origin. For a fixed integer y, let $\Psi_y(\eta) = \Psi_y(\eta(0))$ be the function $\tau_y\Psi$ evaluated on the configuration η^0 defined by

$$\eta^0(x) = \begin{cases} \eta(0), & \text{if } x = 0,\\ 0, & \text{otherwise.}\end{cases}$$

With this notation the sum may be rewritten as

$$\begin{aligned}
&\overline{E}^N_{\overline{\mu}^N}\Bigg[\int_0^\infty dt\, N^{-1}\sum_{|x|\le 2l_0} H\Big(t,\frac{x}{N}\Big)\bigg\{(2l+1)^{-1}\sum_{|y-x|\le l}\big|\Psi_y(\eta_t) - \Psi_y(\xi_t)\big| \\
&\qquad\qquad - \bigg|(2l+1)^{-1}\sum_{|y-x|\le l}[\Psi_y(\eta_t) - \Psi_y(\xi_t)]\bigg|\bigg\}\Bigg]\\
&\quad + \overline{E}^N_{\overline{\mu}^N}\Bigg[\int_0^\infty dt\, 2N^{-1}\sum_{|x|\le 2l_0} H\Big(t,\frac{x}{N}\Big)\bigg\{(2l+1)^{-1}\sum_{|y-x|\le l}\big|\tau_y\Psi(\eta_t) - \Psi_y(\eta_t)\big|\\
&\qquad\qquad + (2l+1)^{-1}\sum_{|y-x|\le l}\big|\tau_y\Psi(\xi_t) - \Psi_y(\xi_t)\big|\bigg\}\Bigg].
\end{aligned}$$

Since Ψ is monotone and, for each y, Ψ_y depends only on the value of $\eta(0)$, the expression inside the brackets in the first expectation vanishes for each N. On the other hand, since Ψ is Lipschitz, repeating the coupling arguments used in the second part of this proof, we show that the second expression converges to 0 as N increases to ∞ because the sum of a finite number of terms is divided by N. □

The third result toward the proof of Proposition 5.1 is a one-block estimate for the uncoupled process. To state this next lemma, we need additional notation. For a bounded cylinder function Ψ, we denote by $\tilde{\Psi}(\rho)$ the expected value of Ψ with respect to the product translation invariant measure ν_ρ^{ti} defined in (1.6):

$$\tilde{\Psi}(\rho) = \nu_\rho^{\mathrm{ti}}[\Psi(\eta)]. \tag{5.6}$$

Recall the terminology introduced just after (1.10) of a sequence of measures associated with a profile.

LEMMA 5.4 (One-block estimate). *Let μ^N be a sequence of product measures on $\mathscr{X}$ associated with a bounded profile. For every positive continuous function H: $\mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ with compact support and for every bounded cylinder function Ψ,*

$$\lim_{l\to\infty}\lim_{N\to\infty} E_{\mu^N}^N\left[\int_0^\infty dt\, N^{-1}\sum_x H\left(t, \frac{x}{N}\right)\left|(2l+1)^{-1}\sum_{|y-x|\le l}\tau_y\Psi(\eta)-\tilde{\Psi}\big(\eta_t^l(x)\big)\right|\right]=0.$$

PROOF. This proof is only sketched since the arguments are by now standard. First of all, it is clear that we need to prove Lemma 5.4 only for product measures associated with profiles that are identically equal to ρ outside some compact set. Fix such a measure μ^N.

To keep the notation simple, for a bounded cylinder function Ψ and a positive integer l, let

$$V_{\Psi,l}(\eta) = \left|(2l+1)^{-1}\sum_{|y|\le l}\tau_y\Psi(\eta) - \tilde{\Psi}\big(\eta_t^l(0)\big)\right|.$$

Recall that we represent by L_h the generator of the space homogeneous zero range process. Recall also that we denoted by $P_h^{N,\rho}$ the probability measure on the path space corresponding to the Markov process with generator L_h accelerated by N and starting from the product measure ν_ρ^{ti}.

Fix a time T such that the support of H is contained in $[0, T] \times \mathbf{R}$. Denote by $P_{\mu^N}^N(T)$ and $P_h^{N,\rho}(T)$ the restrictions of the measures $P_{\mu^N}^N$ and $P_h^{N,\rho}$ to $D([0, T], \mathscr{X})$. By the entropy inequality, for every positive γ, the expected value appearing in the statement of the lemma is bounded above by

$$\frac{1}{\gamma N} H\big(P_{\mu^N}^N(T)\,\big|\,P_h^{N,\rho}(T)\big) + \frac{1}{\gamma N}\log E_h^{N,\rho}\left[\exp\left\{\int_0^T dt\,\gamma\sum_x H\left(t,\frac{x}{N}\right)\tau_x V_{\Psi,l}(\eta_t)\right\}\right].$$

In this last formula $H(P_{\mu^N}^N(T)\,|\,P_h^{N,\rho}(T))$ stands for the entropy of $P_{\mu^N}^N(T)$ with respect to $P_h^{N,\rho}(T)$. A simple computation shows that this entropy has the following explicit form:

$$\int \log\left\{\frac{d\mu^N}{d\nu_\rho^{\mathrm{ti}}}\right\} d\mu^N \\ + E_N\left[N\int_0^T g(\eta_s(0))\{p_N(s,0)\log p_N(s,0) + 1 - p_N(s,0)\}\,ds\right].$$

Since the measure μ^N is associated with a profile equal to ρ outside some compact set, it is easy to show that the last expression is bounded by $C_0(T)N$. On the other hand, following the arguments of [3], we show that, for every $\gamma > 0$,

$$\lim_{l\to\infty}\lim_{N\to\infty}\frac{1}{N}\log E_h^{N,\rho}\left[\exp\left\{\int_0^T dt\,\gamma\sum_x H\left(t,\frac{x}{N}\right)\tau_x V_{\Psi,l}(\eta_t)\right\}\right] = 0.$$

This concludes the proof of the lemma. □

We are now ready to prove the entropy inequality at the microscopic level.

PROOF OF PROPOSITION 5.1. Let $M^H(t)$ be the martingale defined by

$$M^H(t) = N^{-1}\sum_x H\left(t,\frac{x}{N}\right)\left|\eta_t(x) - \xi_t(x)\right| \\ - \int_0^t (\partial_s + N\overline{L}_N)N^{-1}\sum_x H\left(s,\frac{x}{N}\right)\left|\eta_s(x) - \xi_s(x)\right|\,ds.$$

A computation using the results of Lemma 5.2 shows that

$$\lim_{N\to\infty}\overline{E}^N_{\overline{\mu}^N}\left[\left(M^H(t)\right)^2\right] = 0.$$

From Doob's inequality we obtain that, for every $t \geq 0$ and every positive ε,

$$\lim_{N\to\infty}\overline{P}^N_{\overline{\mu}^N}\left[\left|M^H(t)\right| > \varepsilon\right] = 0. \tag{5.7}$$

On the other hand, it is easily seen that, for t sufficiently large, the martingale is bounded below by

$$-\int_0^\infty N^{-1}\sum_x\left\{\partial_s H\left(s,\frac{x}{N}\right)\left|\eta_s(x) - \xi_s(x)\right| \right. \\ \left. + \partial_u H\left(s,\frac{x}{N}\right)p_N(s,x)\left|g(\eta_s(x)) - g(\xi_s(x))\right|\right\}ds - O\left(\frac{1}{N}\right).$$

Since g and $\partial_u H$ are bounded functions and since $p_N(s,x)$ is equal to 1 at all sites $x \neq 0$, $p_N(s,x)$ can be removed in the last formula. Thus the martingale is bounded below by

$$- \int_0^\infty N^{-1} \sum_x \Big\{ \partial_s H\Big(s, \frac{x}{N}\Big) |\eta_s(x) - \xi_s(x)| + \partial_u H\Big(s, \frac{x}{N}\Big) |g(\eta_s(x)) - g(\xi_s(x))| \Big\}\, ds - O\Big(\frac{1}{N}\Big).$$

By the assumption on the initial measure, for every continuous function G: $\mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ with compact support,

$$\lim_{N\to\infty} \overline{E}^N_{\overline{\mu}^N} \Big[\int_0^\infty dt\, N^{-1} \sum_x \Big[G\Big(t, \frac{x}{N}\Big) - (2l+1)^{-1} \sum_{|y-x|\le l} G\Big(t, \frac{y}{N}\Big) \Big] \times \{ |\eta_t(x) - \xi_t(x)| + |g(\eta_t(x)) - g(\xi_t(x))| \} \Big] = 0.$$

Applying this result to the functions $\partial_t H$ and $\partial_u H$ and making a discrete integration by parts, we obtain that, for t sufficiently large, the martingale is bounded below by

$$-\int_0^\infty N^{-1} \sum_x \Big\{ \partial_s H\Big(s, \frac{x}{N}\Big) (2l+1)^{-1} \sum_{|y-x|\le l} |\eta_s(y) - \xi_s(y)| + \partial_u H\Big(s, \frac{x}{N}\Big) (2l+1)^{-1} \sum_{|y-x|\le l} |g(\eta_s(y)) - g(\xi_s(y))| \Big\}\, ds - o_N(1).$$

Therefore, from (5.7) and Lemma 5.3, we obtain that, for every $\varepsilon > 0$,

$$\lim_{N\to\infty} \overline{P}^N_{\overline{\mu}^N} \Big[\int_0^\infty N^{-1} \sum_x \Big\{ \partial_s H\Big(s, \frac{x}{N}\Big) \Big| (2l+1)^{-1} \sum_{|y-x|\le l} [\eta_s(y) - \xi_s(y)] \Big| + \partial_u H\Big(s, \frac{x}{N}\Big) \Big| (2l+1)^{-1} \sum_{|y-x|\le l} [g(\eta_s(y)) - g(\xi_s(y))] \Big| \Big\}\, ds < -\varepsilon \Big] = 0.$$

Finally, applying the one-block estimate stated in Lemma 5.4 to the function $g(\eta(0))$ and recalling that the expectation of this function with respect to the product measure ν_ρ^{ti} is $\varphi(\rho)$, we conclude the proof of the proposition. □

REMARK 5.5. In Proposition 5.1 the assumption (5.3) was necessary to control the remainder in the cutoff made in Lemma 5.3 to reduce the general case to the case of bounded monotone functions. This hypothesis can be relaxed. For instance, (5.4) and the existence of a density ρ_1 such that

$$\overline{\mu}_i^N S_t^N \le \nu_{\rho_1}^{\mathrm{ti}}$$

away from the origin for all $t > 0$ and $i = 1, 2$ are enough to prove Proposition 5.1.

We conclude this section by applying Proposition 5.1 to some special cases where the function H has compact support in $(0,\infty)\times(0,\infty)$. In this case the coupling on sites $x\le 0$ is irrelevant for the proof of Proposition 5.1. In particular, we may couple two processes with different jump rates at the origin.

For fixed rates $p_N^1(t,x)$ and $p_N^2(t,x)$ denote by L_N^* the generator of the basic coupling of two inhomogeneous zero-range processes:

$$
\begin{aligned}
(L_N^* f)(\eta,\xi) = \sum_{x\in\mathbf{Z}} & \min\{p_N^1(t,x)g(\eta(x)),\, p_N^2(t,x)g(\xi(x))\}\\
&\times [f(\eta^{x,x+1},\xi^{x,x+1}) - f(\eta,\xi)]\\
&+\sum_{x\in\mathbf{Z}} \big(p_N^1(t,x)g(\eta(x)) - p_N^2(t,x)g(\xi(x))\big)^+\\
&\quad\times [f(\eta^{x,x+1},\xi) - f(\eta,\xi)]\\
&+\sum_{x\in\mathbf{Z}} \big(p_N^2(t,x)g(\xi(x)) - p_N^1(t,x)g(\eta(x))\big)^+\\
&\quad\times [f(\eta,\xi^{x,x+1}) - f(\eta,\xi)].
\end{aligned}
\tag{5.8}
$$

Repeating the proof of Proposition 5.1, we obtain the following result.

LEMMA 5.6. *Let $\overline{\mu}^N$ be a measure with both marginals restricted to $\mathbf{N}^{\mathbf{N}^*}$ bounded by a translation invariant product measure $\nu_{\rho_0}^{\mathrm{ti}}$:*

$$\overline{\mu}_i^N\big|_{\mathbf{N}^{\mathbf{N}^*}} \le \nu_{\rho_0}^{\mathrm{ti}}$$

for $i=1,2$ and some density ρ_0. Then the conclusions of Proposition 5.1 are valid for every smooth positive function with compact support in $(0,\infty)\times(0,\infty)$ if (η_t,ξ_t) evolves as the Markov process with generator defined in (5.8), accelerated by N.

Notice that, for each $0<\alpha<1$, $\rho\ge 0$, the measures $\mu_{m^*}^N\times\nu_{\rho,\alpha}^0$ satisfy the assumptions of Lemma 5.6.

Fix $p_N^2(t,x)$ to be equal to α if $x=0$ and 1 otherwise. Since $\nu_{\rho,\alpha}^0$ is an invariant measure, (ξ_t) is a stationary process. Moreover, by the law of large numbers, at a macroscopic distance from the origin $\xi^l(x)$ converges in probability to $\nu_{\rho,\alpha}^0[\xi(1)]$. Since the range of $\nu_{\rho,\alpha}^0[\xi(1)]$ as (ρ,α) varies in $\mathbf{R}_+\times[0,1)$ is $\mathbf{R}_+$, from Lemma 5.6 we obtain the following microscopic entropy inequality.

COROLLARY 5.7. *For every smooth positive function H with compact support in $(0,\infty)\times(0,\infty)$, for every nonnegative constant c and for every positive $\varepsilon>0$,*

$$
\begin{aligned}
\lim_{l\to\infty}\lim_{N\to\infty} P_{\mu_{m^*}^N}^N\bigg[\int_0^\infty dt\, N^{-1}\sum_x\bigg\{&\partial_t H\Big(t,\frac{x}{N}\Big)\big|\eta_t^l(x)-c\big|\\
&+\partial_u H\Big(t,\frac{x}{N}\Big)\big|\varphi(\eta_t^l(x))-\varphi(c)\big|\bigg\}\ge -\varepsilon\bigg]=1.
\end{aligned}
$$

6. Young measures. In this section we conclude the proof of Theorem 1. The approach presented here to prove the hydrodynamical behavior of asymmetric processes relies on some results on entropy measure-valued weak solutions of scalar hyperbolic equations. The terminology and the results needed in this section are discussed in the Appendix.

Recall the definition of $\eta^l(x)$ given in (5.1). For each positive integer N and l and each configuration η, define the Young measure $\sigma^{N,l} = \sigma^{N,l}(\eta)$ as the Radon measure on $\mathbf{R}_+^2$ that integrates a continuous function with compact support $H: \mathbf{R}_+^2 \to \mathbf{R}$ as

$$\langle \sigma^{N,l}, H \rangle = \int H(u, q)\sigma^{N,l}(du, dq) = N^{-1} \sum_{x>l} H\left(\frac{x}{N}, \eta^l(x)\right).$$

Notice that the sum starts from $l+1$ to avoid the interference of $\eta(0)$. To keep the notation simple, we use the shorthand $\sigma_t^{N,l}$ to denote $\sigma^{N,l}(\eta_t)$. The first marginal of $\sigma^{N,l}$ is a discrete approximation of the Lebesgue measure and at site x/N the second marginal of $\sigma^{N,l}$ is a Dirac measure at $\eta^l(x)$.

The measures $\sigma_t^{N,l}$ and π_t^N are related by the following relation. For every continuous function $H: \mathbf{R} \to \mathbf{R}$ with compact support in $(0, \infty)$:

$$\int H(u)\, q\, \sigma_t^{N,l}(du, dq) = \int H(u)\, \pi_t^N(du) \tag{6.1}$$

plus error terms uniformly small in N.

The strategy of the proof of the hydrodynamical limit on the right of the origin is simple. We first prove that $\sigma_t^{N,l}$ converges in law to the entropy measure-valued solution on $\mathbf{R}_+$ of the hyperbolic equation

$$\begin{aligned} &\partial_t \rho + \partial_u \varphi(\rho) = 0, \\ &\partial_t \int_{\mathbf{R}_+} \rho(t, u)\, du = \omega(t), \\ &\rho(0, \cdot) = m^*(\cdot) \end{aligned} \tag{6.2}$$

with boundary conditions $m^*(\cdot)$ and $\omega(\cdot)$ given by

$$\omega(t) = \begin{cases} \alpha(t) g(\infty), & \text{if } \beta(t) > 0, \\ \alpha(t) g(\infty) \wedge \varphi(\lambda(t, 0)), & \text{otherwise.} \end{cases} \tag{6.3}$$

We then use (6.1) to show that π_t^N converges to the entropy solution of the same boundary hyperbolic equation.

Recall that we assumed m^* to be continuous with compact support.

To state the main result, we need some notation. Denote by $\mathscr{M}_+(\mathbf{R}_+^2)$ the space of all positive Radon measures on $\mathbf{R}_+^2$ endowed with the vague topology. For a fixed time T_0 and integers N and l, let $R^{N,l}$ denote the probability on $D([0, T_0], \mathscr{M}_+(\mathbf{R}) \times \mathscr{M}_+(\mathbf{R}_+^2))$ corresponding to the process $(\pi_t^N, \sigma_t^{N,l})$ with generator $L_{N,t}$, accelarated by N and starting from the product measure $\mu_{m^*}^N$.

In view of the results of Section 4, in the case where m^* is uniformly Lipschitz continuous, Theorem 1 follows from the following proposition.

PROPOSITION 6.1. *The sequence $R^{N,l}$ converges as $N \uparrow \infty$ and then $l \uparrow \infty$ to the probability measure concentrated on the deterministic path (π_t, σ_t), where $\sigma_t(du, dq) = \sigma_t(u, dq)\,du$ is the entropy measure-valued solution of* (6.2) *and π_t satisfies*

$$\int H(u)\, q\, \sigma_t(du, dq) = \int H(u)\, \pi_t(du)$$

for every continuous function H: $\mathbf{R} \to \mathbf{R}$ with compact support in $(0, \infty)$.

A simple coupling argument permits us to prove Theorem 1 for continuous profiles (in fact, for a much larger class) from the result for uniformly Lipschitz profiles. Indeed, the L^1 norm is contractive for entropy solutions of hyperbolic equations. It is also contractive at a microscopic level since

$$\overline{E}_N\left[N^{-1}\sum_x \left|\eta_t(x) - \xi_t(x)\right|\right]$$

decreases in time if (η_t, ξ_t) evolves according to the generator defined in (5.2). Thus, to prove Theorem 1 for continuous profiles, we just have to approximate the initial data by uniformly Lipschitz continuous profiles.

We now turn to the proof of Proposition 6.1. We start proving the tightness of the sequence $R^{N,l}$.

LEMMA 6.2. *The sequence $R^{N,l}$ is tight. Every limit point R^* of this sequence when $N \uparrow \infty$ and then $l \uparrow \infty$ is concentrated on weakly continuous paths (π_t, σ_t) such that*:

(a) $\pi_t(du) = \rho(t, u)\,du$ *on* $(0, \infty)$;
(b) $\sigma(t, du, dq) = \sigma(t, u, dq)\,du$;
(c) *there exists $\rho_1 = \rho_1(m^*, \alpha)$ such that $\sigma(t, u, [0, \rho_1]^c) = 0$ for every* $(t, u) \in (0, \infty)^2$;
(d) $\rho(t, u) = \int q\sigma(t, u, dq)$ *for every* $(t, u) \in (0, \infty)^2$.

The proof of this lemma relies on Lemma 4.2 and is similar to the proof of Lemma 5.5 in [7].

In view of Theorem A2 in the Appendix, to conclude the proof of Proposition 6.1, we have to prove an entropy inequality and the L^1 convergence to the boundary conditions. The entropy inequality is just a restatement of Corollary 5.7. The convergence in L^1 to m^* is stated in the next lemma.

LEMMA 6.3. *Assume that m^* is uniformly Lipschitz continuous. Every limit point R^* of the sequence $R^{N,l}$ is concentrated on paths σ_t such that*

$$\lim_{t\to 0} \int du \int \left|q - m^*(u)\right| \sigma(t, u, dq) = 0.$$

This proof relies on coupling, on the assumption made on m^* and on the fact that $\mu^N_{m^*}$ is a product measure. The proof is identical to the proof of Lemma 5.6(a) in [7].

We now conclude the proof of Theorem 1 by considering the problem of L^1 convergence to $\omega(\cdot)$.

LEMMA 6.4. *Assume that m^* is uniformly Lipschitz continuous. Every limit point R^* of the sequence $R^{N,l}$ is concentrated on paths σ_t such that*

$$\lim_{u\to 0}\int_0^{T_0} dt \int \left|\varphi(q) - \omega(t)\right| \sigma(t, u, dq) = 0.$$

PROOF. Fix an interval $[T_1, T_2]$ where $\alpha(\cdot)$ is constant, say $\alpha(t) = \alpha_0$. The proof is divided into two steps. We prove separately the L^1 convergence in intervals where $b(\cdot)$ is positive and in intervals where $b(\cdot)$ vanishes.

Consider a subinterval $[t_0, t_1]$ of $[T_1, T_2]$ where $b(\cdot)$ is strictly positive. In this case, from (6.3) we have that

$$\omega(t) = \alpha_0 g(\infty) = \varphi(\rho_{\alpha_0}) \quad \text{for } t \in (t_0, t_1).$$

The second equality defines ρ_{α_0}. For fixed large A and ρ_1, let $\nu^N = \nu^N_{A,\alpha_0,\rho_1}$ denote the product measure on $\mathbf{N}^{\mathbf{Z}}$ with marginals given by

$$\begin{aligned} \nu^N\{\eta;\ \eta(x) = k\} &= \nu^0_{\rho_1,\alpha_0}\{\eta;\ \eta(x) = k\} \quad \text{for } x \neq 0,\ k \in \mathbf{N},\\ \nu^N\{\eta;\ \eta(0) = AN\} &= 1. \end{aligned}$$

For A and ρ_1 sufficiently large it is easy to show that, if ξ_t evolves according to the generator $L_{N,t}$ associated with $(0, \alpha_0)$ defined in (1.2) accelerated by N and starting from ν^N, then $N^{-1}\xi_t(0)$ converges in probability to

$$g(t) = A + [\varphi(\rho_1) - \alpha_0 g(\infty)]t. \tag{6.4}$$

At the right of the origin the measure ν^N looks like the invariant measure $\nu^0_{\rho_1,\alpha_0}$. Since for A and ρ_1 sufficiently large the probability that at some time there are no particles at site 0 converges to 0 as $N \uparrow \infty$, a coupling argument shows that, for every positive continuous function H: $\mathbf{R} \to \mathbf{R}$ with compact support,

$$\lim_{l\to\infty}\lim_{N\to\infty} P^N_{\nu^N}\left[\int_{t_0}^{t_1} \sum_{x>l} H\left(\frac{x}{N}\right)\left|\xi^l_t(x) - \rho_{\alpha_0}\right| dt > \varepsilon\right] = 0$$

for every $\varepsilon > 0$.

In Proposition 5.1 let $\overline{\mu}^N$ be the measure $\mu^N S^N_{t_0} \times \nu^N$; $\overline{\mu}^N$ satisfies the assumptions of Remark 5.5. Thus, from Proposition 5.1 and the hydrodynamical behavior at the left of the origin, for sufficiently large A and ρ_1, and for every smooth positive function H: $\mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ with compact support in $(t_0, t_1) \times \mathbf{R} \to \mathbf{R}$,

$$\lim_{l\to\infty} \lim_{N\to\infty} \overline{P}^N_{\bar{\mu}^N} \bigg[\int_0^\infty dt\, N^{-1} \sum_{x>l} \bigg\{ \partial_t H\Big(t, \frac{x}{N}\Big) |\eta_t^l(x) - \rho_{\alpha_0}| + \partial_u H\Big(t, \frac{x}{N}\Big) |\varphi(\eta_t^l(x)) - \varphi(\rho_{\alpha_0})| \bigg\} + \int_0^\infty dt \int_{-\infty}^0 du \, \{\partial_t H(t,u)[\rho_1 - \lambda(t,u)] + \partial_u H(t,u)[\varphi(\rho_1) - \varphi(\lambda(t,u))]\} + \int_0^\infty dt\, \partial_t H(t,0)[g(t) - b(t)] \ge -\varepsilon \bigg] = 1.$$

Here $g(\cdot)$ is the function defined in (6.4). We used that $\lambda(\cdot, \cdot)$ is bounded above by ρ_1 and that $b(\cdot)$ is bounded above by $g(\cdot)$ for A and ρ_1 large.

Letting $\varepsilon \downarrow 0$, we obtain that every limit point R^* of the sequence $R^{N,l}$ is concentrated on trajectories σ_t such that

$$\int_0^\infty dt \int_0^\infty du \bigg\{ \partial_t H \int |q - \rho_{\alpha_0}| \sigma(t,u,dq) + \partial_u H \int |\varphi(q) - \varphi(\rho_{\alpha_0})| \sigma(t,u,dq) \bigg\} \ge -\int_0^\infty dt \int_{-\infty}^0 du \{\partial_t H(t,u)[\rho_1 - \lambda(t,u)] + \partial_u H(t,u)[\varphi(\rho_1) - \varphi(\lambda(t,u))]\} - \int_0^\infty dt\, \partial_t H(t,0)[g(t) - b(t)].$$

Denote by $\psi\colon \mathbf{R} \to \mathbf{R}_+$ a smooth approximation of the identity:

$$\psi(u) \ge 0, \qquad \int \psi(u)\, du = 1, \qquad \operatorname{supp} \psi \subset [-1, 1].$$

For a positive γ, let ψ_γ denote a rescaling of ψ by γ:

$$\psi_\gamma(u) = \gamma^{-1} \psi(\gamma^{-1} u).$$

Fix $d, e > 0$. Let f_γ be a smooth approximation of the indicator function $\mathbf{1}_{[-d, e]}$ and g_γ a smooth approximation of the indicator function $\mathbf{1}_{[t_0, t_1]}$:

$$f_\gamma(u) = \int_{-\infty}^u [\psi_\gamma(v + d - \gamma) - \psi_\gamma(v - e + \gamma)]\, dv,$$

$$g_\gamma(u) = \int_{-\infty}^u [\psi_\gamma(v - t_0 - \gamma) - \psi_\gamma(v - t_1 + \gamma)]\, dv.$$

Notice that the support of g_γ is contained in $[t_0, t_1]$.

Take H to be equal to $f_\gamma(u) g_{\gamma'}(t)$ in the last inequality. A simple computation shows that the right-hand side converges as $\gamma \downarrow 0$ and $\gamma' \downarrow 0$ to

$$\int_{-d}^0 [\rho_1 - \lambda(t_0, u)]\, du - \int_{-d}^0 [\rho_1 - \lambda(t_1, u)]\, du + \int_{t_0}^{t_1} [\varphi(\rho_1) - \varphi(\lambda(t_0, u))]\, du.$$

This expression is bounded below by $-dC(\rho_1, m^*)$.

On the other hand, the left-hand side is bounded above by

$$\int_0^\infty dt\, \psi_\gamma(t-t_0-\gamma)\int_0^e du \int |q-\rho_{\alpha_0}|\sigma(t,u,dq)$$
$$-\int_0^\infty \psi_{\gamma'}(u-e+\gamma')\, du \int_0^\infty dt\, g_\gamma(t)\int |\varphi(q)-\varphi(\rho_{\alpha_0})|\sigma(t,u,dq).$$

The term line is bounded above by $eC(m^*, \rho_1, \alpha_0)$. Letting $\gamma \downarrow 0$, we obtain that the left-hand side of the last inequality is bounded above by

$$eC(m^*,\rho_1,\alpha) - \int_0^\infty \psi_{\gamma'}(u-e+\gamma')du \int_{t_0}^{t_1} dt \int |\varphi(q)-\varphi(\rho_{\alpha_0})|\sigma(t,u,dq).$$

Since almost all points of measurable functions are Lebesgue points, letting $\gamma' \downarrow 0$ and $d \downarrow 0$, we obtain that, for almost all $e > 0$,

$$\int_{t_0}^{t_1} dt \int |\varphi(q)-\varphi(\rho_{\alpha_0})|\sigma(t,e,dq) \le eC(m^*,\rho_1,\alpha).$$

Redefining $\sigma(t,u,dq)$ in a set of measure 0, we obtain the inequality for every e. This proves the L^1 convergence to $\omega(\cdot)$ in time intervals where $b(\cdot)$ is strictly positive.

Consider now a subinterval $[t_0', t_1']$ of $[T_1, T_2]$ where $b(\cdot)$ vanishes. Since b vanishes, $\varphi(\lambda(s,0)) \le \alpha_0 g(\infty)$ for s in $[t_0', t_1']$. Recall from (6.3) that in this interval

$$\omega(t) = \varphi(\lambda(t,0)).$$

Proposition 5.1 with $\overline{\mu}^N = \mu^N S^N_{t_0'} \times \nu_{\rho,\alpha_0}$ with $\rho < \rho_0$ together with the behavior at the left of the origin gives that every limit point R^* of the sequence $R^{N,l}$ is concentrated on trajectories $\sigma(t,u,dq)$ such that, for every positive smooth function H with compact support in $(t_0', t_1') \times \mathbf{R}$ and every $c \le \rho_{\alpha_0}$,

$$\begin{aligned}\int_0^\infty dt \int_0^\infty du\Big\{&\partial_t H(t,u)\int |q-c|\sigma(t,u,dq)\\ &+\partial_u H(t,u)\int |\varphi(q)-\varphi(c)|\sigma(t,u,dq)\Big\}\\ \ge -\int_0^\infty dt\int_{-\infty}^0 du\,&\{\partial_t H(t,u)|\lambda(t,u)-c|\\ &+\partial_u H(t,u)|\varphi(\lambda(t,u))-\varphi(\rho_1)|\}.\end{aligned} \tag{6.5}$$

Let $\overline{\lambda}$ be an upper bound for $m^*(\cdot)$ and therefore for $\lambda(\cdot,\cdot)$. Set $K_1 = \inf\{\varphi'(u);\ 0 \le u \le \overline{\lambda}\} > 0$. Since $\lambda(t,0) \le \rho_{\alpha_0}$ in (t_0', t_1'), $\lambda(t,u) \le \rho_{\alpha_0}$ in the set $\mathscr{A} = \{t_0' \le t \le t_1'\} \cap \{K_1(t-t_1') \le u \le K_1(t-t_0')\}$.

For any H with compact support in the interior of $\mathscr{A}$, repeating Kružkov's argument to prove that the L^1 norm is contractive for entropy solutions of hyperbolic equations, we show that we may replace c by $\lambda(t,u)$ in formula (6.5) (cf. the proof of Theorem 4.2 in [2] for a similar proof). Thus

$$\int_0^\infty dt \int_0^\infty du \left\{ \partial_t H(t,u) \int |q-\lambda(t,u)| \sigma(t,u,dq) \right.$$
$$\left. + \partial_u H(t,u) \int |\varphi(q)-\varphi(\lambda(t,u))| \sigma(t,u,dq) \right\} \geq 0.$$

Fix $\varepsilon > 0$. There exists $\iota(\varepsilon) > 0$ such that $[0,\iota] \times [t_0'+\varepsilon, t_1'] \subset \mathscr{A}^o$ for $\iota < \iota(\varepsilon)$. We argue now as we did at the end of the first part of the proof.

Consider a sequence $f_\gamma(u) g_{\gamma'}(t)$ of smooth approximations of the indicator function of $[0,\iota] \times [t_0'+\varepsilon, t_1']$ similar to the one considered in the first part of this proof. Since, for every γ and γ',

$$\int_0^\infty dt\, g_{\gamma'}'(t) \int_0^\infty du\, f_\gamma(u) \int |q-c| \sigma(t,u,dq)$$

is bounded by $C(m^*,\alpha)\iota$, letting $\gamma \downarrow 0$, we obtain that

$$\int_0^\infty du\, \psi_{\gamma'}(u-\iota+\gamma') \int_{t_0'}^{t_1'} dt \int |\varphi(q)-\varphi(c)| \sigma(t,u,dq) \leq C(m^*,\alpha)\iota.$$

Thus, for almost all $0 < \iota < \iota(\varepsilon)$,

$$\int_{t_0'}^{t_1'} dt \int |\varphi(q)-\varphi(c)| \sigma(t,\iota,dq) \leq C(m^*,\alpha)\,\iota.$$

We may change $\sigma(t,u,dq)$ in a set of measure 0 in order for this equation to be satisfied for all $\iota < \iota(\varepsilon)$. Since the sequence $\varphi(\lambda(\cdot,\iota))$ converges to $\varphi(\lambda(\cdot,0))$ in $L^1([0,T_0])$ as $\iota \downarrow 0$, the proof is concluded. □

APPENDIX

Terminology and results on weak solutions of hyperbolic equations. In this section we fix the terminology on weak solutions of scalar hyperbolic equations on the semiinfinite line used throughout this article.

Let $\varphi\colon \mathbf{R}_+ \to \mathbf{R}_+$ be a smooth bounded increasing function, let $\omega\colon \mathbf{R}_+ \to \mathbf{R}_+$ be a continuous function and let $m\colon \mathbf{R}_+ \to \mathbf{R}_+$ be a bounded function.

A bounded function $\rho\colon \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$ is an entropy solution of the equation

$$\begin{aligned} &\partial_t \rho + \partial_u \varphi(\rho) = 0, \\ &\partial_t \int_{\mathbf{R}_+} \rho(t,u)\, du = \omega(t), \\ &\rho(0,\cdot) = m(\cdot). \end{aligned} \tag{A.1}$$

if the following hold:

(a) (Entropy inequality) For every positive function $H: \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$ in $C_K^{1,1}((0,\infty)\times(0,\infty))$ and every constant $c \in \mathbf{R}$,

$$\int\int_{\mathbf{R}_+^2} dt\,du\{\partial_t H|\rho - c| + \partial_u H|\varphi(\rho) - \varphi(c)|\} \ge 0.$$

(b) (L^1_{loc} convergence to boundary conditions) For every $A > 0$,

$$\lim_{t\to 0}\int_0^A du\,|\rho(t,u) - m(u)| = 0,$$

$$\lim_{u\to 0}\int_0^A dt\,|\varphi(\rho(t,u)) - \omega(t)| = 0.$$

A slight modification in the proof of Theorem 1 of [5] gives the following result.

THEOREM A.1. *There exists a unique entropy solution of* (A.1).

We now turn to entropy measure-valued solutions. Denote by $\mathscr{P}(\mathbf{R})$ the set of probability measures on $\mathbf{R}$. A measurable function $\sigma: \mathbf{R}_+ \times \mathbf{R}_+ \to \mathscr{P}(\mathbf{R})$ is said to be an entropy measure-valued solution of (A.1) if the following hold:

(a) (Boundedness) There exists a constant K_0 such that

$$\sigma(t,u,[-K_0,K_0]^c) = 0$$

for every (t,u) in $\mathbf{R}_+^2$.

(b) (Entropy inequality) For every positive function $H: \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$ in $C_K^{1,1}((0,\infty)\times(0,\infty))$ and every constant $c \in \mathbf{R}$,

$$\int\int_{\mathbf{R}_+^2} dt\,du\biggl\{\partial_t H\int_{\mathbf{R}}|q-c|\sigma(t,u,dq) + \partial_u H\int_{\mathbf{R}}|\varphi(q)-\varphi(c)|\sigma(t,u,dq)\biggr\} \ge 0.$$

(c) (L^1_{loc} convergence to boundary conditions) For every $A > 0$,

$$\lim_{t\to 0}\int_0^A du\int_{\mathbf{R}}|q - m(u)|\sigma(t,u,dq) = 0,$$

$$\lim_{u\to 0}\int_0^A dt\int_{\mathbf{R}}|\varphi(q) - \omega(t)|\sigma(t,u,dq) = 0.$$

If $\rho(t,u)$ is the entropy solution of (A.1), the measure-valued function $\sigma(t,u,dq)$ defined by

$$\sigma(t,u,dq) = \delta_{\rho(t,u)}(dq)$$

is an entropy measure-valued solution. The natural question is whether this is the unique entropy measure-valued solution of (A.1). A slight modification in the proof of Theorem 4.2 of [2] gives the following result.

THEOREM A.2. *There exists a unique entropy measure-valued solution to equation* (A.1).

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URA CNRS 1378
UNIVERSITE DE ROUEN
76 128 MONT SAINT AIGNAN
FRANCE

INSTITUTO DE MATEMÁTICA PURA E APLICADA
ESTRADA DONA CASTORINA 110
22460 JARDIM BOTÂNICO
RIO DE JANEIRO RJ
BRASIL
E-mail: landim@impa.br