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REGULARITY OF QUASI-STATIONARY MEASURES FOR SIMPLE EXCLUSION IN DIMENSION $d \geq 5$

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We consider the symmetric simple exclusion process on $\mathbb{Z}^d$, for $d \geq 5$, and study the regularity of the quasi-stationary measures of the dynamics conditioned on not occupying the origin. For each $\rho \in]0, 1[$, we establish uniqueness of the density of quasi-stationary measures in $L^2(d\nu_\rho)$, where ν_ρ is the stationary measure of density ρ. This, in turn, permits us to obtain sharp estimates for $P_{\nu_\rho}(\tau > t)$, where τ is the first time the origin is occupied.

1. Introduction. Let $\{\eta_t : t \geq 0\}$ be the symmetric simple exclusion process on $\mathbb{Z}^d$. In this process, there is at most one particle per site (i.e., the state space is $\Omega := \{0, 1\}^{\mathbb{Z}^d}$), and at rate one the contents of neighboring sites are interchanged. The homogeneous Bernoulli product measures, say ν_ρ with density $\rho \in [0, 1]$, are invariant and reversible for this process. Let τ be the first time the origin of $\mathbb{Z}^d$ is occupied by a particle. We are interested in two issues: (i) to estimate the probability that the origin remains empty for large time when the initial configurations are drawn from ν_ρ for each $\rho \in]0, 1[$; (ii) to describe the law of η_t, at large time t, conditioned on $\{\tau > t\}$, the event that the origin is empty up to time t.

When the dimension of the lattice is larger than 4, we show that there exists a measure μ_ρ, such that for any continuous function f,

$$\lim_{t\to\infty} E_{\nu_\rho}[f(\eta_t) \mid \tau > t] = \int f \, d\mu_\rho. \tag{1.1}$$

This establishes the so-called Yaglom limit [14]. Such limiting measures can be intrinsically characterized as fixed points of the semi-groups $\{T_t, t > 0\}$ defined by

$$\big(T_t(\mu), f\big) := E_\mu[f(\eta_t) \mid \tau > t], \qquad t > 0. \tag{1.2}$$

Thus, fixed points of $\{T_t,\ t > 0\}$ are dubbed quasi-stationary measures [12, 6]. Here, we study the regularity of μ_ρ and uniqueness when the dimension $d > 4$. This, in turn, gives us sharp asymptotics for the probability of $P_{\nu_\rho}(\tau > t)$, namely

$$\lim_{t\to\infty} \frac{\exp(-\lambda(\rho).t)}{P_{\nu_\rho}(\tau > t)} = \int \left(\frac{d\mu_\rho}{d\nu_\rho}\right)^2 d\nu_\rho < \infty, \tag{1.3}$$

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where $-\lambda(\rho) < 0$ is the top of the spectrum in $L^2(\nu_\rho)$ of $\bar{L}$, the generator of the simple exclusion process absorbed when hitting $\{\eta : \eta_0 = 1\}$.

We briefly summarize some relevant results of [4]. In dimensions 1 and 2, the Yaglom limit is $\delta_{\underline{0}}$, the measure concentrated on the configuration with no particle (and $\lambda(\rho) = 0$). In dimensions 3 and 4, $\lambda(\rho) > 0$ for $\rho \in]0, 1[$, and $\int_0^t T_s(\nu_\rho)\, ds/t$ converges to a quasi-stationary measure μ_ρ. By analogy with the case of independent random walks [4], we conjecture that the Yaglom limit exists and that μ_ρ is singular with respect to ν_ρ. Thus, it is only for dimensions larger than 4 that we expect regularity of μ_ρ with respect to ν_ρ.

2. Notation and results.

Henceforth, we consider dimensions larger or equal to 5, and $\rho \in]0, 1[$. The symmetric simple exclusion process (SSEP) on the lattice $\mathbb{Z}^d$ can be graphically constructed "à la Harris" [7] as follows. First, fix the initial configuration by assigning to each site of $\mathbb{Z}^d$ a value in $\{0, 1\}$ which indicates if the site is occupied or empty. Then, to each bond—pairs of adjacent sites—associate a Poisson process of intensity 1; Poisson processes of different bonds are independent and independent of the initial configuration. At the time events (*marks*) of each Poisson process, the values of the corresponding sites are interchanged. In this way, each particle jumps when a mark is present; two particles may jump at the same time in opposite directions. By labeling particles, we can trace in time their trajectories: they evolve as the so-called *stirring particles*. This construction is described in Arratia [2]. When the labels of the stirring particles are disregarded one obtains only the occupation numbers; in this case the resulting process, called η_t, has infinitesimal generator

$$Lf(\eta) = \sum_{i \in \mathbb{Z}^d} \sum_{j : j \sim i} [f(\eta^{i,j}) - f(\eta)] \qquad \text{for } \eta \in \{0, 1\}^{\mathbb{Z}^d},$$

where $\eta_k^{i,j} = \eta_k + (\delta_{kj} - \delta_{ki})(\eta_i - \eta_j)$ and $i \sim j$ means that $|i_1 - j_1| + \cdots + |i_d - j_d| = 1$. It is well known that the process is Feller, and the product measures of density ρ in $[0, 1]$, say ν_ρ, are reversible for L (see Chapter VIII of Liggett [10]). In other words, L is an unbounded self-adjoint operator in $L^2(d\nu_\rho)$, and local functions form a core for the domain, say $D(L)$. We denote by P_{ν_ρ} the law of the SSEP with initial measure ν_ρ. Let $\mathcal{A} = \{\eta : \eta_0 = 1\}$ and denote by τ the time of first occurrence of $\mathcal{A}$. As we are interested in the Dirichlet problem on $\mathcal{A}^c$, we introduce $\mathcal{H}_{\mathcal{A}} = \{\varphi \in L^2(\nu_\rho) : \varphi(\eta) = 0 \text{ for } \eta \in \mathcal{A}\}$. Let $\bar{L}$ be the operator defined by

$$\bar{L}f = \mathbb{1}_{\mathcal{A}^c} Lf \qquad \text{for } f \in D(L) \cap \mathcal{H}_{\mathcal{A}}.$$

This corresponds to the simple exclusion dynamics absorbed when hitting the event A. $\bar{L}$ is self-adjoint on $\mathcal{H}_{\mathcal{A}}$ with respect to ν_ρ. We call $\{\bar{S}_t,\ t > 0\}$ the corresponding sub-Markovian semi-group of bounded operators on $L^2(\mathcal{H}_{\mathcal{A}}, \nu_\rho)$. In other words,

$$\forall t > 0, \qquad \bar{S}_t f(\eta) = E^\eta[f(\eta_t)\mathbb{1}_{\{\tau > t\}}].$$

We denote by $T_t(\nu_\rho)$ the probability measure defined by duality on $\varphi \in \mathcal{H}_{\mathcal{A}}$,

$$(T_t(\nu_\rho), \varphi) = \frac{\int \bar{S}_t \varphi \mathbb{1}_{\mathcal{A}^c} \, d\nu_\rho}{P_{\nu_\rho}(\tau > t)} = \int \varphi \frac{\bar{S}_t \mathbb{1}_{\mathcal{A}^c}}{P_{\nu_\rho}(\tau > t)} \, d\nu_\rho.$$

where we have used reversibility to obtain the third term. Thus, if f_t is the density of $T_t(\nu_\rho)$ with respect to ν_ρ

$$\forall t > 0, \qquad f_t(\eta) = \frac{\bar{S}_t \mathbb{1}_{\mathcal{A}^c}(\eta)}{P_{\nu_\rho}(\tau > t)} = \frac{P^\eta(\tau > t)}{\int P^\zeta(\tau > t) \, d\nu_\rho(\zeta)}. \tag{2.1}$$

It was established in [4] that a nontrivial quasistationary measure, say μ_ρ, could be obtained as limit along a Cesàro subsequence of $T_t(\nu_\rho)$. Our main result is the following theorem.

THEOREM 2.1. *If the dimension is larger or equal to* 5, *then* μ_ρ *is absolutely continuous with respect to* ν_ρ. *Moreover, for any integer* $k \geq 1$, $d\mu_\rho / d\nu_\rho \in L^k(\nu_\rho)$ *and*

$$\lim_{t \to \infty} \int \left(\frac{dT_t(\nu_\rho)}{d\nu_\rho} - \frac{d\mu_\rho}{d\nu_\rho} \right)^2 d\nu_\rho = 0. \tag{2.2}$$

REMARK 2.2. This is stronger than establishing the Yaglom limit, that is, $\lim T_t(\nu_\rho) = \mu_\rho$. As a consequence, $f := d\mu_\rho / d\nu_\rho$ belongs to $D(\bar{L})$ and satisfies [in the $L^2(\nu_\rho)$-sense]

$$\bar{L} f + \lambda(\rho) f = 0 \quad \text{and} \quad \bar{S}_t f = e^{-\lambda(\rho) t} f, \tag{2.3}$$

with (see Theorem 2 of [4])

$$\lambda(\rho) = \inf \left\{ \frac{(f, -\bar{L} f)_{\nu_\rho}}{(f, f)_{\nu_\rho}} : f \in D(L) \cap \mathcal{H}_{\mathcal{A}} \right\}. \tag{2.4}$$

Theorem 2.1 is based on a priori bounds through the following lemma, in which we reformulate a general result essentially contained in [4].

LEMMA 2.3. *Let* $\{\bar{S}_t\}$ *be the semi-group of a process absorbed when hitting a set* $\mathcal{A}$. *Assume that* $\{\bar{S}_t\}$ *is reversible with respect to* ν, *and let* $\lambda < \infty$ *be given by* (2.4). *The following two conditions are equivalent*:

$$\text{(i)} \ \sup_{t > 0} \frac{e^{-\lambda t}}{P_\nu(\tau > t)} < \infty \quad \textit{and} \quad \text{(ii)} \ \sup_{t > 0} \int f_t^2 \, d\nu < \infty. \tag{2.5}$$

Moreover, *if either* (i) *or* (ii) *holds*, *then the Yaglom limit* μ *exists and* (2.2) *holds*.

Now, the a priori bounds are a corollary of the following proposition, interesting on its own.

PROPOSITION 2.4. *Let the dimension $d \geq 3$. Let $i \in \mathbb{Z}^d \setminus \{0\}$ and $\eta \in \Omega$ with $\eta_i = 0$. We denote by η^i the configuration identical to η except on i, where its value is* 1. *There is a constant C_d, independent of i and η such that for any $t > 0$,*

$$0 \leq P^{\eta}(\tau > t) - P^{\eta^i}(\tau > t) \leq C_d P^{\eta^i}(\tau > t)\mathbb{P}(H_i < \infty), \tag{2.6}$$

where H_i denotes the first time a symmetric random walk starting at i hits the origin.

Relation (2.6) would be obvious if the particles were independent. Though it is rather intuitive for the symmetric exclusion, our proof is rather long. A sketch of it is as follows. We first write $P^{\eta}(\tau > t)$ in terms of a dual process, say $\{X(\varnothing, t)\}$, which corresponds to a stirring process on $\mathbb{Z}^d \setminus \{0\}$ with birth at the nearest neighbors of the origin and with initial condition an empty configuration. Then, $P^{\eta}(\tau > t) - P^{\eta^i}(\tau > t)$ corresponds to the weight of all paths whose end-points $X(\varnothing, t) = U \cup \{i\}$ with $\eta_j = 0$ for all $j \in U \subset \mathbb{Z}^d \setminus \{0, i\}$. The problem is then to uncouple U from $\{i\}$. We then re-express $P(X(\varnothing, t) = U \cup \{i\})$ in terms of a dual *with finitely many particles*, say $\{\Lambda_t\}$. Note that $\{\Lambda_t\}$ is not the "natural dual" of $\{X(\varnothing, t)\}$ and the correspondence is obtained through a Feynman–Kac formula. Then we show a correlation inequality for the expression in terms of $\{\Lambda_t\}$ by generalizing Andjel's inequality [1].

The results about a priori bounds are the following.

COROLLARY 2.5. *Let the dimension $d \geq 5$.* (i) *There is a product measure $\nu_{\alpha(.)}$ of density $\alpha(i)$, for $i \in \mathbb{Z}^d$, such that, for any $t > 0$,*

$$\nu_{\alpha(.)} \prec T_t(\nu_\rho) \prec \nu_\rho \quad \textit{and} \quad \sum_{i \in \mathbb{Z}^d} \left(1 - \frac{\alpha_i}{\rho}\right)^2 < \infty,$$

where $\prec$ denotes stochastic domination.

(ii) *For any integer $k \geq 1$, there is a positive constant C, such that*

$$\sup_{t>0} \int f_t^k(\eta)\, d\nu_\rho(\eta) \leq C. \tag{2.7}$$

A consequence of Lemma 2.3 is a sharp asymptotic estimate for the tail of τ (compare with [4], Lemma 1).

COROLLARY 2.6. *If the dimension $d \geq 5$, then*

$$\lim_{t \to \infty} \frac{e^{-\lambda(\rho)t}}{P_{\nu_\rho}(\tau > t)} = \int f^2\, d\nu_\rho. \tag{2.8}$$

Finally, we have a uniqueness result and some properties of μ_ρ.

THEOREM 2.7. (i) *The set* $\{\mu \ll \nu_\rho : \mu$ *is quasi stationary and* $d\mu/d\nu_\rho \in L^2(\nu_\rho)\}$ *contains only* μ_ρ.

(ii) *For* $i \in \mathbb{Z}^d$, *define* $\theta_i\colon\ \Omega \to \Omega$ *with* $\theta_i \eta_k = \eta_{k+i}$, *then for any* φ *local* (*i.e., depending on finitely many sites*),

$$\lim_{\|i\|\to\infty} \int \varphi(\theta_i \eta)\, d\mu_\rho(\eta) = \int \varphi\, d\nu_\rho.$$

(iii) *If* ν *is a probability with a continuous density with respect to* ν_ρ, *then*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t T_s(\nu)\, ds = \mu_\rho.$$

The convergence holds in weak-$L^2(\nu_\rho)$.

Proposition 2.4 is proven in Section 3. Section 4 contains the proofs of Corollary 2.5, and of Theorem 2.1. In Section 5, we establish the uniqueness part of Theorem 2.7. In Section 6, we show that in μ_ρ the density at infinity is ρ, and we conclude with the result about the basin of attraction of μ_ρ.

3. Proof of Proposition 2.4.

3.1. *Duality and Feynman–Kac.* We first express $P^\eta(\tau > t)$ using the *dual* process [10, 2] based on the fact that the Poisson clocks associated to bonds are invariant by time reflections. The dual process tracing back-in-time the positions of the stirring particles touching the origin can be described using the graphical construction at the beginning of Section 2. Again at each bond, there is an independent mark process corresponding to the realization of a Poisson process of intensity 1. At each mark between zero and one of its nearest neighbor, say i, a particle is born at i unless i is already occupied (in which case nothing happens); the particles born in this way evolve afterwards as stirring particles on $\mathbb{Z}^d \setminus \{0\}$. The only difference with the previous construction is that now it is imposed to the origin to be occupied at all times—so that when it becomes empty, it is immediately occupied with a newly created particle. Assume that at time 0, the lattice is empty and let $X(\varnothing, t)$ be the set of sites occupied by the stirring particles at time t; all these particles have been created at the origin. Let $\mathbb{P}$ denote averages over the Poisson realizations. If $\mathcal{P}^*$ is the collection of finite subsets of $\mathbb{Z}^d \setminus \{0\}$, then the duality formula reads, for any $t > 0$,

$$P^\eta(\tau > t) = (1 - \eta_0) \sum_{\Lambda \in \mathcal{P}^*} \mathbb{P}\big(X(\varnothing, t) = \Lambda\big) \prod_{j \in \Lambda} (1 - \eta_j). \tag{3.1}$$

Thus, if η is such that $\eta_i = 0$,

$$P^\eta(\tau > t) - P^{\eta^i}(\tau > t) = (1 - \eta_0) \sum_{\Lambda \in \mathcal{P}^*,\ \Lambda \ni i} \mathbb{P}\big(X(\varnothing, t) = \Lambda\big) \prod_{j \in \Lambda} (1 - \eta_j).$$

Assume, for a moment, that for $U \in \mathcal{P}^*$ and $i \notin U$, we have a constant C_d independent of i and t such that

$$\mathbb{P}\big(X(\varnothing,t)=U\cup\{i\}\big)\le C_d\mathbb{P}\big(X(\varnothing,t)=U\big)\mathbb{P}(H_i<\infty), \tag{3.2}$$

where H_i denotes the first time a symmetric random walk starting at i hits the origin. Then, for η such that $\eta_i=0$,

$$\begin{aligned}
&P^{\eta}(\tau>t)-P^{\eta^i}(\tau>t)\\
&\quad\le C_d\mathbb{P}(H_i<\infty)(1-\eta_0)\sum_{U\in\mathcal{P}^*,i\notin U}\mathbb{P}\big(X(\varnothing,t)=U\big)\prod_{j\in U\cup\{i\}}(1-\eta_j)\\
&\quad\le C_d(1-\eta_i)\mathbb{P}(H_i<\infty)P^{\eta^i}(\tau>t).
\end{aligned}$$

Thus, it remains to prove (3.2).

Let $\mathcal{L}^+$ be the generator of $\{X(\varnothing,t),\ t\ge 0\}$, and let S_t^+ be the associated semi-group. We first express the dual of $\{X(\varnothing,t),\ t\ge 0\}$ in terms of a process with finitely many particles. Actually, we are only interested in $S_t^+(\mathbb{1}_\Lambda)(\varnothing):=\mathbb{P}(X(\varnothing,t)=\Lambda)$ for $\Lambda\in\mathcal{P}^*$. Let Λ and A be in $\mathcal{P}^*$. We have, using Δ for the symmetric difference,

$$\mathcal{L}^+(\mathbb{1}_\Lambda)(A)=\sum_{\substack{x\sim y;\ x,y\neq\{0\}\\ |A\Delta\{x,y\}|=|A|}}[\mathbb{1}_\Lambda(A\Delta\{x,y\})-\mathbb{1}_\Lambda(A)]+\sum_{\substack{y\sim 0\\ y\notin A}}[\mathbb{1}_\Lambda(A\cup\{y\})-\mathbb{1}_\Lambda(A)].$$

The first sum corresponds to the stirring process over $\mathbb{Z}^d\setminus\{0\}$, while the second sum corresponds to birth at the origin. We reexpress now the last sum. For simplicity, we omit to write $y\sim 0$. Thus,

$$\sum_{y\notin A}[\mathbb{1}_\Lambda(A\cup\{y\})-\mathbb{1}_\Lambda(A)]=\sum_{\substack{y\notin A\\ y\in\Lambda}}[\mathbb{1}_\Lambda(A\cup\{y\})-\mathbb{1}_\Lambda(A)]-\sum_{\substack{y\notin A\\ y\notin\Lambda}}\mathbb{1}_A(\Lambda).$$

We claim that this expression is equal to

$$\mathcal{C}:=\sum_{y\in\Lambda}[\mathbb{1}_A(\Lambda\setminus\{y\})-\mathbb{1}_A(\Lambda)]-\sum_{y\notin\Lambda}\mathbb{1}_A(\Lambda)+\sum_{y\in\Lambda}\mathbb{1}_A(\Lambda).$$

Indeed, we expand $\mathcal{C}$:

$$\begin{aligned}
\mathcal{C}&=\sum_{\substack{y\in\Lambda\\ y\notin A}}[\mathbb{1}_A(\Lambda\setminus\{y\})-\mathbb{1}_A(\Lambda)]+\sum_{\substack{y\in\Lambda\\ y\in A}}[\mathbb{1}_A(\Lambda\setminus\{y\})-\mathbb{1}_A(\Lambda)]\\
&\quad-\sum_{y\notin\Lambda}\mathbb{1}_A(\Lambda)+\sum_{y\in\Lambda}\mathbb{1}_A(\Lambda)\\
&=\sum_{\substack{y\notin A\\ y\in\Lambda}}[\mathbb{1}_A(\Lambda\setminus\{y\})-\mathbb{1}_A(\Lambda)]-\sum_{y\notin\Lambda}\mathbb{1}_A(\Lambda).
\end{aligned}$$

Thus, calling $\mathcal{N}_0 := \{y \in \mathbb{Z}^d : y \sim 0\}$ and using the self-duality of the stirring part of $\mathcal{L}^+$, we obtain

$$\begin{aligned}\mathcal{L}^+(\mathbb{1}_\Lambda)(A) = &\sum_{\substack{x\sim y;\ x,y\neq 0\\ |\Lambda\Delta\{x,y\}|=|\Lambda|}} [\mathbb{1}_A(\Lambda\Delta\{x,y\}) - \mathbb{1}_A(\Lambda)] \\ &+ \sum_{\substack{y\sim 0\\ y\in\Lambda}} [\mathbb{1}_A(\Lambda\setminus\{y\}) - \mathbb{1}_A(\Lambda)] + V(\Lambda)\mathbb{1}_A(\Lambda),\end{aligned} \tag{3.3}$$

where we set $V(\Lambda) := 2|\mathcal{N}_0 \cap \Lambda| - |\mathcal{N}_0|$. Now let $\mathcal{L}^-$ denote the generator of the stirring process on $\mathbb{Z}^d \setminus \{0\}$ with death when particles jump on the origin. Then, (3.3) can be written as

$$\mathcal{L}^+(\mathbb{1}_\Lambda)(A) = \mathcal{L}^-(\mathbb{1}_A)(\Lambda) + V(\Lambda)\mathbb{1}_A(\Lambda).$$

Thus, if $u(\Lambda, t) := S_t^+(\mathbb{1}_\Lambda)(\varnothing)$, we have

$$\frac{du(\Lambda,t)}{dt} = S_t^+(\mathcal{L}^+\mathbb{1}_\Lambda)(\varnothing) = \mathcal{L}^- u(\Lambda,t) + V(\Lambda)u(\Lambda,t).$$

Now we call $\{\Lambda(t),\ t \geq 0\}$ the process generated by $\mathcal{L}^-$ and we use Feynman–Kac to obtain

$$\begin{aligned}\mathbb{P}\big(X(\varnothing,t) = \Lambda\big) &= \mathbb{E}^\Lambda\bigg[\exp\bigg\{\int_0^t V\big(\Lambda(s)\big)\,ds\bigg\}\mathbb{1}_\varnothing\big(\Lambda(t)\big)\bigg] \\ &= \exp\{-|\mathcal{N}_0|t\}\mathbb{E}^\Lambda\bigg[\exp\bigg(2\int_0^t |\mathcal{N}_0 \cap \Lambda(s)|\,ds\bigg)\mathbb{1}_\varnothing\big(\Lambda(t)\big)\bigg].\end{aligned}$$

Now let $U \in \mathcal{P}^*$ and $i \notin U$. We show in Section 3.2 that if $\Lambda \in \mathcal{P}^*$ and

$$g(\Lambda,t) := \mathbb{E}^\Lambda\bigg[\exp\bigg(2\int_0^t |\mathcal{N}_0 \cap \Lambda(s)|\,ds\bigg)\mathbb{1}_\varnothing\big(\Lambda(t)\big)\bigg] \tag{3.4}$$

then $g(U \cup \{i\}, t) \leq g(U,t)g(\{i\},t)$. Then, in Section 3.3 we prove that $g(\{i\},t) \leq C_d P(H_i < \infty)$ for $d \geq 3$. Inequality (3.2) follows then readily.

3.2. *A generalized correlation inequality.* To make the notation closer to those of Andjel [1], we set $p(x,y) = 1$ when $x \sim y$, and $p(x,y) = 0$ otherwise. Also, we realize our stirring process as an exclusion process: the particles attempt to jump to one of their nearest neighboring sites at the time marks of independent Poisson processes of intensity $2d$; if the site chosen (each neighboring site is chosen with the same probability) for the attempt is occupied, the particle stays still. As we are blind to the labelling of particles, the trajectories are, in law, indistinguishable from our initial stirring process.

We proceed by induction on n to prove that for any sets $A, B \in \mathcal{P}^*$, with $A \cap B = \varnothing$, for any $t > 0$, $\alpha \in \mathbb{R}$, and any n-tuples $0 \le s^1 < s^2 < \cdots < s^n \le t$, the following inequality holds:

$$\begin{aligned} &\mathbb{E}^{A\cup B}\left[\exp\left\{\alpha \sum_{k=1}^{n} |\mathcal{N}_0 \cap \Lambda(s^k)|\right\} \mathbb{1}_{\varnothing}\big(\Lambda(t)\big)\right] \\ &\quad \le \mathbb{E}^{A}\left[\exp\left\{\alpha \sum_{k=1}^{n} |\mathcal{N}_0 \cap \Lambda(s^k)|\right\} \mathbb{1}_{\varnothing}\big(\Lambda(t)\big)\right] \\ &\quad\quad \times \mathbb{E}^{B}\left[\exp\left\{\alpha \sum_{k=1}^{n} |\mathcal{N}_0 \cap \Lambda(s^k)|\right\} \mathbb{1}_{\varnothing}\big(\Lambda(t)\big)\right]. \end{aligned} \tag{3.5}$$

This will be our induction hypothesis at order n. Once (3.2) is proved, inequality (3.5) follows easily as in Proposition 4.1 of [8].

Step $n = 0$. We need to prove that for $A, B \in \mathcal{P}^*$ with $A \cap B = \varnothing$

$$\mathbb{P}^{A\cup B}\big(\Lambda(t) = \varnothing\big) \le \mathbb{P}^{A}\big(\Lambda(t) = \varnothing\big)\mathbb{P}^{B}\big(\Lambda(t) = \varnothing\big). \tag{3.6}$$

Following an idea of Arratia [3], we represent the process $\Lambda(t)$ as limit of a stirring process with no absorption on an enlarged lattice: we link the origin 0 with $\tilde{0}$, the origin of a three dimensional lattice $\tilde{\mathbb{Z}}^3$ isomorphic to $\mathbb{Z}^3$ (here we fix $\mathbb{Z}^3$ for concreteness; any graph supporting the stirring construction, for which the corresponding random walk is transient would fit). On each bond of $\tilde{\mathbb{Z}}^3$ and on the bond $(0, \tilde{0})$, the rates of stirring are set equal to κ large. On the enlarged lattice $\mathbb{Z}^d \cup \tilde{\mathbb{Z}}^3$, the particles perform a conservative stirring process, though with different rates whether they jump across the bonds of $\mathbb{Z}^d$ or across the bonds of $\tilde{\mathbb{Z}}^3$ and $(0, \tilde{0})$. When a particle hits the origin 0, it has a probability going to 1 as $\kappa \to \infty$ to wander in $\tilde{\mathbb{Z}}^3$ up to time t without using bonds of $\mathbb{Z}^d$. We call $U(t)$ the stirring process on $\mathbb{Z}^d \cup \tilde{\mathbb{Z}}^3$, and $\mathbb{P}_\kappa$ the law of the Poisson marks on the enlarged lattice. It is not difficult to show that for any $\Lambda \in \mathcal{P}^*$,

$$\mathbb{P}^{\Lambda}\big(\Lambda(t) = \varnothing\big) = \lim_{\kappa\to\infty} \mathbb{P}_\kappa^{\Lambda}\big(U(t) \subset \tilde{\mathbb{Z}}^3\big). \tag{3.7}$$

Now, for the stirring process on the enlarged lattice $\mathbb{Z}^d \cup \tilde{\mathbb{Z}}^3$, we use a correlation inequality due to Andjel [1]:

$$\mathbb{P}_\kappa^{A\cup B}\big(U(t) \subset \tilde{\mathbb{Z}}^3\big) \le \mathbb{P}_\kappa^{A}\big(U(t) \subset \tilde{\mathbb{Z}}^3\big)\,\mathbb{P}_\kappa^{B}\big(U(t) \subset \tilde{\mathbb{Z}}^3\big). \tag{3.8}$$

Thus, after taking the limit $\kappa \to \infty$, we obtain (3.6).

Step n. Our proof follows essentially Andjel's proof. Our induction hypothesis is that (3.5) is valid for $n-1$ instants of time. Let $0 \le s^1 < s^2 < \cdots < s^n \le t$ be n instants of time, and for each $\Lambda \in \mathcal{P}^*$ let

$$g_n(\Lambda, t; s^1, \dots, s^n) = \mathbb{E}^{\Lambda}\left[\exp\left(\alpha \sum_{i=1}^{n} |\mathcal{N}_0 \cap \Lambda(s^i)|\right) \mathbb{1}_{\varnothing}\big(\Lambda(t)\big)\right].$$

We set $\lambda = 2d(|A|+|B|)$ and we let τ_1 be the first time a particle of $A \cup B$ attempts a jump (i.e., τ_1 is an exponential time of parameter λ). Note that by the Markov property

$$\begin{aligned}
&\mathbb{E}^{A\cup B}\left[\exp\left(\alpha \sum_{i=1}^{n} |\mathcal{N}_0 \cap \Lambda(s^i)|\right) \mathbb{1}_{\varnothing}\big(\Lambda(t)\big)\mathbb{1}_{\{\tau_1 > s^1\}}\right] \\
&\quad = P(\tau_1 > s^1) \exp\{\alpha(|A \cap \mathcal{N}_0| + |B \cap \mathcal{N}_0|)\} \\
&\qquad \times \mathbb{E}^{A\cup B}\left[\exp\left(\alpha \sum_{i=2}^{n} |\mathcal{N}_0 \cap \Lambda(s^i - s^1)|\right) \mathbb{1}_{\varnothing}\big(\Lambda(t - s^1)\big)\right] \\
&\quad = P(\tau_1 > s^1) \exp\{\alpha(|A \cap \mathcal{N}_0| + |B \cap \mathcal{N}_0|)\} \\
&\qquad \times g_{n-1}(A \cup B, t - s^1; s^2 - s^1, \dots, s^n - s^1).
\end{aligned}$$

Following [1], using the shorthand notation $\mathbf{s}$ for $s^1, \dots, s^n$ [and its abuse $\mathbf{s} - \mathbf{u} = (s^1 - u, \dots, s^n - u)$], and writing $A\tilde{\Delta}\{x, y\}$ to mean $A\Delta\{x, y\} \setminus \{0\}$, for we have to account for deaths of particles when they jump on 0, we have

$$\begin{aligned}
&g_n(A \cup B, t; \mathbf{s}) \\
&\quad = P(\tau_1 > s^1) e^{\alpha|A\cap\mathcal{N}_0|} e^{\alpha|B\cap\mathcal{N}_0|} g_{n-1}(A \cup B, t - s^1; s^2 - s^1, \dots, s^n - s^1) \\
&\qquad + \int_0^{s^1} du\, \lambda e^{-\lambda u} \frac{1}{\lambda} \Bigg\{ \left[\sum_{x,y\in A} p(x, y) + \sum_{x,y\in B} p(x, y)\right] \\
&\qquad\qquad \times g_n(A \cup B, t - u; \mathbf{s} - \mathbf{u}) \\
&\qquad\qquad + \left[\sum_{\substack{x\in A \\ y\notin A\cup B}} p(x, y) g_n(A\tilde{\Delta}\{x, y\} \cup B, t - u; \mathbf{s} - \mathbf{u})\right] \\
&\qquad\qquad + \left[\sum_{\substack{x\in B \\ y\notin A\cup B}} p(x, y) g_n(A \cup B\tilde{\Delta}\{x, y\}, t - u; \mathbf{s} - \mathbf{u})\right]
\end{aligned}$$

$$+ \left[\sum_{\substack{x \in A \\ y \in B}} p(x, y) + \sum_{\substack{x \in B \\ y \in A}} p(x, y) \right]$$
$$\times\, g_n(A \cup B, t-u; \mathbf{s}-\mathbf{u}) \Bigg\}$$

Reasoning as if the particles in A were independent from the particles in B, we obtain

$$\begin{aligned}
&g_n(A, t; \mathbf{s}) g_n(B, t; \mathbf{s}) \\
&\quad = P(\tau_1 > s_1) e^{\alpha|A \cap \mathcal{N}_0|} e^{\alpha|B \cap \mathcal{N}_0|} g_{n-1}(A, t - s^1; s^2 - s^1, \ldots) \\
&\qquad \times g_{n-1}(B, t - s^1; s^2 - s^1, \ldots) \\
&\qquad + \int_0^{s^1} du\, \lambda e^{-\lambda u} \frac{1}{\lambda} \Bigg\{ \left[\sum_{x, y \in A} p(x, y) + \sum_{x, y \in B} p(x, y) \right] \\
&\qquad\qquad \times\, g_n(A, t-u; \mathbf{s}-\mathbf{u}) g_n(B, t-u; \mathbf{s}-\mathbf{u}) \\
&\qquad + \left[\sum_{\substack{x \in A \\ y \notin A \cup B}} p(x, y) g_n(A \tilde{\Delta} \{x, y\}, t-u; \mathbf{s}-\mathbf{u}) g_n(B, t-u; \mathbf{s}-\mathbf{u}) \right] \\
&\qquad + \left[\sum_{\substack{x \in B \\ y \notin A \cup B}} p(x, y) g_n(A, t-u; \mathbf{s}-\mathbf{u}) g_n(B \tilde{\Delta} \{x, y\}, t-u; \mathbf{s}-\mathbf{u}) \right] \\
&\qquad + \left[\sum_{\substack{x \in A \\ y \in B}} p(x, y) g_n(A \Delta \{x, y\}, t-u; \mathbf{s}-\mathbf{u}) g_n(B, t-u; \mathbf{s}-\mathbf{u}) \right] \\
&\qquad + \left[\sum_{\substack{x \in B \\ y \in A}} p(x, y) g_n(A, t-u; \mathbf{s}-\mathbf{u}) g_n(B \Delta \{x, y\}, t-u; \mathbf{s}-\mathbf{u}) \right] \Bigg\}.
\end{aligned}$$

Define

$$G_n(t) = \sup_{0 \le s^1 < \cdots < s^n \le t} \sup_{\substack{C \cap D = \varnothing \\ C, D \in \mathcal{P}^*}} g_n(C \cup D, t; \mathbf{s}) - g_n(C, t; \mathbf{s}) g_n(D, t; \mathbf{s}),$$

also set $F_n(t) = \sup\{G_n(s) : 0 \le s \le t\}$. Now the key observation of Andjel ([1], page 720) is that for $x \in A$ and $y \in B$ (so that both $x, y \neq 0$),

$$\begin{aligned}
g_n(A \cup B, t; \mathbf{s}) \le F_n(t) + \tfrac{1}{2} \big(& g_n(A, t; \mathbf{s}) g_n(B \Delta \{x, y\}, t; \mathbf{s}) \\
& + g_n(A \Delta \{x, y\}, t; \mathbf{s}) g_n(B, t; \mathbf{s}) \big).
\end{aligned}$$

Thus, using the induction hypothesis ($F_{n-1}=0$) and the symmetry of $p(.,.)$, we obtain

$$
\begin{aligned}
&g_n(A\cup B,t;\mathbf{s})-g_n(A,t;\mathbf{s})g_n(B,t;\mathbf{s})\\
&\quad\le\int_0^{s^1}e^{-\lambda u}\Bigg[\sum_{x,y\in A}p(x,y)+\sum_{x,y\in B}p(x,y)+\sum_{\substack{x\in A\\ y\notin A\cup B}}p(x,y)\\
&\qquad\qquad+\sum_{\substack{x\in B\\ y\notin A\cup B}}p(x,y)+2\sum_{\substack{x\in A\\ y\in B}}p(x,y)\Bigg]F_n(t-u)\,du\\
&\quad=\int_0^{s^1}e^{-\lambda u}\lambda F_n(t-u)\,du\le F_n(t)\int_0^t\lambda e^{-\lambda u}\,du.
\end{aligned}
$$

Thus, by taking the supremum over $A,B\in\mathcal{P}^*$ with $A\cap B=\varnothing$, we obtain $G_n(t)\le F_n(t)\int_0^t\lambda\exp(-\lambda u)\,du$. Finally, this implies that $F_n(t)=0$, and the proof is complete. $\square$

3.3. *Upper bound* $g(\{i\},t)\le C_d\mathbb{P}(H_i<\infty)$. We first use the classical representation of the trajectories $\{\Lambda(\{i\},t),\ t>0\}$ as sequences of jump times $\{\tau_i,\ i\in\mathbb{N}\}$, which are independent exponential variables of parameter $2d$, associated with the paths of a simple symmetric random walk killed at the origin, say $\{\Lambda_i,\ i\in\mathbb{N}\}$. The processes $\{\Lambda_i,\ i\in\mathbb{N}\}$ and $\{\tau_j,\ j\in\mathbb{N}\}$ are independent. We use the notation E^y and P^y to denote average over paths of $\{\Lambda_i,\ i\in\mathbb{N}\}$ starting on y. Let $T_0=\inf\{n>0:\Lambda_n\in\mathcal{N}_0\}$ with $\mathcal{N}_0=\{y:y\sim 0\}$. When $T_0<\infty$, let T_1 be the first return time to $\mathcal{N}_0$, whereas when $T_0=\infty$, set $T_1=\infty$. Then, the sequence of successive entrance times in $\mathcal{N}_0$, $\{T_2,T_3,\dots\}$ is defined inductively. The number of return to $\mathcal{N}_0$ is called R, i.e., if $T_i=\infty$ but $T_{i-1}<\infty$ then $R=i$. Our walk on $\mathbb{Z}^d_*$ being transient, we have $R<\infty$, a.s. We note also, that by symmetry, for any $y,y'\in\mathcal{N}_0$,

$$P^y(T_0<\infty)=P^{y'}(T_0<\infty)\quad\text{and}\quad\forall k\in\mathbb{N},\qquad P^y(R=k)=P^{y'}(R=k).$$

For convenience, we call $P^{\mathcal{N}_0}(T_0<\infty):=P^y(T_0<\infty)$ and $P^{\mathcal{N}_0}(R=k)=P^y(R=k)$ for $y\in\mathcal{N}_0$. Now,

$$
\begin{aligned}
g(\{i\},t)&\le E^i\left[\mathbb{1}_{\{T_0<\infty\}}\exp\left(2\sum_{i=0}^{R}\tau_{T_i}\right)\right]\\
&=\sum_{k=0}^{\infty}E^i\left[\mathbb{1}_{\{T_0<\infty\}}\mathbb{1}_{\{R=k\}}\exp\left(2\sum_{i=0}^{k}\tau_{T_i}\right)\right]\qquad(3.9)\\
&=P^i(T_0<\infty)\sum_{k=0}^{\infty}P^{\mathcal{N}_0}(R=k)(E[\exp\{2\tau_1\}])^k,
\end{aligned}
$$

where we used the Markov property and induction. Now, by the same arguments,

$$P^{\mathcal{N}_0}(R=k) \le \big(P^{\mathcal{N}_0}(T_0<\infty)\big)^k.$$

On the other hand, the evaluation of $E[\exp(2\tau_1)]$ is easy:

$$E[e^{2\tau_1}] = \int_0^\infty e^{2t} 2de^{-2dt}\,dt = \frac{d}{d-1}. \tag{3.10}$$

Thus, with (3.9) and (3.10), our upper bound follows easily as soon as

$$P^{\mathcal{N}_0}(T_0<\infty) < \frac{d-1}{d}. \tag{3.11}$$

We want to formulate (3.7) in terms of hitting probabilities for the standard random walk, say $\{S_n,\ n\ge 0\}$. We will denote the averages over the standard walk with a tilde. Let $\kappa = \inf\{n>0 : S_n = 0\}$, and note first that

$$P^{\mathcal{N}_0}(T_0<\infty) = \tilde{P}^{\mathcal{N}_0}(T_0<\infty,\ \kappa > T_0).$$

By conditioning on the first move, we obtain

$$\tilde{P}^{\mathcal{N}_0}(T_0<\infty) = \frac{1}{2d} + \tilde{P}^{\mathcal{N}_0}(T_0<\infty,\ \kappa > T_0).$$

Thus, (3.11) is equivalent to $\tilde{P}^{\mathcal{N}_0}(T_0<\infty) < (2d-1)/2d$. We recall that R is the number of return to $\mathcal{N}_0$ for a walk starting on $\mathcal{N}_0$. We can set $S_0=0$ but count what happens only after two steps,

$$R = \sum_{n=2}^{\infty} \mathbb{1}_{\{S_n\in\mathcal{N}_0\}} = \sum_{n\ge 1} \mathbb{1}_{\{T_n<\infty\}}, \tag{3.12}$$

so that

$$\tilde{E}[R] = \sum_{n=2}^{\infty} \tilde{P}(S_n\in\mathcal{N}_0) = \frac{\tilde{P}^{\mathcal{N}_0}(T_0<\infty)}{1-\tilde{P}^{\mathcal{N}_0}(T_0<\infty)}.$$

Thus, (3.11) reads $\tilde{E}[R] < 2d-1$. Now we note that for $n>0$,

$$\begin{aligned}\tilde{P}(S_{n+1}=0) &= \tilde{P}(S_{n+1}=0,\ S_n=y,\ y\in\mathcal{N}_0)\\ &= \sum_{y\in\mathcal{N}_0} \tilde{P}(S_{n+1}=0 \mid S_n=y)\tilde{P}(S_n=y) = \frac{1}{2d}\tilde{P}(S_n\in\mathcal{N}_0).\end{aligned} \tag{3.13}$$

Thus,

$$\sum_{n=2}^{\infty} \tilde{P}(S_n\in\mathcal{N}_0) = 2d\sum_{n=3}^{\infty} \tilde{P}(S_n=0).$$

In dimension 3, it has been established (see [5], page 170, exercise 2.7) that

$$1 + \sum_{n=1}^{\infty} \tilde{P}(S_n = 0) = (\sqrt{6}/32\pi^3)\Gamma(1/24)\Gamma(5/24)\Gamma(6/24)\Gamma(11/24) = 1.516\dots.$$

Thus, in dimension 3, $\tilde{E}[R] \le 6(0.52) < 5$ and condition (3.11) holds. Thus, in dimension 3, $\tilde{E}[R] \le 6(0.52) < 5$ and condition (3.11) holds. We conclude that (3.11) holds for any dimension larger or equal to 3 because the right-hand side of $\tilde{E}[R] < 2d - 1$ increases and the average number of visits to 0 decreases with dimensions. As we have not found a reference to this latter fact, we present a short proof due to Andjel. The number of visits to 0 is a geometric random variable of parameter $P_0(T_0 < \infty)$, so it suffices to show monotonicity for this quantity. Let i be any neighbor of 0; by symmetry $a(d) := P_0(T_0 < \infty) = P_i(T_0 < \infty)$, for dimension $d \ge 3$. Project the d-dimensional walk on a hyperplane orthogonal to i', a neighbor of the origin different from i. The projected walk on Z^{d-1} has transition probabilities $1/2d$ to go to each of its $2(d-1)$ neighbors and $1/d$ not to move. It is clear that $a(d)$ is not larger than the probability of visiting the origin starting at (the projection of) i for the projected walk. This latter probability is actually equal to $a(d-1)$. Indeed, the projected process goes along the same trajectories as the $(d-1)$-dimensional standard walk and the waiting times at each point are geometric random variables with parameter $(d-1)/d$: thus, if the trajectory of the standard walk is such that $\{T < \infty\}$, then the same holds for the projected walk and vice versa.

4. A priori bounds.

PROOF OF COROLLARY 2.5. (i) The proof proceeds along the same lines as the proof of Theorem 3c of [4], once the measure $\nu_{\alpha(.)}$ is defined. We set $\alpha_0 = 0$, and for $i \neq 0$, let α_i be defined by

$$\frac{\alpha_i}{1-\alpha_i}\frac{1-\rho}{\rho} = \frac{1}{1 + C_d \mathbb{P}(H_i < \infty)},$$

where the constant C_d is that of (2.6). Note that $0 < \alpha_i < \rho$. Now, in the proof of Theorem 3c of [4], we showed that $\nu_{\alpha(.)} \ll \nu_\rho$ and $d\nu_{\alpha(.)}/d\nu_\rho \in L^p(\nu_\rho)$ for $p > 1$ as soon as

$$\sum_{i \in \mathbb{Z}^d} \left(1 - \frac{\alpha_i}{\rho}\right)^2 < \infty,$$

or equivalently in $d \ge 3$,

$$\sum_{i \in \mathbb{Z}^d} \mathbb{P}(H_i < \infty)^2 < \infty,$$

which holds as soon as $d \geq 5$.

We rewrite (2.6) on $\{\eta : \eta_i = 0\}$ with $i \neq 0$, denoting by σ_i the action of spin flip at site i [i.e., $(\sigma_i \eta)_k = \eta_k$ if $k \neq i$ and $(\sigma_i \eta)_i = 1 - \eta_i$],

$$\sigma_i f_t \geq \frac{\alpha_i}{1-\alpha_i} \frac{1-\rho}{\rho} f_t \quad \text{or equivalently,} \quad \sigma_i(f_t) \frac{d\nu_{\alpha(.)}}{d\nu_\rho} \geq \sigma_i \left(\frac{d\nu_{\alpha(.)}}{d\nu_\rho} \right) f_t. \tag{4.1}$$

Now, on $\mathcal{A}^c$, we form $\varphi = dT_t(\nu_\rho)/d\nu_{\alpha(.)}$ and we note that φ is increasing. Indeed, if $i \neq 0$ and $\eta_i = 0$, then (4.1) is nothing but $\sigma_i \varphi \geq \varphi$. Now, as a product measure, $\nu_{\alpha(.)}$ satisfies FKG. Thus, for ψ increasing,

$$\begin{aligned} \int \psi \, dT_t(\nu_\rho) &= \int \psi \left(\frac{dT_t(\nu_\rho)}{d\nu_{\alpha(.)}} \right) d\nu_{\alpha(.)} \\ &\geq \int \psi \, d\nu_{\alpha(.)} \int \left(\frac{dT_t(\nu_\rho)}{d\nu_{\alpha(.)}} \right) d\nu_{\alpha(.)} = \int \psi \, d\nu_{\alpha(.)}. \end{aligned}$$

Thus, $\nu_{\alpha(.)} \prec T_t(\nu_\rho)$. The fact that $T_t(\nu_\rho) \prec \nu_\rho$ comes from the fact that f_t is decreasing and ν_ρ satisfies FKG. Now (ii) of Corollary 2.5 follows as in the proof of Theorem 3c of [4]. Using that f_t and $d\nu_{\alpha(.)}/d\nu_\rho$ are decreasing, for $i \geq 1$ and $j \geq 0$,

$$\begin{aligned} \int f_t^i \left(\frac{d\nu_{\alpha(.)}}{d\nu_\rho} \right)^j d\nu_\rho &= \int f_t^{i-1} \left(\frac{d\nu_{\alpha(.)}}{d\nu_\rho} \right)^j dT_t(\nu_\rho) \leq \int f_t^{i-1} \left(\frac{d\nu_{\alpha(.)}}{d\nu_\rho} \right)^j d\nu_{\alpha(.)} \\ &= \int f_t^{i-1} \left(\frac{d\nu_{\alpha(.)}}{d\nu_\rho} \right)^{j+1} d\nu_\rho. \end{aligned}$$

Thus, we obtain by induction, for each $n \geq 1$,

$$\int f_t^n \, d\nu_\rho \leq \int \left(\frac{d\nu_{\alpha(.)}}{d\nu_\rho} \right)^n d\nu_\rho. \tag{4.2}$$

Since the right-hand side of (4.2) is bounded for $d \geq 5$, the corollary follows. □

PROOF OF LEMMA 2.3. In [4], Section 4, we have that $t \mapsto R(t) := e^{-\lambda t}/P_\nu(\tau > t)$ is increasing and $R(0) > 0$. Suppose (i) and let $\lim_{t\to\infty} R(t) = R < \infty$. Now (ii) follows from

$$\begin{aligned} \int f_t^2 \, d\nu &= \int \frac{\bar{S}_t(\mathbb{1}_{\mathcal{A}^c}) \bar{S}_t(\mathbb{1}_{\mathcal{A}^c})}{P_\nu(\tau > t)^2} \, d\nu \\ &= \int \frac{\bar{S}_{2t}(\mathbb{1}_{\mathcal{A}^c})}{P_\nu(\tau > t)^2} \, d\nu = \frac{P_\nu(\tau > 2t)}{P_\nu(\tau > t)^2} = \frac{R(2t)}{R(t)^2} \leq \frac{R}{R(0)^2}. \end{aligned} \tag{4.3}$$

Conversely, Let μ be a limit point of $\{(1/t) \int_0^t T_s(\nu) \, ds\}$ along the subsequence $\{t_n\}$ in weak-$L^2(d\nu)$. By Theorem 1 of [4], μ is a quasi-stationary measure with $P_\mu(\tau > t) = \exp(-\lambda t)$. Thus, (i) follows from

$$\begin{aligned}\lim_{t\to\infty} R(t) &= \lim_{n\to+\infty} \frac{1}{t_n}\int_0^{t_n} \frac{P_\mu(\tau > s)}{P_{\nu_\rho}(\tau > s)}\,ds \\ &= \lim_{n\to+\infty} \frac{1}{t_n}\int_0^{t_n} \int f_s \frac{d\mu}{d\nu_\rho}\,d\nu_\rho\,ds = \int\left(\frac{d\mu}{d\nu_\rho}\right)^2 d\nu_\rho < \infty.\end{aligned} \tag{4.4}$$

Now, if either (i) or (ii) holds, by Remark 3 of [4], we have that the Yaglom limit exists. Now, to show that $\int (f_t - f)^2\,d\nu$ converges to 0, we only need to show that $\int f_t^2\,d\nu$ converges to $\int f^2\,d\nu$. It is easy to see from (4.2) and (4.3) that $\lim \int f_t^2\,d\nu = \lim R(t) = \int (d\mu/d\nu)^2\,d\nu$, and the proof of Lemma 2.3 is concluded. □

5. Proof of Theorem 2.7(i): Uniqueness.

5.1. *Positivity of f*. We first show that ν_ρ-a.s., $f > 0$ on $\{\eta_0 = 0\}$. We introduce a symmetric simple exclusion process on $\mathbb{Z}^d \setminus \{0\}$: there is no site 0, and its adjacent bonds are suppressed. Let ν_ρ^* be the product Bernoulli measure on $\mathbb{Z}^d \setminus \{0\}$ of density ρ. Let $\{S_t^*,\ t \ge 0\}$ be the Markov semi-group of this process. It is known that the process $\{S_t^*,\ t \ge 0\}$, with initial measure ν_ρ^*, is reversible and ergodic: indeed, by Theorem 1.44 on page 377 of [10], ν_ρ^* is an extremal invariant measure and by Theorem B.52, on page 23 of [11], all extremal invariant measures are ergodic. In other words, if for any $t > 0$, $S_t^* g = g$, ν_ρ^*-a.s., then g is constant ν_ρ^*-a.s.

Let $\mathcal{B} := \{\eta : \eta_0 = 0,\ f(\eta) = 0\}$, and note that ν_ρ-a.s.,

$$\bar{S}_t f.\mathbb{1}_{\mathcal{B}} = \exp(-\lambda t) f.\mathbb{1}_{\mathcal{B}} = 0.$$

Now, because $f \ge 0$, we have that for any $\varepsilon > 0$, $\varepsilon \mathbb{1}_{\{f > \varepsilon\}} \le f$. Thus, for $\eta \in \mathcal{B}$, ν_ρ-a.s.,

$$\forall \varepsilon > 0, \qquad P^\eta(\{f(\eta_t) > \varepsilon\},\ \tau > t) = 0 \quad\Longrightarrow\quad P^\eta(\{f(\eta_t) > 0\},\ \tau > t) = 0.$$

Now, if the bonds linking 0 to its neighbors are not marked, up to time t, then $\bar{S}_t$ acts like S_t^*. So, if τ_0 is the first time a Poisson mark appears in one of these bonds, for $\eta \in \mathcal{B}$, ν_ρ-a.s.,

$$P^\eta(f(\eta_t) > 0,\ \tau_0 > t) = 0,$$

because $\tau_0 \le \tau$. Now τ_0 is independent from the Poisson processes on bonds of $\mathbb{Z}^d \setminus \{0\}$, so

$$P^\eta(f(\eta_t) > 0,\ \tau_0 > t) = S_t^* \mathbb{1}_{\{f(.) > 0\}} P(\tau_0 > t) = 0.$$

Now, for any $t > 0$, $P(\tau_0 > t) > 0$. Thus, for $\eta \in \mathcal{B}$, ν_ρ^*-a.s.,

$$S_t^* \mathbb{1}_{\{f(.) > 0\}}(\eta) = 0.$$

In other words, for any $t > 0$, we have ν_ρ^*-a.s.,

$$S_t^* \mathbb{1}_{\mathcal{B}} \geq \mathbb{1}_{\mathcal{B}}.$$

Now ν_ρ^* is reversible for S_t^*, so both expressions have the same mean, and we conclude that ν_ρ^*-a.s., for any $t > 0$, $S_t^* \mathbb{1}_{\mathcal{B}} = \mathbb{1}_{\mathcal{B}}$. By the ergodicity of ν_ρ^*, we conclude that $\mathbb{1}_{\mathcal{B}}$ is ν_ρ^* constant, so that necessarily $\nu_\rho(\mathcal{B}) = 0$.

5.2. *One eigenvalue with a positive eigenvector.* Suppose that $f, f' \in \mathcal{H}_{\mathcal{A}}$ are the densities of two quasi-stationary measures. There are two real numbers $\lambda(\rho)$ and λ' such that f, f' satisfy in an $L^2(\nu_\rho)$ sense

$$\bar{S}_t f = e^{-\lambda(\rho)t} f \quad \text{and} \quad \bar{S}_t f' = e^{-\lambda' t} f'. \tag{5.1}$$

First, we show that $\lambda(\rho) = \lambda'$. We assume that $d\mu_\rho = f\, d\nu_\rho$ corresponds to the Yaglom limit. Thus,

$$\lim_{t\to\infty} \int f' f_t \, d\nu_\rho = \int f' f \, d\nu_\rho \quad \text{and} \quad \lim_{t\to\infty} \int f f_t \, d\nu_\rho = \int f^2 \, d\nu_\rho. \tag{5.2}$$

However, as $f > 0$ on $\{\eta_0 = 0\}$, ν_ρ-a.s.,

$$\frac{e^{-\lambda' t} \int f' \, d\nu_\rho}{P_{\nu_\rho}(\tau > t)} = \frac{\int \mathbb{1}_{A^c} \bar{S}_t f' \, d\nu_\rho}{P_{\nu_\rho}(\tau > t)} = \int f' f_t \, d\nu_\rho \longrightarrow \int f' f \, d\nu_\rho > 0. \tag{5.3}$$

Similarly,

$$\int f f_t \, d\nu_\rho = \frac{e^{-\lambda(\rho)t} \int f \, d\nu_\rho}{P_{\nu_\rho}(\tau > t)} \longrightarrow \int f^2 \, d\nu_\rho. \tag{5.4}$$

Thus, $\lambda(\rho) = \lambda'$.

5.3. *Dual expansion.* We expand f on the countable basis of $L^2(\nu_\rho)$, say $\{H_A,\ A \in \mathcal{P}\}$, where $\mathcal{P}$ is the collection of finite subsets of $\mathbb{Z}^d$ and

$$H_\varnothing = 1 \quad \text{and} \quad H_A(\eta) = \prod_{i\in A} \frac{(\eta_i - \rho)}{\sqrt{\rho(1-\rho)}}. \tag{5.5}$$

Thus, there are real numbers $\{C_A,\ A \in \mathcal{P}\}$ with

$$f = \sum_{A\in\mathcal{P}} C_A H_A \quad \text{and} \quad \int f^2 \, d\nu_\rho = \sum_{A\in\mathcal{P}} C_A^2. \tag{5.6}$$

The constraint that $f \in \mathcal{H}_{\mathcal{A}}$, i.e., $\eta_0 f(\eta) = 0$, is equivalent to

$$\forall A \notin 0, \qquad \int H_A \eta_0 f \, d\nu_\rho = 0. \tag{5.7}$$

Thus,

$$\sqrt{\rho(1-\rho)}\int H_{A\cup\{0\}} f\, d\nu_\rho + \rho \int H_A f\, d\nu_\rho = 0. \tag{5.8}$$

So, for $A \not\ni 0$,

$$\sqrt{\rho(1-\rho)} C_{A\cup\{0\}} + \rho C_A = 0. \tag{5.9}$$

We define, for all $A \in \mathcal{P}^*$,

$$\psi(A) = \left(-\sqrt{\frac{1-\rho}{\rho}}\right)^{|A|} C_A. \tag{5.10}$$

Now condition (5.9) reads $\psi(A \cup \{0\}) = \psi(A)$ for $A \not\ni 0$. Now we express $(-Lf, f)$ in terms of the $\{C_A\}$:

$$(-Lf, f) = \sum_{A\in\mathcal{P}} \sum_{B\sim A} (C_B - C_A)^2, \tag{5.11}$$

where $B \sim A$ if there is $i \in A \setminus B$ and $j \in B \setminus A$ with $A \triangle B = \{i, j\}$. We replace the C_A's by the $\psi(A)$'s and distinguish 0 to eliminate (5.9),

$$\begin{aligned}
(-Lf, f) &= \sum_{A\not\ni 0}\Bigg(\sum_{B\not\ni 0, B\sim A} \gamma^{|A|}\big(\psi(B) - \psi(A)\big)^2 \\
&\qquad + \sum_{B\ni 0, B\sim A} \gamma^{|A|}\big(\psi(B\setminus\{0\}) - \psi(A)\big)^2\Bigg) \\
&\quad + \sum_{A\ni 0}\Bigg(\sum_{B\ni 0, B\sim A} \gamma^{|A|}\big(\psi(B\setminus\{0\}) - \psi(A\setminus\{0\})\big)^2 \\
&\qquad + \sum_{B\not\ni 0, B\sim A} \gamma^{|A|}\big(\psi(B) - \psi(A\setminus\{0\})\big)^2\Bigg) \\
&= \sum_{A\not\ni 0}\Bigg[\sum_{B\in\mathcal{N}_A} \gamma^{|A|}(1+\gamma)\big(\psi(B) - \psi(A)\big)^2 \\
&\qquad + \sum_{B\in\mathcal{N}_A^-} \gamma^{|A|}\big(\psi(B) - \psi(A)\big)^2 \\
&\qquad + \sum_{B\in\mathcal{N}_A^+} \gamma^{|A|+1}\big(\psi(B) - \psi(A)\big)^2\Bigg],
\end{aligned} \tag{5.12}$$

where $\gamma = (1-\rho)/\rho$, $B \in \mathcal{N}_A$ means $B \sim A$ and $B \not\ni 0$, $B \in \mathcal{N}_A^+$ means $B \sim A \cup \{0\}$ and $B \not\ni 0$, and $B \in \mathcal{N}_A^-$ means $B \cup \{0\} \sim A$ and $B \not\ni 0$. With this rewriting, $(-\bar{L}f, f)$ can be thought of as the Dirichlet form, $\mathcal{E}(\psi, \psi)$, of a dynamics with finitely many particles, with creation and annihilation at site 0, with respect to a measure $m(A) = \gamma^{|A|}$. The advantages of this rewriting are threefold: (i) the constraint (5.9) has vanished, (ii) the new dynamics is clearly irreducible and (iii) in studying the minimizers of $\mathcal{E}(\psi, \psi)$, we can assume the $\{\psi(A)\}$ to be nonnegatives. Indeed, note that $\mathcal{E}(\psi, \psi) \geq \mathcal{E}(|\psi|, |\psi|)$ and equality holds only if $\psi(A) \geq 0$ for each $A \in \mathcal{P}^*$. Also, we rewrite the $L^2(\nu_\rho)$ norm of f in terms of the $\{\psi(A), A \in \mathcal{P}^*\}$:

$$|f|^2 = \sum_{A \not\ni 0} C_A^2 + C_{A\cup\{0\}}^2 = (1+\gamma) \sum_{A \not\ni 0} \psi(A)^2 \gamma^{|A|} = (1+\gamma)\|\psi\|_m^2. \tag{5.13}$$

Now $\bar{L} f + \lambda f = 0$ implies that for $A \in \mathcal{P}^*$,

$$\begin{aligned} &(1+\gamma) \sum_{B \in \mathcal{N}_A} \big(\psi(B) - \psi(A)\big) + \gamma \sum_{B \in \mathcal{N}_A^+} \big(\psi(B) - \psi(A)\big) \\ &\qquad + \sum_{B \in \mathcal{N}_A^-} \big(\psi(B) - \psi(A)\big) = -\lambda \psi(A). \end{aligned} \tag{5.14}$$

Thus, $\psi(A) > 0$ for all $A \in \mathcal{P}^*$. Now let $\phi = \psi^2$ and note that for any A and $B \in \mathcal{P}^*$, the functional $\phi \mapsto (\sqrt{\phi(B)} - \sqrt{\phi(A)})^2$ is convex. Assume that ψ and ψ' are two positive normalized minimizers, and let ϕ and ϕ' be their respective squares. Then, for any $\lambda \in [0, 1]$,

$$\psi_\lambda := \sqrt{\lambda\phi + (1-\lambda)\phi'}$$

has $\|\psi_\lambda\|_m = 1$ and

$$\mathcal{E}(\psi_\lambda, \psi_\lambda) \leq \lambda \mathcal{E}(\psi, \psi) + (1-\lambda)\mathcal{E}(\psi', \psi').$$

Thus, the convex inequality is an equality, so that for any $A \in \mathcal{P}^*$ and any $B \in \mathcal{N}_A \cup \mathcal{N}_A^- \cup \mathcal{N}_A^+$,

$$\big(\psi_\lambda(B) - \psi_\lambda(A)\big)^2 = \lambda\big(\psi(B) - \psi(A)\big)^2 + (1-\lambda)\big(\psi'(B) - \psi'(A)\big)^2,$$

which implies, after expanding, that $\phi(A)\phi'(B) = \phi(B)\phi'(A)$. Now, as $\phi(A) > 0$ and the dynamics is irreducible, we conclude that $\phi \equiv \phi'$ and so are the positive square-roots $\psi \equiv \psi'$.

6. Proof of Theorem 2.7(ii) and (iii).

PROOF OF THEOREM 2.7(ii): DENSITY AT INFINITY. The facts that for any $t > 0$, $\nu_{\alpha(.)} \prec T_t(\nu_\rho) \prec \nu_\rho$ with $\alpha(i) \to \rho$ when $\|i\| \to \infty$ implies that for any $A \in \mathcal{P}$,

$$\int \prod_{j \in A} \eta_{j+i} \, d\mu_\rho(\eta) \overset{\|i\|\to\infty}{\to} \rho^{|A|},$$

for $\prod_{j\in A}\eta_{j+i}$ is an increasing function. Now any local function φ can be written as a linear combination of local increasing functions, and the property follows by linearity. □

PROOF OF THEOREM 2.7(iii): BASIN OF ATTRACTION. We show that for any measure ν, any subsequence of the Cesàro limit of $\{T_t(\nu)\}$ contains a further subsequence converging to a quasi-stationary measure, say μ. When the density of ν with respect to ν_ρ, say ϕ, is continuous, we show that $\mu=\mu_\rho$. As in the proof of existence of a quasi-stationary measure (see [4], Lemma 1), we establish first the existence, for any $s>0$, of the following limit:

$$\lim_{t\to\infty}\frac{P_\nu(\tau>t+s)}{P_\nu(\tau>t)}=\exp\bigl(-\lambda(\rho)s\bigr). \tag{6.1}$$

Indeed, recall that

$$P_\nu(\tau>t)=\int \bar{S}_t(\mathbb{1}_{\mathcal{A}^c})\mathbb{1}_{\mathcal{A}^c}\phi\,d\nu_\rho=\int \bar{S}_t(\mathbb{1}_{\mathcal{A}^c}\phi)\,d\nu_\rho, \tag{6.2}$$

so that by the existence of the Yaglom limit,

$$\lim_{t\to\infty}\frac{P_\nu(\tau>t)}{P_{\nu_\rho}(\tau>t)}=\int\phi\,d\mu_\rho>0. \tag{6.3}$$

Thus,

$$\lim_{t\to\infty}\frac{P_\nu(\tau>t+s)}{P_\nu(\tau>t)}=\lim_{t\to\infty}\frac{P_{\nu_\rho}(\tau>t+s)}{P_{\nu_\rho}(\tau>t)}=\exp\bigl(-\lambda(\rho)s\bigr). \tag{6.4}$$

By the weak* compactness, for any sequence $\{t_n\}$, there is a further subsequence (still named $\{t_n\}$ for convenience), and a probability measure μ, such that for any continuous function φ,

$$\frac{1}{t_n}\int_0^{t_n}\bigl(T_t(\nu),\varphi\bigr)\,dt\longrightarrow\int\varphi\,d\mu. \tag{6.5}$$

The same argument as in the proof of Theorem 1 of [4] implies that μ is quasi-stationary. Now, we establish a priori estimates:

$$\begin{aligned}\int\left(\frac{dT_t(\nu)}{d\nu_\rho}\right)^2 d\nu_\rho&=\int\left(\frac{E^\eta[\phi(\eta_t)\mathbb{1}_{\{\tau>t\}}]}{P_\nu(\tau>t)}\right)^2 d\nu_\rho\\ &\le|\phi|_\infty^2\int\left(\frac{P^\eta(\tau>t)}{P_\nu(\tau>t)}\right)^2 d\nu_\rho\\ &=|\phi|_\infty^2\int f_t^2\,d\nu_\rho\left(\frac{P_{\nu_\rho}(\tau>t)}{P_\nu(\tau>t)}\right)^2.\end{aligned} \tag{6.6}$$

This quantity is bounded by Corollary 2.5 and (6.3). By standard arguments, this implies that $\mu\in L^2(\nu_\rho)$, which establishes, by the uniqueness result, that $\mu=\mu_\rho$, so that the Cesàro limit exists and is μ_ρ. □

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REFERENCES

[1] ANDJEL, E. (1988). A correlation inequality for the symmetric exclusion process. *Ann. Probab.* **16** 717–721.

[2] ARRATIA, R. (1983). The motion of a tagged particle in the simple symmetric exclusion system on $\mathbb{Z}$. *Ann. Probab.* **11** 362–373.

[3] ARRATIA, R. (1985). Symmetric exclusion processes: a comparison inequality and a large deviation result. *Ann. Probab.* **13** 53–61.

[4] ASSELAH, A. and DAI PRA, P. (2001). Quasi-stationary measures for conservative dynamics in the infinite lattice. *Ann. Probab.* **29** 1733–1754.

[5] DURRETT, R. (1991). *Probability: Theory and Examples*. Wadsworth and Brooks/Cole, Pacific Grove, CA.

[6] FERRARI, P. A., KESTEN, H. and MARTÍNEZ, S. (1996). R-positivity, quasi-stationary distributions and ratio limit theorems for a class of probabilistic automata. *Ann. Appl. Probab.* **6** 577–616.

[7] HARRIS, T. E. (1978). Additive set valued Markov processes and graphical methods. *Ann. Probab.* **6** 355–378.

[8] LANDIM, C. (1992). Occupation time large deviations for the symmetric simple exclusion process. *Ann. Probab.* **20** 206–231.

[9] LAWLER, G. (1991). *Intersections of Random Walks*. Birkhäuser, Boston.

[10] LIGGETT, T. M. (1985). *Interacting Particle Systems*. Springer, New York.

[11] LIGGETT, T. M. (1999). *Stochastic Interacting Systems*. Springer, New York.

[12] SENETA, E. and VERE-JONES, D. (1966). On quasi-stationary distributions in discrete-time Markov chains with a denumerable infinity of states. *J. Appl. Probab.* **3** 403–434.

[13] VARADHAN, S. R. S. (1994). Regularity of self-diffusion coefficient. In *The Dynkin Festschrift*: *Markov Processes and Their Applications* (M. I. Friedlin, ed.) 387–397. Birkhäuser, Boston.

[14] YAGLOM, A. M. (1947). Certain limit theorems of the theory of branching stochastic processes. *Dokl. Akad. Nauk SSSR* (*N.S.*) **56** 797–798.

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