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CHARACTERIZATION OF STATIONARY MEASURES FOR ONE-DIMENSIONAL EXCLUSION PROCESSES

BY MAURY BRAMSON,[1] THOMAS M. LIGGETT[2] AND THOMAS MOUNTFORD[3]

University of Minnesota, University of California, Los Angeles and University of California, Los Angeles

The product Bernoulli measures ν_α with densities α, $\alpha \in [0, 1]$, are the extremal translation invariant stationary measures for an exclusion process on $\mathbb{Z}$ with irreducible random walk kernel $p(\cdot)$. Stationary measures that are not translation invariant are known to exist for finite range $p(\cdot)$ with positive mean. These measures have particle densities that tend to 1 as $x \to \infty$ and tend to 0 as $x \to -\infty$; the corresponding extremal measures form a one-parameter family and are translates of one another. Here, we show that for an exclusion process where $p(\cdot)$ is irreducible and has positive mean, there are no other extremal stationary measures. When $\sum_{x<0} x^2 p(x) = \infty$, we show that any nontranslation invariant stationary measure is not a blocking measure; that is, there are always either an infinite number of particles to the left of any site or an infinite number of empty sites to the right of the site. This contrasts with the case where $p(\cdot)$ has finite range and the above stationary measures are all blocking measures. We also present two results on the existence of blocking measures when $p(\cdot)$ has positive mean, and $p(y) \le p(x)$ and $p(-y) \le p(-x)$ for $1 \le x \le y$. When the left tail of $p(\cdot)$ has slightly more than a third moment, stationary blocking measures exist. When $p(-x) \le p(x)$ for $x > 0$ and $\sum_{x<0} x^2 p(x) < \infty$, stationary blocking measures also exist.

1. Introduction. Exclusion processes are among the most heavily studied models in the area of interacting particle systems. Much of their theory is described in Chapter VIII of Liggett (1985) and Part III of Liggett (1999). The one-dimensional exclusion process $\eta_\cdot = (\eta_t)_{t \ge 0}$, with random walk kernel $p(\cdot)$, is a continuous time Markov process on $\{0, 1\}^{\mathbb{Z}}$. A configuration $\eta \in \{0, 1\}^{\mathbb{Z}}$ is said to be occupied by a particle at x if $\eta(x) = 1$, and is empty (or has a hole) at x if $\eta(x) = 0$; we use the convention of identifying η with the set of its occupied sites. A particle moves from an occupied site x to an empty site y at rate $p(y - x)$. When the site y is already occupied, such a particle remains at x; there is always at most one particle at a given site. The exclusion process $\eta_\cdot$ is formally defined as

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the Feller process on $\{0,1\}^{\mathbb{Z}}$, with generator

$$\Omega f(\eta) = \sum_{x,y\in\mathbb{Z}} \big(f(\eta_{xy}) - f(\eta)\big) p(y-x)\eta(x)\big(1-\eta(y)\big)$$

for cylindrical functions f, where

$$\eta_{xy}(x) = \eta(y), \quad \eta_{xy}(y) = \eta(x) \quad \text{and} \quad \eta_{xy}(z) = \eta(z) \qquad \text{for } z \neq x, y.$$

We assign the usual local topology to configurations in $\{0,1\}^{\mathbb{Z}}$.

A basic problem is the characterization of stationary measures for exclusion processes. Assume that the random walk kernel $p(\cdot)$ is *irreducible*, by which we will mean that for each $x \in \mathbb{Z}$, $p^{(n)}(x) + p^{(n)}(-x) > 0$ for some $n \in \mathbb{Z}^+$. It is well known that the product Bernoulli measures ν_α with densities α, $\alpha \in [0,1]$, at each site are the extremal translation invariant stationary measures for the process. When the mean $\mu \stackrel{\text{def}}{=} \sum_x x p(x) = 0$, there are no nontranslation invariant stationary measures. [See Liggett (1985) for both results.]

This behavior contrasts with the case where $p(\cdot)$ is nearest neighbor with a bias to the right. The extremal nontranslation invariant stationary measures are then given by the one parameter family of translates of a measure ν on $\{0,1\}^{\mathbb{Z}}$ [Liggett (1976)]. The measure ν is a *blocking measure*; that is, it is concentrated on configurations which are completely occupied by particles far enough to the right and are completely empty far enough to the left. In particular, it is a *profile measure*; that is, $\nu(\{\eta : \eta(x) = 1\}) \to 1$ as $x \to \infty$ and $\nu(\{\eta : \eta(x) = 1\}) \to 0$ as $x \to -\infty$. The existence of stationary blocking measures for a restricted class of $p(\cdot)$ with $\mu > 0$ was shown in Ferrari, Lebowitz and Speer (2001); the existence of stationary blocking measures for finite range $p(\cdot)$ with $\mu > 0$ was shown in Bramson and Mountford (2002). (By symmetry, analogs of the above results of course hold when $\mu < 0$.)

We first present a result in the opposite direction.

THEOREM 1.1. *Assume that $\eta_\cdot$ is an exclusion process whose random walk kernel $p(\cdot)$ has mean $\mu \in (0,\infty)$ and is irreducible. Then, the only possible extremal nontranslation invariant stationary measures consist of a profile measure ν, together with its translates.*

By the preceding discussion, these profile measures exist when $p(\cdot)$ also has finite range, and in that case are blocking measures. Our next result shows that blocking measures need not exist for more general $p(\cdot)$.

THEOREM 1.2. *Assume that $\eta_\cdot$ is an exclusion process whose random walk kernel $p(\cdot)$ has finite mean and satisfies $\sum_{x<0} x^2 p(x) = \infty$. Then, no stationary blocking measures exist.*

When $\mu \in (0, \infty)$, we believe that a stationary profile measure always exists. For $\sum_{x<0} x^2 p(x) = \infty$, Theorem 1.2 would then imply the existence of a stationary profile measure that is not a blocking measure. No such examples are currently known.

We also present two results on the existence of blocking measures under the monotonicity condition on $p(\cdot)$,

$$p(y) \le p(x) \quad \text{and} \quad p(-y) \le p(-x) \qquad \text{for } 1 \le x \le y. \tag{1.1}$$

Under (1.1), we substantially relax the tail behavior on $p(\cdot)$ assumed in Bramson and Mountford (2002). Theorem 1.3 requires slightly more than three moments on the left tail of $p(\cdot)$.

THEOREM 1.3. *Assume that $\eta_{\cdot}$ is an exclusion process whose random walk kernel $p(\cdot)$ has mean $\mu \in (0, \infty)$, and satisfies* (1.1) *and*

$$\sum_{x>0} x^3 (\log x)^{2+\delta} p(-x) < \infty \tag{1.2}$$

for some $\delta > 0$. Then, there exists a stationary blocking measure ν satisfying

$$\sum_{x<0} \nu(\{\eta : \eta(x) = 1\}) < \infty \quad \textit{and} \quad \sum_{x \ge 0} \nu(\{\eta : \eta(x) = 0\}) < \infty. \tag{1.3}$$

Theorem 1.3 does not apply to exclusion processes whose random walk kernels have left tails with between two and three moments. Theorem 1.4 includes these cases under the additional condition on $p(\cdot)$,

$$p(-x) \le p(x) \qquad \text{for } x \ge 1. \tag{1.4}$$

THEOREM 1.4. *Assume that $\eta_{\cdot}$ is an exclusion process whose random walk kernel $p(\cdot)$ has mean $\mu \in (0, \infty)$, and satisfies* (1.1), (1.4) *and* $\sum_{x<0} x^2 p(x) < \infty$. *Then, there exists a stationary blocking measure ν satisfying*

$$\sum_{x} \nu(\{\eta : \eta(x) = 1,\ \eta(x+1) = 0\}) < \infty. \tag{1.5}$$

Combining Theorems 1.2 and 1.4, it follows that for $p(\cdot)$ with positive mean and satisfying (1.1) and (1.4), a stationary blocking measure exists for the corresponding exclusion process exactly when $\sum_{x<0} x^2 p(x) < \infty$. We also note that the proof of Theorem 1.3 requires the third moment assumption (1.2) to establish the bounds on ν in (1.3), which are important for the proof. As discussed at the end of Section 5, it would not be surprising if (1.3) in fact fails in the absence of a third moment. The weaker bounds on ν, in (1.5), still hold under a finite second moment, and are used in the construction of the stationary blocking measure in Theorem 1.4.

The proof of Theorem 1.1 employs only "soft" analysis, making use of a standard coupling together with general properties of exclusion processes that have been available since Liggett (1976). The proof of Theorem 1.2 is short and involves direct computation. The proofs of Theorems 1.3 and 1.4 are analytic in nature. They are quite different than that given in Bramson and Mountford (2002), where a hydrodynamic limit from Rezakhanlou (1991) is used.

We now summarize the contents of the remaining five sections, and sketch some of the main ideas behind the theorems. Our methodology for showing Theorem 1.1 involves coupling two copies of an exclusion process with different stationary initial measures. Sites where the processes differ are referred to as "discrepancies." Their movement is tied to the random walk kernel $p(\cdot)$. When two discrepancies of opposite types meet, they both disappear. If the joint exclusion process is itself stationary, this imposes substantial constraints on the possible joint configurations at any given time, and therefore on the relationship between the corresponding stationary marginal measures. These constraints lead to the characterization of stationary measures in Theorem 1.1.

Sections 2 and 3 are devoted to showing Theorem 1.1. The main result in Section 2 is Proposition 2.5. Assume that the coupling is between exclusion processes, where the initial state of the second process is the translation of the initial state of the first process by one unit to the left, and the processes have stationary initial measures. The proposition states that when $p(\cdot)$ has finite mean and is irreducible, the expected number of discrepancies that visit an interval $[-T, T]$ at any time in $[\sqrt{T}, T]$, is $o(T)$ as $T \to \infty$. To demonstrate Proposition 2.5, we will first bound the density of discrepancies for large times when the stationary measures are the product measures ν_α. We then apply this result to the general case by representing the Cesaro limits of a stationary measure and its translates as convex combinations of ν_α, $\alpha \in [0, 1]$.

Assume now that the joint exclusion process in the coupling is itself stationary. In Proposition 3.2, we apply Proposition 2.5 and lower bounds on the rate at which discrepancies of opposite type meet to show that the joint measure is concentrated on coordinates (η, ξ) with either $\eta \le \xi$ or $\eta \ge \xi$. When the marginal measures are extremal, then either $\eta = \xi$, $\eta < \xi$ or $\eta > \xi$ must always hold. [$\eta \le \xi$ means that $\eta(x) \le \xi(x)$ at each x, and $\eta < \xi$ means that, in addition, $\eta(y) < \xi(y)$ at some y.] In the first case, the measures are ν_α, and in the second case, they are profile measures; the third case cannot occur when $\mu \in (0, \infty)$. This is shown in Propositions 3.3 and 3.4. Similar reasoning can be applied to a jointly stationary exclusion process, whose coordinates each have extremal stationary measures, to show that these measures are, in fact, translates of one another.

In Section 4, we demonstrate Theorem 1.2. The basic idea is that because of the condition $\sum_{x<0} x^2 p(x) = \infty$, particles far to the right of 0 jump frequently enough far to the left of 0, so that a stationary measure must have particles arbitrarily far to the left or empty sites arbitrarily far to the right. Such a measure is not a blocking measure. The computations needed to analyze the jumps of the particles are done in Lemma 4.1 and Proposition 4.2.

We demonstrate Theorem 1.3 in Section 5. The main result in the section is Proposition 5.4, which provides a fairly general sufficient condition for the existence of a stationary blocking measure. Theorem 1.3 follows once we verify the condition (5.20) in the proposition, which is done using Proposition 5.5. The main idea behind the demonstration of Proposition 5.4 is to find a blocking product measure so that the exclusion process with this initial measure moves stochastically to the right with respect to an appropriate partial order on configurations. Both this partial order and the condition (1.1) were introduced in Ferrari, Lebowitz and Speer (2001). The limiting measure as $t \to \infty$ is then the desired stationary blocking measure. Sufficient conditions for the evolution to move the distribution to the right are given in Proposition 5.1, with the nonpositive derivative in (5.6) on the initial measure being the main condition to check. Verification of (5.6) under the condition (5.20) of Proposition 5.4 requires a number of steps. Much of this work is carried out in Lemma 5.2 and Proposition 5.3. We note that the proof of Theorem 1.3 is much shorter than that for the result in Bramson and Mountford (2002) that was cited earlier.

Theorem 1.4 is proved in Section 6. The main result in the section is Proposition 6.5, which is a slight generalization of Theorem 1.4. The argument consists of two main steps. In Proposition 6.1, we choose random walk kernels $p_\varepsilon(\cdot)$, $\varepsilon > 0$, such that $p_\varepsilon(n) \to p(n)$ as $\varepsilon \downarrow 0$. With an appropriate choice of $p_\varepsilon(\cdot)$, Theorem 1.3 implies that the exclusion processes corresponding to $p_\varepsilon(\cdot)$ have stationary blocking measures that satisfy (6.2), which is a uniform version of (1.5). It follows that the exclusion process with random walk kernel $p(\cdot)$ has a stationary blocking measure. Part of the work in showing Proposition 6.1 is carried out in Lemmas 6.2 and 6.3. In order to apply the proposition, we still need to check that (6.2) is satisfied for appropriate random walk kernels $p_\varepsilon(\cdot)$. This is done in Lemma 6.4 and Proposition 6.5. In Lemma 6.4, we establish the identity (6.12), which expresses $\sum_{x<0} x^2 p(x)$ in terms of a subadditive function $M(\cdot)$. This identity is used in Proposition 6.5 to obtain the desired bound (6.2) on the stationary blocking measures. At the end of the section, we summarize how the results of Bramson and Mountford (2002) could be used to prove Theorem 1.4 without assuming (1.1). We would still require assumption (1.4), however.

2. Discrepancies in the coupled process. In order to couple copies of an exclusion process, we use a standard graphical representation. Let $\mathcal{N}^{x,y}$, $x, y \in \mathbb{Z}$, denote a Harris system of independent Poisson point processes, with rates $p(y-x)$ corresponding to the underlying random walk of the exclusion processes. One stipulates that if at $t \in \mathcal{N}^{x,y}$, $\eta_{t-}(x) = 1$ and $\eta_{t-}(y) = 0$, then $\eta_t(x) = 0$ and $\eta_t(y) = 1$, with there otherwise being no change in $\eta_\cdot$. That is, at $t \in \mathcal{N}^{x,y}$, "a particle tries to move from site x to site y." The filtration $(\mathcal{F}_t)_{t\geq 0}$ for the process will be the natural filtration associated with the whole Harris system, together with the initial configurations of the different copies of the exclusion process under consideration.

Let $\eta_\cdot$ and $\xi_\cdot$ be two copies of such an exclusion process. At time 0, we refer to those sites where $\eta_0(x) > \xi_0(x)$ as positive discrepancies and those sites where $\eta_0(x) < \xi_0(x)$ as negative discrepancies. One can check that, as time evolves, a discrepancy moves from x to y at $t \in \mathcal{N}^{x,y}$ if neither process already occupies y, and a discrepancy moves from x to y at $t \in \mathcal{N}^{y,x}$ if both processes already occupy y. When two discrepancies of opposite types meet, they disappear; discrepancies are never created. At a site x where there is no discrepancy, $\eta_t(x) = \xi_t(x)$. Hence, the presence of only a "few" discrepancies at a given time implies the processes are "close" then. We denote by $(X_t^k)_{t \geq 0}$, the process corresponding to the discrepancy initially at k, if it exists; we continue $X_\cdot^k$ after the discrepancy disappears by keeping its position fixed.

Let $\eta_\cdot$ be an exclusion process with a stationary initial measure. Let $\eta_0^1(x) = \eta_0(x+1)$ be the translation of η_0 by one unit to the left and let $\eta_\cdot^1$ denote the corresponding exclusion process which is generated by the same Harris system as $\eta_\cdot$. The main result in this section is Proposition 2.5, which states that when $p(\cdot)$ has finite mean and is irreducible, the expected number of discrepancies that visit the interval $[-T, T]$ at any time in $[\sqrt{T}, T]$ is $o(T)$ as $T \to \infty$. That is, the "local density" of discrepancies is typically small after large times. Such discrepancies can either originate in $[-MT, MT]$ or $[-MT, MT]^c$, for appropriate $M > 1$. Lemmas 2.3 and 2.4 will show that the contribution of discrepancies from $[-MT, MT]$ is small, and Lemma 2.2 will show that the contribution from $[-MT, MT]^c$ is also small.

In Lemma 2.2, we will need to bound the movement of discrepancies. This is easy to do in terms of a random walk, since a discrepancy can move from x to y at time t only if $t \in \mathcal{N}^{x,y}$ or $\mathcal{N}^{y,x}$. We set $\bar{p}(x) = p(x) + p(-x)$.

LEMMA 2.1. *Let $X_\cdot^k$, $k \in \mathbb{Z}$, be a discrepancy of $(\eta_\cdot, \xi_\cdot)$. Then, there exists an increasing random walk $Z_\cdot$ on $\mathbb{Z}$, which is adapted to $\mathcal{F}_\cdot$, with $Z_0 = 0$ and which jumps from x to y, $y > x$, at rate $\bar{p}(y-x)$, so that*

$$X_t^k - X_s^k \leq Z_t - Z_s \qquad \textit{for all } s \leq t.$$

For Lemma 2.2, we fix an M with

$$M \geq \sum_x |x| p(x) + 3. \tag{2.1}$$

Although the result is stated for discrepancies, it is true more generally and relies only on the random walk bounds in Lemma 2.1 and not on the disappearance of discrepancies.

LEMMA 2.2. *Let $\eta_\cdot$ and $\xi_\cdot$ be coupled exclusion processes, with $p(\cdot)$ having finite mean. Then, the expected number of discrepancies for $(\eta_\cdot, \xi_\cdot)$ that originate in $[-Mt, Mt]^c$ and visit $[-t, t]$ by time t is $o(t)$ as $t \to \infty$.*

PROOF. By symmetry, it suffices to show this limit for discrepancies that visit $[-t,\infty)$ by time t and that originate in $(-\infty,-Mt)$. Such discrepancies must cross the site $-Mt$ by time t, going from some $x<-Mt$ to some $y\ge -Mt$. We consider separately the contribution from discrepancies which first cross at y satisfying (a) $y\ge(-M+1)t$ and (b) $y\in[-Mt,(-M+1)t)$. We denote by N_t^1, respectively N_t^2, the number of such discrepancies in each case.

In case (a), the rate at which such crossings occur is bounded above by the sum $\sum_{z\ge t} z\bar{p}(z)$, which equals $\sum_{|z|\ge t}|z|p(z)$. By assumption, this goes to 0 as $t\to\infty$. So, $E[N_t^1]=o(t)$.

After moving to $[-Mt,(-M+1)t)$, the discrepancies in case (b) must subsequently move distance at least $(M-2)t$ to the right. By (2.1), this is at least $t(\sum_z|z|p(z)+1)$. The motion of a given discrepancy is bounded above by that of the increasing random walk $Z_\cdot$ in Lemma 2.1. Applying the strong Markov property to $Z_\cdot$ when the discrepancy first hits $[-Mt,-(M+1)t)$, it follows that

$$E[N_t^2]\le\left(t\sum_z|z|p(z)\right)P\left(Z_t\ge t\left(\sum_z|z|p(z)+1\right)\right). \tag{2.2}$$

The first sum on the right-hand side is finite by assumption. Since $E[Z_1]\le\sum_z|z|p(z)$, it follows from the weak law of large numbers that the probability in (2.2) goes to 0 as $t\to\infty$. So, $E[N_t^2]=o(t)$. The lemma follows from this together with the bound on $E[N_t^1]$. $\square$

Let $(\eta_\cdot,\eta_\cdot^1)$ be the coupled exclusion process, where η_0^1 is the translate of η_0 by one unit to the left. Set $V_t^k=1$ if there is initially a discrepancy at k and this discrepancy has not disappeared by time t, and set $V_t^k=0$ otherwise. Lemma 2.3 states that $E^{\nu_\alpha}[V_t^0]\to 0$ uniformly in α as $t\to\infty$, where ν_α is the measure of η_0. (Since ν_α refers here to just the first coordinate, this is a slight abuse of notation.) This behavior is intuitively fairly clear, since η_t and η_t^1 have the same density of particles, and therefore individual discrepancies will ultimately meet those of the opposite type. The proof makes use of the extremality of ν_α among translation invariant measures, which will ensure that $\lambda(\{\eta=\xi\})=1$ for any measure λ which is a Cesaro limit of $(\eta_\cdot,\eta_\cdot^1)$ over $[0,t]$ as $t\to\infty$.

LEMMA 2.3. *Assume that $p(\cdot)$ is irreducible. Then,*

$$E^{\nu_\alpha}[V_t^0]\to 0 \qquad \text{uniformly in } \alpha\in[0,1] \text{ as } t\to\infty. \tag{2.3}$$

PROOF. It suffices to show that (2.3) holds for each fixed α. To see this, set $f_t(\alpha)=E^{\nu_\alpha}[V_t^0]$. The functions f_t are continuous, with $f_t(\alpha)\downarrow 0$ as $t\uparrow\infty$ for each α. The uniformity in (2.3) therefore follows from the compactness of $[0,1]$.

In order to show $E^{\nu_\alpha}[V_t^0]\to 0$ as $t\to\infty$ for a given α, we instead show that the density of discrepancies of (η_t,η_t^1) decreases to 0 as $t\to\infty$. The two are equivalent, since for each t, (η_t, η_t^1) is translation invariant and ergodic [see, e.g., Liggett (1985), page 38], which implies the existence of a nonrandom limiting density over $[-x, x]$ as $x \to \infty$. For this, we consider the Cesaro average of the measures corresponding to (η_s, η_s^1) over $[0, t]$; that is, the measure λ_t on $(\{0,1\} \times \{0,1\})^{\mathbb{Z}}$ given by

$$E^{\lambda_t}[g(\eta, \xi)] = \frac{1}{t} \int_0^t E^{\nu_\alpha}[g(\eta_s, \eta_s^1)]\, ds \tag{2.4}$$

for bounded continuous functions g. It suffices to show that the weak limit λ of any converging subsequence λ_{t_n} as $t_n \to \infty$ has no discrepancies; that is, λ is concentrated on $\eta = \xi$.

The measure λ is stationary in time (although, conceivably not ergodic). Since λ_t is translation invariant, so is λ. Also, since $p(\cdot)$ is irreducible, λ assigns no mass to configurations (η, ξ) that have discrepancies of both types; otherwise, the density of discrepancies would decrease over time [see, e.g., Liggett (1985), page 385]. So, λ is a convex combination $\lambda = \sum_{i=1}^3 c_i \lambda^i$ of measures λ^1, λ^2 and λ^3, concentrated respectively on $\eta = \xi$, $\eta < \xi$ and $\eta > \xi$.

These three measures are each translation invariant. Consequently, so are their marginals. But, η_0 and η_0^1 were assumed to have measure ν_α, which is stationary, and thus the marginals of λ are also ν_α. They are the convex combination of the marginals of λ^1, λ^2 and λ^3. Since ν_α is extremal among translation invariant measures, it follows that the marginals of λ^i are also ν_α when $c_i > 0$. Since the first and second marginals have different densities for both λ^2 and λ^3, $c_2 = c_3 = 0$ must therefore hold. So, λ is concentrated on $\eta = \xi$, as desired. $\square$

Let $(\eta_\cdot, \eta_\cdot^1)$ be a coupled exclusion process, where η_0 is stationary with distribution ν. Lemma 2.4 states that the expected number of discrepancies that are initially in $[-N, N]$ and persist up until time t is $o(N)$ as $t \to \infty$ and $N \to \infty$. The proof uses Lemma 2.3, and our ability to write a Cesaro limit of ν and its translates as a convex combination of the product measures ν_α.

LEMMA 2.4. *Let $\eta_\cdot$ be an exclusion process with a stationary initial measure ν. Assume that $p(\cdot)$ is irreducible. Then,*

$$\frac{1}{N} \sum_{k=-N}^{N} E^\nu[V_t^k] \to 0 \qquad \textit{as } t \to \infty \textit{ and } N \to \infty. \tag{2.5}$$

PROOF. Let ν^k denote the translation of the measure ν by k units to the left, and let $\bar{\nu}^N$ be the measure which is the Cesaro average of $\nu^{-N}, \dots, \nu^N$. Suppose that $\bar{\nu}$ is the weak limit of $\bar{\nu}^{N_i}$ along some subsequence N_i. Since $f(\eta) = E^\eta[V_t^0]$ is continuous in η for a given t,

$$E^{\bar{\nu}}[V_t^0] = \lim_{i\to\infty} E^{\bar{\nu}^{N_i}}[V_t^0] = \lim_{i\to\infty} \frac{1}{2N_i + 1} \sum_{k=-N_i}^{N_i} E^\nu[V_t^k]. \tag{2.6}$$

The measure $\bar{\nu}$ is both stationary and translation invariant, and $p(\cdot)$ is irreducible. So, $\bar{\nu}$ is a convex combination of ν_α, $\alpha \in [0,1]$. It therefore follows from Lemma 2.3, that $E^{\bar{\nu}}[V_t^0] \to 0$ as $t \to \infty$. Since $E^{\nu}[V_t^k]$ is decreasing in t for each k, it follows from this limit and (2.6) that

$$\frac{1}{N_i} \sum_{k=-N_i}^{N_i} E^{\nu}[V_t^k] \to 0 \qquad \text{as } t \to \infty \text{ and } i \to \infty.$$

This implies (2.5). $\square$

Proposition 2.5 is the main result of this section; it is an important ingredient in the proof of Proposition 3.2. The proposition considers the same coupling of exclusion processes as in Lemma 2.4, and bounds the expected number of discrepancies to visit $[-T, T]$ at any time in $[\sqrt{T}, T]$, for large T. The proof is a simple application of Lemmas 2.2 and 2.4.

PROPOSITION 2.5. *Let $\eta_\cdot$ be an exclusion process with a stationary initial measure. Assume that $p(\cdot)$ has finite mean and is irreducible. Then, the expected number of discrepancies for $(\eta_\cdot, \eta_\cdot^1)$ that visit $[-T, T]$ at any time in $[\sqrt{T}, T]$ is $o(T)$ as $T \to \infty$.*

PROOF. It follows from Lemma 2.4, by setting $t = \sqrt{T}$ and $N = [MT]$, that the expected number of discrepancies that originate in $[-MT, MT]$ and still exist at time $\sqrt{T}$ is $o(T)$, for given M. It follows from Lemma 2.2, by setting $t = T$, that the expected number of discrepancies that originate in $[-MT, MT]^c$ and visit $[-T, T]$ by time T is $o(T)$, for M chosen as in (2.1). The assertion of the proposition is an immediate consequence of these two observations. $\square$

3. The stationary measures. In this section, we prove Theorem 1.1. The argument requires several steps, which we now sketch. Consider an extremal stationary measure ν for the exclusion process. Let ν^1 denote its translate one unit to the left. In Proposition 3.2, we couple ν and ν^1 using a stationary joint measure λ, and show that $\lambda(\{(\eta, \xi) : \eta \le \xi \text{ or } \eta \ge \xi\}) = 1$. For this, we need to show that positive and negative discrepancies cannot both exist for any configuration under λ. Proposition 2.5 and Lemma 3.1 provide the main ingredients.

It follows from Proposition 3.2 that λ is a convex combination of stationary measures λ^1, λ^2 and λ^3 concentrated on $\eta = \xi$, $\eta < \xi$ and $\eta > \xi$, respectively. Because of the extremality of ν, it is in fact not difficult to show that λ must be equal to one of these three measures. When $\lambda = \lambda^1$, one has $\nu = \nu^1$; that is, ν is translation invariant. So, $\nu = \nu_\alpha$ for some $\alpha \in [0,1]$. When $\lambda = \lambda^2$, it follows from Proposition 3.3 that ν is a profile measure. In Proposition 3.4, we show that $\lambda = \lambda^3$

is not possible when $\mu > 0$. So, if ν is extremal and stationary but not translation invariant, it must, in fact, be a profile measure.

In order to complete the proof of Theorem 1.1, we need to show that any extremal stationary profile measure ν' is a translate of a given ν. Proposition 3.5, the analog of Proposition 3.2, compares ν and ν', and shows that under any stationary joint measure γ, one has $\gamma(\{(\eta, \xi) : \eta \le \xi \text{ or } \eta \ge \xi\}) = 1$. The same reasoning as before shows γ is either concentrated on $\eta = \xi$, $\eta < \xi$ or $\eta > \xi$. Comparison of translates ν^n of ν with ν', using Propositions 3.3 and 3.5, shows that $\nu^n = \nu'$ for some n.

For Propositions 3.2 and 3.5, we will need to know that the probability that discrepancies of opposite types at given sites disappear by a fixed time is bounded away from 0. To show this, let $A^{x,y}$ denote the set of configurations (η, ξ) that have discrepancies of opposite types at x and y. Let $a^{x,y}$ be the infimum, over all $(\eta, \xi) \in A^{x,y}$, of the probability that at least one of these discrepancies disappears by time 1 for a coupled exclusion process with initial configuration (η, ξ).

LEMMA 3.1. *For a coupled exclusion process with irreducible $p(\cdot)$, $a^{x,y} > 0$ for all $x \neq y$.*

PROOF. Since $p(\cdot)$ is irreducible, there exist $x_0, \dots, x_m$, with $x_0 = x$ and $x_m = y$, so that $p(x_i - x_{i-1}) > 0$ for each i [or $p(x_{i-1} - x_i) > 0$ for each i, in which case the proof is the same]. Fix these $x_0, \dots, x_m$, set $L = \{x_0, \dots, x_m\}$ and

$$G = \{(x_{i-1}, x_i) : 1 \le i \le m\} \cup \{(z, w) : z \notin L, w \notin L\}.$$

With positive probability, no event in $\mathcal{N}^{u,v}$ will occur for any $(u, v) \in G^c$ by time 1.

Assume now that none of these events, $(u, v) \in G^c$, occurs by time 1. When $t \in \mathcal{N}^{x_{i-1}, x_i}$, we will say there is a potential move to the right from x_{i-1} at time t. We consider the event of $\binom{m+1}{2}$ potential moves to the right, that are respectively from x_{m-1}; x_{m-2}, x_{m-1}; x_{m-3}, x_{m-2}, x_{m-1}; $\dots$; x_0, $x_1, \dots, x_{m-1}$, and that are stipulated to occur by time 1, and in the order in which they are listed. If no other potential moves on L occur during this time, this sequence results in a configuration in which all particles are as far to the right as possible; that is, its coordinates have the form

$$0\,0\,0\,0\,0\,0\,1\,1\,1\,1\,1$$

$$0\,0\,0\,1\,1\,1\,1\,1\,1\,1\,1$$

on L. In particular, at most one type of discrepancy can remain on L. So, at least one of the discrepancies initially at x and y will have disappeared. The probability that there are events in the Poisson processes $\mathcal{N}^{x_{i-1}, x_i}$ that occur in exactly this order by time 1 with no other potential moves on L is positive, and depends only on L, and not on the configuration. Since these Poisson processes are independent of $\mathcal{N}^{u,v}$ for $(u, v) \in G^c$, it follows that $a^{x,y} > 0$. $\square$

Let $\eta_\cdot$ be an exclusion process with an extremal stationary measure ν as its initial measure and denote by λ_t the measure at time t of the coupled process $(\eta_\cdot, \eta_\cdot^1)$. We write $\bar{\lambda}_T$ for the Cesaro average of λ_t over $[\sqrt{T}, T-1]$. Suppose that λ is the weak limit of $\bar{\lambda}_T$ along some subsequence T_i, where $T_i \to \infty$ as $i \to \infty$. Then, λ is stationary with respect to the coupled pair of exclusion processes corresponding to its coordinates; the marginals of λ are ν and ν^1.

In order to show that ν is either a product measure or a profile measure, we will use λ. Proposition 3.2 is a first step in this direction. It follows with some work from Proposition 2.5 and Lemma 3.1.

PROPOSITION 3.2. *Assume that $p(\cdot)$ has finite mean μ and is irreducible. Then*, $\lambda(\{(\eta, \xi) : \eta \le \xi \textit{ or } \eta \ge \xi\}) = 1$.

PROOF. We need to show that $\lambda(A^{x,y}) = 0$ for each x and y. Let $w_{t_1,t_2}^{x,y}$ be the expected number of discrepancies for the coupled exclusion process $(\eta_\cdot, \eta_\cdot^1)$, where η_0 has the stationary measure ν, that visit either x or y and disappear during $[t_1, t_2]$. By Proposition 2.5,

$$\lim_{T\to\infty} \frac{1}{T} w_{\sqrt{T},T}^{x,y} = 0. \tag{3.1}$$

On the other hand,

$$w_{t_1,t_2}^{x,y} \ge a^{x,y} E^\nu\left[\int_{t_1}^{t_2-1} \mathbb{1}_{\{(\eta_t,\eta_t^1)\in A^{x,y}\}}\, dt\right], \tag{3.2}$$

where $A^{x,y}$ and $a^{x,y}$ are given before Lemma 3.1 and $\mathbb{1}_G$ is the indicator function of G. To see (3.2), we note that if N_{t_1,t_2} is the maximal number of points in $[t_1, t_2]$ at which $A^{x,y}$ occurs, that are each at least distance 1 apart, then $N_{t_1,t_2} \ge \int_{t_1}^{t_2} \mathbb{1}_{\{(\eta_t,\eta_t^1)\in A^{x,y}\}}\, dt$. The inequality then follows by repeated application of the strong Markov property.

On the subsequence T_i on which λ is defined,

$$\frac{1}{T_i - 1 - \sqrt{T_i}} E^\nu\left[\int_{\sqrt{T_i}}^{T_i-1} \mathbb{1}_{\{(\eta_t,\eta_t^1)\in A^{x,y}\}}\, dt\right] \to \lambda(A^{x,y}) \qquad \text{as } i \to \infty.$$

Together with (3.2), this implies

$$\liminf_{T\to\infty} \frac{1}{T} w_{\sqrt{T},T}^{x,y} \ge a^{x,y}\lambda(A^{x,y}). \tag{3.3}$$

In order for (3.1) and (3.3) to be consistent, $\lambda(A^{x,y}) = 0$ must hold, as desired. $\square$

It follows from Proposition 3.2 that λ is a convex combination $\sum_{i=1}^3 c_i\lambda^i$ of measures λ^1, λ^2 and λ^3 concentrated on $\eta = \xi$, $\eta < \xi$ and $\eta > \xi$, respectively. It is not difficult to see that λ must equal one of these measures: the three measures are stationary, and hence so are their marginals. The marginals of λ are ν and ν^1, which are stationary and extremal. Hence, the corresponding marginals of λ^i are also ν and ν^1, when $c_i > 0$. This is not simultaneously possible for both marginals unless $c_i = 1$ for some i, because of the inequalities between η and ξ for the sets defining λ^i.

When $\lambda = \lambda^1$, one has $\nu = \nu^1$, and hence ν is translation invariant as well as being stationary and extremal. So, $\nu = \nu_\alpha$ for some $\alpha \in [0, 1]$. We need to investigate the other two cases. The following result provides information on the behavior of $\nu(\{\eta : \eta(x) = 1\})$ as $x \to \pm\infty$ when either $\lambda = \lambda^2$ or $\lambda = \lambda^3$.

PROPOSITION 3.3. (a) *Suppose* $\lambda(\{(\eta, \xi) : \eta < \xi\}) = 1$. *Then*,

$$\lim_{x\to\infty} \nu\bigl(\{\eta : \eta(x) = 1\}\bigr) = \lim_{x\to-\infty} \nu\bigl(\{\eta : \eta(x) = 0\}\bigr) = 1. \tag{3.4}$$

Moreover, $\lim_{n\to\infty}(E^{\nu^1}[\sum_{x=-n}^n \eta(x)] - E^{\nu}[\sum_{x=-n}^n \eta(x)]) = 1$.

(b) *Suppose* $\lambda(\{(\eta, \xi) : \eta > \xi\}) = 1$. *Then*,

$$\lim_{x\to\infty} \nu\bigl(\{\eta : \eta(x) = 0\}\bigr) = \lim_{x\to-\infty} \nu\bigl(\{\eta : \eta(x) = 1\}\bigr) = 1. \tag{3.5}$$

Moreover, $\lim_{n\to\infty}(E^{\nu^1}[\sum_{x=-n}^n \eta(x)] - E^{\nu}[\sum_{x=-n}^n \eta(x)]) = -1$.

PROOF. We will show (a); (b) follows from (a) by symmetry. Set

$$f_n(\eta, \xi) = \sum_{x=-n}^{n} \mathbb{1}_{\{\eta(x)<\xi(x)\}}. \tag{3.6}$$

By assumption, $f_n(\eta, \xi) \ge 0$ with $f_n(\eta, \xi) \ge 1$ a.s. for sufficiently large n. Therefore,

$$\lim_{n\to\infty} E^{\lambda}[f_n(\eta, \xi)] \ge 1. \tag{3.7}$$

However, the marginals of λ are ν and ν^1, and ν^1 is the translate of ν, so $E[f_n(\eta, \xi)]$ can be written as a telescoping series,

$$\begin{aligned} E^{\lambda}[f_n(\eta, \xi)] &= \sum_{x=-n}^{n} \bigl[\lambda\bigl(\{\xi(x) = 1\}\bigr) - \lambda\bigl(\{\eta(x) = 1\}\bigr)\bigr] \\ &= \sum_{x=-n}^{n} \bigl[\nu^1\bigl(\{\eta : \eta(x) = 1\}\bigr) - \nu\bigl(\{\eta : \eta(x) = 1\}\bigr)\bigr] \\ &= \nu\bigl(\{\eta : \eta(n+1) = 1\}\bigr) - \nu\bigl(\{\eta : \eta(-n) = 1\}\bigr). \end{aligned}$$

Together with (3.7), this implies that (3.4) holds and $\lim_{n\to\infty} E^{\lambda}[f_n(\eta, \xi)] = 1$. [In fact, $\lim_{n\to\infty} f_n(n, \xi) = 1$ a.s.] This limit immediately implies the other limit in (a). $\square$

When $\mu > 0$, we can rule out the case $\lambda = \lambda^3$ in Proposition 3.4. The argument applies the asymptotics in (3.5) to a coupling of ν with $\nu_{1/2}$. [Any choice of ν_α, $\alpha \in (0, 1)$, can be used.]

PROPOSITION 3.4. *Suppose* $\lambda(\{(\eta, \xi) : \eta > \xi\}) = 1$ *and that* μ *exists. Then*, $\mu \leq 0$.

PROOF. There exists a coupled exclusion process $(\eta_\cdot, \xi_\cdot)$, with stationary initial measure γ having marginal measures ν and $\nu_{1/2}$ on its first and second coordinates. [See Liggett (1985), page 383.] Let $D_n(\eta, \xi)$ denote the number of discrepancies for (η, ξ) that lie in $(-n, n)$. We will consider the rate of change of $E^\gamma[D_n(\eta_t, \xi_t)]$ at $t = 0$. Since γ is stationary, this rate is zero. On the other hand, the following computation will show that it is strictly negative for large n unless $\mu \leq 0$.

We will show that c_n, the difference of the rates at which discrepancies move from $(-\infty, -n]$ to $(-n, n)$ and from $(-n, n)$ to $(-\infty, -n]$, is strictly negative for large enough n, if $\mu > 0$. By symmetry, the analog also holds for the intervals $[n, \infty)$ and $(-n, n)$. (One obtains the same process by replacing particles by holes, holes by particles, and substituting $-x$ for x.) Since the disappearance of discrepancies can only decrease D_n further,

$$\frac{d}{dt} E^\gamma[D_n(\eta_t, \xi_t)]\Big|_{t=0} \leq 2c_n < 0$$

for large n, if $\mu > 0$. However, this contradicts the stationarity of γ, and so one must have $\mu \leq 0$ as claimed.

Let $x \in (-\infty, -n]$ and $y \in (-n, n)$. Under an event in $\mathcal{N}^{x,y}$, a discrepancy moves from x to y when one coordinate at x and neither at y is occupied, and a discrepancy moves from y to x when both coordinates at x and one coordinate at y are occupied. Analogous behavior occurs under an event in $\mathcal{N}^{y,x}$. Consequently,

$$\begin{aligned} c_n = \sum_{z>0} \Bigg[p(z) \sum_{\substack{y-x=z \\ x \leq -n,\ |y|<n}} & \Big(\gamma\big(\{\eta(x) \neq \xi(x),\ \eta(y) = \xi(y) = 0\}\big) \\ & \qquad - \gamma\big(\{\eta(x) = \xi(x) = 1,\ \eta(y) \neq \xi(y)\}\big) \Big) \\ + p(-z) \sum_{\substack{y-x=z \\ x \leq -n,\ |y|<n}} & \Big(\gamma\big(\{\eta(x) \neq \xi(x),\ \eta(y) = \xi(y) = 1\}\big) \\ & \qquad - \gamma\big(\{\eta(x) = \xi(x) = 0,\ \eta(y) \neq \xi(y)\}\big) \Big) \Bigg]. \end{aligned}$$

By part (b) of Proposition 3.3 and the choice of measure $\nu_{1/2}$ for the second coordinate of γ, the probabilities of the first and fourth sets in the display go to 0 as

$x, y \to -\infty$, and the probabilities of the middle two sets go to $1/4$ as $x, y \to -\infty$. The functions inside the brackets are dominated by $z(p(z) + p(-z))$, which is integrable by assumption. So, by dominated convergence,

$$\lim_{n\to\infty} c_n = -\frac{1}{4}\sum_{z>0} z\big(p(z) - p(-z)\big) = -\frac{1}{4}\sum_z zp(z) < 0. \qquad \square$$

After Proposition 3.2, we concluded that λ is either concentrated on configurations of the form $\eta = \xi$, $\eta < \xi$ or $\eta > \xi$. In the first case, the extremal stationary measure ν is translation invariant (and hence equal to ν_α, for some α). By Proposition 3.4, the last case is not possible when $\mu \in (0, \infty)$. So, any extremal non-translation invariant stationary measure corresponds to the case $\eta < \xi$, and hence by part (a) of Proposition 3.3, is a profile measure. Thus all extremal stationary measures are either homogeneous product measures or profile measures. In order to complete the proof of Theorem 1.1, we only need to show that these profile measures, if they exist, form a one parameter family of translates.

The reasoning is similar to that already used to compare ν with its translate ν^1. Let ν and ν' be extremal stationary profile measures for the exclusion process. Let γ be a stationary measure for the coupled exclusion process that has marginals ν and ν'. Then, the following analog of Proposition 3.2 holds.

PROPOSITION 3.5. *Assume that $p(\cdot)$ has finite mean and is irreducible. Then*, $\gamma\{(\eta, \xi) : \eta \le \xi$ *or* $\eta \ge \xi\} = 1$.

PROOF. We need to show that $\gamma(A^{x,y}) = 0$ for each x and y, where $A^{x,y}$ is the set of configurations (η, ξ) that have discrepancies of opposite types at x and y.

Since ν and ν' are both profile measures, the expected number of discrepancies that originate in $[-MT, MT]$ is $o(T)$ as $T \to \infty$, for given M. Choosing M as in (2.1), it follows from Lemma 2.2 that the expected number of discrepancies that originate in $[-MT, MT]^c$ and visit $[-T, T]$ by time T is $o(T)$. So, the expected number of discrepancies that visit $[-T, T]$ by time T is $o(T)$.

On the other hand, the expected number of discrepancies at x or y at time t, that disappear by time $t + 1$, is at least $a^{x,y}\gamma(A^{x,y})$, where $a^{x,y}$ is the infimum given before Lemma 3.1. So, for large T, the expected number of discrepancies that visit $[-T, T]$ and disappear by time T is at least $a^{x,y}\gamma(A^{x,y})T$. By Lemma 3.1, $a^{x,y} > 0$. Because of the previous paragraph, it follows that $\gamma(A^{x,y}) = 0$. Since this holds for each x and y, the proposition follows. $\square$

Using the same reasoning as immediately following Proposition 3.2, it follows from Proposition 3.5 that either $\gamma(\{\eta = \xi\}) = 1$, $\gamma(\{\eta < \xi\}) = 1$ or $\gamma(\{\eta > \xi\}) = 1$. An elementary translation argument together with Proposition 3.3 enables us to finish the proof of Theorem 1.1.

PROOF THAT ν' IS A TRANSLATE OF ν. Denote by ν^n the translate of ν by n units to the left. Let γ^n denote the stationary measure before Proposition 3.5 with marginals ν^n and ν'. Then, either $\gamma^n(\{\eta = \xi\}) = 1$, $\gamma^n(\{\eta < \xi\}) = 1$ or $\gamma^n(\{\eta > \xi\}) = 1$ for a given n.

As $n \to \infty$ or $n \to -\infty$, ν^n converges weakly to point masses on $\eta \equiv 1$ or $\eta \equiv 0$, respectively. So, for some n_0, $\gamma^n(\{\eta < \xi\}) = 1$ holds for $n = n_0 - 1$ but not for $n = n_0$. In particular,

$$\begin{aligned}
&\lim_{m\to\infty}\left(E^{\nu'}\left[\sum_{x=-m}^{m}\eta(x)\right] - E^{\nu^{n_0-1}}\left[\sum_{x=-m}^{m}\eta(x)\right]\right) \geq 1, \\
&\lim_{m\to\infty}\left(E^{\nu'}\left[\sum_{x=-m}^{m}\eta(x)\right] - E^{\nu^{n_0}}\left[\sum_{x=-m}^{m}\eta(x)\right]\right) \leq 0.
\end{aligned} \tag{3.8}$$

On the other hand, by part (a) of Proposition 3.3,

$$\lim_{m\to\infty}\left(E^{\nu^{n_0}}\left[\sum_{x=-m}^{m}\eta(x)\right] - E^{\nu^{n_0-1}}\left[\sum_{x=-m}^{m}\eta(x)\right]\right) = 1.$$

It follows from this that the limit on the first line of (3.8) must be 1, and that on the second line must be 0. So, of the three possibilities for $n = n_0$ in the first paragraph, only $\gamma^{n_0}(\{\eta = \xi\}) = 1$ can hold. That is, $\nu^{n_0} = \nu'$, and so ν' is a translate of ν. $\square$

4. Nonexistence of blocking measures. Here, we demonstrate Theorem 1.2, namely that stationary blocking measures cannot exist for an exclusion process $\eta_{\cdot}$ when its random walk kernel $p(\cdot)$ has finite mean and satisfies

$$\sum_{x<0} x^2 p(x) = \infty. \tag{4.1}$$

The basic idea will be to show that, on account of (4.1), particles far to the right of 0 jump frequently enough far to the left of 0, so that a stationary measure must have particles arbitrarily far to the left or holes arbitrarily far to the right. Hence, such a measure will not be a blocking measure.

Let Y_t^k denote the position of the particle or hole at time t that was initially at k. In both cases, a jump from either x to y or from y to x occurs at most at rate $p(y-x) + p(x-y)$. Let $Z_{\cdot}$ denote an increasing random walk with $Z_0 = 0$, and which jumps from x to y, $y > x$, at this rate. For an appropriate copy of $Z_{\cdot}$ which is adapted to $\mathcal{F}_{\cdot}$,

$$|Y_t^k - Y_s^k| \leq Z_t - Z_s \qquad \text{for all } s \leq t.$$

Note that Z_1 has mean $\sum_x |x| p(x) < \infty$. We choose M so that

$$M > \sum_x |x| p(x), \tag{4.2}$$

and let τ denote a stopping time for $\eta_\cdot$. It follows by applying the strong Markov property and the law of large numbers to $Z_\cdot$, that

$$P\bigl(|Y_u^k - Y_\tau^k| \geq Ms \text{ for some } u \in [\tau, \tau+s] \mid \mathcal{F}_\tau\bigr) \leq 1/3 \qquad \text{a.s.} \tag{4.3}$$

for large enough s, which does not depend on k or the choice of τ.

Let A_L denote the set of η with either $\eta(x) = 1$ for some $x \leq -L$ or $\eta(x) = 0$ for some $x \geq L$. For m and r with $m \leq r$, define

$$\tau_r(m) = \inf\{t \in [2^r - 2^m, 2^r] : \eta_t \in A_{(M+1)2^m}\},$$

with $\tau_r(m) = \infty$ if no such t exists. Set $\tau_{\ell,r} = \min_{m \in [\ell,r]} \tau_r(m)$, where $\ell \leq r$.

On the event $\tau_{\ell,r} \leq 2^r$, $\eta_{\tau_{\ell,r}}$ either has a particle to the left of $-(M+1)2^\ell$ or a hole to the right of $(M+1)2^\ell$. Using (4.3), the following lemma shows that, for large enough ℓ, this event usually persists at time 2^r.

LEMMA 4.1. *Let $\eta_\cdot$ be an exclusion process for which $p(\cdot)$ has finite mean. Then, for $r \geq \ell$ and large ℓ,*

$$P(\eta_{2^r} \in A_{2^\ell} \mid \mathcal{F}_{\tau_{\ell,r}}) \geq 2/3 \qquad \textit{a.s. on } \{\tau_{\ell,r} \leq 2^r\}. \tag{4.4}$$

PROOF. Assume that $\tau_{\ell,r} = \tau_r(m_0) \leq 2^r$, and let $Y_{\tau_{\ell,r}}^k$ be the position at this time of a particle to the left of $-(M+1)2^{m_0}$ or a hole to the right of $(M+1)2^{m_0}$. Since $\tau_{\ell,r}$ is a stopping time, it follows from (4.3) that

$$P\bigl(|Y_{2^r}^k - Y_{\tau_{\ell,r}}^k| \geq M2^{m_0} \mid \mathcal{F}_{\tau_{\ell,r}}\bigr) \leq 1/3 \qquad \text{a.s.}$$

for large enough ℓ, on this set. Off of the exceptional set in the display, $Y_{2^r}^k$ lies to the left of -2^{m_0} or to the right of 2^{m_0}, depending on whether $Y_\cdot^k$ is a particle or a hole. Since $\ell \leq m_0$, one has $\eta_{2^r} \in A_{2^\ell}$, which implies (4.4). □

Using Lemma 4.1, we obtain the following behavior of $\eta_\cdot$ for large times starting from an arbitrary initial state.

PROPOSITION 4.2. *Let $\eta_\cdot$ be an exclusion process for which $p(\cdot)$ has finite mean and satisfies* (4.1). *For given L, there exists a t_L not depending on η_0, so that for all $t \geq t_L$,*

$$P^{\eta_0}\bigl(\{\eta_t(x) = 1 \textit{ for some } x \leq -L \textit{ or } \eta_t(x) = 0 \textit{ for some } x \geq L\}\bigr) \geq 1/3. \tag{4.5}$$

Any $\eta_\cdot$ with stationary initial measure ν must satisfy the bound given in (4.5) (with η_0 replaced by ν) at all times, including at time 0. Such a measure is obviously not a blocking measure, and so Theorem 1.2 follows immediately from Proposition 4.2.

PROOF OF PROPOSITION 4.2. By restarting $\eta_\cdot$, it suffices to show (4.5) at $t = 2^r$, for appropriate r depending on L. We may also assume $L = 2^\ell$, where ℓ is large.

Let $B_{\ell,r}$ be the event that for some $m \in [\ell, r]$, there are

$$x \in (-\infty, -(M+1)2^m], \qquad y \in [(M+1)2^m, (M+2)2^m] \quad \text{and} \quad t \in [2^r - 2^m, 2^r], \tag{4.6}$$

with $t \in \mathcal{N}^{y,x}$. If $B_{\ell,r}$ occurs at time t, then $\eta_t(x) = 1$ or $\eta_t(y) = 0$ (or both), irrespective of the configuration at $t-$. In each case, $\eta_t \in A_{(M+1)2^m}$, and therefore $\tau_{\ell,r} \le 2^r$. We will show that for each ℓ, $P^{\eta_0}(B_{\ell,r}) \to 1$ uniformly in η_0 as $r \to \infty$. Together with Lemma 4.1, this implies that, uniformly in η_0,

$$\lim_{\ell\to\infty} \liminf_{r\to\infty} P^{\eta_0}(\eta_{2^r} \in A_{2^\ell}) \ge 2/3. \tag{4.7}$$

The bound in (4.5) follows by choosing ℓ and r sufficiently large in (4.7).

To show that $P^{\eta_0}(B_{\ell,r}) \to 1$ uniformly in η_0 as $r \to \infty$, we observe that

$$P^{\eta_0}(B_{\ell,r}) = 1 - \exp\left\{-\sum_{m=\ell}^{r} 2^m \sum_{y=(M+1)2^m}^{(M+2)2^m} \sum_{z=(M+1)2^m+y}^{\infty} p(-z)\right\}. \tag{4.8}$$

The two inner sums give the rate at which an event occurs in $\mathcal{N}^{y,x}$ for x and y chosen as in (4.6), with $z = y - x$, and the factor 2^m is the allocated amount of time for a given m. The outer sum follows from the independence of the subevents corresponding to different m, since the intervals $[(M+1)2^m, (M+2)2^m]$ are disjoint. Replacing y in (4.8) by its maximum $(M+2)2^m$ implies that

$$P^{\eta_0}(B_{\ell,r}) \ge 1 - \exp\left\{-\sum_{m=\ell}^{r} 2^{2m} \sum_{z=(2M+3)2^m}^{\infty} p(-z)\right\} \tag{4.9}$$

for all η_0. But, it follows from (4.1) that

$$\sum_{m=0}^{\infty} 2^{2m} \sum_{z=2^{m+j}}^{\infty} p(-z) = \infty$$

for a given j. [For $z \in [2^{m-1}, 2^m)$, $p(-z)$ has coefficient $1 + \cdots + 2^{2(m-j-1)} > 2^{-2(j+1)} \cdot 2^{2m}$ after reversing the order of summation.] So, the right-hand side of (4.9) goes to 1 as $r \to \infty$. $\square$

We point out that any stationary profile measure for which $p(\cdot)$ has finite mean and satisfies (4.1) is, in fact, concentrated on configurations with both an infinite number of particles to the left of any site and an infinite number of holes to the right of the site. To see this, note that the sets of configurations F_1 and F_2 satisfying these two events are each invariant with respect to the exclusion process. So, any extremal stationary profile measure ν with $\nu(F_1) + \nu(F_2) > 0$ will satisfy

$\nu(F_1) = 1$ or $\nu(F_2) = 1$; we need to show both must hold. Suppose that $\nu(F_i) = 1$ holds for a given i. The measure ν' obtained from ν by interchanging particles and holes, and substituting $-x$ for x is also an extremal stationary profile measure. By Theorem 1.1, for irreducible $p(\cdot)$, ν' must be a translate of ν, which can only be the case if $\nu(F_j) = 1$ for the other index j. With some additional work, the conclusion still holds when $p(\cdot)$ is not irreducible, since one can then apply Theorem 1.1 on the appropriate sublattices.

5. Existence of blocking measures under third moment assumptions. In this section and the next, we construct stationary blocking measures for the exclusion process under the monotonicity assumption (1.1) on the random walk kernel $p(\cdot)$. Proposition 5.4 is the main result in this section and provides criteria for the existence of a stationary blocking measure; together with Proposition 5.5, it implies Theorem 1.3. Our construction is based on monotonicity ideas from Ferrari, Lebowitz and Speer (2001) and from the proof of the contact process critical value bound in Holley and Liggett (1978). [See also of Liggett (1985), page 274, for the latter result.]

We begin with a few comments about the connections between our approach and that of Holley and Liggett (1978). To find an upper bound for the contact process critical value, it is necessary to prove that the process has a nontrivial stationary distribution for appropriate parameter values. The Holley–Liggett proof does so by finding an initial distribution (a renewal measure in that case) for which the distribution at time t is increasing in t in an appropriate sense. The usual stochastic monotonicity was not the appropriate sense there, and it is not the appropriate sense here. Ferrari, Lebowitz and Speer (2001) provides the right definition of monotonicity for us. We choose the initial distribution to be a product measure, since it is easy to compute with. The details of the proof here are quite different from those in the Holley–Liggett paper.

Let Ξ be defined by

$$\Xi = \left\{\eta : \sum_{x<0} \eta(x) < \infty,\ \sum_{x \geq 0} [1 - \eta(x)] < \infty\right\}.$$

The exclusion process corresponding to $p(\cdot)$ is a (countable state) Markov chain on Ξ. A stationary blocking measure is simply a stationary distribution for this Markov chain. Ferrari, Lebowitz and Speer (2001) defined a partial order on Ξ by saying that $\eta \preceq \zeta$ if η is obtained from ζ by moving finitely many particles to the right. More precisely, this means that

$$\sum_x [\eta(x) - \zeta(x)] = 0, \tag{5.1}$$

(note that only finitely many terms in this sum are nonzero, since $\eta, \zeta \in \Xi$) and that

$$\sum_{x \leq u} \eta(x) \leq \sum_{x \leq u} \zeta(x) \tag{5.2}$$

for all u. In the presence of (5.1), (5.2) is equivalent to

$$\sum_{x\geq u}[1-\eta(x)] \leq \sum_{x\geq u}[1-\zeta(x)] \tag{5.3}$$

for all u. We will say that a function F on Ξ is increasing if $\eta \preceq \zeta$ implies $F(\eta) \leq F(\zeta)$.

In their paper, Ferrari, Lebowitz and Speer prove a more general form of the following statement (see their Lemma 4.2): If $p(\cdot)$ satisfies (1.1) and η_0 and ζ_0 satisfy $\eta_0 \preceq \zeta_0$, then the processes η_t and ζ_t with these initial configurations can be coupled so that $\eta_t \preceq \zeta_t$ for all $t \geq 0$. An easy consequence of this is the following proposition. Here and later on, we denote by $q(\eta,\zeta)$ the transition rates of the exclusion process on Ξ; that is,

$$\begin{aligned} q(\eta,\eta_{xy}) &= p(y-x)\eta(x)[1-\eta(y)] + p(x-y)\eta(y)[1-\eta(x)], \\ q(\eta,\eta) &= -\sum_{x,y} p(y-x)\eta(x)[1-\eta(y)], \end{aligned} \tag{5.4}$$

and $q(\eta,\zeta)=0$ otherwise.

PROPOSITION 5.1. *Assume that the random walk kernel $p(\cdot)$ has a finite mean.*

(a) *Suppose that $p(\cdot)$ satisfies* (1.1). *If F is a bounded increasing function on Ξ, then so is the function $\eta \to E^{\eta}[F(\eta_t)]$ for any $t \geq 0$.*

(b) *Suppose that F is bounded, and ν is a probability measure on Ξ that satisfies* (1.3). *Then*

$$\frac{d}{dt}E^{\nu}[F(\eta_t)]\Big|_{t=0} = \sum_{\eta,\zeta\in\Xi} \nu(\eta)q(\eta,\zeta)F(\zeta). \tag{5.5}$$

If $p(\cdot)$ also satisfies (1.1) *and*

$$\frac{d}{dt}E^{\nu}[F(\eta_t)]\Big|_{t=0} \leq 0 \tag{5.6}$$

for all bounded increasing functions F, then $E^{\nu}[F(\eta_t)]$ is nonincreasing in t for each such F. Furthermore, there exists a stationary blocking measure π that satisfies

$$\int F\,d\pi \leq \int F\,d\nu \tag{5.7}$$

for all nonnegative increasing functions F.

PROOF. For part (a), use the coupling (η_t, ζ_t) provided by Ferrari, Lebowitz and Speer and take expected values of both sides of the inequality $F(\eta_t) \leq F(\zeta_t)$.

Turning to part (b), note first that (1.3) implies that ν concentrates on Ξ. Furthermore,

$$\begin{aligned}\sum_{\eta\neq\zeta}\nu(\eta)q(\eta,\zeta)&=\sum_{\eta}\nu(\eta)\sum_{x,y}p(y-x)\eta(x)[1-\eta(y)]\\ &\le\sum_{\eta}\nu(\eta)\biggl[\sum_{x\le 0}\eta(x)+\sum_{y\ge 0}[1-\eta(y)]+\sum_{y<0<x}p(y-x)\biggr],\end{aligned}\tag{5.8}$$

which is finite by (1.3) and the finite mean assumption on $p(\cdot)$. By the Kolmogorov backward equation for continuous time Markov chains,

$$\frac{d}{dt}E^{\eta}[F(\eta_t)]=\sum_{\zeta}q(\eta,\zeta)E^{\zeta}[F(\eta_t)]$$

for $t\ge 0$, $\eta\in\Xi$, and bounded F. Multiplying both sides of this expression by $\nu(\eta)$ and summing gives

$$\sum_{\eta}\nu(\eta)\frac{d}{dt}E^{\eta}[F(\eta_t)]=\sum_{\eta,\zeta}\nu(\eta)q(\eta,\zeta)E^{\zeta}[F(\eta_t)],\tag{5.9}$$

and if one first takes absolute values and the supremum on t,

$$\sum_{\eta}\nu(\eta)\sup_{t\ge 0}\left|\frac{d}{dt}E^{\eta}[F(\eta_t)]\right|\le\|F\|_\infty\sum_{\eta,\zeta}\nu(\eta)|q(\eta,\zeta)|=2\|F\|_\infty\sum_{\eta\neq\zeta}\nu(\eta)q(\eta,\zeta).$$

[The last equality comes from $\sum_\zeta q(\eta,\zeta)=0$.] Note that the double series on the right converges since (5.8) is finite. This justifies the exchange of order of summations above, and since we now have

$$\sum_{\eta}\nu(\eta)\sup_{t\ge 0}\left|\frac{d}{dt}E^{\eta}[F(\eta_t)]\right|<\infty,$$

the order of the derivative and the sum on η in (5.9) can be interchanged. This gives (5.5).

Now,

$$\frac{d}{dt}E^{\nu}[F(\eta_t)]=\sum_{\eta,\zeta}\nu(\eta)q(\eta,\zeta)E^{\zeta}[F(\eta_t)]\le 0,$$

where the equality comes from (5.9), and the inequality comes from (5.5) and (5.6) applied to the function $\zeta\to E^{\zeta}[F(\eta_t)]$, which is increasing by part (a). So, we conclude that $E^{\nu}[F(\eta_t)]\downarrow$ in t. If h is any bounded increasing function on $\mathbb{Z}_+^k$, then $F(\eta)=h(\sum_{x\le u_1}\eta(x),\dots,\sum_{x\le u_k}\eta(x))$ is a bounded increasing function on Ξ. Therefore, for the process with initial distribution ν, the distribution of $\sum_{x\le u}\eta_t(x)$ is stochastically decreasing in t, jointly in u. The same is true for the distribution of $\sum_{x\ge u}[1-\eta_t(x)]$. It follows that the limiting distribution π

of η_t exists, and is concentrated on Ξ. This distribution is stationary, and satisfies (5.7). $\square$

In order to use Proposition 5.1, we must find measures ν that satisfy (1.3) and (5.6). If $\alpha(\cdot)$ is a function on the integers that satisfies $0 < \alpha(x) < 1$ for all x, let ν_α be the product measure on $\{0, 1\}^{\mathbb{Z}}$ with marginals

$$\nu_\alpha(\{\eta : \eta(x) = 1\}) = \alpha(x).$$

These are natural first guesses for ν, since product measures are convenient for calculations. We need to find a useful expression for the left-hand side of (5.6) with this choice for ν. This is the purpose of the next result.

LEMMA 5.2. *Suppose that $p(\cdot)$ has a finite mean and $\alpha(\cdot)$ satisfies*

$$\sum_{x<0} \alpha(x) < \infty, \qquad \sum_{x\geq 0} [1 - \alpha(x)] < \infty. \tag{5.10}$$

For a bounded F on Ξ and $x, y \in \mathbb{Z}$, define

$$\phi(x, y) = \sum_\eta \nu_\alpha(\eta)\eta(x)[1 - \eta(y)][F(\eta_{xy}) - F(\eta)].$$

Then,

$$\frac{d}{dt} E^{\nu_\alpha}[F(\eta_t)]\Big|_{t=0} = \sum_{x,y} p(y - x)\phi(x, y). \tag{5.11}$$

Furthermore, if F is also increasing, then $\phi(x, y)$ has the same sign as $x - y$.

PROOF. First note that (5.10) is just (1.3) in this case. By (5.5),

$$\begin{aligned}
\frac{d}{dt} E^{\nu_\alpha}[F(\eta_t)]\Big|_{t=0} &= \sum_{\eta,\zeta} \nu_\alpha(\eta) q(\eta, \zeta) F(\zeta) \\
&= \sum_{x,y,\eta} p(y - x)\nu_\alpha(\eta)\eta(x)[1 - \eta(y)][F(\eta_{xy}) - F(\eta)] \\
&= \sum_{x,y} p(y - x)\phi(x, y),
\end{aligned}$$

which is (5.11). To check that the manipulations involving these series are justified, we need to know that the series on the right-hand side of (5.11) converges absolutely. This follows from

$$|\phi(x, y)| \leq 2\alpha(x)[1 - \alpha(y)]\|F\|_\infty, \tag{5.12}$$

the finite mean of $p(\cdot)$ and (5.10), since

$$\sum_{x,y} p(y - x)\alpha(x)[1 - \alpha(y)] \leq \sum_{x\leq 0} \alpha(x) + \sum_{y\geq 0} [1 - \alpha(y)] + \sum_{y<0<x} p(y - x) < \infty.$$

This proves (5.11).

For the final statement in the lemma, note that for η, x and y for which $\eta(x)=1$ and $\eta(y)=0$, $\eta_{xy}\preceq\eta$ if $x<y$ and $\eta\preceq\eta_{xy}$ if $x>y$. Therefore, if F is increasing, the summands of $\phi(x,y)$ have the same sign as $x-y$. $\square$

Note that if F is increasing, the right-hand side of (5.11) is negative if $p(x)=0$ for all $x<0$, as should be expected, since then particles can only move to the right. In general, in order to verify (5.6), we need to cancel out positive terms in (5.11) against negative terms. Here is the strategy. Proposition 5.3 below gives inequalities that allow us to replace $\phi(x,y)$ in (5.11) by sums of multiples of $\phi(z,z+1)$ or $\phi(z+1,z)$, $z\in\mathbb{Z}$, according to whether $x<y$ or $x>y$. It also relates $\phi(z,z+1)$ to $\phi(z+1,z)$. Consequently, the right-hand side of (5.11) can be bounded by an expression that just involves $\phi(z+1,z)$, which is nonnegative. Proposition 5.4 says that if the coefficients of $\phi(z+1,z)$ in the resulting expression have the right sign, then the right-hand side of (5.11) will be nonpositive, and hence (5.6) holds. Proposition 5.5 provides a relatively easy way of checking the main hypothesis of Proposition 5.4.

To begin this development, define $\beta(x)=\alpha(x)/[1-\alpha(x)]$. Note that $\beta(\cdot)$ is increasing whenever $\alpha(\cdot)$ is.

PROPOSITION 5.3. *Suppose F is a bounded increasing function on Ξ and $\alpha(\cdot)$ is increasing on $\mathbb{Z}$. Then, for $n\geq 1$ and $x\in\mathbb{Z}$,*

$$\phi(x,x+n)\frac{\beta(x+n)}{\beta(x)}\leq\sum_{z=x}^{x+n-1}\phi(z,z+1)\frac{\beta(z+1)}{\beta(z)}\tag{5.13}$$

and

$$\phi(x+n,x)\frac{\beta(x)}{\beta(x+n)}\leq\sum_{z=x}^{x+n-1}\phi(z+1,z)\frac{\beta(z)}{\beta(z+1)}.\tag{5.14}$$

Furthermore, for any z,

$$\phi(z+1,z)\beta(z)=-\phi(z,z+1)\beta(z+1)\geq 0.\tag{5.15}$$

PROOF. We begin with (5.15), which is the easiest part to prove. Let ζ have distribution ν_α. Then, since ν_α is a product measure,

$$\begin{aligned}\phi(z,z+1)=\alpha(z)[1-\alpha(z+1)]\{&E[F(\zeta)\mid\zeta(z)=0,\ \zeta(z+1)=1]\\&-E[F(\zeta)\mid\zeta(z)=1,\zeta(z+1)=0]\}\end{aligned}\tag{5.16}$$

and

$$\begin{aligned}\phi(z+1,z)=\alpha(z+1)[1-\alpha(z)]\{&E[F(\zeta)\mid\zeta(z)=1,\ \zeta(z+1)=0]\\&-E[F(\zeta)\mid\zeta(z)=0,\zeta(z+1)=1]\}.\end{aligned}\tag{5.17}$$

By the definition of $\beta(\cdot)$,

$$\alpha(z)[1-\alpha(z+1)]\beta(z+1) = \alpha(z+1)[1-\alpha(z)]\beta(z)$$
$$= \alpha(z)\alpha(z+1).$$

Therefore, multiplying (5.16) and (5.17) by $\beta(z+1)$ and $\beta(z)$, respectively, we conclude that the equality in (5.15) holds. The nonnegativity in (5.15) comes from the last statement of Lemma 5.2.

To prove (5.13), it is enough to show that

$$\phi(x, x+j+1)\frac{\beta(x+j+1)}{\beta(x)} - \phi(x, x+j)\frac{\beta(x+j)}{\beta(x)} \le \phi(x+j, x+j+1)\frac{\beta(x+j+1)}{\beta(x+j)}$$

for $j \ge 0$. To see this, note that (5.13) is obtained by summing these inequalities over $0 \le j \le n-1$, since the sum of the left-hand side telescopes and $\phi(x,x)=0$. Similarly, to prove (5.14), we want to show that

$$\phi(x+j+1, x)\frac{\beta(x)}{\beta(x+j+1)} - \phi(x+j, x)\frac{\beta(x)}{\beta(x+j)} \le \phi(x+j+1, x+j)\frac{\beta(x+j)}{\beta(x+j+1)}$$

for $0 \le j \le n-1$. Both of these inequalities follow from the following subadditivity statement: If $u < v < w$ or $w < v < u$, then

$$\phi(u, w)\frac{\beta(w)}{\beta(u)} \le \phi(u, v)\frac{\beta(v)}{\beta(u)} + \phi(v, w)\frac{\beta(w)}{\beta(v)}. \tag{5.18}$$

To check (5.18), define

$$g_{abc} = E\big[F(\zeta) \mid \zeta(u)=a, \zeta(v)=b, \zeta(w)=c\big] \tag{5.19}$$

for $a, b, c \in \{0, 1\}$. Then, arguing as in (5.16) and (5.17),

$$\frac{\phi(u, w)}{\alpha(u)[1-\alpha(w)]} = \alpha(v)[g_{011} - g_{110}] + [1-\alpha(v)][g_{001} - g_{100}],$$
$$\frac{\phi(u, v)}{\alpha(u)[1-\alpha(v)]} = \alpha(w)[g_{011} - g_{101}] + [1-\alpha(w)][g_{010} - g_{100}]$$

and

$$\frac{\phi(v, w)}{\alpha(v)[1-\alpha(w)]} = \alpha(u)[g_{101} - g_{110}] + [1-\alpha(u)][g_{001} - g_{010}].$$

Therefore, substituting $\alpha(\cdot)$ in for $\beta(\cdot)$,

$$\begin{aligned}
\phi(u,v)\frac{\beta(v)}{\beta(u)} &+ \phi(v,w)\frac{\beta(w)}{\beta(v)} - \phi(u,w)\frac{\beta(w)}{\beta(u)} \\
&= [1-\alpha(u)]\alpha(v)\alpha(w)[g_{011}-g_{101}] \\
&\quad + [1-\alpha(u)]\alpha(v)[1-\alpha(w)][g_{010}-g_{100}] \\
&\quad + \alpha(u)[1-\alpha(v)]\alpha(w)[g_{101}-g_{110}] \\
&\quad + [1-\alpha(u)][1-\alpha(v)]\alpha(w)[g_{001}-g_{010}] \\
&\quad - [1-\alpha(u)]\alpha(v)\alpha(w)[g_{011}-g_{110}] \\
&\quad - [1-\alpha(u)][1-\alpha(v)]\alpha(w)[g_{001}-g_{100}] \\
&= \alpha(w)[\alpha(u)-\alpha(v)][g_{101}-g_{110}] \\
&\quad + [1-\alpha(u)][\alpha(v)-\alpha(w)][g_{010}-g_{100}].
\end{aligned}$$

All the terms on the last line of this equality are nonnegative, since both α and F are increasing. To see this, note that since F is increasing, $g_{101} \le g_{110}$ if $v < w$ and $g_{101} \ge g_{110}$ if $w < v$, while $g_{010} \le g_{100}$ if $u < v$ and $g_{010} \ge g_{100}$ if $v < u$. This proves (5.18), and hence (5.13) and (5.14). $\square$

Combining Lemma 5.2 and Propositions 5.1 and 5.3, we obtain Proposition 5.4, which is the main result of this section. It provides a fairly general sufficient condition for the existence of a stationary blocking measure. The proposition given after the proof of Proposition 5.4 facilitates the verification of the condition (5.20) in certain cases.

The condition (5.20) may be a bit hard to absorb. To understand it, note that if $\beta(\cdot)$ were a constant, (5.20) would simply say that the mean of $p(\cdot)$ is nonnegative, since each of the inner sums is then equal to n. Of course, $\beta(\cdot)$ cannot be constant in view of (5.10). However, when the mean of $p(\cdot)$ is strictly positive, it should be possible to choose a function $\beta(\cdot)$ that is almost constant in such a way that (5.20) still holds. We will see later that this is often the case.

PROPOSITION 5.4. *Suppose that $p(\cdot)$ has a finite first moment and satisfies* (1.1). *Assume that $\alpha(\cdot)$ is increasing and satisfies* (5.10). *If*

$$\sum_{n=1}^{\infty} p(-n) \sum_{x=z-n+1}^{z} \frac{\beta(x+n)}{\beta(x)} \le \sum_{n=1}^{\infty} p(n) \sum_{x=z-n+1}^{z} \frac{\beta(x)}{\beta(x+n)} \tag{5.20}$$

for all $z \in \mathbb{Z}$, then there is a stationary blocking measure π that satisfies

$$\sum_{x \le u} \pi(\{\eta : \eta(x) = 1\}) \le \sum_{x \le u} \alpha(x)$$

and

$$\sum_{x\ge u}\pi\bigl(\{\eta:\eta(x)=0\}\bigr)\le\sum_{x\ge u}[1-\alpha(x)]$$

for all u.

PROOF. By Proposition 5.1, it suffices to check (5.6) for $\nu=\nu_\alpha$. We therefore need to check that the right-hand side of (5.11) is nonpositive for all bounded increasing F. By (5.13) and (5.14) of Proposition 5.3,

$$\begin{aligned}\sum_{x,y}p(y-x)\phi(x,y)&=\sum_{n=1}^{\infty}\left\{p(n)\sum_x\phi(x,x+n)+p(-n)\sum_y\phi(y+n,y)\right\}\\ &\le\sum_{n=1}^{\infty}\left\{p(n)\sum_x\sum_{z=x}^{x+n-1}\phi(z,z+1)\frac{\beta(x)\beta(z+1)}{\beta(x+n)\beta(z)}\right.\\ &\qquad\left.+p(-n)\sum_y\sum_{z=y}^{y+n-1}\phi(z+1,z)\frac{\beta(y+n)\beta(z)}{\beta(y)\beta(z+1)}\right\}.\end{aligned}\tag{5.21}$$

The series on the right-hand side of (5.21) converges absolutely. To see this, first consider the sum involving $p(n)$. All terms are nonpositive by Lemma 5.2, and the absolute value of the sum is bounded above by $2\|F\|_\infty$ times

$$\sum_{n=1}^{\infty}p(n)\sum_x\sum_{z=x}^{x+n-1}\alpha(z)[1-\alpha(z+1)]\frac{\beta(x)\beta(z+1)}{\beta(x+n)\beta(z)}.$$

Recalling the definition of $\beta(\cdot)$, and using its monotonicity to replace $\beta(x)/\beta(x+n)$ by one, this expression is at most

$$\sum_{n=1}^{\infty}p(n)\sum_x\sum_{z=x}^{x+n-1}\alpha(z+1)[1-\alpha(z)]=\sum_{n=1}^{\infty}np(n)\sum_z\alpha(z+1)[1-\alpha(z)],$$

which is finite by (5.10) and the finite first moment assumption on $p(\cdot)$. The sum involving $p(-n)$ contains only nonnegative terms, and is at most $2\|F\|_\infty$ times

$$\sum_{n=1}^{\infty}p(-n)\sum_z\sum_{y=z-n+1}^{z}\alpha(z)[1-\alpha(z+1)]\frac{\beta(y+n)}{\beta(y)}.$$

By substituting x for y and applying (5.20), this is at most

$$\begin{aligned}&\sum_z\alpha(z)[1-\alpha(z+1)]\sum_{n=1}^{\infty}p(n)\sum_{x=z-n+1}^{z}\frac{\beta(x)}{\beta(x+n)}\\ &\quad\le\sum_z\alpha(z)[1-\alpha(z+1)]\sum_{n=1}^{\infty}np(n),\end{aligned}$$

which is again finite.

Using (5.15), we get

$$\gamma(z)=\phi(z+1,z)\beta(z)=-\phi(z,z+1)\beta(z+1). \tag{5.22}$$

Interchanging the order of summation on the right-hand side of (5.21) yields

$$\sum_z \gamma(z)\sum_{n=1}^{\infty}\left\{-p(n)\sum_{x=z-n+1}^{z}\frac{\beta(x)}{\beta(z)\beta(x+n)}+p(-n)\sum_{y=z-n+1}^{z}\frac{\beta(y+n)}{\beta(z+1)\beta(y)}\right\}.$$

Since F is increasing, $\phi(z+1,z)\geq 0$ by the last statement of Lemma 5.2; consequently, $\gamma(z)\geq 0$. Since $\alpha(\cdot)$ is increasing, so is $\beta(\cdot)$, which allows us to replace the term $\beta(z)$ by $\beta(z+1)$. Application of (5.20) shows that, after summing over n, the first term in the above display dominates the second, and hence the quantity in the display is nonpositive, as required. $\square$

REMARK. The stationary blocking measure constructed in Proposition 5.4 not only has a finite expected number of particles to the left of the origin, but this number has finite exponential moments of all orders. To see this, apply (5.7) to the function $F(\eta)=\exp\{\sigma\sum_{x<0}\eta(x)\}$, for $\sigma>0$. Then,

$$\int F\,d\pi\leq\int F\,d\nu_\alpha=\prod_{x<0}[1+(e^\sigma-1)\alpha(x)]\leq\exp\left[(e^\sigma-1)\sum_{x<0}\alpha(x)\right]<\infty.$$

In the final inequality, we have used $1+u\leq e^u$.

In order to apply Proposition 5.4, we need an easily checkable sufficient condition for (5.20). This is the purpose of Proposition 5.5. Theorem 1.3 follows immediately from Propositions 5.4 and 5.5 by setting

$$g(x)=(1+x)[\log(e+x)]^{1+\delta/2},$$

$\delta>0$, in the latter result. Note that in order for the $\alpha(\cdot)$ provided in Proposition 5.5 to satisfy (5.10), g must satisfy

$$\int_0^\infty \frac{1}{g(t)}\,dt<\infty,$$

which explains the choice of $g(\cdot)$ above.

PROPOSITION 5.5. *Let $g(\cdot)$ be an increasing function on $[0,\infty)$ that satisfies $g(0)=1$ and*

$$\frac{d^2}{dx^2}\log g(x)\leq 0 \tag{5.23}$$

for all $x\geq 0$. For $\varepsilon>0$, define $\beta_\varepsilon(n)=g(\varepsilon n)$ and $\beta_\varepsilon(-n)=1/g(\varepsilon n)$ for $n\geq 0$. If $p(\cdot)$ has strictly positive mean and

$$\sum_{n=1}^{\infty} n[g(n)]^2 p(-n)<\infty, \tag{5.24}$$

then $\beta_\varepsilon(\cdot)$ satisfies (5.20) for sufficiently small $\varepsilon>0$.

PROOF. Note that (5.23) implies $\beta_\varepsilon(n)$ is logconcave for $n \geq 0$, and hence

$$\frac{\beta_\varepsilon(1)}{\beta_\varepsilon(0)} \geq \frac{\beta_\varepsilon(2)}{\beta_\varepsilon(1)} \geq \frac{\beta_\varepsilon(3)}{\beta_\varepsilon(2)} \geq \cdots.$$

Using these bounds, we will first check that

$$\sum_{x=z-n+1}^{z} \frac{\beta_\varepsilon(x+n)}{\beta_\varepsilon(x)} \leq n\big(\beta_\varepsilon(n)\big)^2 \tag{5.25}$$

for all z by considering the cases $x \geq 0$, $x \leq -n$ and $-n < x < 0$ separately. In the first case,

$$\frac{\beta_\varepsilon(x+n)}{\beta_\varepsilon(x)} \leq \frac{\beta_\varepsilon(n)}{\beta_\varepsilon(0)}$$

by the logconcavity of $\beta(\cdot)$, over $[0,\infty)$. In the second case, x and $x+n$ are both nonpositive, so

$$\frac{\beta_\varepsilon(x+n)}{\beta_\varepsilon(x)} = \frac{\beta_\varepsilon(-x)}{\beta_\varepsilon(-x-n)} \leq \frac{\beta_\varepsilon(n)}{\beta_\varepsilon(0)}.$$

In the third case,

$$\frac{\beta_\varepsilon(x+n)}{\beta_\varepsilon(x)} = \beta_\varepsilon(x+n)\beta_\varepsilon(-x) \leq \beta_\varepsilon(x+2n)\beta_\varepsilon(-x) \leq \big(\beta_\varepsilon(n)\big)^2,$$

where the first inequality follows from the monotonicity of $\beta_\varepsilon(\cdot)$ and the second inequality from its logconcavity over $[0,\infty)$. Since $\beta_\varepsilon(0) = 1$, (5.25) follows.

These three cases also imply that

$$\sum_{x=z-n+1}^{z} \frac{\beta_\varepsilon(x)}{\beta_\varepsilon(x+n)} \geq n\big(\beta_\varepsilon(n)\big)^{-2} \tag{5.26}$$

for all z. On account of (5.25) and (5.26), (5.20) will hold provided that

$$\sum_{n=1}^{\infty} n\big(\beta_\varepsilon(n)\big)^2 p(-n) \leq \sum_{n=1}^{\infty} n\big(\beta_\varepsilon(n)\big)^{-2} p(n). \tag{5.27}$$

However, the limits of the two sides of this inequality, as $\varepsilon \downarrow 0$, are $\sum_{n=1}^{\infty} np(-n)$ and $\sum_{n=1}^{\infty} np(n)$, respectively; the first of these bounds requires (5.24) and the dominated convergence theorem. Since the mean of $p(\cdot)$ is positive, (5.27) holds for sufficiently small positive ε. This completes the proof. □

We conclude this section with several remarks. First, Proposition 5.4 provides explicit bounds on the tails of the stationary blocking measure. Here is an example. Take $g(x) = (1+x)^2$ in Proposition 5.5. Suppose $p(\cdot)$ has finite positive mean, and satisfies (1.1) and

$$\sum_{n=1}^{\infty} n^5 p(-n) < \infty. \tag{5.28}$$

Choose $\varepsilon > 0$ sufficiently small so that

$$\sum_{n=1}^{\infty} n(1+\varepsilon n)^4 p(-n) \le \sum_{n=1}^{\infty} n(1+\varepsilon n)^{-4} p(n).$$

This is possible by (5.28) and the dominated convergence theorem, since the mean of $p(\cdot)$ is strictly positive. This last inequality is (5.27), so (5.20) is satisfied. In this case, $\alpha(-n) = 1/[1+(1+\varepsilon n)^2]$ for $n \ge 0$. Therefore, by Proposition 5.4, the blocking measure π satisfies

$$\begin{aligned}\sum_{x<-N} \pi\bigl(\{\eta : \eta(x) = 1\}\bigr) &\le \sum_{n>N} \frac{1}{1+(1+\varepsilon n)^2} \\ &\le \int_N^{\infty} \frac{dx}{(1+\varepsilon x)^2} = \frac{1}{\varepsilon(1+\varepsilon N)}\end{aligned}$$

for $N \ge 0$.

Second, when combined with Theorem 4.1 of Ferrari, Lebowitz and Speer (2001), our results can be used to show that stationary blocking measures exist even in certain cases in which the monotonicity assumption (1.1) is not satisfied. In particular, if the kernel $p^*(\cdot)$ is defined by

$$p^*(n) = c \min\{p(1), \dots, p(n)\} \quad \text{and} \quad p^*(-n) = c \max\{p(-n), p(-n-1), \dots\}$$

for $n \ge 1$, where the constant c is chosen so that $\sum_x p^*(x) = 1$, then $p^*(\cdot)$ satisfies (1.1). If $p^*(\cdot)$ satisfies the assumptions of Proposition 5.4 for an appropriately chosen $\alpha(\cdot)$, then it follows that the process corresponding to $p(\cdot)$ has a stationary blocking measure.

Finally, the results of this section say nothing about kernels whose negative tails have a finite second moment but infinite third moment. We suspect that stationary blocking measures exist in this case, and, in fact, we will prove this under an additional assumption in the next section. It is possible that a proof that applies more generally could be carried out along the lines of the present section, using an initial measure ν that is not a product measure, or perhaps by weakening condition (5.20).

As pointed out earlier, the stationary measures constructed in this section not only concentrate on Ξ, but also have the property that the number of particles to the left of the origin has exponential moments of all orders. Perhaps a second moment assumption is the right condition for the existence of stationary blocking measures, but a third moment assumption is the right condition for the existence of stationary measures with a finite expected number of particles to the left of the origin. If that is the case, it would explain why using ν_α as the initial distribution for the process in this section forces us to assume third moments, since for independent Bernoulli random variables $\eta(x)$, $\sum_x \eta(x) < \infty$ a.s. is equivalent to $\sum_x E[\eta(x)] < \infty$.

6. Existence of blocking measures under second moment assumptions. This section is devoted to the proof of Theorem 1.4, which asserts the existence of a stationary blocking measure under the assumptions (1.1) and (1.4), when the left tail of the random walk kernel $p(\cdot)$ has a finite second moment. This result supports the possibility discussed at the end of the last section, and provides a partial converse to Theorem 1.2. As far as we know, the stationary blocking measure of the process may not have a finite expected number of particles to the left of the origin.

The main idea is to approximate $p(\cdot)$ by random walk kernels $p_\varepsilon(\cdot)$, $\varepsilon > 0$, that satisfy the hypotheses of Theorem 1.3, and to show that any limit π of the stationary blocking measures π_ε thus obtained is a stationary blocking measure for $p(\cdot)$. This program is carried out in Proposition 6.1 under an assumption that is checked in Lemma 6.4 and Proposition 6.5.

In Proposition 6.1, we will need an upper bound on the expected number of blocks of particles in configurations with the stationary blocking measures π_ε; this is given by (6.2). We will also need the upper bound on $p_\varepsilon(\cdot)$ in (6.1). Here and later on, we use Ξ_0 to denote the subset of Ξ,

$$\Xi_0 = \left\{\eta \in \Xi : \sum_{x<0} \eta(x) = \sum_{x\geq 0} [1-\eta(x)]\right\}.$$

Note that Ξ is the union of translates of Ξ_0, and that Ξ_0 is closed for the exclusion process. So, if there is a stationary blocking measure for the process, it can be taken to concentrate on Ξ_0.

PROPOSITION 6.1. *Assume that the random walk kernel $p(\cdot)$ has mean $\mu \in (0,\infty)$, and that the random walk kernels $p_\varepsilon(\cdot)$, $\varepsilon > 0$, satisfy*

$$\lim_{\varepsilon\downarrow 0} p_\varepsilon(n) = p(n) \quad \textit{and} \quad p_\varepsilon(n) \leq Cp(n) \tag{6.1}$$

for each n, where C is a constant. Also, suppose that for each ε, the exclusion process with kernel $p_\varepsilon(\cdot)$ has a stationary measure π_ε that concentrates on Ξ_0, with

$$\limsup_{\varepsilon\downarrow 0} \sum_{x\in\mathbb{Z}} \pi_\varepsilon(\{\eta : \eta(x) = 1,\ \eta(x+1) = 0\}) < \infty. \tag{6.2}$$

Then the exclusion process with kernel $p(\cdot)$ has a stationary blocking measure on Ξ_0 that satisfies (1.5).

By taking the limit along an appropriate sequence of the measures π_ε, it is not difficult to obtain a stationary measure π for the exclusion process with kernel $p(\cdot)$. [See Liggett (1985), page 18.] However, it is not at all obvious that π concentrates on Ξ (or Ξ_0) and the demonstration of this is fairly long. We therefore provide some intuition before proving Proposition 6.1.

An example of a possible limit π that we need to rule out is the measure that puts mass $\frac{1}{2}$ on each of the two configurations $\eta \equiv 1$ and $\eta \equiv 0$. Suppose that such a measure is, in fact, the limit of π_ε as $\varepsilon \downarrow 0$. Then, for small $\varepsilon > 0$, there will be a large block of particles extending in both directions from the origin, with probability close to $\frac{1}{2}$ (with the corresponding statement also holding for a large block of holes). Since the configuration is in Ξ_0 and there are many particles to the left of the origin, there must also be many holes to the right of the origin. If (6.2) holds, it follows that there must be some site z for which $[z - N, z)$ is completely occupied by particles and $[z + K, z + K + N)$ is completely empty, where N is large but K is not; sites in $[z, z + K)$ may be either occupied or empty. However, the limiting distribution for an exclusion process with positive drift whose initial configuration is a finite perturbation of "all particles on the left followed by all holes on the right" is the product measure with density $\frac{1}{2}$. In particular, this measure has infinitely many blocks of particles. Since π_ε is stationary, this will contradict (6.2). Hence, π cannot be as assumed above, with mass concentrated on $\eta \equiv 1$ and $\eta \equiv 0$.

We now proceed to the proof of Proposition 6.1. We begin with two lemmas. We let $A(K, N)$, $K, N \in \mathbb{Z}^+$, denote the set of configurations η on which, for some $z \in \mathbb{Z}$, $\eta(x + z) = 1$ for $x \in [-N, 0)$ and $\eta(x + z) = 0$ for $x \in [K, K + N)$.

LEMMA 6.2. *Assume that the kernel $p(\cdot)$ has mean $\mu \in (0, \infty)$, that the kernels $p_\varepsilon(\cdot)$, $\varepsilon > 0$, satisfy* (6.1) *for some C, and that the corresponding exclusion processes have stationary measures π_ε. Then, for given K,*

$$\sum_{x \in \mathbb{Z}} \pi_\varepsilon\bigl(\{\eta : \eta(x) = 1,\ \eta(x + 1) = 0\}\bigr) > \frac{K}{5} \pi_\varepsilon\bigl(A(K, N)\bigr) \tag{6.3}$$

for large enough N and sufficiently small ε.

PROOF. By Theorem 1.1 of Liggett (1977), if η is any configuration for which $\eta(x) = 1$ for $x < 0$ and $\eta(x) = 0$ for $x \geq K$, the exclusion process with kernel $p(\cdot)$ and initial configuration η has a measure at time t that converges to the homogeneous product measure with density $\frac{1}{2}$ as $t \to \infty$. Since there are only finitely many such configurations η for a given K, there exists a $t > 0$ so that for each such η,

$$\sum_{x=0}^{K-1} P^\eta\bigl(\eta_t(x) = 1,\ \eta_t(x + 1) = 0\bigr) > \frac{K}{5}. \tag{6.4}$$

Let P_ε denote probabilities with respect to the exclusion process with kernel $p_\varepsilon(\cdot)$. By the first part of (6.1) and Theorem 2.12 on page 17 of Liggett (1985),

$$\lim_{\varepsilon \downarrow 0} P_\varepsilon^\eta\bigl(\eta_t(x) = 1,\ \eta_t(x + 1) = 0\bigr) = P^\eta\bigl(\eta_t(x) = 1,\ \eta_t(x + 1) = 0\bigr)$$

for each η and t. Therefore, for sufficiently small ε, (6.4) holds with P^η replaced by P_ε^η.

It follows from this last inequality that, for large enough N and sufficiently small ε,

$$\sum_{x=0}^{K-1} P_\varepsilon^\eta\big(\eta_t(x)=1,\ \eta_t(x+1)=0\big) > \frac{K}{5} \tag{6.5}$$

for all $\eta \in A(K,N)$. To see this, let $Z_\cdot$ be the random walk appearing in Lemma 2.1. By the second part of (6.1), $P_\varepsilon(Z_t = x) \le e^{Ct} P(Z_t = x)$. Also, let $(\eta_\cdot, \xi_\cdot)$ be a coupled pair of exclusion processes with initial configuration (η, ξ) satisfying $\eta(x) = \xi(x)$ for $x \in [-N, K+N]$. It follows from Lemma 2.1 that

$$\begin{aligned} P_\varepsilon\big(\eta_t(x) \ne \xi_t(x) \text{ for some } x \in [0,K)\big) &\le 2\sum_{x=N}^{\infty} P_\varepsilon(Z_t \ge x) \\ &\le 2e^{Ct} \sum_{x=N}^{\infty} P(Z_t \ge x). \end{aligned} \tag{6.6}$$

Since $p(\cdot)$ has finite mean, (6.5) follows from (6.6) by choosing N sufficiently large.

The measure π_ε is stationary for the exclusion process with kernel $p_\varepsilon(\cdot)$. Together with (6.5), this implies that

$$\begin{aligned} &\sum_{x\in\mathbb{Z}} \pi_\varepsilon\big(\{\eta : \eta(x)=1,\ \eta(x+1)=0\}\big) \\ &\qquad = \sum_{x\in\mathbb{Z}} \int P_\varepsilon^\eta\big(\eta_t(x)=1,\ \eta_t(x+1)=0\big)\pi_\varepsilon(d\eta) \\ &\qquad > \frac{K}{5}\pi_\varepsilon\big(A(K,N)\big), \end{aligned}$$

which is (6.3). $\square$

It will be convenient to introduce the following random variables corresponding to the stationary measures π_ε in Lemma 6.2, in order to interpret (6.3). Let L_ε be the number of finite blocks of particles (or equivalently, of holes) for a configuration in Ξ. Let $X_\varepsilon(i)$ and $Y_\varepsilon(i)$, $i = 1, \dots, L_\varepsilon$, be the lengths of the ith finite blocks of particles and holes, starting from the left. (The length of a block of k consecutive elements is taken to be k.) Note that the ith block of particles lies to the left of the ith block of holes.

Since the space of probability measures on $\{0,1\}^{\mathbb{Z}}$ is compact, there is a sequence of ε's tending to 0 on which π_ε converges weakly to some probability measure π on $\{0,1\}^{\mathbb{Z}}$. By taking an appropriate subsequence, we can also ensure that the random vector

$$\big(L_\varepsilon, X_\varepsilon(1), \dots, X_\varepsilon(L_\varepsilon), 0, \dots, Y_\varepsilon(1), \dots, Y_\varepsilon(L_\varepsilon), 0, \dots\big)$$

converges in distribution to a random vector

$$\big(L, X(1), \dots, X(L), 0, \dots, Y(1), \dots, Y(L), 0, \dots\big)$$

in the weak topology, as ε tends to 0. (These vectors are elements of $\bar{\mathbb{Z}}^+ \times (\bar{\mathbb{Z}}^+)^{\mathbb{Z}^+} \times (\bar{\mathbb{Z}}^+)^{\mathbb{Z}^+}$, where $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$ and $\bar{\mathbb{Z}}^+ = \mathbb{Z}^+ \cup \{\infty\}$.)

Suppose now that (6.2) holds for the measures π_ε. This implies that the distributions of L_ε are tight as $\varepsilon \downarrow 0$, and so $L < \infty$ a.s. We make no assertion about the random variables $X(1), \dots, X(L)$ and $Y(1), \dots, Y(L)$ at this point; they may be infinite. In particular, they might not correspond to the blocks of particles and holes of π, and $\pi(\Xi) \neq 1$ is possible. However, because of Lemma 6.2, the following limiting behavior must hold.

LEMMA 6.3. *Assume that the kernel $p(\cdot)$ has mean $\mu \in (0, \infty)$, and that the kernels $p_\varepsilon(\cdot)$, $\varepsilon > 0$, satisfy* (6.1) *for some C, and the corresponding exclusion processes have stationary measures π_ε that concentrate on Ξ_0 and satisfy* (6.2). *Then, the random vectors defined above satisfy*

$$P\big(\exists\, 1 \le i \le j \le L : X(i) = Y(j) = \infty\big) = 0. \tag{6.7}$$

PROOF. One can check that

$$\begin{aligned} \pi_\varepsilon\big(A(K, N)\big) \ge P\Bigg(&\exists\, 1 \le i \le j \le L_\varepsilon : X_\varepsilon(i) \ge K + N,\ Y_\varepsilon(j) \ge K + N, \\ &\sum_{i<k\le j} X_\varepsilon(k) + \sum_{i \le k < j} Y_\varepsilon(k) \le K\Bigg). \end{aligned} \tag{6.8}$$

The event on the right-hand side of (6.8) says that for some $i \le j$, the ith finite block of particles and the jth finite block of holes are each large, but the sum of the lengths of intervals in between is relatively small. Passing to the limit in (6.3) and (6.8), as ε tends to zero along the chosen subsequence, implies that for every K there is an N so that

$$\begin{aligned} &P\Bigg(\exists\, 1 \le i \le j \le L : X(i), Y(j) \ge K + N,\ \sum_{i<k\le j} X(k) + \sum_{i \le k < j} Y(k) \le K\Bigg) \\ &\quad \le \frac{5}{K} \limsup_{\varepsilon \downarrow 0} \sum_{x \in \mathbb{Z}} \pi_\varepsilon\big(\{\eta : \eta(x) = 1,\ \eta(x+1) = 0\}\big). \end{aligned}$$

Letting $K \to \infty$ and applying (6.2) furthermore implies that

$$P\Bigg(\exists\, 1 \le i \le j \le L : X(i) = Y(j) = \infty,\ \sum_{i<k\le j} X(k) + \sum_{i \le k < j} Y(k) < \infty\Bigg) = 0.$$

One may increase i and decrease j so that both sums are automatically finite. This implies (6.7). $\square$

We now complete the demonstration of Proposition 6.1.

PROOF OF PROPOSITION 6.1. Let π be the measure introduced before Lemma 6.2. On account of the limit in (6.1), π is stationary with respect to the random walk kernel $p(\cdot)$ [Liggett (1985), page 18]. We need to show that π is concentrated on Ξ. Once we know this, it follows that π must assign a positive probability to some translate of Ξ_0, which, after conditioning and shifting, produces a stationary blocking measure on Ξ_0. The bound (1.5) follows by applying Fatou's lemma to (6.2).

To show that $\pi(\Xi) = 1$, we use the random variables $L_\varepsilon, X_\varepsilon(i)$ and $Y_\varepsilon(i)$ that were introduced before Lemma 6.3. For a given $\eta \in \Xi_0$, define $k_\varepsilon \le L_\varepsilon$, $X_\varepsilon \in [0, X_\varepsilon(k_\varepsilon + 1))$ and $Y_\varepsilon \in [0, Y_\varepsilon(k_\varepsilon)]$ so that

$$\sum_{x<0} \eta(x) = \sum_{i=1}^{k_\varepsilon} X_\varepsilon(i) + X_\varepsilon \quad \text{and} \quad \sum_{x\ge 0} [1 - \eta(x)] = Y_\varepsilon + \sum_{i=k_\varepsilon+1}^{L_\varepsilon} Y_\varepsilon(i). \tag{6.9}$$

In words, k_ε is the number of blocks of particles lying entirely within $(-\infty, 0)$, X_ε is the number of particles to the left of the origin in the $(k_\varepsilon + 1)$st block and Y_ε is the number of empty sites to the right of the origin in the k_εth block. Note that either $X_\varepsilon = 0$ or $Y_\varepsilon = 0$ must always hold, and that $X_\varepsilon = 0$ when $k_\varepsilon = L_\varepsilon$.

Since $\eta \in \Xi_0$, the left sides of the two equations in (6.9) are equal, and so

$$\sum_{i=1}^{k_\varepsilon} X_\varepsilon(i) + X_\varepsilon = Y_\varepsilon + \sum_{i=k_\varepsilon+1}^{L_\varepsilon} Y_\varepsilon(i). \tag{6.10}$$

On account of Lemma 6.3, the random variables on both sides of (6.10) are tight as $\varepsilon \downarrow 0$ along the subsequence defined before the lemma. Consequently, so are $\sum_{x<0} \eta(x)$ and $\sum_{x\ge 0}[1 - \eta(x)]$ with respect to the measures π_ε. It follows that the limit π concentrates on Ξ. $\square$

In order to apply Proposition 6.1, we need to be able to verify (6.2). An important step is carried out in the following lemma, where we introduce the subadditive function $M(\cdot)$, and show that it satisfies (6.12). This equality figures centrally in the proof of Proposition 6.5, and explains how the second moment assumption arises. Note that the $M(1)$ defined below is the expected number of finite blocks of particles relative to the measure ν.

LEMMA 6.4. *Assume that the random walk kernel $p(\cdot)$ has finite mean with $\sum_{n=1}^{\infty} n^2 p(-n) < \infty$, and that ν is a stationary measure that satisfies* (1.3). *For $n \in \mathbb{Z}$, set*

$$M(n) = \sum_{x\in\mathbb{Z}} \nu\big(\{\eta : \eta(x) = 1,\ \eta(x+n) = 0\}\big).$$

Then, $M(-n) = M(n) + n$ *for* $n \geq 1$,

$$M(m+n) \leq M(m) + M(n) \tag{6.11}$$

for $m, n \geq 1$ *and*

$$\sum_{n=1}^{\infty} n^2 p(-n) = \sum_{n=1}^{\infty} n M(n)[p(n) - p(-n)]. \tag{6.12}$$

PROOF. First note that $M(n)$ is finite for each n by (1.3), since

$$M(n) \leq \sum_{x \leq 0} \nu(\{\eta : \eta(x) = 1\}) + \sum_{x \geq n} \nu(\{\eta : \eta(x) = 0\}).$$

To check that $M(-n) = M(n) + n$ for $n \geq 1$, write

$$\begin{aligned}
&M(-n) - M(n) \\
&\quad = \sum_x [\nu(\{\eta : \eta(x) = 0, \eta(x+n) = 1\}) - \nu(\{\eta : \eta(x) = 1, \eta(x+n) = 0\})] \\
&\quad = \sum_x [\nu(\{\eta : \eta(x+n) = 1\}) - \nu(\{\eta : \eta(x) = 1\})] \\
&\quad = \lim_{N \to \infty} \sum_{x=-N}^{N} [\nu(\{\eta : \eta(x+n) = 1\}) - \nu(\{\eta : \eta(x) = 1\})] \\
&\quad = \lim_{N \to \infty} \left[\sum_{x=N+1}^{N+n} \nu(\{\eta : \eta(x) = 1\}) - \sum_{x=-N}^{-N+n-1} \nu(\{\eta : \eta(x) = 1\}) \right] = n.
\end{aligned}$$

The final equality above is a consequence of the fact that ν is a blocking measure. For (6.11), write

$$\begin{aligned}
&\nu(\{\eta : \eta(x) = 1, \eta(x+m+n) = 0\}) \\
&\quad \leq \nu(\{\eta : \eta(x) = 1, \eta(x+n) = 0\}) + \nu(\{\eta : \eta(x+n) = 1, \eta(x+m+n) = 0\})
\end{aligned}$$

and sum on x.

We still need to show (6.12). In a stationary blocking measure, the net rate at which particles go from the left of x to the right of x is zero. Therefore, for fixed x,

$$\begin{aligned}
&\sum_{u \leq x < v} p(v-u) \nu(\{\eta : \eta(u) = 1, \eta(v) = 0\}) \\
&\qquad = \sum_{u \leq x < v} p(u-v) \nu(\{\eta : \eta(u) = 0, \eta(v) = 1\}).
\end{aligned}$$

Summing over x gives

$$\sum_{n=1}^{\infty} n p(n) M(n) = \sum_{n=1}^{\infty} n p(-n) M(-n).$$

The right-hand side above is finite, since $M(-n) = M(n) + n \le n[M(1) + 1]$ by (6.11), and the negative tails of $p(\cdot)$ have a finite second moment. Now use $M(-n) = M(n) + n$. $\square$

Using Proposition 6.1 and Lemma 6.4, we obtain the following sufficient condition for the existence of a stationary blocking measure. Theorem 1.4 is an immediate consequence, since (6.13) trivially follows from (1.4) and $\mu > 0$. The proposition applies in certain other cases as well. For example, if $p(n) \ge p(-n)$ for $n \ne 2$, then (6.13) holds if $p(1) + 4p(2) > p(-1) + 4p(-2)$.

PROPOSITION 6.5. *Suppose that the random walk kernel* $p(\cdot)$ *has mean* $\mu \in (0, \infty)$, *and satisfies* (1.1) *and* $\sum_{n=1}^{\infty} n^2 p(-n) < \infty$. *Also, suppose that*

$$\sum_{n=1}^{\infty} n\gamma(n)[p(n) - p(-n)] > 0 \tag{6.13}$$

for all strictly positive subadditive sequences $\gamma(\cdot)$. *Then, a stationary blocking measure satisfying* (1.5) *exists.*

PROOF. Let $p_\varepsilon(\cdot)$, $\varepsilon > 0$, be the random walk kernel

$$p_\varepsilon(n) = C(\varepsilon)e^{-\varepsilon|n|}p(n),$$

where $C(\varepsilon)$ is the normalizing constant that makes $\sum_n p_\varepsilon(n) = 1$. Note that $p_\varepsilon(\cdot)$ satisfies (1.1), and that $\sum_n np_\varepsilon(n) > 0$ for small enough ε. Theorem 1.3 therefore implies the existence of a stationary blocking measure π_ε for the exclusion process corresponding to $p_\varepsilon(\cdot)$ that satisfies (1.3), when ε is small. Without loss of generality, we may assume that π_ε is concentrated on Ξ_0. All of the conditions of Proposition 6.1 are clearly satisfied, except for (6.2). To conclude that a stationary blocking measure for $p(\cdot)$ that satisfies (1.5) exists, it therefore suffices to verify (6.2).

Let $M_\varepsilon(n)$ be defined as in the statement of Lemma 6.4 relative to π_ε, and write (6.12) as

$$\sum_{n=1}^{\infty} n^2 p_\varepsilon(-n) = M_\varepsilon(1) \sum_{n=1}^{\infty} n \frac{M_\varepsilon(n)}{M_\varepsilon(1)}[p_\varepsilon(n) - p_\varepsilon(-n)]. \tag{6.14}$$

Condition (6.2) says that $\limsup_{\varepsilon\downarrow 0} M_\varepsilon(1) < \infty$. Since the left-hand side of (6.14) is bounded by assumption, to show (6.2) it suffices to show that

$$\liminf_{\varepsilon\downarrow 0} \sum_{n=1}^{\infty} n \frac{M_\varepsilon(n)}{M_\varepsilon(1)}[p_\varepsilon(n) - p_\varepsilon(-n)] > 0. \tag{6.15}$$

Take any sequence ε_k tending to 0 for which

$$\gamma_1(n) = \lim_{k\to\infty} \frac{M_{\varepsilon_k}(n)}{M_{\varepsilon_k}(1)}$$

exists for each $n \ge 1$. By (6.11), $M_\varepsilon(n) \le nM_\varepsilon(1)$. So,

$$\liminf_{k\to\infty} \sum_{n=1}^{\infty} n \frac{M_{\varepsilon_k}(n)}{M_{\varepsilon_k}(1)} [p_{\varepsilon_k}(n) - p_{\varepsilon_k}(-n)] \ge \sum_{n=1}^{\infty} n\gamma_1(n)[p(n) - p(-n)] \tag{6.16}$$

by Fatou's lemma [applied to the terms involving $p(n)$] and the dominated convergence theorem [applied to the terms involving $p(-n)$]. By (6.11), $M_\varepsilon(\cdot)$ is subadditive, and therefore, so is $\gamma_1(\cdot)$. In view of (6.13), (6.15) and (6.16), the demonstration of (6.2) reduces to showing that $\gamma_1(n) > 0$ for all $n \ge 1$.

To show this, it suffices to check that

$$\pi_\varepsilon(\{\eta : \eta(x) = 1,\ \eta(x+n) = 0\}) \ge \delta(n)\pi_\varepsilon(\{\eta : \eta(x) = 1, \eta(x+1) = 0\}) \tag{6.17}$$

for $\varepsilon \le 1$, where $\delta(n) \in (0, 1]$ is independent of ε and x, since summing (6.17) over x gives $M_\varepsilon(n) \ge \delta(n)M_\varepsilon(1)$. For (6.17), it suffices to check that

$$\begin{aligned} &\pi_\varepsilon(\{\eta : \eta(x) = 1,\ \eta(x+1) = 1,\ \eta(x+n) = 0\}) \\ &\qquad \ge \delta(n)\pi_\varepsilon(\{\eta : \eta(x) = 1,\ \eta(x+1) = 0,\ \eta(x+n) = 1\}). \end{aligned} \tag{6.18}$$

Because of (1.1), we may assume that $p(-1) > 0$. [Otherwise, $p(x) = 0$ for all $x < 0$, and the proposition holds trivially.] Allowing only a specified sequence of jumps of size 1 to the left by particles over times $t \in (0, 1]$, one can check that the event on the left side of (6.18) occurs at time 1, with at least a fixed fraction of the probability that the event on the right occurs at time 0, where the bound is uniform over $\varepsilon \le 1$. Since π_ε is stationary, this implies (6.18), and completes the proof of the proposition. □

We recall that the bound (1.5) for the blocking measures π produced in this section is weaker than the bound (1.3) for the blocking measure in Section 5. In particular, although we know that the expected number of blocks of particles for π is finite, we do not know whether the expected number of particles to the left of the origin is finite. This issue was also addressed at the end of Section 5.

In this section, we have chosen to assume (1.1), so that in the approximation argument used in the proof of Proposition 6.5, we could use Theorem 1.3. This choice makes this paper relatively self-contained, and in particular, makes it independent of the harder arguments in Bramson and Mountford (2002). One could instead use their results to prove Theorem 1.4 without assumption (1.1). One would then truncate $p(n)$, instead of using the approximation in the proof of Proposition 6.5. This approach would be straightforward, except for the fact that they did not prove that the stationary blocking measures they constructed satisfy (1.3). However, note that (1.3) was used in Lemma 6.4 only to guarantee that $M(n) < \infty$. But if $\mu > 0$ and (1.4) holds, this can be proved directly using the technique of the proof of Lemma 6.4. In the final argument in that proof, sum on $|x| \le N$ instead of on all x. Then (6.12) is replaced by the inequality

$$\sum_{n=1}^{\infty} nM_N(n)[p(n) - p(-n)] \le \sum_{n=1}^{\infty} n^2 p(-n)$$

for the following truncated version $M_N(n)$ of $M(n)$:

$$M_N(n) = \sum_{x\in\mathbb{Z}} \nu\big(\{\eta : \eta(x) = 1,\ \eta(x+n) = 0\}\big)\frac{\#([-N, N] \cap [x - n^-, x + n^+))}{|n|}.$$

(Here n^- and n^+ denote the negative and positive parts of n, respectively.) Note that $M_N(n) \le 2N + 1$ for each $n \ne 0$, and that $M_N(n) \uparrow M(n)$ as $N \uparrow \infty$. The above inequality can be then used in place of (6.12).

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SCHOOL OF MATHEMATICS
UNIVERSITY OF MINNESOTA
206 CHURCH ST. SE
MINNEAPOLIS, MINNESOTA 55455
E-MAIL: bramson@math.umn.edu

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA, LOS ANGELES
405 HILGARD AVENUE
LOS ANGELES, CALIFORNIA 90095
E-MAIL: tml@math.ucla.edu
malloy@math.ucla.edu