

## NECESSARY AND SUFFICIENT CONDITIONS FOR THE CONDITIONAL CENTRAL LIMIT THEOREM

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Following Lindeberg's approach, we obtain a new condition for a stationary sequence of square-integrable and real-valued random variables to satisfy the central limit theorem. In the adapted case, this condition is weaker than any projective criterion derived from Gordin's theorem [*Dokl. Akad. Nauk SSSR* **188** (1969) 739–741] about approximating martingales. Moreover, our criterion is equivalent to the *conditional central limit theorem*, which implies stable convergence (in the sense of Rényi) to a mixture of normal distributions. We also establish functional and triangular versions of this theorem. From these general results, we derive sufficient conditions which are easier to verify and may be compared to other results in the literature. To be complete, we present an application to kernel density estimators for some classes of discrete time processes.

**1. Introduction.** In 1963 Rényi introduced the concept of stable convergence of random variables. This notion is more precise than convergence in distribution and may be useful in several contexts, especially in connection with random normalization [this fact was first pointed out by Smith (1945) in the particular case of sums of Rademacher random variables]. Aldous and Eagleson (1978) made clear the equivalence between stability and weak- $\mathbb{L}^1$  convergence of some functions of the variables, and proposed some powerful tools to establish stability of limit theorems. Further, adapting a result of McLeish (1974), they gave sufficient conditions for a sequence of martingale differences to converge stably to a mixture of normal distributions. Their results among many others have been used and developed by Hall and Heyde [(1980), Chapter 3] to provide an elegant and rather complete contribution to martingale central limit theory.

Some of these results have been extended to general sequences by providing strong enough conditions to ensure that the partial sums behave asymptotically like a martingale. In this context, McLeish (1975b, 1977) used the concept of mixingale, while Peligrad (1981) followed Gordin's approach. The common feature of these works is the application of Theorem 19.4 in Billingsley (1968): firstly they prove tightness of the partial sum process and secondly they identify the limit by using a suitable characterization of the Wiener process. They obtain *mixing* convergence of the partial sum process to a Brownian motion, which

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coincides with stable convergence provided that the conditional variance of the partial sums with respect to the past  $\sigma$ -algebra is asymptotically constant (cf. Remark 5, Section 2 for the relation between stability and mixing).

In this paper we focus on the central limit question for strictly stationary sequences indexed by  $\mathbb{Z}$ . In Theorem 1 we propose a simple criterion which implies stable convergence of the normalized partial sums to a mixture of normal distributions. More precisely, we show that this criterion is necessary and sufficient to obtain a stronger result than stable convergence. We shall see that this new type of convergence to a mixture of normal distributions, close to the one introduced by Touati [(1993), Theorem 3, H-2], is satisfied for a wide class of stationary sequences.

Before stating Theorem 1, we need some preliminary notations.

NOTATION 1. Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a probability space, and  $T: \Omega \mapsto \Omega$  be a bijective bimeasurable transformation preserving the probability  $\mathbb{P}$ . An element  $A$  is said to be invariant if  $T(A) = A$ . We denote by  $\mathcal{I}$  the  $\sigma$ -algebra of all invariant sets. The probability  $\mathbb{P}$  is ergodic if each element of  $\mathcal{I}$  has measure 0 or 1. Finally, let  $\mathcal{H}$  be the space of continuous real functions  $\varphi$  such that  $x \rightarrow |(1 + x^2)^{-1}\varphi(x)|$  is bounded.

THEOREM 1. Let  $\mathcal{M}_0$  be a  $\sigma$ -algebra of  $\mathcal{A}$  satisfying  $\mathcal{M}_0 \subseteq T^{-1}(\mathcal{M}_0)$  and define the nondecreasing filtration  $(\mathcal{M}_i)_{i \in \mathbb{Z}}$  by  $\mathcal{M}_i = T^{-i}(\mathcal{M}_0)$ . Let  $X_0$  be a  $\mathcal{M}_0$ -measurable, square integrable and centered random variable. Define the sequence  $(X_i)_{i \in \mathbb{Z}}$  by  $X_i = X_0 \circ T^i$ , and  $S_n = X_1 + \dots + X_n$ . The following statements are equivalent:

- s1. There exists a nonnegative  $\mathcal{M}_0$ -measurable random variable  $\eta$  such that, for any  $\varphi$  in  $\mathcal{H}$  and any positive integer  $k$ ,

$$\lim_{n \rightarrow \infty} \left\| \mathbb{E} \left( \varphi(n^{-1/2} S_n) - \int \varphi(x \sqrt{\eta}) g(x) dx \mid \mathcal{M}_k \right) \right\|_1 = 0$$

where  $g$  is the distribution of a standard normal.

- s2. (a) The sequence  $(n^{-1} S_n^2)_{n > 0}$  is uniformly integrable;
- (b) the sequence  $\|\mathbb{E}(n^{-1/2} S_n \mid \mathcal{M}_0)\|_1$  tends to 0 as  $n$  tends to infinity;
- (c) there exists a nonnegative  $\mathcal{M}_0$ -measurable random variable  $\eta$  such that  $\|\mathbb{E}(n^{-1} S_n^2 - \eta \mid \mathcal{M}_0)\|_1$  tends to 0 as  $n$  tends to infinity.

Moreover the random variable  $\eta$  satisfies  $\eta = \eta \circ T$  almost surely.

A stationary sequence  $(X \circ T^i)_{i \in \mathbb{Z}}$  of random variables is said to satisfy the conditional central limit theorem (CCLT for short) if it verifies s1.

Before presenting the applications of Theorem 1, let us compare Condition s2 with Theorem 9.5 in Jakubowski (1993) (taking  $B_n = \sqrt{n}$  therein), where

necessary and sufficient conditions for the usual CLT are given. There are two distinct sets of conditions in this theorem: on one hand Condition B and on the other hand Conditions (9.5), (9.6) and (9.7). First, it is clear that s2(a) is stronger than Conditions (9.5) and (9.6). Second, s2(a), together with s2(c), implies (9.7) with  $\sigma^2 = \mathbb{E}(\eta)$ . Since Jakubowski's result only deals with pure Gaussian limit, we infer from the two preceding remarks that s2 implies Condition B *if and only if* the random variable  $\eta$  is constant. This means that the two results are of a different nature and that Theorem 1 may not be derived by using Jakubowski's result (another reason is that the latter does not necessarily imply stable convergence). In fact, Condition B is a kind of mixing property involving the whole past and the whole future of the sequence, so that random variances are forbidden (this is also the case when considering classical mixing coefficients). Note also that to obtain Condition B from s2 in the case where  $\eta$  is constant seems as difficult as to prove s1 directly.

We now give sufficient conditions for the CCLT to hold. Note first that criterion s2 is satisfied for stationary sequences of martingale differences: indeed, in that case, s2(a) follows from Doob's maximal inequality, s2(b) is straightforward and s2(c) is a consequence of the  $\mathbb{L}^1$ -ergodic theorem. Now, as first noticed by Gordin (1969), it is often possible to approximate the partial sums of a stationary process by a naturally related martingale with stationary differences. Such an approximation provides a possible approach to obtain sufficient conditions for the CCLT, as shown by Proposition 1:

**PROPOSITION 1.** *Let  $(\mathcal{M}_i)_{i \in \mathbb{Z}}$  and  $(X_i)_{i \in \mathbb{Z}}$  be as in Theorem 1. Let  $H_i$  be the Hilbert space of  $\mathcal{M}_i$ -measurable, centered and square integrable functions. For all integer  $j$  less than  $i$ , denote by  $H_i \ominus H_j$  the orthogonal of  $H_j$  into  $H_i$ . Let  $Q$  be the set of all functions from  $H_i \ominus H_j$  where  $-\infty < j \leq i < \infty$ . If*

$$(1.1) \quad \inf_{f \in Q} \limsup_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left\| \sum_{i=1}^n X_0 \circ T^i - f \circ T^i \right\|_2 = 0,$$

*then s2 (hence s1) holds.*

**REMARK 1.** Condition (1.1) is due to Gordin (1969). However, compared to Gordin's theorem, we have to restrict ourselves to adapted sequences (which means that  $X_i$  is  $\mathcal{M}_i$ -measurable). Note that, as in Eagleson (1975), we do not require  $\mathbb{P}$  to be ergodic.

It follows from Proposition 1 that any projective criterion derived from (1.1) leads to the CCLT. For instance, Conditions (1) and (2) of Theorem 1 in Heyde (1974) yield (1.1), and so does Condition (3.15) in Dürr and Goldstein (1986). From Theorem 2 in Heyde (1974) we know that (1.1) is satisfied as soon as

$$(1.2) \quad \sum_{i=0}^n P_0(X_i) \text{ converges in } \mathbb{L}^2 \text{ to } g \quad \text{and} \quad \lim_{n \rightarrow \infty} n^{-1/2} \|S_n\|_2 = \|g\|_2,$$

where  $P_0$  is the projection operator onto  $H_0 \ominus H_{-1}$ . Volný [(1993), Theorems 5 and 6] proposed sufficient conditions based upon the sequence  $(P_0(X_i))_{i \geq 0}$  for (1.2) to hold. From these conditions, we easily infer (cf. Section 6.2) that (1.2) is satisfied as soon as there exists a sequence  $(L_k)_{k > 0}$  of positive numbers such that

$$(1.3) \quad \sum_{i > 0} \left( \sum_{k=1}^i L_k \right)^{-1} < \infty \quad \text{and} \quad \sum_{k > 0} L_k \|\mathbb{E}(X_k | \mathcal{M}_0)\|_2^2 < \infty.$$

According to the definition by McLeish (1975a), (1.3) is a mixingale-type condition. Note that it is close to optimality, since the choice  $L_k = 1$  is not strong enough to imply weak convergence of  $n^{-1/2} S_n$  (see Proposition 7, Section 6).

We now turn to the functional version of Theorem 1. As usual, to prove tightness of the normalized Donsker line, we need to control the maximum of partial sums. More precisely, under the condition

$$s2^* \left\{ \begin{array}{l} \text{(b) and (c) of s2 hold, and (a) is replaced by} \\ \text{(a*): } \frac{1}{n} \left( \max_{1 \leq i \leq n} |S_i| \right)^2 \text{ is uniformly integrable,} \end{array} \right.$$

we obtain the functional version of the CCLT described in Theorem 3, Section 2. Note that if  $\eta$  is constant, the usual functional central limit theorem follows from  $s2^*$  by applying Theorem 19.4 in Billingsley (1968). Note also that if the convergence rate to zero in  $s2(c)$  is of order  $n^{-\theta}$  for some positive  $\theta$  and the variables have finite  $(2 + \delta)$ -moments, then it follows from Theorem 2.1 in Serfling (1968) that  $s2(a^*)$  is automatically satisfied. Assuming that the same rate holds for  $s2(b)$ , Eberlein (1986a) has obtained strong invariance principles with order of approximation  $O(t^{1/2-\kappa})$  for some positive  $\kappa$ .

Once again, there exists a large class of stationary sequences satisfying  $s2^*$ , as shown by Propositions 2 and 3.

**PROPOSITION 2.** *Let  $(\mathcal{M}_i)_{i \in \mathbb{Z}}$  and  $(X_i)_{i \in \mathbb{Z}}$  be as in Theorem 1. Condition (1.3) implies  $s2^*$ .*

The key for Proposition 2 is to establish a maximal inequality which together with (1.3) implies  $s2(a^*)$ . This can be done by following McLeish’s approach for nonstationary mixingales [cf. McLeish (1975b), Lemma (6.3)]. In the particular case of stationary and adapted sequences, Proposition 2 improves on McLeish’s results in three ways: First, Condition (1.3) is satisfied if either Condition (2.5) in McLeish (1977) holds or  $(X_n, \mathcal{M}_n)$  is a mixingale of size  $-1/2$  [cf. McLeish (1975b), Definitions (2.1) and (2.4)]. Second, the extra condition in Theorem (2.6) of the latter may be removed (in fact it may be replaced by the weaker condition  $s2(c)$ , which follows from (1.3) as we have already noticed). Third, we obtain a

stronger result in terms of convergence, the functional CCLT implying *mixing*-convergence as soon as  $\eta$  is constant, which we do not require here. To get an idea of the wide range of applications of mixingales, we refer to McLeish (1975a, b, 1977) and Hall and Heyde [(1980), Section 2.3]. See also Eberlein (1986b) for a survey of results concerning mixingales and other generalizations of martingales.

The second condition [Condition (1.4) below] has a different structure and is not obtained via martingale approximations, although many results in that field may be derived from it.

**PROPOSITION 3.** *Let  $(\mathcal{M}_i)_{i \in \mathbb{Z}}$ ,  $(X_i)_{i \in \mathbb{Z}}$  and  $S_n$  be as in Theorem 1. Consider the condition*

$$(1.4) \quad \sum_{k=1}^n X_0 \mathbb{E}(X_k | \mathcal{M}_0) \quad \text{converges in } \mathbb{L}^1.$$

*If (1.4) is satisfied, then  $s_2^*$  holds and the sequence  $\mathbb{E}(X_0^2 | \mathcal{I}) + 2\mathbb{E}(X_0 S_n | \mathcal{I})$  converges in  $\mathbb{L}^1$  to  $\eta$ .*

Condition (1.4) was introduced by Dedecker and Rio (2000) to obtain the usual functional central limit theorem. In the adapted case, Condition (1.4) is weaker than the  $\mathbb{L}^2$ -criterion of Gordin (1969)

$$(1.5) \quad \sum_{k=1}^n \mathbb{E}(X_k | \mathcal{M}_0) \quad \text{converges in } \mathbb{L}^2,$$

which has been extended to nonstationary sequences by Peligrad (1981). If  $X_i = g(\xi_i)$  is a function of a stationary Markov chain  $(\xi_i)_{i \in \mathbb{Z}}$  with transition kernel  $K$  and marginal distribution  $\mu$ , condition (1.4) becomes

$$(1.6) \quad \sum_{k=0}^n g K^k g \quad \text{converges in } \mathbb{L}^1(\mu),$$

and improves on classical results based upon the Poisson equation. Condition (1.6) was simultaneously discovered by Chen (1999) in the particular case of positive Harris Chain. Finally the application of (1.4) to strongly mixing sequences leads to the conditional and nonergodic version of the invariance principle of Doukhan, Massart and Rio (1994), whose optimality is discussed in Bradley (1997).

The paper is organized as follows. In Section 2 we state the conditional central limit theorem (Theorem 2) and its functional version (Theorem 3) in the more general context of triangular arrays. As a consequence, we derive in Corollary 1 the stable convergence of the normalized partial sums to a mixture of normal distributions. The proofs of these results are postponed to Sections 3, 4 and 5 respectively. Section 6 is devoted to the applications of Theorem 1: we prove Propositions 1, 2 and 3 in Sections 6.1, 6.2 and 6.3 respectively. In Section 7 we give the analogous of Propositions 2 and 3 for triangular arrays. In Section 8 we explain how to apply these results to kernel density estimators.

**2. Main results.** Theorem 1 presented in the Introduction is a straightforward consequence of the following theorem for triangular arrays with stationary rows.

**THEOREM 2.** *For each positive integer  $n$ , let  $\mathcal{M}_{0,n}$  be a  $\sigma$ -algebra of  $\mathcal{A}$  satisfying  $\mathcal{M}_{0,n} \subseteq T^{-1}(\mathcal{M}_{0,n})$ . Define the nondecreasing filtration  $(\mathcal{M}_{i,n})_{i \in \mathbb{Z}}$  by  $\mathcal{M}_{i,n} = T^{-i}(\mathcal{M}_{0,n})$  and  $\mathcal{M}_{i,\text{inf}} = \sigma(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \mathcal{M}_{i,k})$ . Let  $X_{0,n}$  be a  $\mathcal{M}_{0,n}$ -measurable and square integrable random variable and define the sequence  $(X_{i,n})_{i \in \mathbb{Z}}$  by  $X_{i,n} = X_{0,n} \circ T^i$ . Finally, for any  $t$  in  $[0, 1]$ , write  $S_n(t) = X_{1,n} + \dots + X_{[nt],n}$ . Suppose that  $n^{-1/2}X_{0,n}$  converges in probability to zero as  $n$  tends toward infinity. The following statements are equivalent:*

S1. *There exists a nonnegative  $\mathcal{M}_{0,\text{inf}}$ -measurable random variable  $\eta$  such that, for any  $\varphi$  in  $\mathcal{H}$ , any  $t$  in  $[0, 1]$  and any positive integer  $k$ ,*

$$S1(\varphi): \lim_{n \rightarrow \infty} \left\| \mathbb{E} \left( \varphi(n^{-1/2}S_n(t)) - \int \varphi(x\sqrt{t\eta})g(x) dx \mid \mathcal{M}_{k,n} \right) \right\|_1 = 0$$

where  $g$  is the distribution of a standard normal.

S2. (a)

$$\lim_{t \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{E} \left( \frac{S_n^2(t)}{nt} \left( 1 \wedge \frac{|S_n(t)|}{\sqrt{n}} \right) \right) = 0.$$

(b) *For any  $t$  in  $[0, 1]$ , the sequence  $\|\mathbb{E}(n^{-1/2}S_n(t) \mid \mathcal{M}_{0,n})\|_1$  tends to 0 as  $n$  tends to infinity.*

(c) *There exists a nonnegative  $\mathcal{M}_{0,\text{inf}}$ -measurable random variable  $\eta$  such that, for any  $t$  in  $[0, 1]$ , the sequence  $\|\mathbb{E}(n^{-1}S_n^2(t) - t\eta \mid \mathcal{M}_{0,n})\|_1$  tends to 0 as  $n$  tends to infinity.*

Moreover, the random variable  $\eta$  satisfies  $\eta = \eta \circ T$  almost surely.

**REMARK 2.** An assumption on the asymptotic negligibility of the variables  $n^{-1/2}X_{0,n}$  seems to be natural in the context of triangular arrays [see, e.g., the discussion in Hall and Heyde (1980), page 53].

We now turn to the functional version of Theorem 2. Denote by  $\mathcal{H}^*$  the space of continuous functions  $\varphi$  from  $(C([0, 1]), \|\cdot\|_{\infty})$  to  $\mathbb{R}$  such that  $x \rightarrow |(1 + \|x\|_{\infty}^2)^{-1}\varphi(x)|$  is bounded.

**THEOREM 3.** *Let  $X_{i,n}$ ,  $\mathcal{M}_{i,n}$  and  $S_n(t)$  be as in Theorem 2. For any  $t$  in  $[0, 1]$ , set  $U_n(t) = S_n(t) + (nt - [nt])X_{[nt]+1,n}$  and define  $\overline{S}_n(t) = \sup_{0 \leq s \leq t} |S_n(s)|$ . The following statements are equivalent:*

S1\*. *There exists a nonnegative  $\mathcal{M}_{0,\text{inf}}$ -measurable random variable  $\eta$  such that, for any  $\varphi$  in  $\mathcal{H}^*$  and any positive integer  $k$ ,*

$$S1^*(\varphi): \lim_{n \rightarrow \infty} \left\| \mathbb{E} \left( \varphi(n^{-1/2}U_n) - \int \varphi(x\sqrt{\eta})W(dx) \mid \mathcal{M}_{k,n} \right) \right\|_1 = 0,$$

where  $W$  is the distribution of a standard Wiener process.

S2\*. (b) and (c) of S2 hold, and (a) is replaced by:

(a\*) the sequence  $(n^{-1}(\bar{S}_n(1))^2)_{n>0}$  is uniformly integrable, and

$$\lim_{t \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{E} \left( \frac{(\bar{S}_n(t))^2}{nt} \left( 1 \wedge \frac{\bar{S}_n(t)}{\sqrt{n}} \right) \right) = 0.$$

Moreover the random variable  $\eta$  satisfies  $\eta = \eta \circ T$  almost surely.

REMARK 3. If we omit the assumption that  $(n^{-1}(\bar{S}_n(1))^2)_{n>0}$  is uniformly integrable, we still obtain S1\*( $\varphi$ ) for any bounded function  $\varphi$  of  $\mathcal{H}^*$ . Note that Theorem 3 remains valid if we replace (a\*) by the stronger condition

$$\lim_{M \rightarrow \infty} \sup_{t \in [0, 1]} \limsup_{n \rightarrow \infty} \mathbb{E} \left( \frac{(\bar{S}_n(t))^2}{nt} I_{\bar{S}_n(t) \geq M\sqrt{nt}} \right) = 0.$$

REMARK 4. Let  $D([0, 1])$  be the space of caldlag functions equipped with the Skorohod distance  $d$ . Denote by  $\mathcal{H}^*(D)$  the space of continuous functions  $\varphi$  from  $(D([0, 1]), d)$  to  $\mathbb{R}$  such that the function  $x \rightarrow |(1 + \|x\|_\infty^2)^{-1} \varphi(x)|$  is bounded. Applying Theorem 15.5 in Billingsley (1968), we can also obtain that S2\* holds if and only if, for any  $\varphi$  in  $\mathcal{H}^*(D)$  and any positive integer  $k$ ,

$$\lim_{n \rightarrow \infty} \left\| \mathbb{E} \left( \varphi(n^{-1/2} S_n) - \int \varphi(x\sqrt{\eta}) W(dx) \mid \mathcal{M}_{k,n} \right) \right\|_1 = 0.$$

The following result is an important consequence of Theorems 2 and 3:

COROLLARY 1. Let  $X_{i,n}, \mathcal{M}_{i,n}, S_n(t)$  be as in Theorem 2 and  $U_n(t)$  as in Theorem 3. Suppose that the sequence  $(\mathcal{M}_{0,n})_{n \geq 1}$  is nondecreasing. If Condition S2 (resp. S2\*) is satisfied, then, for any  $\varphi$  in  $\mathcal{H}$  [resp.  $\mathcal{H}^*$ ], the sequence  $\varphi(n^{-1/2} S_n(t))$  [resp.  $\varphi(n^{-1/2} U_n)$ ] converges weakly in  $\mathbb{L}^1$  to  $\int \varphi(x\sqrt{t\eta}) g(x) dx$  [resp.  $\int \varphi(x\sqrt{\eta}) W(dx)$ ].

REMARK 5. Corollary 1 implies that the sequence  $n^{-1/2} S_n(t)$  converges stably to a mixture of normal distributions. We refer to Aldous and Eagleson (1978) for a complete exposition of the concept of stability [introduced by Rényi (1963)] and its connection to weak  $\mathbb{L}^1$ -convergence. Note that stable convergence is a useful tool to establish weak convergence of joint distributions [see again Aldous and Eagleson, or Hall and Heyde (1980), Chapter 3, Section 3.2(vi)]. If furthermore  $\eta$  is constant, then the convergence is mixing. If  $\mathbb{P}$  is ergodic, this result is a consequence of Theorem 4 in Eagleson (1976) (see Application 4.2 therein). For a review of mixing results see Csörgő and Fischler (1973).

**3. Proof of Theorem 2.** The fact that S1 implies S2 is obvious. In this section, we focus on the consequences of condition S2. We start with some preliminary results.

3.1. *Definitions and preliminary lemmas.*

DEFINITION 1. Let  $\mu$  be a signed measure on a metric space  $(S, \mathcal{B}(S))$ . Denote by  $|\mu|$  the total variation measure of  $\mu$ , and by  $\|\mu\| = |\mu|(S)$  its norm. We say that a family  $\Pi$  of signed measures on  $(S, \mathcal{B}(S))$  is tight if for every positive  $\varepsilon$  there exists a compact set  $K$  such that  $|\mu|(K^c) < \varepsilon$  for any  $\mu$  in  $\Pi$ . Denote by  $\mathcal{S}(S)$  the set of continuous and bounded functions from  $S$  to  $\mathbb{R}$ . We say that a sequence of signed measures  $(\mu_n)_{n>0}$  converges weakly to a signed measure  $\mu$  if for any  $\varphi$  in  $\mathcal{S}(S)$ ,  $\mu_n(\varphi)$  tends to  $\mu(\varphi)$  as  $n$  tends to infinity.

LEMMA 1. Let  $(\mu_n)_{n>0}$  be a sequence of signed measure on  $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ , and set  $\hat{\mu}_n(t) = \mu_n(\exp(i\langle t, \cdot \rangle))$ . Assume that the sequence  $(\mu_n)_{n>0}$  is tight and that  $\sup_{n>0} \|\mu_n\| < \infty$ . The following statements are equivalent:

1. The sequence  $(\mu_n)_{n>0}$  converges weakly to the null measure.
2. For any  $t$  in  $\mathbb{R}^d$ ,  $\hat{\mu}_n(t)$  tends to zero as  $n$  tends to infinity.

The proof of Lemma 1 will be done in the Appendix.

DEFINITION 2. Define the set  $R(\mathcal{M}_{k,n})$  of Rademacher  $\mathcal{M}_{k,n}$ -measurable random variables:  $R(\mathcal{M}_{k,n}) = \{2\mathbb{I}_A - 1 : A \in \mathcal{M}_{k,n}\}$ . Recall that  $g$  is the  $\mathcal{N}(0, 1)$ -distribution and that  $W$  is the Wiener measure on  $C([0, 1])$ . For the random variable  $\eta$  introduced in Theorem 2 and any bounded random variable  $Z$ , let:

1.  $\nu_n[Z]$  be the image measure of  $Z.\mathbb{P}$  by the variable  $n^{-1/2}S_n(t)$ .
2.  $\nu_n^*[Z]$  be the image measure of  $Z.\mathbb{P}$  by the process  $n^{-1/2}U_n$ .
3.  $\nu[Z]$  be the image measure of  $g.\lambda \otimes Z.\mathbb{P}$  by the variable  $\phi$  from  $\mathbb{R} \otimes \Omega$  to  $\mathbb{R}$  defined by  $\phi(x, \omega) = x\sqrt{t\eta(\omega)}$ .
4.  $\nu^*[Z]$  be the image measure of  $W \otimes Z.\mathbb{P}$  by the variable  $\phi$  from  $C([0, 1]) \otimes \Omega$  to  $C([0, 1])$  defined by  $\phi(x, \omega) = x\sqrt{\eta(\omega)}$ .

LEMMA 2. Let  $\mu_n[Z_n] = \nu_n[Z_n] - \nu[Z_n]$  and  $\mu_n^*[Z_n] = \nu_n^*[Z_n] - \nu^*[Z_n]$ . For any  $\varphi$  in  $\mathcal{H}$  (resp.  $\mathcal{H}^*$ ), the statement S1( $\varphi$ ) [resp. S1\*( $\varphi$ )] is equivalent to:

S3( $\varphi$ ) [resp. S3\*( $\varphi$ ): for any  $Z_n$  in  $R(\mathcal{M}_{k,n})$ , the sequence  $\mu_n[Z_n](\varphi)$  [resp.  $\mu_n^*[Z_n](\varphi)$ ] tends to zero as  $n$  tends to infinity.

PROOF. We prove that S1( $\varphi$ )  $\Leftrightarrow$  S3( $\varphi$ ), the \*-case being unchanged. For  $Z_n$  in  $R(\mathcal{M}_{k,n})$  and  $\varphi$  in  $\mathcal{H}$ , we have

$$\begin{aligned} |\mu_n[Z_n](\varphi)| &= \left| \mathbb{E} \left( Z_n \left( \varphi(n^{-1/2}S_n(t)) - \int \varphi(x\sqrt{t\eta})g(x) dx \right) \right) \right| \\ &\leq \left\| \mathbb{E} \left( \varphi(n^{-1/2}S_n(t)) - \int \varphi(x\sqrt{t\eta})g(x) dx \mid \mathcal{M}_{k,n} \right) \right\|_1. \end{aligned}$$

Consequently, S1( $\varphi$ ) implies S3( $\varphi$ ). Now to prove that S3( $\varphi$ ) implies S1( $\varphi$ ), choose

$$A(n, \varphi) = \left\{ \mathbb{E} \left( \varphi(n^{-1/2} S_n(t)) - \int \varphi(x\sqrt{t\eta})g(x) dx \mid \mathcal{M}_{k,n} \right) \geq 0 \right\},$$

and  $Z_n^\varphi = 2\mathbb{I}_{A(n,\varphi)} - 1$ . Obviously

$$\mu_n[Z_n^\varphi](\varphi) = \left\| \mathbb{E} \left( \varphi(n^{-1/2} S_n(t)) - \int \varphi(x\sqrt{t\eta})g(x) dx \mid \mathcal{M}_{k,n} \right) \right\|_1,$$

and S3( $\varphi$ ) implies S1( $\varphi$ ).  $\square$

3.2. *Invariance of  $\eta$ .* We first prove that if S2 holds, the random variables  $\eta$  satisfies  $\eta = \eta \circ T$  almost surely (or equivalently that  $\eta$  is measurable with respect to the  $\mathbb{P}$ -completion of  $\mathcal{L}$ ). From S2(c) and both the facts that  $(X_{i,n})_{i \in \mathbb{Z}}$  is strictly stationary and  $\mathcal{M}_{0,n} \subseteq \mathcal{M}_{1,n}$ , we have for any  $t$  in  $]0, 1]$ ,

$$(3.1) \quad \lim_{n \rightarrow \infty} \left\| \mathbb{E} \left( \eta \circ T - \frac{S_n^2(t) \circ T}{nt} \mid \mathcal{M}_{0,n} \right) \right\|_1 = 0.$$

On the other hand, defining  $\psi(x) = x^2(1 - (1 \wedge |x|))$  and using the fact that  $T$  preserves  $\mathbb{P}$ , we have

$$(3.2) \quad \begin{aligned} \left\| \frac{S_n^2(t)}{nt} - \frac{S_n^2(t) \circ T}{nt} \right\|_1 &\leq 2 \left\| \frac{S_n^2(t)}{nt} \left( 1 \wedge \frac{|S_n(t)|}{\sqrt{n}} \right) \right\|_1 \\ &\quad + \frac{1}{t} \left\| \psi \left( \frac{S_n(t)}{\sqrt{n}} \right) - \psi \left( \frac{S_n(t) \circ T}{\sqrt{n}} \right) \right\|_1. \end{aligned}$$

To control the second term on the right-hand side, note that the function  $\psi$  is 3-Lipschitz and bounded by 1. It follows that for each positive  $\varepsilon$ ,

$$\left\| \psi \left( \frac{S_n(t)}{\sqrt{n}} \right) - \psi \left( \frac{S_n(t) \circ T}{\sqrt{n}} \right) \right\|_1 \leq 3\varepsilon + 2\mathbb{P}(|X_{0,n} - X_{[nt],n}| > \sqrt{n}\varepsilon).$$

Using that  $n^{-1/2}X_{0,n}$  converges in probability to 0, we derive that

$$\lim_{n \rightarrow \infty} \left\| \psi \left( \frac{S_n(t)}{\sqrt{n}} \right) - \psi \left( \frac{S_n(t) \circ T}{\sqrt{n}} \right) \right\|_1 = 0$$

and the second term on the right-hand side in (3.2) tends to 0 as  $n$  tends to infinity. This fact, together with inequality (3.2) and Condition S2(a), yields

$$\lim_{t \rightarrow 0} \limsup_{n \rightarrow \infty} \left\| \frac{S_n^2(t)}{nt} - \frac{S_n^2(t) \circ T}{nt} \right\|_1 = 0,$$

which, together with S2(c), implies that

$$(3.3) \quad \lim_{t \rightarrow 0} \limsup_{n \rightarrow \infty} \left\| \mathbb{E} \left( \eta - \frac{S_n^2(t) \circ T}{nt} \mid \mathcal{M}_{0,n} \right) \right\|_1 = 0.$$

Combining (3.1) and (3.3), it follows that  $\lim_{n \rightarrow \infty} \|\mathbb{E}(\eta - \eta \circ T \mid \mathcal{M}_{0,n})\|_1 = 0$ , which implies that

$$\lim_{n \rightarrow \infty} \left\| \mathbb{E} \left( \eta - \eta \circ T \mid \bigcap_{k \geq n} \mathcal{M}_{0,k} \right) \right\|_1 = 0.$$

Applying the martingale convergence theorem, we obtain

$$(3.4) \quad \lim_{n \rightarrow \infty} \left\| \mathbb{E} \left( \eta - \eta \circ T \mid \bigcap_{k \geq n} \mathcal{M}_{0,k} \right) \right\|_1 = \|\mathbb{E}(\eta - \eta \circ T \mid \mathcal{M}_{0,\text{inf}})\|_1 = 0.$$

According to S2(c), the random variable  $\eta$  is  $\mathcal{M}_{0,\text{inf}}$ -measurable. Therefore, (3.4) implies that  $\mathbb{E}(\eta \circ T \mid \mathcal{M}_{0,\text{inf}}) = \eta$ . The fact that  $\eta \circ T = \eta$  almost surely is a direct consequence of the following elementary result, whose proof will be done in the Appendix.

LEMMA 3. *Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a probability space,  $X$  a nonnegative random variable, and  $\mathcal{M}$  a  $\sigma$ -algebra of  $\mathcal{A}$ . If the random variable  $\mathbb{E}(X \mid \mathcal{M})$  has the same law as  $X$ , then  $\mathbb{E}(X \mid \mathcal{M}) = X$  almost surely.*

3.3. S2 implies S1. We now turn to the main proof of the paper. First, note that we can restrict ourselves to bounded functions of  $\mathcal{H}$ : if S2 implies S1( $h$ ) for any continuous and bounded function  $h$  then we easily infer from S2(c) that  $n^{-1} S_n^2(t)$  is uniformly integrable for any  $t$  in  $[0, 1]$ , which implies that S1 extends to the whole space  $\mathcal{H}$ .

NOTATION 2. Let  $B_1^3(\mathbb{R})$  be the class of three-times continuously differentiable functions from  $\mathbb{R}$  to  $\mathbb{R}$  such that  $\max(\|h^{(i)}\|_\infty, i \in \{0, 1, 2, 3\}) \leq 1$ .

Suppose now that S1( $h$ ) holds for any  $h$  in  $B_1^3(\mathbb{R})$ . Applying Lemma 2, this is equivalent to say that S3( $h$ ) holds for any  $h$  in  $B_1^3(\mathbb{R})$ , which obviously implies that S3( $h$ ) holds for  $h_t = \exp(it)$ . Using that the probability  $\nu_n[1]$  is tight (since it converges weakly to  $\nu[1]$ ) and that  $|\mu_n[Z_n]| \leq \nu_n[1] + \nu[1]$ , we infer that  $\mu_n[Z_n]$  is tight, and Lemma 1 implies that S3( $h$ ) [and therefore S1( $h$ )] holds for any continuous bounded function  $h$ .

On the other hand, from the asymptotic negligibility of  $n^{-1/2} X_{0,n}$  we infer that, for any positive integer  $k$ ,  $n^{-1/2}(S_n(t) - S_n(t) \circ T^k)$  converges in probability to

zero. Consequently, since any function  $h$  belonging to  $B_1^3(\mathbb{R})$  is 1-Lipschitz and bounded, we have

$$\lim_{n \rightarrow \infty} \|h(n^{-1/2}S_n(t)) - h(n^{-1/2}S_n(t) \circ T^k)\|_1 = 0,$$

and  $S1(h)$  is equivalent to

$$\lim_{n \rightarrow \infty} \left\| \mathbb{E} \left( h(n^{-1/2}S_n(t) \circ T^k) - \int h(x\sqrt{t\eta})g(x) dx \mid \mathcal{M}_{k,n} \right) \right\|_1 = 0.$$

Now, since both  $\eta$  and  $\mathbb{P}$  are invariant by  $T$ , we infer that Theorem 2 is a straightforward consequence of Proposition 4 below:

PROPOSITION 4. *Let  $X_{i,n}$  and  $\mathcal{M}_{i,n}$  be defined as in Theorem 2. If S2 holds, then, for any  $h$  in  $B_1^3(\mathbb{R})$  and any  $t$  in  $[0, 1]$ ,*

$$\lim_{n \rightarrow \infty} \left\| \mathbb{E} \left( h(n^{-1/2}S_n(t)) - \int h(x\sqrt{t\eta})g(x) dx \mid \mathcal{M}_{0,n} \right) \right\|_1 = 0$$

where  $g$  is the distribution of a standard normal.

PROOF. We prove the result for  $S_n(1)$ , the proof of the general case being unchanged. Without loss of generality, suppose that there exists a sequence  $(\varepsilon_i)_{i \in \mathbb{Z}}$  of  $\mathcal{N}(0, 1)$ -distributed and independent random variables, independent of  $\mathcal{M}_{\infty, \infty} = \sigma(\bigcup_{k,n} \mathcal{M}_{k,n})$ .

NOTATION 3. Let  $i, p$  and  $n$  be three integers such that  $1 \leq i \leq p \leq n$ . Set  $q = [n/p]$  and define

$$U_{i,n} = X_{iq-q+1,n} + \dots + X_{iq,n}, \quad V_{i,n} = \frac{1}{\sqrt{n}}(U_{1,n} + U_{2,n} + \dots + U_{i,n}),$$

$$\Delta_i = \varepsilon_{iq-q+1} + \dots + \varepsilon_{iq}, \quad \Gamma_i = \sqrt{\frac{\eta}{n}}(\Delta_i + \Delta_{i+1} + \dots + \Delta_p).$$

NOTATION 4. Let  $g$  be any function from  $\mathbb{R}$  to  $\mathbb{R}$ . For  $k$  and  $l$  in  $[1, p]$  and any positive integer  $n \geq p$ , set  $g_{k,l;n} = g(V_{k,n} + \Gamma_l)$ , with the conventions  $g_{k,p+1;n} = g(V_{k,n})$  and  $g_{0,l;n} = g(\Gamma_l)$ . Afterward, we shall apply this notation to the successive derivatives of the function  $h$ . For brevity we shall omit the index  $n$ .

Let  $s_n = \sqrt{\eta}(\varepsilon_1 + \dots + \varepsilon_n)$ . Since  $(\varepsilon_i)_{i \in \mathbb{Z}}$  is independent of  $\mathcal{M}_{\infty, \infty}$ , we have, integrating with respect to  $(\varepsilon_i)_{i \in \mathbb{Z}}$ ,

$$(3.5) \quad \begin{aligned} & \mathbb{E} \left( h(n^{-1/2}S_n(1)) - \int h(x\sqrt{\eta})g(x)dx \mid \mathcal{M}_{0,n} \right) \\ &= \mathbb{E}(h(n^{-1/2}S_n(1)) - h(V_{p,n}) \mid \mathcal{M}_{0,n}) \\ & \quad + \mathbb{E}(h(V_{p,n}) - h(\Gamma_1) \mid \mathcal{M}_{0,n}) + \mathbb{E}(h(\Gamma_1) - h(n^{-1/2}s_n) \mid \mathcal{M}_{0,n}). \end{aligned}$$

Here, note that  $|n^{-1/2}S_n(1) - V_{p,n}| \leq n^{-1/2}(|X_{n-p+2,n}| + \dots + |X_{n,n}|)$ . Using the asymptotic negligibility of  $n^{-1/2}X_{0,n}$ , we infer that  $n^{-1/2}S_n(1) - V_{p,n}$  converges in probability to zero. Since furthermore  $h$  is 1-Lipschitz and bounded, we conclude that

$$(3.6) \quad \lim_{n \rightarrow \infty} \|h(n^{-1/2}S_n(1)) - h(V_{p,n})\|_1 = 0,$$

and the same arguments yield

$$(3.7) \quad \lim_{n \rightarrow \infty} \|h(\Gamma_1) - h(n^{-1/2}s_n)\|_1 = 0.$$

In view of (3.6) and (3.7), it remains to control the second term in the right-hand side of (3.5). To this end, we use Lindeberg’s decomposition, as shown in Dedecker and Rio (2000):

$$(3.8) \quad h(V_{p,n}) - h(\Gamma_1) = \sum_{i=1}^p (h_{i,i+1} - h_{i-1,i+1}) + \sum_{i=1}^p (h_{i-1,i+1} - h_{i-1,i}).$$

Now, applying Taylor’s integral formula we get that

$$\begin{aligned} h_{i,i+1} - h_{i-1,i+1} &= \frac{1}{\sqrt{n}}U_{i,n}h'_{i-1,i+1} + \frac{1}{2n}U_{i,n}^2h''_{i-1,i+1} + R_i, \\ h_{i-1,i+1} - h_{i-1,i} &= -\sqrt{\frac{\eta}{n}}\Delta_i h'_{i-1,i+1} - \frac{\eta}{2n}\Delta_i^2 h''_{i-1,i+1} + r_i, \end{aligned}$$

where

$$(3.9) \quad |R_i| \leq \frac{U_{i,n}^2}{n} \left(1 \wedge \frac{|U_{i,n}|}{\sqrt{n}}\right) \quad \text{and} \quad |r_i| \leq \frac{\eta \Delta_i^2}{n} \left(1 \wedge \frac{\sqrt{\eta}|\Delta_i|}{\sqrt{n}}\right).$$

Since  $\Delta_i$  is centered and independent of  $\sigma(\mathcal{M}_{\infty,\infty} \cup \sigma(h'_{i-1,i+1}))$ , we have  $\mathbb{E}(\sqrt{\eta}\Delta_i h'_{i-1,i+1} | \mathcal{M}_{0,n}) = \mathbb{E}(\Delta_i)\mathbb{E}(\sqrt{\eta}h'_{i-1,i+1} | \mathcal{M}_{0,n}) = 0$ . It follows that

$$(3.10) \quad \mathbb{E}(h(V_p) - h(\Gamma_1) | \mathcal{M}_{0,n}) = D_1 + D_2 + D_3,$$

where

$$\begin{aligned} D_1 &= \sum_{i=1}^p \mathbb{E}(n^{-1/2}U_{i,n}h'_{i-1,i+1} | \mathcal{M}_{0,n}), \\ D_2 &= \frac{1}{2} \sum_{i=1}^p \mathbb{E}(n^{-1}(U_{i,n}^2 - \eta \Delta_i^2)h''_{i-1,i+1} | \mathcal{M}_{0,n}), \\ D_3 &= \sum_{i=1}^p \mathbb{E}(R_i + r_i | \mathcal{M}_{0,n}). \end{aligned}$$

CONTROL OF  $D_3$ . From (3.9) and the fact that  $T$  preserves  $\mathbb{P}$ , we get

$$\sum_{i=1}^p \|R_i\|_1 \leq \mathbb{E} \left( \frac{S_n^2(1/p)}{n/p} \left( 1 \wedge \frac{|S_n(1/p)|}{\sqrt{n}} \right) \right),$$

and S2(a) implies that

$$(3.11) \quad \lim_{p \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{i=1}^p \|R_i\|_1 = 0.$$

Moreover, since for  $t \in [0, 1]$ , the sequence  $(\eta/n)^{-1/2}(\varepsilon_1 + \dots + \varepsilon_{[nt]})$  obviously satisfies S2(a), the same argument applies to  $\sum_{i=1}^p \|r_i\|_1$ . Finally,

$$(3.12) \quad \lim_{p \rightarrow \infty} \limsup_{n \rightarrow \infty} \|D_3\|_1 = 0.$$

CONTROL OF  $D_1$ . To prove that  $\|D_1\|_1$  tends to zero as  $n$  tends to infinity, it suffices to show that, for any positive integer  $i$  less than  $p$ ,

$$(3.13) \quad \lim_{n \rightarrow \infty} \|\mathbb{E}(n^{-1/2}U_{i,n}h'_{i-1,i+1}|\mathcal{M}_{0,n})\|_1 = 0.$$

Denote by  $\mathbb{E}_\varepsilon$  the integration with respect to the sequence  $(\varepsilon_i)_{i \in \mathbb{Z}}$ .

Set  $l(i, n) = (i - 1)[n/p]$ . Bearing in mind the definition of  $h'_{i-1,i+1}$  and integrating with respect to  $(\varepsilon_i)_{i \in \mathbb{Z}}$  we deduce that the random variable  $\mathbb{E}_\varepsilon(h'_{i-1,i+1})$  is  $\mathcal{M}_{l(i,n),n}$ -measurable and bounded by one. Now, since the  $\sigma$ -algebra  $\mathcal{M}_{0,n}$  is included into  $\mathcal{M}_{l(i,n),n}$ , we obtain

$$\|\mathbb{E}(n^{-1/2}U_{i,n}h'_{i-1,i+1}|\mathcal{M}_{0,n})\|_1 \leq \|\mathbb{E}(n^{-1/2}U_{i,n}|\mathcal{M}_{l(i,n),n})\|_1.$$

Using that  $T$  preserves  $\mathbb{P}$ , the latter equals  $\|\mathbb{E}(n^{-1/2}S_n(1/p)|\mathcal{M}_{0,n})\|_1$  and S2(b) implies that (3.13) holds.

CONTROL OF  $D_2$ . To prove that  $\|D_2\|_1$  tends to zero as  $n$  tends to infinity, it suffices to show that, for any positive integer  $i$  less than  $p$ ,

$$(3.14) \quad \lim_{n \rightarrow \infty} \|\mathbb{E}(n^{-1}(U_{i,n}^2 - \eta\Delta_i^2)h''_{i-1,i+1}|\mathcal{M}_{0,n})\|_1 = 0.$$

Integrating with respect to  $(\varepsilon_i)_{i \in \mathbb{Z}}$ , we have

$$\|\mathbb{E}((U_{i,n}^2 - \eta\Delta_i^2)h''_{i-1,i+1}|\mathcal{M}_{0,n})\|_1 = \|\mathbb{E}((U_{i,n}^2 - \eta[np^{-1}])\mathbb{E}_\varepsilon(h''_{i-1,i+1})|\mathcal{M}_{0,n})\|_1.$$

Since  $n^{-1}[np^{-1}]$  converges to  $p^{-1}$ , (3.14) will be proved if for any positive integer  $i$  less than  $p$ ,

$$\lim_{n \rightarrow \infty} \|\mathbb{E}((n^{-1}U_{i,n}^2 - \eta p^{-1})\mathbb{E}_\varepsilon(h''_{i-1,i+1})|\mathcal{M}_{0,n})\|_1 = 0.$$

Arguing as for the control of  $D_1$ , we have

$$\|\mathbb{E}((n^{-1}U_{i,n}^2 - \eta p^{-1})\mathbb{E}_\varepsilon(h''_{i-1,i+1})|\mathcal{M}_{0,n})\|_1 \leq \|\mathbb{E}(n^{-1}U_{i,n}^2 - \eta p^{-1}|\mathcal{M}_{l(i,n),n})\|_1.$$

Since both  $\eta$  and  $\mathbb{P}$  are invariant by the transformation  $T$ , the latter equals  $\|\mathbb{E}(n^{-1}S_n^2(1/p) - \eta p^{-1}|\mathcal{M}_{0,n})\|_1$  and S2(c) implies that (3.14) holds.

END OF THE PROOF OF PROPOSITION 4. From (3.12), (3.13) and (3.14) we infer that, for any  $h$  in  $B_1^3(\mathbb{R})$ ,

$$\lim_{p \rightarrow \infty} \limsup_{n \rightarrow \infty} \|D_1 + D_2 + D_3\|_1 = 0.$$

This fact, together with (3.5), (3.6), (3.7) and (3.10), implies Proposition 4.  $\square$

**4. Proof of Theorem 3.** Once again, it suffices to prove that  $S2^*$  implies  $S1^*$ . Suppose that  $S1^*(\varphi)$  holds for any bounded function  $\varphi$  of  $\mathcal{H}^*$ . Since  $(n^{-1}(S_n^*(1))^2)_{n>0}$  is uniformly integrable,  $S1^*(\varphi)$  obviously extends to the whole space  $\mathcal{H}^*$ .

Consequently, we can restrict ourselves to the space of continuous bounded functions from  $C([0, 1])$  to  $\mathbb{R}$ . According to Lemma 2, the proof of Theorem 2 will be complete if we show that, for any  $Z_n$  in  $R(\mathcal{M}_{k,n})$ , the sequence  $\mu^*[Z_n]$  converges weakly to the null measure as  $n$  tends to infinity.

**DEFINITION 3.** For  $0 \leq t_1 < \dots < t_d \leq 1$ , define the functions  $\pi_{t_1 \dots t_d}$  and  $Q_{t_1 \dots t_d}$  from  $C([0, 1])$  to  $\mathbb{R}^d$  by the equalities  $\pi_{t_1 \dots t_d}(x) = (x(t_1), \dots, x(t_d))$  and  $Q_{t_1 \dots t_d}(x) = (x(t_1), x(t_2) - x(t_1), \dots, x(t_d) - x(t_{d-1}))$ . For any signed measure  $\mu$  on  $(C([0, 1]), \mathcal{B}(C([0, 1])))$  and any function  $f$  from  $C([0, 1])$  to  $\mathbb{R}^d$ , denote by  $\mu f^{-1}$  the image measure of  $\mu$  by  $f$ .

Let  $\mu$  and  $\nu$  be two signed measures on  $(C([0, 1]), \mathcal{B}(C([0, 1])))$ . Recall that if  $\mu \pi_{t_1 \dots t_d}^{-1} = \nu \pi_{t_1 \dots t_d}^{-1}$  for any integer  $d$  and any  $d$ -tuple such that  $0 \leq t_1 < \dots < t_d \leq 1$ , then  $\mu = \nu$ . Consequently Theorem 2 is a straightforward consequence of the two following items:

1. Relative compactness: for any  $Z_n$  in  $R(\mathcal{M}_{k,n})$ , the family  $(\mu_n^*[Z_n])_{n>0}$  is relatively compact with respect to the topology of weak convergence.
2. Finite dimensional convergence: for any any  $Z_n$  in  $R(\mathcal{M}_{k,n})$ , any positive integer  $d$  and any  $d$ -tuple  $0 \leq t_1 < \dots < t_d \leq 1$ , the sequence  $\mu_n^*[Z_n] \pi_{t_1 \dots t_d}^{-1}$  converges weakly to the null measure as  $n$  tends to infinity.

**4.1. Finite dimensional convergence.** Clearly it is equivalent to take  $Q_{t_1 \dots t_d}$  instead of  $\pi_{t_1 \dots t_d}$  in item 2. The following lemma shows that finite dimensional convergence is a consequence of Condition S2. The stronger condition  $S2^*$  is only required for tightness.

**LEMMA 4.** For any  $a$  in  $\mathbb{R}^d$  define  $f_a$  from  $\mathbb{R}^d$  to  $\mathbb{R}$  by  $f_a(x) = \langle a, x \rangle$ . If S2 holds then, for any  $a$  in  $\mathbb{R}^d$ , any  $d$ -tuple  $0 \leq t_1 < \dots < t_d \leq 1$  and any  $Z_n$  in  $R(\mathcal{M}_{k,n})$ , the sequence  $\mu_n^*[Z_n](f_a \circ Q_{t_1 \dots t_d})^{-1}$  converges weakly to the null measure.

Write  $\mu_n^*[Z_n](f_a \circ Q_{t_1 \dots t_d})^{-1}(\exp(i \cdot)) = \mu_n^*[Z_n]Q_{t_1 \dots t_d}^{-1}(\exp(i \langle a, \cdot \rangle))$ . According to Lemma 4, the latter converges to zero as  $n$  tends to infinity. Taking  $Z_n = 1$ , we infer that the probability measure  $\nu_n^*[1]Q_{t_1 \dots t_d}^{-1}$  converges weakly to the probability measure  $\nu^*[1]Q_{t_1 \dots t_d}^{-1}$  and hence is tight. Since  $|\mu^*[Z_n]Q_{t_1 \dots t_d}^{-1}| \leq \nu_n^*[1]Q_{t_1 \dots t_d}^{-1} + \nu^*[1]Q_{t_1 \dots t_d}^{-1}$ , the sequence  $(\mu^*[Z_n]Q_{t_1 \dots t_d}^{-1})_{n>0}$  is tight. Consequently we can apply Lemma 1 to conclude that  $\mu^*[Z_n]Q_{t_1 \dots t_d}^{-1}$  converges weakly to the null measure.

PROOF OF LEMMA 4. According to Lemma 2, we have to prove the property  $S1^*(\varphi \circ f_a \circ Q_{t_1 \dots t_d})$  for any continuous bounded function  $\varphi$ . Arguing as in Section 3.3, we can restrict ourselves to the class of function  $B_1^3(\mathbb{R})$ . Let  $h$  be any element of  $B_1^3(\mathbb{R})$  and write

$$\begin{aligned} & h \circ f_a \circ Q_{t_1 \dots t_d}(n^{-1/2}U_n) - \int h \circ f_a \circ Q_{t_1 \dots t_d}(x\sqrt{\eta})W(dx) \\ &= \sum_{\ell=1}^d h_\ell \left( a_\ell \left( \frac{U_n(t_\ell) - U_n(t_{\ell-1})}{\sqrt{n}} \right) \right) - \int h_\ell(a_\ell x \sqrt{(t_\ell - t_{\ell-1})\eta})g(x) dx, \end{aligned}$$

where the random variable  $h_\ell(x)$  is equal to

$$\int h \left( \sum_{i=1}^{\ell-1} a_i \left( \frac{U_n(t_i) - U_n(t_{i-1})}{\sqrt{n}} \right) + x + \sum_{i=\ell+1}^d a_i x_i \sqrt{(t_i - t_{i-1})\eta} \right) \prod_{i=\ell+1}^d g(x_i) dx_i.$$

Note that for any  $\omega$  in  $\Omega$ , the random function  $h_\ell$  belongs to  $B_1^3(\mathbb{R})$ . To complete the proof of Lemma 4, it suffices to see that, for any positive integers  $k$  and  $\ell$ , the sequence

$$\begin{aligned} & \left\| \mathbb{E} \left( h_\ell \left( a_\ell \left( \frac{U_n(t_\ell) - U_n(t_{\ell-1})}{\sqrt{n}} \right) \right) \right. \right. \\ (4.1) \quad & \left. \left. - \int h_\ell(a_\ell x \sqrt{(t_\ell - t_{\ell-1})\eta})g(x) dx \mid \mathcal{M}_{k,n} \right) \right\|_1 \end{aligned}$$

tends to zero as  $n$  tends to infinity. Since  $h_\ell$  is 1-Lipshitz and bounded, we infer from the asymptotic negligibility of  $n^{-1/2}X_{0,n}$  that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left\| h_\ell \left( a_\ell \left( \frac{U_n(t_\ell) - U_n(t_{\ell-1})}{\sqrt{n}} \right) \right) \right. \\ (4.2) \quad & \left. - h_\ell \left( a_\ell \left( \frac{S_n(t_\ell - t_{\ell-1}) \circ T^{[nt_{\ell-1}] + 1}}{\sqrt{n}} \right) \right) \right\|_1 = 0. \end{aligned}$$

Denote by  $g_\ell$  the random function  $g_\ell = h_\ell \circ T^{-[nt_{\ell-1}]^{-1}}$ . Combining (4.1), (4.2) and the fact that  $\mathcal{M}_{k-1-[nt_{\ell-1}],n} \subseteq \mathcal{M}_{k,n}$ , we infer that it suffices to prove that

$$(4.3) \quad \lim_{n \rightarrow \infty} \left\| \mathbb{E} \left( g_\ell(a_\ell n^{-1/2} S_n(u)) - \int g_\ell(a_\ell x \sqrt{u\eta}) g(x) dx \mid \mathcal{M}_{k,n} \right) \right\|_1 = 0.$$

Since the random functions  $g_\ell$  is  $\mathcal{M}_{0,n}$ -measurable (4.3) can be proved exactly as property S1 [see Section (3.3)]. This completes the proof of Lemma 4.  $\square$

*4.2. Relative compactness.* In this section we shall prove that the sequence  $(\mu_n^*[Z_n])_{n>0}$  is relatively compact with respect to the topology of weak convergence. That is, for any increasing function  $f$  from  $\mathbb{N}$  to  $\mathbb{N}$ , there exists an increasing function  $g$  with value in  $f(\mathbb{N})$  and a signed measure  $\mu$  on  $(C([0, 1]), \mathcal{B}(C([0, 1])))$  such that  $(\mu_{g(n)}^*[Z_{g(n)}])_{n>0}$  converges weakly to  $\mu$ .

Let  $Z_n^+$  (resp.  $Z_n^-$ ) be the positive (resp. negative) part of  $Z_n$ , and write

$$\mu_n^*[Z_n] = \mu_n^*[Z_n^+] - \mu_n^*[Z_n^-] = \nu_n^*[Z_n^+] - \nu_n^*[Z_n^-] - \nu^*[Z_n^+] + \nu^*[Z_n^-],$$

where  $\nu_n^*[Z]$  and  $\nu^*[Z]$  are defined in 2. and 4. of Definition 2. Obviously, it is enough to prove that each sequence of finite positive measures  $(\nu_n^*[Z_n^+])_{n>0}$ ,  $(\nu_n^*[Z_n^-])_{n>0}$ ,  $(\nu^*[Z_n^+])_{n>0}$  and  $(\nu^*[Z_n^-])_{n>0}$  is relatively compact. We prove the result for the sequence  $(\nu_n^*[Z_n^+])_{n>0}$ , the other cases being similar.

Let  $f$  be any increasing function from  $\mathbb{N}$  to  $\mathbb{N}$ . Choose an increasing function  $l$  with value in  $f(\mathbb{N})$  such that

$$\lim_{n \rightarrow \infty} \mathbb{E}(Z_{l(n)}^+) = \liminf_{n \rightarrow \infty} \mathbb{E}(Z_{f(n)}^+).$$

We must sort out two cases:

1. If  $\mathbb{E}(Z_{l(n)}^+)$  converges to zero as  $n$  tends to infinity, then, taking  $g = l$ , the sequence  $(\nu_{g(n)}^*[Z_{g(n)}^+])_{n>0}$  converges weakly to the null measure.
2. If  $\mathbb{E}(Z_{l(n)}^+)$  converges to a positive real number as  $n$  tends to infinity, we introduce, for  $n$  large enough, the probability measure  $p_n$  defined by  $p_n = (\mathbb{E}(Z_{l(n)}^+))^{-1} \nu_{l(n)}^*[Z_{l(n)}^+]$ . Obviously if  $(p_n)_{n>0}$  is relatively compact with respect to the topology of weak convergence, then there exists an increasing function  $g$  with value in  $l(\mathbb{N})$  [and hence in  $f(\mathbb{N})$ ] and a measure  $\nu$  such that  $(\nu_{g(n)}^*[Z_{g(n)}^+])_{n>0}$  converges weakly to  $\nu$ . Since  $(p_n)_{n>0}$  is a family of probability measures, relative compactness is equivalent to tightness. Here we apply Theorem 8.2 in Billingsley (1968): to derive the tightness of the sequence  $(p_n)_{n>0}$  it is enough to show that, for each positive  $\varepsilon$ ,

$$(4.4) \quad \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} p_n(x : w(x, \delta) \geq \varepsilon) = 0,$$

where  $w(x, \delta)$  is the modulus of continuity of the function  $x$ . According to the definition of  $p_n$ , we have

$$p_n(x : w(x, \delta) \geq \varepsilon) = \frac{1}{\mathbb{E}(Z_{l(n)}^+)} Z_{l(n)}^+ \cdot \mathbb{P} \left( w \left( \frac{U_{l(n)}}{\sqrt{l(n)}}, \delta \right) \geq \varepsilon \right).$$

Since both  $\mathbb{E}(Z_{l(n)}^+)$  converges to a positive number and  $Z_{l(n)}^+$  is bounded by one, we infer that (4.4) holds if

$$(4.5) \quad \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}\left(w\left(\frac{U_{l(n)}}{\sqrt{l(n)}}, \delta\right) \geq \varepsilon\right) = 0.$$

From Theorem 8.3 and inequality (8.16) in Billingsley (1968), it suffices to prove that, for any positive  $\varepsilon$ ,

$$(4.6) \quad \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{\delta} \mathbb{P}\left(\frac{\bar{S}_{l(n)}(\delta)}{\sqrt{l(n)}\delta} \geq \frac{\varepsilon}{\sqrt{\delta}}\right) = 0.$$

We conclude by noting that (4.6) follows straightforwardly from S2(a\*) and Markov’s inequality.

CONCLUSION. In both cases there exists an increasing function  $g$  with value in  $f(\mathbb{N})$  and a measure  $\nu$  such that  $(\nu_{g(n)}^*[Z_{g(n)}^+])_{n>0}$  converges weakly to  $\nu$ . Since this is true for any increasing function  $f$  with value in  $\mathbb{N}$ , we conclude that the sequence  $(\nu_n^*[Z_n^+])_{n>0}$  is relatively compact with respect to the topology of weak convergence. Of course, the same arguments apply to the sequences  $(\nu_n^*[Z_n^-])_{n>0}$ ,  $(\nu^*[Z_n^+])_{n>0}$  and  $(\nu^*[Z_n^-])_{n>0}$ , which implies the relative compactness of the sequence  $(\mu_n^*[Z_n])_{n>0}$ .

**5. Proof of Corollary 1.** We have to prove that if S2 holds, then, for any bounded random variable  $Z$ , any  $t$  in  $[0, 1]$  and any  $\varphi$  in  $\mathcal{H}$ ,

$$(5.1) \quad \lim_{n \rightarrow \infty} \mathbb{E}(Z\varphi(n^{-1/2}S_n(t))) = \mathbb{E}\left(Z \int \varphi(x\sqrt{t\eta})g(x) dx\right).$$

Since  $n^{-1}S_n^2(t)$  is uniformly integrable, we need only prove (5.1) for continuous bounded functions. Recall that  $\mathcal{M}_{\infty,\infty} = \sigma(\bigcup_{k,n} \mathcal{M}_{k,n})$ . Since both  $S_n(t)$  and  $\eta$  are  $\mathcal{M}_{\infty,\infty}$ -measurable, we can and do suppose that so is  $Z$ .

Set  $Z_{k,n} = \mathbb{E}(Z|\mathcal{M}_{k,n})$ , and use the decomposition

$$\mathbb{E}(Z\varphi(n^{-1/2}S_n(t))) - \mathbb{E}\left(Z \int \varphi(x\sqrt{t\eta})g(x) dx\right) = T_1 + T_2 + T_3,$$

where

$$\begin{aligned} T_1 &= \mathbb{E}((Z - Z_{k,n})\varphi(n^{-1/2}S_n(t))), \\ T_2 &= \mathbb{E}\left(Z_{k,n}\left(\varphi(n^{-1/2}S_n(t)) - \int \varphi(x\sqrt{t\eta})g(x) dx\right)\right), \\ T_3 &= \mathbb{E}\left((Z_{k,n} - Z) \int \varphi(x\sqrt{t\eta})g(x) dx\right). \end{aligned}$$

By assumption, the array  $\mathcal{M}_{k,n}$  is nondecreasing in  $k$  and  $n$ . Since the random variable  $Z$  is  $\mathcal{M}_{\infty,\infty}$ -measurable, the martingale convergence theorem implies that  $\lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \|Z_{k,n} - Z\|_1 = 0$ . Consequently,

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} |T_1| = \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} |T_3| = 0.$$

On the other hand, Theorem 2 implies that  $T_2$  tends to zero as  $n$  tends to infinity, which completes the proof of Corollary 1.

**6. Applications of Theorem 1.** In this section we prove Propositions 1, 2 and 3.

6.1. *Proof of Proposition 1.* From Theorem 1 in Volný (1993), we know that (1.1) is equivalent to the existence of a random variable  $m$  in  $H_0 \ominus H_{-1}$  such that

$$(6.1) \quad \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left\| \sum_{i=1}^n X_0 \circ T^i - m \circ T^i \right\|_2 = 0.$$

Let  $W_n = m \circ T + \dots + m \circ T^n$ . Since  $(m \circ T^i)_{i \in \mathbb{Z}}$  is a stationary sequence of martingale differences with respect to the filtration  $(\mathcal{M}_i)_{i \in \mathbb{Z}}$ , it satisfies s2. More precisely,  $n^{-1} \mathbb{E}(W_n^2 | \mathcal{M}_0)$  converges to  $\eta = \mathbb{E}(m^2 | \mathcal{I})$  in  $\mathbb{L}^1$ . Now, we shall use (6.1) to see that the sequence  $(X_i)_{i \in \mathbb{Z}}$  also satisfies s2.

PROOF OF s2(b). From (6.1) it is clear that  $n^{-1/2} \|\mathbb{E}(S_n | \mathcal{M}_0)\|_2$  tends to zero as  $n$  tends to infinity.  $\square$

PROOF OF s2(c). To see that  $n^{-1} \mathbb{E}(S_n^2 | \mathcal{M}_0)$  converges to  $\eta$  in  $\mathbb{L}^1$ , write

$$(6.2) \quad \frac{1}{n} \|\mathbb{E}(S_n^2 - W_n^2 | \mathcal{M}_0)\|_1 \leq \frac{1}{n} \|S_n^2 - W_n^2\|_1 \leq \frac{\|S_n + W_n\|_2 \|S_n - W_n\|_2}{\sqrt{n}}.$$

From (6.1) the latter tends to zero as  $n$  tends to infinity and therefore  $(X_i)_{i \in \mathbb{Z}}$  satisfies s2(c) with  $\eta = \mathbb{E}(m^2 | \mathcal{I})$ .  $\square$

PROOF OF s2(a). Using both that  $n^{-1} W_n^2$  is uniformly integrable and that the function  $x \rightarrow (1 \wedge |x|)$  is 1-Lipschitz, we have, for any positive real  $M$ ,

$$(6.3) \quad \lim_{n \rightarrow \infty} \left( \frac{W_n^2}{n} \left| \left( 1 \wedge \frac{|S_n|}{M\sqrt{n}} \right) - \left( 1 \wedge \frac{|W_n|}{M\sqrt{n}} \right) \right| \right) = 0.$$

Since  $|x^2(1 \wedge |y|) - z^2(1 \wedge |t|)| \leq |x^2 - z^2| + z^2|(1 \wedge |y|) - (1 \wedge |t|)|$ , we infer from (6.2) and (6.3) that

$$\lim_{n \rightarrow \infty} \left\| \frac{S_n^2}{n} \left( 1 \wedge \frac{|S_n|}{M\sqrt{n}} \right) - \frac{W_n^2}{n} \left( 1 \wedge \frac{|W_n|}{M\sqrt{n}} \right) \right\|_1 = 0.$$

Now, the uniform integrability of  $n^{-1}W_n^2$  yields

$$\lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} \left( \frac{S_n^2}{n} \left( 1 \wedge \frac{|S_n|}{M\sqrt{n}} \right) \right) = 0,$$

which means exactly that  $n^{-1}S_n^2$  is uniformly integrable. This completes the proof of Proposition 1.  $\square$

6.2. *Proof of Proposition 2.* Let  $P_i$  be the projection operator onto  $H_i \ominus H_{i-1}$ : for any function  $f$  in  $\mathbb{L}^2(\mathbb{P})$ ,  $P_i(f) = \mathbb{E}(f|\mathcal{M}_i) - \mathbb{E}(f|\mathcal{M}_{i-1})$ . We first recall a result due to Volný [(1993), Theorem 6] [see Theorem 5 of the same paper or Annexe A, Corollary 2 in Dedecker (1998) for weaker conditions].

PROPOSITION 5. *Let  $(\mathcal{M}_i)_{i \in \mathbb{Z}}$  and  $(X_i)_{i \in \mathbb{Z}}$  be as in Theorem 1. Define the  $\sigma$ -algebra  $\mathcal{M}_{-\infty} = \bigcap_{i \in \mathbb{Z}} \mathcal{M}_i$  and consider the condition*

$$(6.4) \quad \mathbb{E}(X_0|\mathcal{M}_{-\infty}) = 0 \quad \text{and} \quad \sum_{i>0} \|P_0(X_i)\|_2 < \infty.$$

Condition (6.4) implies (1.2).

We now prove that (1.3) implies (6.4) and hence (1.2). First, we have the orthogonal decomposition

$$(6.5) \quad X_k = \mathbb{E}(X_k|\mathcal{M}_{-\infty}) + \sum_{i=0}^{\infty} P_{k-i}(X_k).$$

Since (1.3) implies that  $\mathbb{E}(X_k|\mathcal{M}_{-\infty}) = 0$ , we infer from (6.5) and the stationarity of  $(X_i)_{i \in \mathbb{Z}}$  that

$$\sum_{k>0} L_k \|\mathbb{E}(X_k|\mathcal{M}_0)\|_2^2 = \sum_{k>0} L_k \sum_{i \leq 0} \|P_i(X_k)\|_2^2 = \sum_{i>0} \left( \sum_{k=1}^i L_k \right) \|P_0(X_i)\|_2^2.$$

Setting  $a_i = L_1 + \dots + L_i$ , we infer that (1.3) is equivalent to

$$(6.6) \quad \mathbb{E}(X_0|\mathcal{M}_{-\infty}) = 0, \quad \sum_{i>0} a_i \|P_0(X_i)\|_2^2 < \infty \quad \text{and} \quad \sum_{i>0} \frac{1}{a_i} < \infty.$$

Now, Hölder's inequality in  $\ell^2$  gives

$$\sum_{i>0} \|P_0(X_i)\|_2 \leq \left( \sum_{i>0} \frac{1}{a_i} \right)^{1/2} \left( \sum_{i>0} a_i \|P_0(X_i)\|_2^2 \right)^{1/2} < \infty,$$

and (6.4) [hence (1.2)] follows from (1.3).

Now, to complete the proof of Proposition 2, it remains to show that (1.3) implies s2(a\*). This is a direct consequence of Proposition 8 in Section 7 whose proof will be done by applying the following maximal inequality:

**PROPOSITION 6.** *Let  $(X_i)_{i \in \mathbb{Z}}$  be a sequence of square-integrable and centered random variables, adapted to a nondecreasing filtration  $(\mathcal{M}_i)_{i \in \mathbb{Z}}$ . Define the  $\sigma$ -algebra  $\mathcal{M}_{-\infty} = \bigcap_{i \in \mathbb{Z}} \mathcal{M}_i$  and the random variables  $S_n = X_1 + \dots + X_n$  and  $S_n^* = \max\{0, S_1, \dots, S_n\}$ . For any positive integer  $i$ , let  $(Y_{i,j})_{j \geq 1}$  be the martingale  $Y_{i,j} = \sum_{k=1}^j P_{k-i}(X_k)$  and  $Y_{i,n}^* = \max\{0, Y_{i,1}, \dots, Y_{i,n}\}$ . Let  $\lambda$  be any nonnegative real number and  $\Gamma(i, k, \lambda) = \{Y_{i,k}^* > \lambda\}$ . Assume that the sequence is regular: for any integer  $k$ ,  $\mathbb{E}(X_k | \mathcal{M}_{-\infty}) = 0$ . For any two sequences of nonnegative numbers  $(a_i)_{i \geq 0}$  and  $(b_i)_{i \geq 0}$  such that  $K = \sum a_i^{-1}$  is finite and  $\sum b_i = 1$  we have*

$$\mathbb{E}((S_n^* - \lambda)_+^2) \leq 4K \sum_{i=0}^{\infty} a_i \left( \sum_{k=1}^n \mathbb{E}(P_{k-i}^2(X_k) \mathbb{1}_{\Gamma(i,k,b_i\lambda)}) \right).$$

**PROOF.** The proof is adapted from McLeish (1975b). From decomposition (6.5) with  $\mathbb{E}(X_k | \mathcal{M}_{-\infty}) = 0$ , we have  $S_j = \sum_{i \geq 0} Y_{i,j}$  and therefore

$$(S_j - \lambda)_+ \leq \sum_{i \geq 0} (Y_{i,j} - b_i \lambda)_+.$$

Applying Hölder’s inequality and taking the maximum on both side, we get

$$(S_n^* - \lambda)_+^2 \leq K \sum_{i \geq 0} a_i (Y_{i,n}^* - b_i \lambda)_+^2.$$

Taking the expectation and applying Proposition 1(a) of Dedecker and Rio (2000) to the martingale  $(Y_{i,n})_{n \geq 1}$ , we obtain Proposition 6.  $\square$

To be complete, we would like to mention that condition (1.3) is close to optimality, as shown by Proposition 7.

**PROPOSITION 7.** *There exists a sequence  $(X_i)_{i \in \mathbb{Z}}$  satisfying the assumptions of Theorem 1, such that*

$$n^{-1} \mathbb{E}(S_n^2) \text{ converges to } \sigma^2 \quad \text{and} \quad \sum_{k > 0} \|\mathbb{E}(X_k | \mathcal{M}_0)\|_2^2 < \infty,$$

*but the random variables  $n^{-1/2} S_n$  do not converge in distribution.*

See Dedecker [(1998), Annexe A.3] for a proof.

6.3. Proof of Proposition 3.

PROOF OF s2(a\*). Let  $\bar{S}_n = \max\{|S_1|, \dots, |S_n|\}$ . From Proposition 1 in Dedecker and Rio (2000), we infer that  $(n^{-1}(\bar{S}_n)^2)_{n>0}$  is uniformly integrable as soon as (1.4) holds.  $\square$

PROOF OF s2(c). The fact that  $\mathbb{E}(n^{-1}S_n^2|\mathcal{M}_0)$  converges in  $\mathbb{L}^1$  to  $\eta$  has been already proved in Dedecker and Rio [(2000), Section 4, Control of  $D_2$ ].  $\square$

PROOF OF s2(b). Using first the stationarity of  $(X_i)_{i \in \mathbb{Z}}$  and next the orthogonal decomposition (6.5), we obtain

$$(6.7) \quad \sum_{i=0}^{\infty} \|P_0(X_i)\|_2^2 = \sum_{i=0}^{\infty} \|P_{-i}(X_0)\|_2^2 \leq \|X_0\|_2^2.$$

Now, from the decomposition

$$\frac{X_{-1}}{\sqrt{n}} \mathbb{E}(S_n|\mathcal{M}_0) = \frac{X_{-1}}{\sqrt{n}} \mathbb{E}(S_n|\mathcal{M}_{-1}) + \frac{X_{-1}}{\sqrt{n}} \sum_{i=1}^n P_0(X_i),$$

we infer that

$$(6.8) \quad \frac{1}{\sqrt{n}} \|X_{-1} \mathbb{E}(S_n|\mathcal{M}_0)\|_1 \leq \frac{1}{\sqrt{n}} \|X_0 \mathbb{E}(S_n \circ T|\mathcal{M}_0)\|_1 + \frac{\|X_0\|_2}{\sqrt{n}} \sum_{i=1}^n \|P_0(X_i)\|_2.$$

By (1.4), the first term on the right-hand side tends to zero as  $n$  tends to infinity. On the other hand, we infer from (6.7) and Cauchy–Schwarz’s inequality that  $n^{-1/2} \sum_{i=1}^n \|P_0(X_i)\|_2$  vanishes as  $n$  goes to infinity, and so does the left-hand term in (6.8). By induction, we can prove that for any positive integer  $k$ ,

$$(6.9) \quad \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \|X_{-k} \mathbb{E}(S_n|\mathcal{M}_0)\|_1 = 0.$$

Now

$$(6.10) \quad \begin{aligned} \frac{1}{\sqrt{n}} \|\mathbb{E}(|X_0|\mathcal{I}) \mathbb{E}(S_n|\mathcal{M}_0)\|_1 &\leq \frac{1}{\sqrt{n}} \left\| \mathbb{E}(S_n|\mathcal{M}_0) \left( \mathbb{E}(|X_0|\mathcal{I}) - \frac{1}{k} \sum_{i=1}^k |X_{-i}| \right) \right\|_1 \\ &+ \frac{1}{\sqrt{n}} \left\| \mathbb{E}(S_n|\mathcal{M}_0) \frac{1}{k} \sum_{i=1}^k |X_{-i}| \right\|_1. \end{aligned}$$

From (6.9), the second term on the right-hand side tends to zero as  $n$  tends to infinity. Applying first Cauchy–Schwarz’s inequality and next the  $\mathbb{L}^2$ -ergodic

theorem, we easily deduce that the first term on the right-hand side is as small as we wish by choosing  $k$  large enough. Therefore

$$(6.11) \quad \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \|\mathbb{E}(|X_0| | \mathcal{I}) \mathbb{E}(S_n | \mathcal{M}_0)\|_1 = 0.$$

Set  $A = \{\mathbb{I}_{\mathbb{E}(|X_0| | \mathcal{I}) > 0}\}$  and  $B = A^c = \{\mathbb{I}_{\mathbb{E}(|X_0| | \mathcal{I}) = 0}\}$ . For any positive real  $m$ , we have

$$(6.12) \quad \begin{aligned} \frac{1}{\sqrt{n}} \|\mathbb{I}_A \mathbb{E}(S_n | \mathcal{M}_0)\|_1 &\leq \frac{1}{m\sqrt{n}} \|\mathbb{E}(|X_0| | \mathcal{I}) \mathbb{E}(S_n | \mathcal{M}_0)\|_1 \\ &+ \frac{1}{\sqrt{n}} \|\mathbb{I}_{0 < \mathbb{E}(|X_0| | \mathcal{I}) < m} \mathbb{E}(S_n | \mathcal{M}_0)\|_1. \end{aligned}$$

From (6.11), the first term on the right-hand side tends to zero as  $n$  tends to infinity. Letting  $m$  goes to zero we infer that the second term on the right-hand side of (6.12) is as small as we wish. Consequently

$$(6.13) \quad \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \|\mathbb{I}_A \mathbb{E}(S_n | \mathcal{M}_0)\|_1 = 0.$$

On the other hand, noting that  $\mathbb{E}(|X_0| | \mathbb{I}_B) = 0$ , we infer that  $X_0$  is zero on the set  $B$ . Since  $B$  is invariant by  $T$ ,  $X_k$  is zero on  $B$  for any  $k$  in  $\mathbb{Z}$ . Now arguing as in Claim 1(b) in Dedecker and Rio (2000), we obtain  $\mathbb{E}(\mathbb{E}(|S_n| | \mathcal{M}_0) | \mathcal{I}) = \mathbb{E}(|S_n| | \mathcal{I})$ . These two facts lead to

$$(6.14) \quad \|\mathbb{I}_B \mathbb{E}(S_n | \mathcal{M}_0)\|_1 \leq \mathbb{E}(\mathbb{I}_B \mathbb{E}(\mathbb{E}(|S_n| | \mathcal{M}_0) | \mathcal{I})) \leq \mathbb{E}(|S_n| | \mathbb{I}_B) \leq 0$$

Collecting (6.13) and (6.14), we conclude that  $n^{-1/2} \|\mathbb{E}(S_n | \mathcal{M}_0)\|_1$  tends to zero as  $n$  tends to infinity. This completes the proof.  $\square$

**7. Applications of Theorem 3.** In this section we extend Propositions 2 and 3 to the case of triangular arrays. In the next section, we shall see how to apply these results to Kernel density estimators.

Consider first the following condition, close to (1.3): there exists a sequence  $(L_k)_{k>0}$  of positive numbers such that

$$(7.1) \quad \sum_{i>0} \left( \sum_{k=1}^i L_k \right)^{-1} < \infty \quad \text{and} \quad \lim_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{k=N}^{\infty} L_k \|\mathbb{E}(X_{k,n} | \mathcal{M}_{0,n})\|_2^2 = 0.$$

If (7.1) is satisfied, define  $Q(N, X)$  and  $N_1(X)$  as follows:

$$Q(N, X) = \limsup_{n \rightarrow \infty} \sum_{k=N}^{\infty} L_k \|\mathbb{E}(X_{k,n} | \mathcal{M}_{0,n})\|_2^2$$

and  $N_1(X) = \inf\{N > 0: Q(N) = 0\}$ . If  $N_1(X)$  is finite, we say that the array  $(X_{i,n})$  is asymptotically  $(N_1(X) - 1)$ -conditionally centered of type 1 (as usual, it is  $m$ -conditionally centered if  $\mathbb{E}(X_{m+1,n} | \mathcal{M}_{0,n}) = 0$ ).

PROPOSITION 8. Let  $X_{i,n}$  and  $\mathcal{M}_{i,n}$  be as in Theorem 2. Define the random variables  $V_n(t) = X_{1,n}^2 + \dots + X_{n,[nt]}^2$ . Assume that (7.1) is satisfied, and that Lindeberg's condition holds: for any positive  $\varepsilon$ ,  $\lim_{n \rightarrow \infty} \mathbb{E}(X_{0,n}^2 \mathbb{I}_{|X_{0,n}| > \varepsilon \sqrt{n}}) = 0$ . Assume furthermore that

$$(7.2) \quad \lim_{M \rightarrow \infty} \sup_{t \in [0,1]} \limsup_{n \rightarrow \infty} \mathbb{E} \left( \frac{V_n(t)}{nt} \mathbb{I}_{V_n(t) \geq Mnt} \right) = 0.$$

Then  $S2(a^*)$  is satisfied.

REMARK 6. Let us compare Proposition 8 with Theorem (2.4) in McLeish (1977). In the particular case of triangular arrays, condition (2.2)(b) in McLeish with  $\sigma_{n,i}^2 = n^{-1}$  (which seems to be the usual case) is equivalent with our notation to the assumption that  $X_{0,n}$  is uniformly integrable. It is easy to verify that this assumption ensures that both Lindeberg's condition and (7.2) hold. However, in many interesting cases, the sequence  $X_{0,n}$  is not uniformly integrable while both Lindeberg's condition and (7.2) are satisfied (it is the case, for instance, when considering Kernel estimators).

As a consequence of Proposition 8, we obtain the invariance principle:

COROLLARY 2. Let  $X_{i,n}$  and  $\mathcal{M}_{i,n}$  be as in Theorem 2. Assume that both (7.1) and Lindeberg's condition are satisfied. Assume furthermore that, for each integer  $0 \leq k < N_1(X)$ , there exists an  $\mathcal{M}_{0,\text{inf}}$ -measurable random variable  $\lambda_k$  such that for any  $t$  in  $]0, 1]$ ,

$$(7.3) \quad \frac{1}{nt} \sum_{i=1}^{[nt]} X_{i+k,n} X_{i,n} \text{ converges in } \mathbb{L}^1 \text{ to } \lambda_k.$$

Then Condition  $S2^*$  holds with  $\eta = \lambda_0 + 2 \sum_{k=1}^{N_1(X)-1} \lambda_k$ .

REMARK 7. Let us discuss the measurability assumption on  $\lambda_k$ . Starting from (7.3), one can easily show first that  $\lambda_k$  is invariant by  $T$ , and next that it is a limit of  $\mathcal{M}_{0,n}$ -measurables random variables. Suppose furthermore that the sequence  $(\mathcal{M}_{0,n})_{n \geq 1}$  is nondecreasing, then  $\lambda_k$  is a limit of  $\mathcal{M}_{0,\text{inf}}$ -measurable random variables. In that case, the assumption that  $\lambda_k$  is  $\mathcal{M}_{0,\text{inf}}$ -measurable is useless, being automatically satisfied.

REMARK 8. Note that if  $X_{i,n} = X_i$  then both Lindeberg's condition and assumption (7.3) are satisfied, so that Corollary 2 extends Proposition 2 to the case of triangular arrays.

The next condition is the natural extension of Condition (1.4):

$$(7.4) \quad \lim_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{N \leq m \leq n} \left\| X_{0,n} \sum_{k=N}^m \mathbb{E}(X_{k,n} | \mathcal{M}_{0,n}) \right\|_1 = 0.$$

If (7.4) is satisfied, define  $R(N, X)$  and  $N_2(X)$  as follows:

$$R(N, X) = \limsup_{n \rightarrow \infty} \sup_{N \leq m \leq n} \left\| X_{0,n} \sum_{k=N}^m \mathbb{E}(X_{k,n} | \mathcal{M}_{0,n}) \right\|_1,$$

and  $N_2(X) = \inf\{N > 0 : R(N, X) = 0\}$ . If  $N_2(X)$  is finite, we say that the array  $(X_{i,n})$  is asymptotically  $(N_2(X) - 1)$ -conditionally centered of type 2.

**PROPOSITION 9.** *Let  $X_{i,n}$ ,  $\mathcal{M}_{i,n}$  be as in Theorem 2 and  $V_n(t)$  as in Proposition 8. If conditions (7.2) and (7.4) are satisfied then S2(a\*) holds.*

As a consequence we obtain the following invariance principle:

**COROLLARY 3.** *Let  $X_{i,n}$  and  $\mathcal{M}_{i,n}$  be as in Theorem 2. Assume that (7.4) and S2(b) are satisfied. Assume furthermore that, for each  $0 \leq k < N_2(X)$ , there exists an  $\mathcal{M}_{0,\text{inf}}$ -measurable random variable  $\lambda_k$  such that (7.3) holds. Then Condition S2\* holds with*

$$\eta = \lambda_0 + 2 \sum_{k=1}^{N_2(X)-1} \lambda_k.$$

**REMARK 9.** Let us have a look to a particular case, for which  $N_1(X) = 1$  (resp.  $N_2(X) = 1$ ). Conditions (7.1) [resp. (7.4)] and (7.3) are satisfied if condition R1 (resp. R1') and R2 below are fulfilled:

- R1.  $\lim_{n \rightarrow \infty} \sum_{k=1}^n L_k \|\mathbb{E}(X_{k,n} | \mathcal{M}_{0,n})\|_2^2 = 0.$
- R1'.  $\lim_{n \rightarrow \infty} \sum_{k=1}^n \|X_{0,n} \mathbb{E}(X_{k,n} | \mathcal{M}_{0,n})\|_1 = 0.$
- R2. For any  $t$  in  $]0, 1]$ ,  $\frac{1}{nt} \sum_{i=1}^{[nt]} X_{i,n}^2$  converges in  $\mathbb{L}^1$  to  $\lambda.$

In the stationary case, these results extend on classical results for triangular arrays of martingale differences (see for instance Hall and Heyde [(1980), Theorem 3.2]), for which Condition R1 (resp. R1') is automatically satisfied. We shall see in Section 8.3 that this particular case is sufficient to improve on many results in the context of Kernel estimators. In the same way, Corollary 2 (or 3) provides sufficient conditions for asymptotically  $m$ -conditionally centered arrays of type 1 (or 2) to satisfy the functional CCLT.

7.1. *Proofs of Proposition 8 and Corollary 2.*

**PROOF OF PROPOSITION 8.** With the same notation as in Proposition 6, define, for  $b_i = 2^{-i-1}$ ,

$$T_{i,n}(t) = \sum_{k=1}^{[nt]} P_{k-i}^2(X_{k,n}) \quad \text{and} \quad c_{i,n}(t, M) = \mathbb{E} \left( \frac{T_{i,n}(t)}{nt} \mathbb{I}_{\Gamma(i, [nt], b_i M \sqrt{nt})} \right).$$

Define  $S_n^*(t) = \sup_{0 \leq s \leq t} \{0, S_n(s)\}$ . From Proposition 6, we have

$$(7.5) \quad \frac{1}{nt} \mathbb{E} \left( (S_n^*(t) - M\sqrt{nt})_+^2 \right) \leq 4K \sum_{i=0}^{\infty} a_i c_{i,n}(t, M).$$

Note that, by stationarity,

$$c_{i,n}(t, M) \leq c_{i,n}(t, 0) \leq \|P_0(X_{i,n})\|_2^2.$$

Now, taking  $a_i = L_1 + \dots + L_i$ , we have

$$(7.6) \quad \sum_{i=N}^{\infty} a_i \|P_0(X_{i,n})\|_2^2 = \sum_{k=N}^{\infty} L_k \|\mathbb{E}(X_{k,n} | \mathcal{M}_{0,n})\|_2^2 + \sum_{k=1}^{N-1} L_k \|\mathbb{E}(X_{N,n} | \mathcal{M}_{0,n})\|_2^2,$$

and from (7.1) and the definition of  $N_1(X)$ , we infer that

$$(7.7) \quad \lim_{N \rightarrow N_1(X)} \limsup_{n \rightarrow \infty} \sum_{i=N}^{\infty} a_i c_{i,n}(t, 0) = 0.$$

Suppose we can prove that, for any  $0 \leq i < N_1(X)$ ,

$$(7.8) \quad \lim_{M \rightarrow \infty} \sup_{t \in [0,1]} \limsup_{n \rightarrow \infty} \mathbb{E} \left( \frac{T_{i,n}(t)}{nt} \mathbb{I}_{T_{i,n}(t) \geq Mnt} \right) = 0.$$

Since by stationarity  $\mathbb{E}(T_{i,n}(t)) = [nt] \|P_0(X_{i,n})\|_2^2$ , equation (7.8) implies first that  $\sup_{n>0} \|P_0(X_{i,n})\|_2^2 \leq B_i < \infty$ , which together with Doob's inequality yield the upper bounds

$$(7.9) \quad \text{for } 0 \leq i < N_1(X), \quad \sup_{t \in [0,1]} \sup_{n>0} \mathbb{P}(\Gamma(i, [nt], b_i M \sqrt{nt})) \leq \frac{4B_i}{b_i^2 M^2}.$$

According to (7.7), we can find a finite integer  $N(\varepsilon) \leq N_1(X)$  such that

$$\limsup_{n \rightarrow \infty} 4K \sum_{i=N(\varepsilon)}^{\infty} a_i c_{i,n}(t, 0) \leq \varepsilon.$$

Now, since  $c_{i,n}(t, M) \leq c_{i,n}(t, 0)$  we obtain from (7.5) that

$$(7.10) \quad \limsup_{n \rightarrow \infty} \frac{1}{nt} \mathbb{E} \left( (S_n^*(t) - M\sqrt{nt})_+^2 \right) \leq \varepsilon + \limsup_{n \rightarrow \infty} \sum_{i=0}^{N(\varepsilon)-1} a_i c_{i,n}(t, M).$$

Here note that (7.8), together with (7.9), yields

$$\text{for } 0 \leq i < N_1(X), \quad \lim_{M \rightarrow \infty} \sup_{t \in [0,1]} \limsup_{n \rightarrow \infty} c_{i,n}(t, M) = 0$$

so that, from (7.10),

$$(7.11) \quad \lim_{M \rightarrow \infty} \sup_{t \in [0,1]} \limsup_{n \rightarrow \infty} \frac{1}{nt} \mathbb{E} \left( (S_n^*(t) - M\sqrt{nt})_+^2 \right) \leq \varepsilon.$$

Of course the same arguments apply to the sequence  $(-X_i)_{i \in \mathbb{Z}}$  so that (7.11) holds for  $\bar{S}_n(t)$ . This being true for any positive  $\varepsilon$ , S2 (a\*) follows.

To complete the proof, it remains to show that Lindeberg's condition, together with assumption (7.2), implies (7.8). Since

$$\begin{aligned} P_{k-i}^2(X_{k,n}) &\leq 2(\mathbb{E}^2(X_{k,n}|\mathcal{M}_{k-i,n}) + \mathbb{E}^2(X_{k,n}|\mathcal{M}_{k-i-1,n})) \\ &\leq 2((\mathbb{E}(X_{k,n}^2|\mathcal{M}_{k-i,n}) + \mathbb{E}(X_{k,n}^2|\mathcal{M}_{k-i-1,n})), \end{aligned}$$

we infer that (7.8) holds if, setting  $U_{i,n}(t) = \sum_{k=1}^{[nt]} \mathbb{E}(X_{k,n}^2|\mathcal{M}_{k-i,n})$ ,

$$(7.12) \quad \text{for any } i \geq 0, \quad \lim_{M \rightarrow \infty} \sup_{t \in [0,1]} \limsup_{n \rightarrow \infty} \mathbb{E} \left( \frac{U_{i,n}(t)}{nt} \mathbb{1}_{U_{i,n}(t) \geq Mnt} \right) = 0.$$

Now if Lindeberg's condition holds, classical arguments ensure that

$$\lim_{n \rightarrow \infty} \frac{1}{nt} \|U_{i,n}(t) - U_{i+1,n}(t)\|_1 = 0,$$

so that if (7.12) holds for  $i$ , it holds for  $i + 1$ . For  $i = 0$ , note that (7.12) is exactly (7.2) since  $U_{0,n}(t) = V_n(t)$ . This ends the proof of Proposition 8.  $\square$

PROOF OF COROLLARY 2. Since Condition (7.3) implies (7.2), S2(a\*) follows from Proposition 8. It remains to prove S2(b) and (c).

From (7.6) we infer that (7.1) is equivalent to

$$(7.13) \quad \lim_{N \rightarrow N_1(X)} \limsup_{n \rightarrow \infty} \sum_{i=N}^{\infty} a_i \|P_0(X_{i,n})\|_2^2 = 0.$$

Since the sequence  $a_i^{-1}$  is nonincreasing and such that  $\sum a_i^{-1} < \infty$ , it follows that  $a_i^{-1} = o(i^{-1})$ , which implies that  $i = O(a_i)$ . Consequently (7.13) holds for  $a_i = i$  and from (7.6) again we obtain

$$(7.14) \quad \lim_{N \rightarrow N_1(X)} \limsup_{n \rightarrow \infty} \sum_{k=N}^{\infty} \|\mathbb{E}(X_{k,n}|\mathcal{M}_{0,n})\|_2^2 = 0.$$

Starting from (7.14), we first prove S2(b). From (7.3) with  $k = 0$  we easily infer that, for each positive  $N$ ,  $n^{-1/2} \|\mathbb{E}(X_{1,n} + \dots + X_{N-1,n}|\mathcal{M}_{0,n})\|_2$  tends to zero as  $n \rightarrow \infty$ . Therefore, to prove S2(b) it suffices to see that

$$(7.15) \quad \lim_{N \rightarrow N_1(X)} \limsup_{n \rightarrow \infty} \frac{1}{n} \left\| \sum_{k=N}^{nt} \mathbb{E}(X_{k,n}|\mathcal{M}_{0,n}) \right\|_2^2 = 0.$$

Since  $2\mathbb{E}(X_{i,n}|\mathcal{M}_{0,n})\mathbb{E}(X_{j,n}|\mathcal{M}_{0,n}) \leq \mathbb{E}^2(X_{i,n}|\mathcal{M}_{0,n}) + \mathbb{E}^2(X_{j,n}|\mathcal{M}_{0,n})$ ,

$$\frac{1}{n} \left\| \sum_{k=N}^n \mathbb{E}(X_{k,n}|\mathcal{M}_{0,n}) \right\|_2^2 \leq \sum_{k=N}^{nt} \|\mathbb{E}(X_{k,n}|\mathcal{M}_{0,n})\|_2^2,$$

and (7.15) follows from (7.14).

We now prove S2(c). For any finite integer  $0 \leq N \leq N_1(X)$ , define the variable  $\eta_N = \lambda_0 + 2(\lambda_1 + \dots + \lambda_{N-1})$  and the two sets

$$\Lambda_N = [1, [nt]]^2 \cap \{(i, j) \in \mathbb{Z}^2 : |i - j| < N\}$$

and

$$\bar{\Lambda}_N = [1, [nt]]^2 \cap \{(i, j) \in \mathbb{Z}^2 : j - i \geq N\},$$

so that

$$(7.16) \quad \left\| \mathbb{E} \left( \frac{S_n^2(t)}{nt} - \eta_N \mid \mathcal{M}_{0,n} \right) \right\|_1 \leq \left\| \eta_N - \frac{1}{nt} \sum_{\Lambda_N} X_{i,n} X_{j,n} \right\|_1 + \frac{2}{nt} \left\| \sum_{\bar{\Lambda}_N} \mathbb{E}(X_{i,n} X_{j,n} \mid \mathcal{M}_{0,n}) \right\|_1.$$

From (7.3), we can easily prove that the first term on the right-hand side goes to zero as  $n$  tends to infinity. It remains to control the second term on the right-hand side. Set  $Y_{i,n} = X_{i,n} - \mathbb{E}(X_{i,n} \mid \mathcal{M}_{0,n})$  and write

$$(7.17) \quad \frac{1}{nt} \left\| \sum_{\bar{\Lambda}_N} \mathbb{E}(X_{i,n} X_{j,n} \mid \mathcal{M}_{0,n}) \right\|_1 \leq \frac{1}{nt} \left\| \sum_{\bar{\Lambda}_N} \mathbb{E}(Y_{i,n} Y_{j,n} \mid \mathcal{M}_{0,n}) \right\|_1 + \frac{1}{nt} \sum_{\bar{\Lambda}_N} \|\mathbb{E}(X_{i,n} \mid \mathcal{M}_{0,n}) \mathbb{E}(X_{j,n} \mid \mathcal{M}_{0,n})\|_1.$$

Arguing as for (7.15), we infer from (7.14) that the second term on the right-hand side in (7.17) is as small as we wish by choosing  $N$  large enough. Next, by using the operators  $P_l$ , we have

$$\begin{aligned} \frac{1}{nt} \left\| \sum_{\bar{\Lambda}_N} \mathbb{E}(Y_{i,n} Y_{j,n} \mid \mathcal{M}_{0,n}) \right\|_1 &\leq \frac{1}{nt} \sum_{i=1}^{nt} \sum_{k=N}^{\infty} \|\mathbb{E}(Y_{i,n} Y_{i+k,n} \mid \mathcal{M}_{0,n})\|_1 \\ &\leq \frac{1}{nt} \sum_{i=1}^{nt} \sum_{k=N}^{\infty} \sum_{l=1}^i \|P_l(X_{i,n}) P_l(X_{i+k,n})\|_1 \end{aligned}$$

Using Cauchy–Schwarz’s inequality, we obtain that the last term is less than

$$\frac{1}{nt} \sum_{i=1}^{nt} \sum_{l=-\infty}^i \|P_l(X_{i,n})\|_2 \left( \sum_{k=N}^{\infty} \|P_l(X_{i+k,n})\|_2 \right),$$

and by stationarity, we conclude that

$$\frac{1}{nt} \left\| \sum_{\bar{\Lambda}_N} \mathbb{E}(Y_{i,n} Y_{j,n} | \mathcal{M}_{0,n}) \right\|_1 \leq \left( \sum_{i=0}^{\infty} \|P_0(X_{i,n})\|_2 \right) \left( \sum_{k=N}^{\infty} \|P_0(X_{k,n})\|_2 \right).$$

The last inequality, together with (7.13) and Cauchy–Schwarz’s inequality, yields

$$\lim_{N \rightarrow N_1(X)} \limsup_{n \rightarrow \infty} \frac{1}{nt} \left\| \sum_{\bar{\Lambda}_N} \mathbb{E}(Y_{i,n} Y_{j,n} | \mathcal{M}_{0,n}) \right\|_1 = 0,$$

and S2(c) follows. This completes the proof of Corollary 2.  $\square$

7.2. Proofs of Proposition 9 and Corollary 3.

PROOF OF PROPOSITION 9. Let  $S_n^*(t)$  be as in (7.5) and define the set  $G(t, M, n)$  by  $G(t, M, n) = \{S_n^*(t) > M\sqrt{nt}\}$ . From Proposition 1 in Dedecker and Rio (2000), we have, for any positive integer  $N$ ,

$$\begin{aligned} \frac{1}{nt} \mathbb{E} \left( (S_n^*(t) - M\sqrt{nt})_+^2 \right) &\leq 8 \mathbb{E} \left( \mathbb{I}_{G(t, M, n)} \frac{1}{nt} \sum_{k=1}^{nt} \sum_{i=0}^{N-1} |X_{k,n} X_{k+i,n}| \right) \\ (7.18) \qquad \qquad \qquad &+ 8 \sup_{N \leq m \leq nt} \left\| X_{0,n} \sum_{k=N+1}^m \mathbb{E}(X_{k,n} | \mathcal{M}_{0,n}) \right\|_1. \end{aligned}$$

From (7.4) we can choose  $N(\varepsilon)$  large enough so that the second term on the right-hand side is less than  $\varepsilon$ . Taking  $M = 0$  in (7.18) and using Assumption (7.2), we infer that there exists a finite real  $B$  such that

$$\limsup_{n \rightarrow \infty} \frac{1}{nt} \mathbb{E}((S_n^*(t))^2) \leq B \quad \text{so that} \quad \limsup_{n \rightarrow \infty} \mathbb{P}(G(t, M, n)) \leq \frac{B}{M^2}.$$

The last bound, together with (7.2), implies that

$$\lim_{M \rightarrow \infty} \sup_{t \in [0,1]} \limsup_{n \rightarrow \infty} \mathbb{E} \left( \mathbb{I}_{G(t, M, n)} \frac{1}{nt} \sum_{k=1}^{nt} \sum_{i=0}^{N-1} |X_{k,n} X_{k+i,n}| \right) = 0,$$

so that, from (7.18),

$$(7.19) \quad \lim_{M \rightarrow \infty} \sup_{t \in [0,1]} \limsup_{n \rightarrow \infty} \frac{1}{nt} \mathbb{E} \left( (S_n^*(t) - M\sqrt{nt})_+^2 \right) \leq \varepsilon.$$

Of course the same arguments apply to the sequence  $(-X_i)_{i \in \mathbb{Z}}$  so that (7.19) holds for  $\bar{S}_n(t)$ . This being true for any positive  $\varepsilon$ , S2(a\*) follows.  $\square$

PROOF OF COROLLARY 3. Since Condition (7.3) implies (7.2), S2(a\*) follows from Proposition 9. It remains to prove S2(c). Starting from inequality (7.16) for any finite integer  $N \leq N_2(X)$ , we have to show that

$$(7.20) \quad \lim_{N \rightarrow N_2(X)} \limsup_{n \rightarrow \infty} \frac{1}{nt} \left\| \sum_{\Lambda_N} \mathbb{E}(X_{i,n} X_{j,n} | \mathcal{M}_{0,n}) \right\|_1 = 0.$$

Using first the inclusion  $\mathcal{M}_{0,n} \subseteq \mathcal{M}_{i,n}$  for any positive  $i$  and second the stationarity of the sequence, we obtain successively

$$\begin{aligned} \frac{1}{nt} \left\| \sum_{\Lambda_N} \mathbb{E}(X_{i,n} X_{j,n} | \mathcal{M}_{0,n}) \right\|_1 &\leq \frac{1}{nt} \left\| \sum_{\Lambda_N} X_{i,n} \mathbb{E}(X_{j,n} | \mathcal{M}_{i,n}) \right\|_1 \\ &\leq \sup_{N \leq m \leq n} \left\| X_{0,n} \sum_{k=N}^m \mathbb{E}(X_{k,n} | \mathcal{M}_{0,n}) \right\|_1 \end{aligned}$$

and (7.20) follows from (7.4). This completes the proof of Corollary 3.  $\square$

**8. Kernel estimators.** Let  $Y$  be a real-valued random variable with unknown density  $f$ , and define the stationary sequence  $(Y_i)_{i \in \mathbb{Z}} = (Y \circ T^i)_{i \in \mathbb{Z}}$ . We wish to estimate  $f$  at point  $x$  from the data  $Y_1, \dots, Y_n$ . To this aim, we shall consider as usual [cf. Rosenblatt (1956b)] the kernel-type estimator of  $f$

$$f_n(x) = \frac{1}{nh_n} \sum_{i=1}^n K\left(\frac{x - Y_i}{h_n}\right).$$

It is well-known that the study of the bias  $\mathbb{E}(f_n(x)) - f(x)$  does not depend on the dependence properties of the process  $(Y_i)_{i \in \mathbb{Z}}$ , but on the regularity of  $f$ . Consequently, we only deal with the asymptotic behavior of  $f_n(x) - \mathbb{E}(f_n(x))$ .

8.1. *A general result.*

DEFINITION 4. We say that a Borel measurable function  $K$  from  $\mathbb{R}$  to  $\mathbb{R}$  is a kernel if

$$(8.1) \quad \|K\|_\infty < \infty, \quad \lim_{|u| \rightarrow \infty} |u|K(u) = 0 \quad \text{and} \quad \int K(u) du = 1.$$

DEFINITION 5. For any  $x$  in  $\mathbb{R}^d$  and any positive real  $M$ , denote by  $\mathbb{L}_{M,x}^1$  the space of integrable functions from  $\mathbb{R}^d$  to  $\mathbb{R}$  whose support is included in the cube  $[x_1 - M, x_1 + M] \times \dots \times [x_d - M, x_d + M]$ . If  $f$  belongs to  $\mathbb{L}_{M,x}^1$ , denote by  $\|f\|_{1,M}(x)$  its norm. For any positive measure  $\mu$  on  $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ , define the two quantities  $\|\mu\|_{\infty,M}(x) = \sup\{\mu(|f|) : \|f\|_{1,M}(x) \leq 1\}$  and  $\|\mu\|_\infty(x) = \lim_{M \rightarrow 0} \|\mu\|_{\infty,M}(x)$ . If  $\|\mu\|_\infty(x) < \infty$ , we say that  $\mu$  belongs to  $\mathbb{L}_x^\infty$ .

For each  $i$  in  $\mathbb{Z}$ , denote by  $\mu_i$  the law of  $(Y_0, Y_i)$ . We make the following assumptions:

- A1.  $Y$  has density  $f$  which is continuous at  $x$ .
- A2. For each  $i$  in  $\mathbb{Z}$ ,  $\mu_i$  belongs to  $\mathbb{L}_{(x,x)}^\infty$ .

PROPOSITION 10. *Let  $K$  be a kernel, and  $h_n$  be a sequence of positive numbers such that  $h_n$  tends to zero and  $nh_n$  tends to infinity as  $n$  tends to infinity. Let  $Y_i$  be a strictly stationary sequence satisfying A1 and A2. Define successively  $\mathcal{M}_{i,n} = \mathcal{M}_i = \sigma(Y_j, j \leq i)$ ,*

$$(8.2) \quad X_{i,n} = \frac{1}{\sqrt{h_n}} \left\{ K\left(\frac{x - Y_i}{h_n}\right) - \mathbb{E}\left(K\left(\frac{x - Y_i}{h_n}\right)\right) \right\},$$

and  $U_n$  as in Theorem 3. Assume that either (7.1) or (7.4) and S2(b) holds. If furthermore

$$(8.3) \quad \lim_{n \rightarrow \infty} \sup_{k \leq n} \frac{1}{n} \sum_{i=1}^k \text{Cov}(X_{0,n}^2, X_{i,n}^2) = 0,$$

then the process  $U_n$  satisfies S1\* with  $\eta = f(x) \|K\|_2^2$ .

PROOF. It is a consequence of either Corollary 2 or Corollary 3. In order to apply Corollary 2, we need to prove that  $X_{0,n}$  satisfies Lindeberg's condition. Define  $K_n(x) = h_n^{-1} K(h_n^{-1}x)$ . Since Assumption A1 holds, classical arguments ensure that  $\mathbb{E}(K_n(x - Y))$  converges to  $f(x)$ , and consequently  $\sqrt{h_n} \mathbb{E}(K_n(x - Y))$  tends to 0 as  $n$  tends to infinity. Now, recall that

$$X_{0,n} = \sqrt{h_n} (K_n(x - Y) - \mathbb{E}(K_n(x - Y))).$$

Therefore it suffices to show that the sequence  $\sqrt{h_n} K_n(x - Y)$  satisfies Lindeberg's condition. The density  $f$  being continuous at  $x$ , there exist two positive reals  $M$  and  $C$  such that for any  $y$  in  $[x - M, x + M]$ ,  $f(y)$  is less than  $C$ . Setting  $K[M] = K \mathbb{I}_{[-M, M]^c}$ , we have

$$\begin{aligned} \frac{1}{h_n} \mathbb{E} \left( K^2 \left( \frac{x - Y}{h_n} \right) \mathbb{I}_{\sqrt{h_n} |K_n(x - Y)| > \varepsilon \sqrt{n}} \right) &\leq \frac{1}{h_n} \|K[M/h_n]\|_\infty^2 \\ &+ C \int K^2(z) \mathbb{I}_{|K(z)| > \varepsilon \sqrt{nh_n}} dz. \end{aligned}$$

Note that  $uK^2(u)$  tends to zero as  $|u|$  tends to infinity, which implies that the first term on right hand vanishes as  $h_n$  tends to zero. On the other hand, since  $\|K\|_2$  is finite, the second term tends to zero as  $nh_n$  tends to infinity.

To complete the proof, it remains to see that (8.3) implies (7.3) with  $\lambda_0 = f(x)\|K\|_2^2$  and  $\lambda_k = 0$  for any positive integer  $k$ . To prove the second point, note that for any  $k > 0$  ( $\star$  denoting the convolution),

$$(8.4) \quad \|X_{0,n}X_{k,n}\|_1 \leq h_n(\mu_k \star |K_n \otimes K_n|(x, x) + 3(f \star |K_n|(x))^2).$$

Again, choose  $M$  such that for any  $y$  in  $[x - M, x + M]$ ,  $f(y)$  is less than  $C$ . Splitting the integral in four parts over the sets  $[x - M, x + M]^2$ ,  $[x - M, x + M] \times [x - M, x + M]^c$ ,  $[x - M, x + M]^c \times [x - M, x + M]$  and  $([x - M, x + M]^c)^2$ , we get the upper bound

$$(8.5) \quad \begin{aligned} \mu_k \star |K_n \otimes K_n|(x, x) &\leq \|K\|_1^2 \|\mu_k\|_{\infty, M}(x) \\ &+ \frac{2C}{h_n} \|K[M/h_n]\|_{\infty} \|K\|_1 + \frac{1}{h_n^2} \|K[M/h_n]\|_{\infty}^2. \end{aligned}$$

Now Assumption A2 ensures that for  $M$  small enough, the first term on the right-hand side is finite. The two other terms on the right-hand side tending to zero as  $n$  tends to infinity, we infer that  $\sup_{n>0} \mu_k \star |K_n \otimes K_n|(x, x) < \infty$ . Both this fact and (8.4) imply that, for any positive  $k$ ,  $\|X_{0,n}X_{k,n}\|_1$  tends to 0 as  $n$  tends to infinity, so that (7.3) holds with  $\lambda_k = 0$ .

It remains to see that (7.3) holds for  $k = 0$  with  $\lambda_0 = f(x)\|K\|_2^2$ . Since the function  $\|K\|_2^{-2}K^2$  is a kernel, classical arguments ensure that  $\mathbb{E}(X_{0,n}^2)$  converges to  $f(x)\|K\|_2^2$ . Now

$$\frac{1}{(nt)^2} \left\| \sum_{i=1}^{[nt]} X_{i,n} - \mathbb{E}(X_{0,n}) \right\|_2^2 = \frac{[nt]}{(nt)^2} \text{Var}(X_{0,n}^2) + \frac{2}{nt} \sum_{k=1}^{[nt]} \frac{1}{nt} \sum_{i=1}^{k-1} \text{Cov}(X_{0,n}^2, X_{i,n}^2).$$

Using that  $\|K\|_4^{-4}K^4$  is a kernel, we infer that  $\text{Var}(X_{0,n}^2) = O(h_n^{-1})$ , so that  $n^{-1} \text{Var}(X_{0,n}^2)$  tends to zero as  $nh_n$  tends to infinity. Since, furthermore, (8.3) implies that the second term on the right-hand side tends to zero as  $n$  tends to infinity, the result follows.  $\square$

8.2. *Application to mixing sequences.* In this section, we give three different applications of Proposition 10 to the case of mixing sequences.

DEFINITION 6. Let  $\mathcal{U}$  and  $\mathcal{V}$  be two  $\sigma$ -algebras of  $\mathcal{A}$ . The strong mixing coefficient of Rosenblatt (1956a) is defined by

$$(8.6) \quad \alpha(\mathcal{U}, \mathcal{V}) = \sup\{|\mathbb{P}(U)\mathbb{P}(V) - \mathbb{P}(U \cap V)| : U \in \mathcal{U}, V \in \mathcal{V}\}.$$

The  $\phi$ -mixing coefficient introduced by Ibragimov (1962) can be defined by

$$(8.7) \quad \phi(\mathcal{U}, \mathcal{V}) = \sup\{\|\mathbb{P}(V|\mathcal{U}) - \mathbb{P}(V)\|_{\infty}, V \in \mathcal{V}\}.$$

Between those coefficients, the following relation holds:  $2\alpha(\mathcal{U}, \mathcal{V}) \leq \phi(\mathcal{U}, \mathcal{V})$ .

DEFINITION 7. Let  $Y$  and  $\varepsilon$  be two real valued random variables, and define  $(Y_i, \varepsilon_i)_{i \in \mathbb{Z}} = (Y \circ T^i, \varepsilon \circ T^i)_{i \in \mathbb{Z}}$ . We say that  $(Y_i, \varepsilon_i)_{i \in \mathbb{Z}}$  is a functional autoregressive process if there exists a Borel-measurable function  $\psi$  such that  $Y_i = \psi(Y_{i-1}) + \varepsilon_i$ , where  $\varepsilon_1$  is independent of the  $\sigma$ -algebra  $\mathcal{M}_0 = \sigma(Y_i, i \leq 0)$ .

Consider the two following assumptions, which are stronger than A2:

- A3. Assumption A2 holds and moreover  $\limsup_{M \rightarrow 0} \sup_{i \in \mathbb{Z}} \|\mu_i\|_{\infty, M}(x, x) < \infty$ .
- A4.  $(Y_i, \varepsilon_i)_{i \in \mathbb{Z}}$  is a functional autoregressive process and  $\varepsilon$  has density  $h$  which is continuous and bounded.

COROLLARY 4. Let  $(Y_i)_{i \in \mathbb{Z}}$  be a strictly stationary sequence. Let  $h_n, (X_{i,n})_{i \in \mathbb{Z}}$  and  $(\mathcal{M}_i)_{i \in \mathbb{Z}}$  be as in Proposition 10, and define the process  $U_n$  as in Theorem 3. Write  $\phi_{\infty,1}(k) = \phi(\mathcal{M}_0, \sigma(Y_k))$  and  $\alpha_{\infty,1}(k) = \alpha(\mathcal{M}_0, \sigma(Y_k))$ . Consider the three following conditions:

- (i)  $\sum_{k>0} \phi_{\infty,1}(k) < \infty$ , (ii)  $\sum_{k>n} \alpha_{\infty,1}(k) = o\left(\frac{1}{n}\right)$  and (iii)  $\sum_{k>0} \alpha_{\infty,1}(k) < \infty$ .

If either [A1, A2, (i)], [A1, A3, (ii)] or [A4, (iii)] holds, then the process  $U_n$  satisfies S1\* with  $\eta = f(x) \|K\|_2^2$ .

REMARK 10. The mixing rate (ii) was first required by Robinson (1983) for the more stringent coefficients  $\alpha_{\infty,\infty}(k) = \alpha(\mathcal{M}_0, \sigma(Y_i, i \geq k))$ . To understand the difference between  $\alpha_{\infty,\infty}$  and  $\alpha_{\infty,1}$ , note that the convergence of  $\alpha_{\infty,\infty}(n)$  to zero implies that  $\mathcal{M}_\infty$  is independent of  $\mathcal{L}$ , while the same property for  $\alpha_{\infty,1}(n)$  does not even implies that  $\sigma(Y_0, Y_1)$  is independent of  $\mathcal{L}$ . This means in particular that a large class of nonergodic processes are concerned by Corollary 4. For such processes, it may be surprising that the variance term  $\eta$  is degenerate. In fact, this is due to Assumption A2 which implies that, for any positive integer  $i$ , the covariance between  $X_{0,n}$  and  $X_{i,n}$  tends to zero as  $n$  tends to infinity.

REMARK 11. Let us recall some recent results about mixing rates for autoregressive processes [cf. Tuominen and Tweedie (1994) and Ango Nzé (1998)]. Suppose that  $(Y_i, \varepsilon_i)_{i \in \mathbb{Z}}$  is a functional autoregressive process and that:

1. The sequence  $(\varepsilon_i)_{i \in \mathbb{Z}}$  is i.i.d., the density  $h$  of  $\varepsilon$  satisfies  $\|h\|_\infty(0) > 0$ , and there exists  $S \geq 1$  such that  $\mathbb{E}|\varepsilon|^S < \infty$ .
2.  $\psi$  is continuous and there exist a positive real  $R$  and  $\delta$  in  $]0, 1[$  such that for any real  $x$  more than  $R$ ,  $|\psi(x)| \leq |x|(1 - |x|^{-\delta})$ .

Then the sequence  $(Y_i)_{i \in \mathbb{Z}}$  is arithmetically absolutely regular with rate  $n^{1-S/\delta}$ . In particular, condition (iii) is satisfied as soon as  $S > 2\delta$ .

8.3. *Proof of Corollary 4.* We shall prove that if either [A1, A2, (i)], [A1, A3, (ii)] or [A4, (iii)] holds, then Condition (8.3), (7.4) and S2(b) hold, so that Proposition 10 applies.

We begin with a preliminary lemma concerning the autoregressive model, the proof of which is omitted.

LEMMA 5. *Let  $(Y_i, \varepsilon_i)_{i \in \mathbb{Z}}$  be a functional autoregressive process, as defined in Definition 7, and suppose that  $\varepsilon$  satisfies A4. Then the sequence  $(Y_i)_{i \in \mathbb{Z}}$  satisfies A1 and A2.*

To check (8.3), (7.4) and S2(b), we need to control covariances. This can be done with the help of the two following inequalities: if  $X$  and  $Y$  are two real random variables respectively  $\mathcal{U}$  and  $\mathcal{V}$ -measurable, then we have

$$(8.8) \quad |\text{Cov}(X, Y)| \leq 4\alpha(\mathcal{U}, \mathcal{V})\|X\|_\infty\|Y\|_\infty,$$

and for any conjugate exponents  $p, q$

$$(8.9) \quad |\text{Cov}(X, Y)| \leq 2\phi^{1/p}(\mathcal{U}, \mathcal{V})\phi^{1/q}(\mathcal{V}, \mathcal{U})\|X\|_p\|Y\|_q.$$

(8.8) is due to Ibragimov (1962) and (8.9) due to Peligrad (1983).

PROOF OF CONDITION (8.3). We consider each case:

(a) Condition [A1, A2, (i)]. From inequality (8.9), we have

$$|\text{Cov}(X_{0,n}^2, X_{k,n}^2)| \leq 2\phi_{\infty,1}(k)\|X_{0,n}^2\|_1\|X_{k,n}\|_\infty^2 \leq \frac{8}{h_n}\phi_{\infty,1}(k)\mathbb{E}(X_{0,n}^2)\|K\|_\infty^2,$$

and (8.3) follows from the inequality

$$\frac{1}{n} \sum_{k=1}^n |\text{Cov}(X_{0,n}^2, X_{k,n}^2)| \leq 8\mathbb{E}(X_{0,n}^2)\|K\|_\infty \left( \frac{1}{nh_n} \sum_{k=1}^\infty \phi_{\infty,1}(k) \right).$$

(b) Condition [A4, (iii)]. Recall that  $Y_i = \psi(Y_{i-1}) + \varepsilon_i$ , and define

$$(8.10) \quad Z_{k-1,n} = \int \frac{1}{h_n} \left\{ K\left(\frac{x - \psi(Y_{k-1}) - z}{h_n}\right) - \mathbb{E}\left(K\left(\frac{x - Y_k}{h_n}\right)\right) \right\}^2 h(z) dz.$$

Since  $\varepsilon_k$  is independent of  $\mathcal{M}_{k-1}$ ,  $\text{Cov}(X_{0,n}^2, X_{k,n}^2) = \text{Cov}(X_{0,n}^2, Z_{k-1,n})$ . Applying inequality (8.8), we obtain

$$|\text{Cov}(X_{0,n}^2, X_{k,n}^2)| \leq \frac{16}{h_n}\alpha_{\infty,1}(k-1)\|K\|_\infty^2\|Z_{k-1,n}\|_\infty.$$

Since  $h$  is bounded, we easily deduce from (8.10) that there exists a constant  $C$  such that  $\|Z_{k-1,n}\|_\infty \leq C$ . We conclude the proof as in (a).

(c) Condition [A1, A3, (ii)]. Write first

$$\begin{aligned} |\text{Cov}(X_{0,n}^2, X_{k,n}^2)| &\leq \|X_{0,n}^2 X_{k,n}^2\|_1 + \|X_{0,n}^2\|_1^2 \\ &\leq \frac{4\|K\|_\infty}{h_n} \|X_{0,n} X_{k,n}\|_1 + \|X_{0,n}^2\|_1^2. \end{aligned}$$

From (8.4), (8.5) and Assumption A3, we infer that  $\sup_{k>0} \|X_{0,n} X_{k,n}\|_1 = O(h_n)$ , so that there exists a constant  $C$  such that

$$(8.11) \quad \sup_{n>0, k>0} |\text{Cov}(X_{0,n}^2, X_{k,n}^2)| \leq C.$$

On the other hand, once again applying inequality (8.8), we have

$$(8.12) \quad |\text{Cov}(X_{0,n}^2, X_{k,n}^2)| \leq \frac{64}{h_n^2} \alpha_{\infty,1}(k) \|K\|_\infty^4.$$

Now (8.11) and (8.12) together yield

$$\frac{1}{n} \sum_{k=1}^n |\text{Cov}(X_{0,n}^2, X_{k,n}^2)| \leq \frac{C}{nh_n} + \frac{64\|K\|_\infty^4}{nh_n^2} \sum_{k=\lceil 1/h_n \rceil}^{\infty} \alpha_{1,\infty}(k),$$

and (8.3) follows from Assumption (ii) of Corollary 4 and the fact that  $nh_n$  tends to infinity.  $\square$

PROOF OF CONDITION (7.4). We shall prove that the sequence is asymptotically 0-conditionally centered of type 2, and more precisely that Condition R1' of Remark 9 holds:

$$(8.13) \quad \lim_{n \rightarrow \infty} \sum_{k=1}^n \|X_{0,n} \mathbb{E}(X_{k,n} | \mathcal{M}_{0,n})\|_1 = 0.$$

(a) Condition [A1, A2, (i)]. Set  $s(k, n) = \mathbb{I}_{\mathbb{E}(X_{k,n} | \mathcal{M}_0) > 0} - \mathbb{I}_{\mathbb{E}(X_{k,n} | \mathcal{M}_0) \leq 0}$ . Since

$$(8.14) \quad \mathbb{E}|X_{0,n} \mathbb{E}(X_{k,n} | \mathcal{M}_0)| = \text{Cov}(|X_{0,n}|s(k, n), X_{k,n}),$$

we obtain from (8.9) the upper bound

$$\mathbb{E}|X_{0,n} \mathbb{E}(X_{k,n} | \mathcal{M}_0)| \leq 4\phi_{\infty,1}(k) \frac{1}{\sqrt{h_n}} \|X_{0,n}\|_1 \|K\|_\infty.$$

Since (i) holds and  $\sup_{n>0} h_n^{-1/2} \|X_{0,n}\|_1$  is finite, the last inequality yields

$$(8.15) \quad \sum_{k=1}^{\infty} \sup_{n>0} \mathbb{E}|X_{0,n} \mathbb{E}(X_{k,n} | \mathcal{M}_0)| < \infty.$$

On the other hand, from inequality (8.4) we infer that

$$(8.16) \quad \mathbb{E}|X_{0,n} \mathbb{E}(X_{k,n} | \mathcal{M}_0)| \leq h_n (\mu_k \star |K_n \otimes K_n|(x, x) + 3(f \star |K_n|(x))^2),$$

and the last bound is a  $O(h_n)$  because of Assumption A2 and (8.5). Since (i) is satisfied, (8.13) follows from (8.15), (8.16) and the dominated convergence theorem.

(b) Condition [A4, (iii)]. Since  $Y_i = \psi(Y_{i-1}) + \varepsilon_i$ , we obtain from (8.14)

$$\begin{aligned} & \mathbb{E}|X_{0,n}\mathbb{E}(X_{k,n}|\mathcal{M}_0)| \\ &= \sqrt{h_n} \text{Cov}\left(|X_{0,n}|s(k, n), \int K(u)h(x - h_nu - \psi(Y_{k-1})) du\right) \end{aligned}$$

and we conclude as in (a) by using (8.16) and the inequality

$$\mathbb{E}|X_{0,n}\mathbb{E}(X_{k,n}|\mathcal{M}_0)| \leq 8\alpha_{\infty,1}(k-1)\|K\|_{\infty}\|h\|_{\infty}\|K\|_1.$$

(c) Condition [A1, A3, (ii)]. From (8.16) and Assumption A3, we infer that there exists a constant  $C$  such that

$$(8.17) \quad \sup_{k>0} \mathbb{E}|X_{0,n}\mathbb{E}(X_{k,n}|\mathcal{M}_0)| \leq Ch_n.$$

On the other hand, from (8.14) and inequality (8.8) we have

$$(8.18) \quad \mathbb{E}|X_{0,n}\mathbb{E}(X_{k,n}|\mathcal{M}_0)| \leq 4\alpha_{\infty,1}(k)\|X_{0,n}\|_{\infty}^2 \leq \frac{16}{h_n}\alpha_{\infty,1}(k)\|K\|_{\infty}^2.$$

Now (8.17) and (8.18) together yield

$$\sum_{k=1}^n \mathbb{E}|X_{0,n}\mathbb{E}(X_{k,n}|\mathcal{M}_0)| \leq Ch_n \left[ \frac{\varepsilon}{h_n} \right] + \frac{16\|K\|_{\infty}^2}{h_n} \sum_{k=\lceil \varepsilon/h_n \rceil}^{\infty} \alpha_{1,\infty}(k),$$

for any positive  $\varepsilon$ . Hence (8.13) follows straightforwardly from (ii).  $\square$

PROOF OF S2(b). Assume that (iii) is satisfied. Using again  $s(k, n)$ , we have

$$\|\mathbb{E}(S_n(t)|\mathcal{M}_0)\|_1 \leq \sum_{k=1}^{\lfloor nt \rfloor} \text{Cov}(X_{k,n}, s(k, n)).$$

Since  $s(k, n)$  is  $\mathcal{M}_0$ -measurable and bounded by 1, inequality (8.8) leads to

$$\text{Cov}(X_{k,n}, s(k, n)) \leq 4\alpha_{\infty,1}(k)\|X_{0,n}\|_{\infty} \leq \frac{8}{\sqrt{h_n}}\alpha_{\infty,1}(k)\|K\|_{\infty}.$$

Finally,

$$\frac{1}{\sqrt{n}}\|\mathbb{E}(S_n(t)|\mathcal{M}_0)\|_1 \leq \frac{8\|K\|_{\infty}}{\sqrt{nh_n}} \sum_{k=1}^{\infty} \alpha_{\infty,1}(k),$$

which tends to zero as  $nh_n$  tends to infinity. To conclude, note that (i) and (ii) are both stronger than (iii) and hence imply S2(b).  $\square$

APPENDIX

PROOF OF LEMMA 1.  $1 \Rightarrow 2$  is obvious. It remains to prove that  $2 \Rightarrow 1$ . We proceed in 3 steps.

*Step 1.* Let  $\mathcal{D}(\mathbb{R}^d)$  be the space of functions from  $\mathbb{R}^d$  to  $\mathbb{C}$  which are infinitely derivable with compact support. Let  $\varphi$  be any element of  $\mathcal{D}(\mathbb{R}^d)$  and set  $\bar{\varphi}(t) = \hat{\varphi}(-t)$ . From the Plancherel equality, we have  $\mu_n(\varphi) = (2\pi)^{-d} \hat{\mu}_n(\bar{\varphi})$ . The function  $\bar{\varphi}$  being infinitely derivable and fast decreasing, it belongs to  $\mathbb{L}^1(\lambda)$ . Since  $|\hat{\mu}_n|$  converges to zero everywhere and is bounded by  $\sup_{n>0} \|\mu_n\|$ , the dominated convergence theorem implies that  $\hat{\mu}_n(\bar{\varphi})$  tends to zero as  $n$  tends to infinity. Consequently, for any  $\varphi$  in  $\mathcal{D}(\mathbb{R}^d)$ ,  $\mu_n(\varphi)$  converges to zero as  $n$  tends to infinity.

*Step 2.* Let  $\varphi$  be any function from  $\mathbb{R}^d$  to  $\mathbb{R}$ , continuous and with compact support. For any positive  $\varepsilon$ , there exists  $\varphi_\varepsilon$  in  $\mathcal{D}(\mathbb{R}^d)$  such that  $\|\varphi - \varphi_\varepsilon\|_\infty \leq \varepsilon$ . Since furthermore  $\sup_{n>0} \|\mu_n\|$  is finite, we infer from Step 1 that  $\mu_n(\varphi)$  tends to zero as  $n$  tends to infinity.

*Step 3.* For any nonnegative integer  $k$ , let  $f_k$  be a positive and continuous function from  $\mathbb{R}^d$  to  $\mathbb{R}$  satisfying:  $\|f_k\|_\infty \leq 1$ ,  $f(x) = 1$  for any  $x$  in  $[-k, k]^d$ ,  $f(x) = 0$  for any  $x$  in  $([-k - 1, k + 1]^d)^c$ .

For any continuous bounded function  $\varphi$ , write

$$|\mu_n(\varphi)| \leq |\mu_n(\varphi f_k)| + \|\varphi\|_\infty |\mu_n|(([-k, k]^d)^c).$$

From Step 2 the first term on the right-hand side tends to zero as  $n$  tends to infinity. Since the sequence  $(\mu_n)_{n>0}$  is tight, the second term on the right-hand side is as small as we wish by choosing  $k$  large enough. This completes the proof of Lemma 1.  $\square$

PROOF OF LEMMA 3. For any real number  $m$ , consider  $A_1 = \{X \leq m\}$ ,  $A_2 = \{\mathbb{E}(X|\mathcal{M}) \leq m\}$ ,  $C_1 = A_1 \cap A_2^c$  and  $C_2 = A_2 \cap A_1^c$ . Since by assumption the random variables  $X$  and  $\mathbb{E}(X|\mathcal{M})$  are identically distributed, it follows that  $\mathbb{P}(A_1) = \mathbb{P}(A_2)$ ,  $\mathbb{P}(C_1) = \mathbb{P}(C_2)$  and  $\mathbb{E}(X\mathbb{I}_{A_1}) = \mathbb{E}(X\mathbb{I}_{A_2})$ . This implies in particular that  $\mathbb{E}((X - m)\mathbb{I}_{C_1}) = \mathbb{E}((X - m)\mathbb{I}_{C_2})$ . These terms having opposite signs, they are zero. Since  $X - m$  is positive on  $C_2$ , it follows that  $C_2$  and consequently  $A_1 \Delta A_2$  have probability zero ( $\Delta$  denoting the symmetric difference). Now it is easily seen that

$$(A.1) \quad \mathbb{E}((\mathbb{E}(X|\mathcal{M}))^2 \mathbb{I}_{A_1}) = \mathbb{E}((\mathbb{E}(X|\mathcal{M}))^2 \mathbb{I}_{A_2}) = \mathbb{E}(X^2 \mathbb{I}_{A_1})$$

and

$$(A.2) \quad \mathbb{E}(X\mathbb{E}(X|\mathcal{M})\mathbb{I}_{A_1}) = \mathbb{E}(X\mathbb{E}(X|\mathcal{M})\mathbb{I}_{A_2}) = \mathbb{E}((\mathbb{E}(X|\mathcal{M}))^2 \mathbb{I}_{A_2}).$$

According to (A.1) and (A.2), we obtain

$$(A.3) \quad \begin{aligned} \|(X - \mathbb{E}(X|\mathcal{M}))\mathbb{I}_{A_1}\|_2^2 &= \|X\mathbb{I}_{A_1}\|_2^2 + \|\mathbb{E}(X|\mathcal{M})\mathbb{I}_{A_1}\|_2^2 - 2\mathbb{E}(X\mathbb{E}(X|\mathcal{M})\mathbb{I}_{A_1}) \\ &= 0. \end{aligned}$$

Since (A.3) is true for any real  $m$ , it follows that  $X = \mathbb{E}(X|\mathcal{M})$  almost surely, and Lemma 3 is proved.  $\square$

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