The Annals of Probability
1997, Vol. 25, No. 3, 1135–1159

BACKWARDS SDE WITH RANDOM TERMINAL TIME AND APPLICATIONS TO SEMILINEAR ELLIPTIC PDE

BY R. W. R. DARLING AND ETIENNE PARDOUX

University of South Florida and Université de Provence

Suppose $\{\Im_t\}$ is the filtration induced by a Wiener process W in R^d, τ is a finite $\{\Im_t\}$ stopping time (terminal time), ξ is an $\Im_\tau$-measurable random variable in R^k (terminal value) and $f(\cdot, y, z)$ is a coefficient process, depending on $y \in R^k$ and $z \in L(R^d; R^k)$, satisfying $(y - \tilde{y})[f(s, y, z) - f(s, \tilde{y}, z)] \le -a|y - \tilde{y}|^2$ (f need not be Lipschitz in y), and $|f(s, y, z) - f(s, y, \bar{z})| \le b\|z - \bar{z}\|$, for some real a and b, plus other mild conditions. We identify a Hilbert space, depending on τ and on the number $\gamma \equiv b^2 - 2a$, in which there exists a unique pair of adapted processes (Y, Z) satisfying the stochastic differential equation

$$dY(s) = 1_{\{s \le \tau\}}[Z(s)\, dW(s) - f(s, Y(s), Z(s))\, ds]$$

with the given terminal condition $Y(\tau) = \xi$, provided a certain integrability condition holds. This result is applied to construct a continuous viscosity solution to the Dirichlet problem for a class of semilinear elliptic PDE's.

1. Introduction.

1.1. *Backwards stochastic differential equations.* Suppose W is a Wiener process in R^d with natural complete right-continuous filtration $\{\Im_t\}$, τ is a finite $\{\Im_t\}$ stopping time, ξ an $\Im_\tau$-measurable random variable in R^k, and we are given a coefficient

$$f: \Omega \times R_+ \times R^k \times L(R^d; R^k) \to R^k, \tag{1}$$

such that $f(\cdot, y, z)$ is a progressively measurable process in R^k for each (y, z) in $R^k \times L(R^d; R^k)$. We wish to find a progressively measurable solution (Y, Z) with values in $R^k \times L(R^d; R^k)$ of the equation

$$Y(t) = \xi + \int_{t\wedge\tau}^{\tau} f(s, Y(s), Z(s))\, ds - \int_{t\wedge\tau}^{\tau} Z(s)\, dW(s), \qquad t \ge 0, \tag{2}$$

satisfying certain integrability criteria, to be described later. We refer to (2) as a backwards stochastic differential equation (BSDE), with terminal time τ and terminal value ξ.

Received January 1996; revised November 1996.
AMS 1991 *subject classifications.* Primary 60H20; secondary 35J60, 90A09.
Key words and phrases. Stochastic differential equation, monotonicity, stopping time, Brownian motion, viscosity solution, semilinear elliptic PDE.

1.2. *Classical result.* The "classical" result of Pardoux and Peng (1990) is that when τ is replaced by a constant time $T > 0$, $\xi \in L^2((\Omega, \Im_T, P); R^k)$, and f is uniformly Lipschitz in y and z and satisfies (3), there exists a unique progressively measurable solution (Y, Z) satisfying (8) and (9) below. A concise proof by a fixed point argument is given in Barles, Buckdahn and Pardoux (1996). For other results outside the Lipschitz context, see Pardoux and Peng (1994) and Darling (1995).

1.3. *Reasons for studying random terminal times.* Pardoux, Pradeilles and Rao (1995) have described how the value at time 0 of a solution Y of (2) for *constant* terminal time may be used to construct a viscosity solution to a system of semilinear *parabolic* PDE. Peng (1991) also describes how the solution Y of (2) for an unbounded *random* terminal time is related to semilinear *elliptic* PDE. Viscosity solutions for such equations will be constructed by stochastic methods below.

2. Results for constant terminal time using monotonicity in *y*.

2.1. *Conditions on f.* The aim of this section is to establish an existence and uniqueness result for a "classical" BSDE (in the sense that the terminal value is given at a fixed, i.e., nonrandom, terminal time), but with coefficients which are not necessarily Lipschitz with respect to both variables. More precisely, we are given the following:

1. A fixed terminal time $T > 0$;
2. A terminal value $\xi \in L^2((\Omega, \Im_T, P); R^k)$;
3. A coefficient f as in (1) with the following five properties:

$$\mathrm{E}\left[\int_0^T |f(t,0,0)|^2\,dt\right] < \infty; \tag{3}$$

$$(y - \tilde{y})[f(s,y,z) - f(s,\tilde{y},z)] \le -a|y-\tilde{y}|^2, \tag{4}$$

for some real (positive or negative) a;

$$|f(s,y,z) - f(s,y,\bar{z})| \le b\|z - \bar{z}\| \tag{5}$$

for some positive b, where $\|z\|^2 \equiv \mathrm{Tr}(zz^T)$; and for some positive κ,

$$|f(s,y,z)| \le |f(s,0,z)| + \kappa(1 + |y|); \tag{6}$$

$$y \to f(s,y,z) \text{ is continuous.} \tag{7}$$

We refer to (4) as the monotonicity condition. Note that an f which is Lipschitz in y has property (4) with a negative a, but the converse is not generally true.

THEOREM 2.2 (Existence and uniqueness for constant terminal time). *Under conditions described in* 2.1, *the BSDE* (2) *has a unique progressively measurable solution* $\{(Y(t), Z(t)) : 0 \le t \le T\}$ *such that*

$$\mathrm{E}\left[\int_0^T \|Z(t)\|^2\,dt\right] < \infty. \tag{8}$$

Moreover, the solution satisfies

$$\mathrm{E}\left|\sup\left\{|Y(t)|^2 : 0 \le t \le T\right\}\right| < \infty, \tag{9}$$

$$\mathrm{E}\left[\int_0^T Y(t) \cdot Z(t)\, dW(t)\right] = 0. \tag{10}$$

PROOF. First we show how (9) and (10) follow from the existence of a solution (Y, Z) under the given conditions. Since $Y(0)$ is deterministic, (9) follows from (2), (3), (5), (6), (8) and Burkholder's inequality. Moreover the continuous local martingale

$$M_t \equiv \int_0^t Y(s) \cdot Z(s)\, dW(s)$$

satisfies $\mathrm{E}[\langle M \rangle_t^{1/2}] < \infty$ from (8) and (9); hence it is a uniformly integrable martingale, and so $\mathrm{E}[M_t] = 0$ for all t (see the end of the proof of Lemma 4.3). □

2.2.1. *Uniqueness*. This is a special case of the proof of Section 5.1 of Proposition 3.2, noting that the function spaces $M_\gamma^2(0, \tau)$ are the same for all γ in the case of nonrandom terminal time, giving condition (8) on Z.

2.2.2. *Existence*. Note that (Y, Z) solves BSDE (2) if and only if $(\hat{Y}(t), \hat{Z}(t)) \equiv (e^{\lambda t} Y(t), e^{\lambda t} Z(t))$ solves the BSDE

$$\hat{Y}(t) = e^{\lambda T}\xi + \int_t^T \left[e^{\lambda s} f\left(s, e^{-\lambda s}\hat{Y}(s), e^{-\lambda s}\hat{Z}(s)\right) - \lambda \hat{Y}(s)\right] ds$$
$$- \int_t^T \hat{Z}(s)\, dW(s).$$

If we choose $\lambda = -a$, we have that

$$\hat{f}(s, y, z) \equiv e^{-as} f(s, e^{as} y, e^{as} z) + ay \tag{11}$$

satisfies (4) with $a = 0$, and (5) with the same constant b as f. Hence we can and will assume the conditions of the theorem are satisfied with $a = 0$. Let us admit for a moment the following proposition.

PROPOSITION 2.3. *Given an $L(R^d; R^k)$-valued progressively measurable process $\{V(t), 0 \le t \le T\}$ which satisfies*

$$\mathrm{E}\left[\int_0^T \|V(t)\|^2\, dt\right] < \infty,$$

there exists a unique pair $\{Y(t), Z(t)): 0 \le t \le T\}$ of progressively measurable processes with values in $R^k \times L(R^d; R^k)$ satisfying

$$\mathrm{E}\left[\int_0^T \|Z(t)\|^2\, dt\right] < \infty,$$

$$Y(t) = \xi + \int_t^T f(s, Y(s), V(s))\, ds - \int_t^T Z(s)\, dW(s), \qquad 0 \le t \le T.$$

Using Proposition 2.3, we can construct a sequence (Y_n, Z_n) as follows: $(Y_0, Z_0) = (0, 0)$, and for $n \geq 1$,

$$Y_{n+1}(t) = \xi + \int_t^T f(s, Y_{n+1}(s), Z_n(s))\, ds - \int_t^T Z_{n+1}(s)\, dW(s), \qquad 0 \leq t \leq T.$$

Let $\Delta Y \equiv Y_{n+1} - Y_n$, and $\Delta Z \equiv Z_{n+1} - Z_n$. Itô's formula for $X(t) \equiv e^{\theta t}|\Delta Y(t)|^2$ on the time interval $[0, T]$ gives, for any $\theta \in R$,

$$\begin{aligned} &\mathrm{E}\left[e^{\theta t}|\Delta Y(t)|^2 + \int_t^T e^{\theta s}\left(\theta|\Delta Y(s)|^2 + \|\Delta Z(s)\|^2\right) ds\right] \\ &\quad = 2\mathrm{E}\left[\int_t^T e^{\theta s}\Delta Y[f(s, Y_{n+1}, Z_n) - f(s, Y_n, Z_{n-1})]\, ds\right] \\ &\quad \leq 2b\mathrm{E}\left[\int_t^T e^{\theta s}|\Delta Y|\,\|Z_n - Z_{n-1}\|\, ds\right] \end{aligned}$$

using (4) with $a = 0$, and (5), dropping some of the s variables. For any $c > 0$, this is

$$\leq cb^2\mathrm{E}\left[\int_t^T e^{\theta s}|\Delta Y|^2\, ds\right] + \frac{1}{c}\mathrm{E}\left[\int_t^T e^{\theta s}\|Z_n - Z_{n-1}\|^2\, ds\right]. \tag{12}$$

Choosing $c = 2$ and $\theta = 2b^2$, we deduce

$$\mathrm{E}\left[\int_0^T e^{\theta s}\|Z_{n+1} - Z_n\|^2\, ds\right] \leq \tfrac{1}{2}\mathrm{E}\left[\int_0^T e^{\theta s}\|Z_n - Z_{n-1}\|^2\, ds\right].$$

Hence the sequence $\{Z_n\}$ is Cauchy in $L^2(\Omega \times [0, T],\ dP \times e^{\theta t}\, dt,\ L(R^d; R^k))$, and tends to a limit Z. Choosing $c = 1$ and $\theta = 2b^2$ in (12), we obtain

$$\mathrm{E}\left[\int_t^T e^{\theta s}|\Delta Y|^2\, ds\right] \leq \mathrm{E}\left[\int_t^T e^{\theta s}\|Z_n - Z_{n-1}\|^2\, ds\right]\Big/ b^2 \leq c' 2^{-n}$$

and so $\{Y_n\}$ is Cauchy in $L^2(\Omega \times [0, T],\ dP \times e^{\theta t}\, dt,\ R^k)$, and hence has a limit Y. The methods of Step 5 of Section 5.2 prove that (Y, Z) solves (2).

PROOF OF PROPOSITION 2.3. Uniqueness follows from that of 2.2.1; we prove existence. Let us write $h(s, y)$ in place of the random vector $f(s, y, V(s))$. Our assumptions imply that

$$\mathrm{E}\left[\int_0^T |h(t, 0)|^2\, dt\right] < \infty; \tag{13}$$

$$|h(s, y)| \leq |h(s, 0)| + \kappa(1 + |y|); \tag{14}$$

$$(y - \tilde{y}) \cdot [h(s, y) - h(s, \tilde{y})] \leq 0 \tag{15}$$

[see (11) for the rationale for taking $a = 0$]. We first approximate h by $\tilde{f}_n$, which coincides with f for $|y| \leq n$, is bounded, and satisfies (14) and (15), and then define

$$f_n(t, y) \equiv \left(\rho_n^* \tilde{f}_n\right)(t, y),$$

where $\{\rho_n\}$ is sequence of smooth functions which approximate the Dirac measure at zero; thus f_n also satisfies (14) and (15) and is Lipschitz in y. Thus by the standard result of Pardoux and Peng (1990), the BSDE

$$Y_n(t) = \xi + \int_t^T f_n(s, Y_n(s))\, ds - \int_t^T Z_n(s)\, dW(s)$$

has a unique solution (Y_n, Z_n) satisfying (8) and (9). Moreover,

$$\begin{aligned} |Y_n(t)|^2 &+ \int_t^T \|Z_n(s)\|^2\, ds \\ &= |\xi|^2 + 2\int_t^T Y_n(s)\cdot f_n(s, Y_n(s))\, ds - 2\int_t^T Y_n(s)\cdot Z_n(s)\, dW(s); \end{aligned}$$

$$\mathrm{E}\left[|Y_n(t)|^2 + \int_t^T \|Z_n(s)\|^2\, ds\right] \le \mathrm{E}\left[|\xi|^2\right] + C\mathrm{E}\left[\int_t^T \left(1 + |Y_n(s)|^2\right) ds\right].$$

It then follows from standard estimates that

$$\sup_n \mathrm{E}\left[\sup\left\{|Y_n(t)|^2 : 0 \le t \le T\right\} + \int_0^T \|Z_n(s)\|^2\, ds\right] < \infty. \tag{16}$$

Let

$$U_n(s) \equiv f_n(s, Y_n(s)).$$

From (13), (16) and (14) for f_n, we have that

$$\sup_n \mathrm{E}\left[\int_0^T |U_n(s)|^2\, ds\right] < \infty.$$

Hence there exists a subsequence

$$\left(Y_{n(j)}, Z_{n(j)}, U_{n(j)}\right)$$

which converges weakly in $L^2(\Omega \times [0, T], dP \times dt, R^k \times L(R^d; R^k) \times R^k)$ to a limit (Y, Z, U). Any $\eta \in L^2(\Omega, \mathfrak{I}_T, P; R^k)$ has an Itô representation of the form

$$\eta = \mathrm{E}[\eta] + \int_0^T \chi_s\, dW(s),$$

and therefore

$$\begin{aligned} \mathrm{E}\left[\eta \int_0^T Z_{n(j)}(s)\, dW(s)\right] &= \mathrm{E}\left[\int_0^T \chi_s Z_{n(j)}(s)\, ds\right] \to \mathrm{E}\left[\int_0^T \chi_s Z(s)\, ds\right] \\ &= \mathrm{E}\left[\eta \int_0^T Z(s)\, dW(s)\right], \end{aligned}$$

proving that $\int_0^T Z_{n(j)}(s)\, dW(s) \to \int_0^T Z(s)\, dW(s)$ weakly in $L^2(\Omega, \mathfrak{I}_T, P; R^k)$. The same is true for $\int_t^T Z_n(s)\, dW(s)$, and we have, by taking a weak limit,

$$Y(t) = \xi + \int_t^T U(s)\, ds - \int_t^T Z(s)\, dW(s).$$

It remains to show that $U(t) = h(t, Y(t))$. Let $\{X_t, 0 \le t \le T\}$ be any element of $L^2(\Omega \times [0, T], dP \times dt, R^k)$. We note that from (15) for f_n,

$$\mathrm{E}\left[\int_0^T (Y_n(t) - X_t) \cdot [f_n(t, Y_n(t)) - f_n(t, X_t)]\, dt\right] \le 0. \tag{17}$$

Also, since $f_n(\cdot, X) \to h(\cdot, X)$ strongly in $L^2(\Omega \times [0, T], dP \times dt, R^k)$,

$$\mathrm{E}\left[\int_0^T (Y_n(t) - X_t) \cdot [f_n(t, X_t) - h(t, X_t)]\, dt\right] \to 0, \qquad n \to \infty. \tag{18}$$

Moreover,

$$\mathrm{E}\left[\int_0^T Y_n(t) \cdot f_n(t, Y_n(t))\, dt\right] = \tfrac{1}{2}\mathrm{E}\left[|Y_n(0)|^2 - |\xi|^2 + \int_0^T \|Z_n(t)\|^2\, dt\right]. \tag{19}$$

Now

$$Y_{n(j)}(0) = \xi + \int_0^T U_{n(j)}(s)\, ds - \int_0^T Z_{n(j)}(s)\, dW(s)$$

and converges weakly in L^2. Since it is deterministic, $Y_{n(j)}(0) \to Y(0)$ in R^k. However, since the mapping

$$Z \to \mathrm{E}\left[\int_0^T \|Z(t)\|^2\, dt\right]$$

is convex and continuous on $L^2(\Omega \times [0, T], dP \times dt, R^k)$, it is lower semicontinuous for the weak topology, and so (19) implies

$$\begin{aligned} &\liminf_{j \to \infty} \mathrm{E}\left[\int_0^T Y_{n(j)}(t) \cdot f_{n(j)}(t, Y_{n(j)}(t))\, dt\right] \\ &\quad \ge \tfrac{1}{2}\mathrm{E}\left[|Y(0)|^2 - |\xi|^2 + \int_0^T \|Z(t)\|^2\, dt\right] \\ &\quad = \mathrm{E}\left[\int_0^T Y(t) \cdot U(t)\, dt\right]. \end{aligned} \tag{20}$$

Combining these results, we deduce

$$\begin{aligned} &\mathrm{E}\left[\int_0^T (Y(t) - X_t) \cdot [U(t) - h(t, X_t)]\, dt\right] \\ &\quad \le \liminf_{j \to \infty} \mathrm{E}\left[\int_0^T (Y_{n(j)}(t) - X_t) \cdot [f_{n(j)}(t, Y_{n(j)}(t)) - f_{n(j)}(t, X_t)]\, dt\right] \le 0, \end{aligned}$$

where the first inequality comes from (20) together with weak convergence, and the second from (17) and (18). Choosing $X_t \equiv Y(t) + \varepsilon\varphi_t$ for an arbitrary $\varepsilon > 0$ and $\varphi \in L^2(\Omega \times [0, T], dP \times dt, R^k)$, dividing by ε, and letting $\varepsilon \to 0$, we obtain

$$\mathrm{E}\left[\int_0^T \varphi_t [U(t) - h(t, Y_t)]\, dt\right] \ge 0.$$

On taking $\varphi_t \equiv -[U(t) - h(t, Y_t)]$, the identity $U(t) = h(t, Y(t))$ follows. □

3. Results for random terminal time. In order to clarify the integrability condition (25) below, we shall replace (6) by

$$|f(s,y,z)| \le |f(s,0,z)| + \kappa(|y| + \kappa'), \tag{21}$$

where $\kappa \ge 0$, and $\kappa' = 0$ or 1, which together with (5) gives

$$|f(s,y,z)| \le |f(s,0,0)| + \kappa|y| + b\|z\| + \kappa\kappa'. \tag{22}$$

3.1. *Function space notation*. For any real number θ, and any Euclidean space V, $M_\theta^2(0,\tau;V)$ will denote the Hilbert space of progressively measurable processes X, with values in V, such that

$$\|X\|_\theta^2 \equiv \mathrm{E}\left[\int_0^\tau e^{\theta s}|X(s)|^2\,ds\right] < \infty. \tag{23}$$

Obviously $M_\theta^2(0,\tau;V) \supseteq M_\rho^2(0,\tau;V)$ for $\theta \le \rho$. We shall state the existence and uniqueness results separately since their assumptions are quite different. In all these results, we define, with reference to (4) and (5),

$$\gamma \equiv b^2 - 2a. \tag{24}$$

Existence and uniqueness results for BSDE with random terminal time were given already by Peng (1991), but under stronger assumptions than ours.

PROPOSITION 3.2 (Uniqueness). *If* (4), (5), *and* (21) *hold, and* $f(\cdot,0,0) \in M_\theta^2(0,\tau;R^k)$ *for all* $\theta < \gamma$, *then* (2) *has at most one solution* (Y,Z) *in* $M_\gamma^2(0,\tau;R^k \times L(R^d;R^k))$.

PROPOSITION 3.3 (Existence). *Suppose f satisfies* (4), (5), (7), *and* (21), *with* $\kappa' = 0$ *or* 1, *and that, for some* $\rho > \gamma$,

$$\mathrm{E}\left[e^{\rho\tau}\left(|\xi|^2 + \kappa'\right) + \int_0^\tau e^{\rho s}|f(s,0,0)|^2\,ds\right] < \infty. \tag{25}$$

Then there exists a solution (Y,Z) *of* (2) *in* $M_\rho^2(0,\tau;R^k \times L(R^d;R^k))$.

THEOREM 3.4 (Combined existence and uniqueness for random terminal time). *Suppose f satisfies* (4), (5), (7) *and* (21), *and* (25) *holds for some* $\rho > \gamma$. *Then there exists a unique solution* (Y,Z) *of* (2) *in* $M_\gamma^2(0,\tau;R^k \times L(R^d;R^k))$, *and this solution actually belongs to* $M_\rho^2(0,\tau;R^k \times L(R^d;R^k))$; *moreover*

$$\mathrm{E}\left[\sup\left\{e^{\rho s}|Y(s)|^2 : 0 \le s \le \tau\right\}\right] < \infty. \tag{26}$$

PROOF. Proposition 3.3 proves the existence of a solution (Y,Z) to (2) in $M_\rho^2(0,\tau;R^k \times L(R^d;R^k))$. Property (26) follows from Proposition 4.3. This solution (Y,Z) a fortiori belongs to $M_\gamma^2(0,\tau;R^k \times L(R^d;R^k))$, since $\rho > \gamma$, and by Proposition 3.2, (Y,Z) is the only solution in that space. □

EXAMPLE 3.5 (Linear coefficient: the question of nonuniqueness). Consider the special case where $A \in L(R^k;R^k)$, $B \in R^d$, and

$$f(t,y,z) = Ay + zB. \tag{27}$$

Writing the time variable as a subscript, one solution to (2) under appropriate integrability conditions is

$$(28)\qquad Y_t = Q_t^{-1}\zeta_t, \qquad Z_t = Q_t^{-1}(\eta_t - \zeta_t \otimes B),$$

where $Q_t \equiv e^{tA}\exp\{B \cdot W_t - |B|^2 t/2\}$, and

$$(29)\qquad \zeta_t \equiv \mathrm{E}\left[Q_\tau \xi | \Im_t\right], \qquad Q_\tau \xi = \zeta_0 + \int_0^\tau \eta_s\, dW_s.$$

Are there other solutions outside the appropriate integrability class? Consider the case where $k = 1$, W is one-dimensional, $\tau \equiv \inf\{t\colon W(t) = 1\} < \infty$ and

$$f(s, y, z) \equiv z.$$

Take the bounded terminal value

$$\xi \equiv \exp(-1 - \tau/2).$$

Observe that for this example $\gamma = 1$, so although $\kappa' = 0$ and $f(s, 0, 0) = 0$, condition (25) for existence does not hold. Nevertheless, there is a trick for finding multiple solutions to (2): if $Q_t \equiv \exp\{W_t - t/2\}$, Itô's formula shows that $M_t \equiv Q_t Y_t$ is a continuous local martingale with terminal value $M_\tau = \exp\{1 - \tau/2\}\xi = e^{-\tau}$. Now we construct two different continuous local martingales M with $M_\tau = e^{-\tau}$, and two corresponding Y's by the formula $Y_t = M_t/Q_t$.

3.5.1. *The unique solution in the desired integrability class.* Let $M_t \equiv \exp\{W_t\sqrt{2} - t - \sqrt{2}\}$, so

$$Y_t = \exp\{W_t(\sqrt{2} - 1) - t/2 - \sqrt{2}\},$$

which gives $Y_0 = e^{-\sqrt{2}}$ and $dY_t = Z_t(dW_t - dt)$ with

$$Z_t = Y_t(\sqrt{2} - 1).$$

Moreover, $\gamma = 1$ for this example, and it is indeed true that $(Y, Z) \in M_1^2(0, \tau; R \times R)$, since, for $\theta \equiv 2(\sqrt{2} - 1)$, Fubini's theorem gives

$$\mathrm{E}\left[\int_0^\tau e^s |Y_s|^2\, ds\right] = C\int_0^\infty dt \int_{-\infty}^1 P(W_t > r, \tau > t) e^{r\theta}\, dr < \infty,$$

where boundedness of the integral is a straightforward calculation based on the formula for the joint density of W_t and $S_t \equiv \sup\{W_s\colon 0 \le s \le t\}$, given for example in Revuz and Yor (1991). By Proposition 3.2, (Y, Z) is the *only* solution in $M_1^2(0, \tau; R \times R)$.

3.5.2. *A continuum of solutions outside the desired integrability class.* Let $\tilde{M}_t \equiv \exp\{-W_t\sqrt{2} - t + \sqrt{2}\}$, so

$$\tilde{Y}_t = \exp\{-W_t(\sqrt{2} + 1) - t/2 + \sqrt{2}\},$$

which gives $\tilde{Y}_0 = e^{\sqrt{2}}$, and $d\tilde{Y}_t = \tilde{Z}_t(dW_t - dt)$ with

$$\tilde{Z}_t = -\tilde{Y}_t(\sqrt{2} + 1).$$

For any real $\alpha \neq 0$, $\alpha \tilde{M}_t + (1-\alpha) M_t$ likewise gives a solution different from (Y, Z), and one which therefore cannot belong to $M_1^2(0, \tau; R \times R)$.

4. Some technical results.

4.1. *Itô representation of the terminal value.* Suppose that λ is some real number such that $e^{\lambda\tau}\xi$ is in L^2 and admits an Itô representation

$$e^{\lambda\tau}\xi = \mathrm{E}\left[e^{\lambda\tau}\xi\right] + \int_0^{\tau} \eta(s)\, dW(s). \tag{30}$$

Let

$$\zeta(t) \equiv \mathrm{E}\left[e^{\lambda\tau}\xi | \Im_t\right] = \mathrm{E}\left[e^{\lambda\tau}\xi\right] + \int_0^{t\wedge\tau} \eta(s)\, dW(s). \tag{31}$$

LEMMA 4.1. *For any $\theta \geq 0$ such that $e^{(\theta/2+\lambda)\tau}\xi$ is in L^2, the processes ζ and η above satisfy*

$$\|\eta\|_\theta^2 + \theta \|\zeta\|_\theta^2 = \mathrm{E}\left[|\exp((\theta/2+\lambda)\tau)\xi|^2\right] - \left|\mathrm{E}\left[e^{\lambda\tau}\xi\right]\right|^2. \tag{32}$$

PROOF. Let $\chi_t \equiv \exp(\theta t/2)\zeta(t)$, so that

$$d\chi_t = 1_{\{t \leq \tau\}}\left[\theta \exp(\theta t/2)\zeta(t)\, dt/2 + e^{\theta t/2}\eta(t)\, dW(t)\right],$$

and

$$\begin{aligned} |\chi_t|^2 = |\chi_0|^2 &+ \int_0^{t\wedge\tau} e^{\theta s}\left(\|\eta(s)\|^2 + \theta|\zeta(s)|^2\right) ds \\ &+ 2\int_0^{t\wedge\tau} \exp(\theta s/2)\chi_s \cdot \eta(s)\, dW(s). \end{aligned} \tag{33}$$

Let $\tau(n) \equiv \inf\{t: |\chi_t| \geq n\} \wedge n \wedge \tau$, and define a continuous local martingale M^n by

$$M_t^n \equiv \int_0^{t\wedge\tau(n)} \exp(\theta s/2)\chi_s \cdot \eta(s)\, dW(s).$$

Evidently its quadratic variation process satisfies

$$\langle M^n \rangle_t \leq e^{\theta n} n^2 \int_0^{\tau} \|\eta(s)\|^2\, ds \in L^1(P),$$

using the fact that $\|\eta\|_0^2 = \mathrm{E}[|e^{\lambda\tau}\xi|^2] - |\mathrm{E}[e^{\lambda\tau}\xi]|^2 < \infty$, by assumption. By Burkholder's inequality, M^n is a uniformly integrable martingale for each n, and hence we may take expectations in (33) to obtain

$$\begin{aligned} &\mathrm{E}\left[\int_0^{\tau(n)} e^{\theta s}\left(\|\eta(s)\|^2 + \theta|\zeta(s)|^2\right) ds\right] \\ &\qquad = \mathrm{E}\left[\left|\mathrm{E}\left[\exp(\lambda\tau + \theta\tau(n)/2)\xi | \Im_{\tau(n)}\right]\right|^2\right] - |\mu|^2, \end{aligned}$$

where $\mu \equiv \mathrm{E}[e^{\lambda\tau}\xi]$. Let $n \to \infty$, and apply monotone convergence and L^2 martingale convergence of $\mathrm{E}[\exp((\theta/2+\lambda)\tau)\xi | \Im_{\tau(n)}] \to \exp((\theta/2+\lambda)\tau)\xi$, to obtain (32). □

4.2. *Some algebraic inequalities.*

4.2.1. *Weak bound.* If (5) and (21) hold, then for any $\delta > 0$,

$$
\begin{aligned}
-2y \cdot f(s,y,z) &\le 2\kappa|y|^2 + 2|y|\big(|f(s,0,0)| + b\|z\| + \kappa\kappa'\big) \\
&\le \alpha|y|^2 + \|z\|^2 + \delta^{-1}|f(s,0,0)|^2 + (\kappa\kappa')^2,
\end{aligned} \tag{34}
$$

where $\alpha \equiv 2\kappa + b^2 + \delta + 1$.

4.2.2. *Strong bound.* Suppose f and $\tilde{f}$ both satisfy (4) and (5) with the same constants a and b. Writing $f(y,z)$ in place of $f(s,y,z)$, and so on, for brevity,

$$
\begin{aligned}
&2(y-\tilde{y}) \cdot \big[f(y,z) - \tilde{f}(\tilde{y},\tilde{z})\big] \\
&\quad = 2(y-\tilde{y}) \cdot \big[f(y,z) - \tilde{f}(y,z) + \tilde{f}(y,z) - \tilde{f}(y,\tilde{z}) + \tilde{f}(y,\tilde{z}) - \tilde{f}(\tilde{y},\tilde{z})\big] \\
&\quad \le -2a|y-\tilde{y}|^2 + 2|y-\tilde{y}|\big[|f(y,z) - \tilde{f}(y,z)| + b\|z-\tilde{z}\|\big].
\end{aligned}
$$

For any $\varepsilon \ge 0$ and $\delta > 0$, this is less than or equal to

$$
\big[b^2(1+\varepsilon) + \delta - 2a\big]|y-\tilde{y}|^2 + \frac{\|z-\tilde{z}\|^2}{1+\varepsilon} + \frac{|f(y,z) - \tilde{f}(y,z)|^2}{\delta}. \tag{35}
$$

In the special case where $y = 0$, $z = 0$, and $f = 0$, we have for any $\delta > 0$ that

$$
\begin{aligned}
&2\tilde{y} \cdot \tilde{f}(s,\tilde{y},\tilde{z}) \\
&\qquad \le \big[b^2(1+\varepsilon) + \delta - 2a\big]|\tilde{y}|^2 + \frac{\|\tilde{z}\|^2}{1+\varepsilon} + \delta^{-1}|\tilde{f}(s,0,0)|^2.
\end{aligned} \tag{36}
$$

PROPOSITION 4.3 (Integrability properties). *If* (2) *has a solution* $(Y,Z) \in M_\theta^2(0,\tau; R^k \times L(R^d; R^k))$ *for some real* θ, *if* f *satisfies* $f(\cdot,0,0) \in M_\theta^2(0,\tau; R^k)$, (5) *and* (21), *and if* $\kappa' \mathrm{E}[e^{\theta\tau}] < \infty$, *then*

$$
\mathrm{E}\Big|\sup\big\{e^{\theta s}|Y(s)|^2 : 0 \le s \le \tau\big\}\Big| < \infty, \tag{37}
$$

and $\{M_t,\ t \ge 0\}$ *is a uniformly integrable martingale, where*

$$
M_t \equiv \int_0^{t\wedge\tau} e^{\theta s} Y(s) \cdot Z(s)\, dW(s). \tag{38}
$$

PROOF. Itô's formula and (34) imply that

$$
\begin{aligned}
&e^{\theta(t\wedge\tau)}|Y(t\wedge\tau)|^2-|Y(0)|^2\\
&\quad=\int_0^{t\wedge\tau}e^{\theta s}\Big[\theta|Y(s)|^2+\|Z(s)\|^2-2Y(s)\cdot f(s,Y(s),Z(s))\Big]\,ds\\
&\qquad+\int_0^{t\wedge\tau}2e^{\theta s}Y(s)\cdot Z(s)\,dW(s)\\
&\quad\le C\int_0^{t\wedge\tau}e^{\theta s}\Big[|Y(s)|^2+\|Z(s)\|^2+|f(s,0,0)|^2+\kappa'\Big]\,ds\\
&\qquad+\int_0^{t\wedge\tau}2e^{\theta s}Y(s)\cdot Z(s)\,dW(s).
\end{aligned}\tag{39}
$$

Let $\tau(n)\equiv\inf\{t:|Y(t)|\ge n\}\wedge n\wedge\tau$. From the assumptions of the proposition and Burkholder's inequality, it follows that

$$
\begin{aligned}
&\mathrm{E}\Big[\sup_{0\le s\le\tau(n)}\big\{e^{\theta s}|Y(s)|^2\big\}\Big]\\
&\quad\le C\left\{1+\mathrm{E}\left[\left(\int_0^{\tau(n)}e^{2\theta s}|Y(s)|^2\|Z(s)\|^2\,ds\right)^{1/2}\right]\right\}\\
&\quad\le C\left\{1+\mathrm{E}\left[\sup_{0\le s\le\tau(n)}\big\{e^{\theta s/2}|Y(s)|\big\}\left(\int_0^{\tau(n)}e^{\theta s}\|Z(s)\|^2\,ds\right)^{1/2}\right]\right\}\\
&\quad\le\tfrac12\mathrm{E}\Big[\sup_{0\le s\le\tau(n)}\big\{e^{\theta s}|Y(s)|^2\big\}\Big]+C+C'\|Z\|_\theta^2.
\end{aligned}
$$

Thus $\mathrm{E}|\sup_{0\le s\le\tau(n)}\{e^{\theta s}|Y(s)|^2\}|\le C''$ and (37) follows from monotone convergence, on letting $n\to\infty$. As for the second assertion, Burkholder's inequality gives

$$
\begin{aligned}
&\mathrm{E}\left[\sup\left\{\left|\int_0^{t\wedge\tau}e^{\theta s}Y(s)\cdot Z(s)\,dW(s)\right|:t\ge0\right\}\right]\\
&\quad\le C\,\mathrm{E}\left[\left|\int_0^{\tau}e^{2\theta s}|Y(s)|^2\|Z(s)\|^2\,ds\right|^{1/2}\right]\\
&\quad\le C\,\mathrm{E}\left[\sup_{0\le s\le\tau}\big\{e^{\theta s/2}|Y(s)|\big\}\left\{\int_0^{\tau}e^{\theta s}\|Z(s)\|^2\,ds\right\}^{1/2}\right]\\
&\quad\le\frac{C}{2}\left\{\mathrm{E}\Big[\sup_{0\le s\le\tau}\big\{e^{\theta s}|Y(s)|^2\big\}\Big]+\|Z\|_\theta^2\right\}<\infty
\end{aligned}
$$

using (37), giving uniform integrability of $\{M_t,\ t\ge0\}$. □

In the next result, we adopt the convention that, for any solution to (2),

$$1_{\{s>\tau\}}Y(s)=\xi,\qquad 1_{\{s>\tau\}}Z(s)=0,\qquad 1_{\{s>\tau\}}f(s,y,z)=0. \tag{40}$$

PROPOSITION 4.4 (Stability with respect to perturbations). *Suppose* (τ,ξ,f) *and* (τ',ξ',f') *are triples for which the conditions of Proposition* 3.3 *are satisfied, with the same* a, b *and* $\rho>b^2-2a$. *Let* $\Delta Y\equiv Y-Y'$, *where* $(Y,Z)\in M^2_\rho(0,\tau;R^k\times L(R^d;R^k))$ *and* $(Y',Z')\in M^2_\rho(0,\tau';R^k\times L(R^d;R^k))$ *are the solutions to* (2) *corresponding to* (τ,ξ,f) *and* (τ',ξ',f'), *respectively. If* $b^2-2a<\theta\le\rho$, *there exist positive numbers* β, δ *such that*

$$\begin{aligned}|\Delta Y(0)|^2&+\beta\,\mathrm{E}\left[\int_0^{\tau\vee\tau'}e^{\theta s}\left(|\Delta Y(s)|^2+\|\Delta Z(s)\|^2\right)ds\right]\\&\le \mathrm{E}|\exp(\theta\tau/2)\xi-\exp(\theta\tau'/2)\xi'|^2\\&\quad+\delta^{-1}\mathrm{E}\left[\int_0^{\tau\vee\tau'}e^{\theta s}|f(s,Y(s),Z(s))-f'(s,Y(s),Z(s))|^2\,ds\right]\end{aligned}\tag{41}$$

PROOF. First note that for any stopping time $\sigma\le\tau\wedge\tau'$,

$$\begin{aligned}&\exp(\theta\sigma/2)\Delta Y(\sigma)\\&\quad=\exp(\theta\tau/2)\xi-\exp(\theta\tau'/2)\xi'\\&\qquad+\int_\sigma^\tau\exp(\theta s/2)\left[f(s,Y(s),Z(s))-\frac{\theta Y(s)}{2}\right]ds\\&\qquad-\int_\sigma^{\tau'}\exp(\theta s/2)\left[f'(s,Y'(s),Z'(s))-\frac{\theta Y'(s)}{2}\right]ds\\&\qquad-\int_\sigma^\tau\exp(\theta s/2)Z(s)\,dW(s)+\int_\sigma^{\tau'}\exp(\theta s/2)Z'(s)\,dW(s).\end{aligned}$$

In view of (40), this can be written as

$$\begin{aligned}&\exp(\theta\sigma/2)\Delta Y(\sigma)\\&\quad=\exp(\theta\tau/2)\xi-\exp(\theta\tau'/2)\xi'-\int_\sigma^{\tau\vee\tau'}\exp(\theta s/2)\Delta Z(s)\,dW(s)\\&\qquad+\int_\sigma^{\tau\vee\tau'}\exp(\theta s/2)\bigg[f(s,Y(s),Z(s))\\&\qquad\qquad-f'(s,Y'(s),Z'(s))-\frac{\theta\Delta Y(s)}{2}\bigg]ds.\end{aligned}$$

Using Itô's formula we obtain

$$\begin{aligned}&e^{\theta\sigma}|\Delta Y(\sigma)|^2+\int_\sigma^{\tau\vee\tau'}e^{\theta s}\left(\theta|\Delta Y(s)|^2+\|\Delta Z(s)\|^2\right)ds\\&\quad=|\exp(\theta\tau/2)\xi-\exp(\theta\tau'/2)\xi'|^2-2\int_\sigma^{\tau\vee\tau'}e^{\theta s}\Delta Y(s)\cdot\Delta Z(s)\,dW(s)\\&\qquad+2\int_\sigma^{\tau\vee\tau'}e^{\theta s}\Delta Y(s)\cdot[f(s,Y(s),Z(s))-f'(s,Y'(s),Z'(s))]\,ds.\end{aligned}\tag{42}$$

Using (35), take $\delta > 0$ and $\varepsilon > 0$ sufficiently small so that $\alpha > 0$, where $\alpha \equiv \theta - [b^2(1+\varepsilon) + \delta - 2a]$; now

$$
\begin{aligned}
&2(y - y') \cdot [f(s, y, z) - f'(s, y', z')] \\
&\quad \le (\theta - \alpha)|\Delta y|^2 + \frac{\|\Delta z\|^2}{1+\varepsilon} + \frac{|f(s, y, z) - f'(s, y, z)|^2}{\delta}.
\end{aligned} \tag{43}
$$

We use this inequality in the right side of (42). On taking expectations, the stochastic integral term vanishes by Proposition 4.3, and we obtain

$$
\begin{aligned}
&\mathrm{E}\biggl[e^{\theta\sigma}|\Delta Y(\sigma)|^2 \\
&\qquad + \int_\sigma^{\tau\vee\tau'} e^{\theta s}\biggl(\alpha|\Delta Y(s)|^2 + \biggl(\frac{\varepsilon}{1+\varepsilon}\biggr)\|\Delta Z(s)\|^2\biggr)\,ds\biggr] \\
&\quad \le \mathrm{E}\biggl[|\exp(\theta\tau/2)\xi - \exp(\theta\tau'/2)\xi'|^2 \\
&\qquad + \delta^{-1}\int_\sigma^{\tau\vee\tau'} e^{\theta s}|f(s, Y(s), Z(s)) - f'(s, Y(s), Z(s))|^2\,ds\biggr]. \qquad \square
\end{aligned} \tag{44}
$$

Corollary 4.4.1 (Solution bounds). *Under the conditions of Proposition* 3.3, *for any* $t \ge 0$, *and any* θ *such that* $b^2 - 2a < \theta \le \rho$,

$$
\begin{aligned}
&\mathrm{E}\Bigl[e^{\theta(t\wedge\tau)}|Y(t\wedge\tau)|^2\Bigr] + \mathrm{E}\biggl[\int_{t\wedge\tau}^{\tau} e^{\theta s}\biggl(\alpha|Y(s)|^2 + \biggl(\frac{\varepsilon}{1+\varepsilon}\biggr)\|Z(s)\|^2\biggr)\,ds\biggr] \\
&\quad \le \mathrm{E}\biggl[e^{\theta\tau}|\xi|^2 + \delta^{-1}\int_{t\wedge\tau}^{\tau} e^{\theta s}|f(s, 0, 0)|^2\,ds\biggr].
\end{aligned}
$$

The proof follows from (36) (taking one of the coefficients to be 0) in the same way that the previous proof used (35).

Corollary 4.4.2 (Comparison theorem). *For the case* $k = 1$, $\xi \le \xi'$, $f \le f'$, $\tau = \tau'$, *we have* $Y(t) \le Y'(t)$ *a.s.*

Proof. Let $\Delta Y^+ \equiv 1_{\{Y - Y' > 0\}}(Y - Y')$. By the reasoning of the last proof,

$$
\begin{aligned}
&\mathrm{E}\Bigl[e^{\theta(t\wedge\tau)}|\Delta Y^+(t\wedge\tau)|^2\Bigr] + \mathrm{E}\biggl[\int_{t\wedge\tau\wedge\tau'}^{\tau\wedge\tau'} e^{\theta s}\Bigl(\theta|\Delta Y^+(s)|^2 + 1_{\{\Delta Y > 0\}}\|\Delta Z(s)\|^2\Bigr)\,ds\biggr] \\
&\quad = 2\mathrm{E}\biggl[\int_{t\wedge\tau}^{\tau} e^{\theta s}\Delta Y^+[f(s, Y(s), Z(s)) - f'(s, Y'(s), Z'(s))]\,ds\biggr].
\end{aligned}
$$

Since $f(s, y, z) \le f'(s, y, z)$, the $f(s, Y(s), Z(s))$ can be replaced by $f'(s, Y(s), Z(s))$, and an application of (43), with $f = f'$, $\alpha = 0$ and $\varepsilon = 0$, shows that this is

$$
\le \mathrm{E}\biggl[\int_{t\wedge\tau\wedge\tau'}^{\tau\wedge\tau'} e^{\theta s}\Bigl(\theta|\Delta Y^+(s)|^2 + 1_{\{\Delta Y > 0\}}\|\Delta Z(s)\|^2\Bigr)\,ds\biggr].
$$

Thus ΔY^+ is zero a.s. $\square$

5. Proofs of results for random terminal time.

5.1. *Proof of uniqueness.* Suppose (Y, Z) and $(\tilde{Y}, \tilde{Z})$ are two solutions in $M_\gamma^2(0, \tau)$. Let $\Delta Y \equiv Y - \tilde{Y}$ and $\Delta Z \equiv Z - \tilde{Z}$. Note that (44) is still valid for $\theta < \gamma$; the terminal values coincide, and $f = f'$, so taking $\delta = 0$ and $\eta \equiv b^2(1+\varepsilon) - 2a - \theta > 0$,

$$\begin{aligned} &\mathrm{E}\left[e^{\theta(t\wedge\tau)}|\Delta Y(t\wedge\tau)|^2 + \int_{t\wedge\tau}^{\tau} e^{\theta s}\left(\frac{\varepsilon}{1+\varepsilon}\right)\|\Delta Z(s)\|^2\,ds\right] \\ &\qquad \le \mathrm{E}\left[\eta\int_{t\wedge\tau}^{\tau} e^{\theta s}|\Delta Y(s)|^2\,ds\right]. \end{aligned} \tag{45}$$

Taking $\varepsilon = 0$ gives $\eta = \gamma - \theta$, and

$$\mathrm{E}\left[e^{(\gamma-\eta)(t\wedge\tau)}|\Delta Y(t\wedge\tau)|^2\right] \le \eta\mathrm{E}\left[\int_{t\wedge\tau}^{\tau} e^{(\gamma-\eta)s}|\Delta Y(s)|^2\,ds\right].$$

Now let $\eta \to 0$ and use dominated convergence to obtain

$$\mathrm{E}\left[e^{\gamma(t\wedge\tau)}|\Delta Y(t\wedge\tau)|^2\right] = 0$$

as desired. Since t was arbitrary, this proves $\Delta Y = 0$. Taking $\varepsilon = 1$ in (45) gives

$$(1/2)\mathrm{E}\left[\int_0^{\tau} e^{\theta s}\|\Delta Z(s)\|^2\,ds\right] \le c\mathrm{E}\left[\int_0^{\tau} e^{\theta s}|\Delta Y(s)|^2\,ds\right] = 0,$$

showing that $\Delta Z = 0$ in the appropriate sense. □

5.2. *Proof of existence.* *Step* 1. Let $\lambda \equiv \gamma/2$, for γ as in (24). Since $e^{\lambda\tau}\xi$ is in L^2 by (25), Theorem 2.2 supplies solutions $(\hat{Y}_n, \hat{Z}_n)$ in $M_0^2(0, n; R^k \times L(R^d; R^k))$ of the backward SDE on $0 \le t \le n$:

$$\begin{aligned} \hat{Y}_n(t) &= \mathrm{E}\left[e^{\lambda\tau}\xi|\Im_n\right] + \int_{t\wedge\tau}^{n\wedge\tau}\left[e^{\lambda s}f\left(s, e^{-\lambda s}\hat{Y}_n, e^{-\lambda s}\hat{Z}_n\right) - \lambda\hat{Y}_n\right]ds \\ &\qquad - \int_{t\wedge\tau}^{n\wedge\tau}\hat{Z}_n\,dW, \end{aligned} \tag{46}$$

where $\hat{Y}_n(s)$ is abbreviated to $\hat{Y}_n$ in the integrands, and so on. Extend these processes to the whole time axis by defining, with reference to (30) and (31),

$$\hat{Y}_n(t) \equiv \zeta(t) = \mathrm{E}\left[e^{\lambda\tau}\xi|\Im_t\right], \qquad t > n, \tag{47}$$

$$\hat{Z}_n(t) \equiv \eta(t), \qquad t > n. \tag{48}$$

Define for all $t \ge 0$,

$$Y_n(t) \equiv e^{-\lambda t}\hat{Y}_n(t), \qquad Z_n(t) \equiv e^{-\lambda t}\hat{Z}_n(t). \tag{49}$$

Since $d\zeta(t) = \eta(t)\,dW(t)$, we see by Itô's formula that

(50)
$$\begin{aligned} dY_n(s) &= Z_n(s)\,dW(s) - 1_{\{s\le\tau\}} f(s, Y_n(s), Z_n(s))\,ds, \qquad 0 \le s \le n,\\ dY_n(s) &= Z_n(s)\,dW(s) - 1_{\{s\le\tau\}}\lambda Y_n(s)\,ds, \qquad s > n. \end{aligned}$$

In other words

$$Y_n(t) = \xi + \int_{t\wedge\tau}^{\tau} f_n(s, Y_n(s), Z_n(s))\,ds - \int_{t\wedge\tau}^{\tau} Z_n(s)\,dW(s), \qquad 0 \le t < \infty,$$

where

(51)
$$f_n(s, y, z) \equiv 1_{\{s\le n\}} f(s, y, z) + \lambda 1_{\{s>n\}} y.$$

Step 2. Fix $m > n$, and let $\Delta Y \equiv Y_m - Y_n$, $\Delta Z \equiv Z_m - Z_n$, both of which are zero when $t > m$. We now apply (44), taking $\delta > 0$ and $\varepsilon > 0$ sufficiently small so that $\phi > 0$, where $\phi \equiv \rho - [b^2(1+\varepsilon) + \delta - 2a]$; taking $t \le m$, and noting that both solutions coincide at the terminal time $m\wedge\tau$, we have

$$\begin{aligned} &\mathrm{E}\left[e^{\rho(t\wedge\tau)}|\Delta Y(t\wedge\tau)|^2 + \int_{t\wedge\tau}^{m\wedge\tau} e^{\rho s}\left(\phi|\Delta Y(s)|^2 + \left(\frac{\varepsilon}{1+\varepsilon}\right)\|\Delta Z(s)\|^2\right)ds\right]\\ &\quad\le \mathrm{E}\left[\delta^{-1}\int_{t\wedge\tau}^{m\wedge\tau} e^{\rho s}|f_m(s, Y_n(s), Z_n(s)) - f_n(s, Y_n(s), Z_n(s))|^2\,ds\right]\\ &\quad\le \mathrm{E}\left[\delta^{-1}\int_{n\wedge\tau}^{m\wedge\tau} e^{\rho s}|f(s, Y_n(s), Z_n(s)) - \lambda Y_n(s)|^2\,ds\right] \end{aligned}$$

(52)
$$\quad\le c\mathrm{E}\left[\int_{n\wedge\tau}^{m\wedge\tau} e^{\rho s}\left\{\kappa' + |Y_n(s)|^2 + \|Z_n(s)\|^2 + |f(s,0,0)|^2\right\}ds\right],$$

using (51) and (22). The assumption (25) ensures that

(53)
$$\lim_{n,m\to\infty} \mathrm{E}\left[\int_{n\wedge\tau}^{m\wedge\tau} e^{\rho s}\left\{\kappa' + |f(s,0,0)|^2\right\}ds\right] = 0.$$

As for the other summands in (52), take $\theta = \rho - 2\lambda > 0$, and observe that, in the notation of (47), (48) and (49),

(54)
$$\begin{aligned} &\mathrm{E}\left[\int_{n\wedge\tau}^{m\wedge\tau} e^{\rho s}\left\{|Y_n(s)|^2 + \theta\|Z_n(s)\|^2\right\}ds\right]\\ &\qquad = \mathrm{E}\left[\int_{n\wedge\tau}^{m\wedge\tau} e^{\theta s}\left\{|\zeta(s)|^2 + \theta\|\eta(s)\|^2\right\}ds\right]. \end{aligned}$$

However, by Lemma 4.1.1 and assumption (25),

(55)
$$\mathrm{E}\left[\int_0^{\tau} e^{\theta s}\left\{|\zeta(s)|^2 + \theta\|\eta(s)\|^2\right\}ds\right] = \mathrm{E}\left[e^{\rho\tau/2}|\xi|^2\right] - \left|\mathrm{E}\left[e^{\lambda\tau}\xi\right]\right|^2 < \infty.$$

It follows from (52)–(55) that

$$\begin{aligned}&\mathrm{E}\bigg[e^{\rho(t\wedge\tau)}|\Delta Y(t\wedge\tau)|^2\\&\qquad+\int_{t\wedge\tau}^{m\wedge\tau}e^{\rho s}\bigg(\phi|\Delta Y(s)|^2+\bigg(\frac{\varepsilon}{1+\varepsilon}\bigg)\|\Delta Z(s)\|^2\bigg)\,ds\bigg]\to 0\end{aligned}\tag{56}$$

as $n, m \to \infty$ with $n < m$. Thus $\{(Y_n, Z_n)\}$ is a Cauchy sequence in the Hilbert space $M_\rho^2(0,\tau; R^k \times L(R^d; R^k))$, converging to some $(Y, Z) \in M_\rho^2(0,\tau; R^k \times L(R^d; R^k))$. Moreover $\Delta Y(t\wedge\tau) = \Delta Y(t)$, and (56) implies that, for each t,

$$e^{-|\rho|t}\mathrm{E}\Big[|\Delta Y(t)|^2\Big] \le \mathrm{E}\Big[e^{\rho(t\wedge\tau)}|\Delta Y(t\wedge\tau)|^2\Big] \to 0 \quad \text{as } n, m \to \infty$$

and so for every t, $\{Y_n(t),\ n = 1, 2, \dots\}$ has a limit in L^2, and we may assume

$$Y(t) = \lim_{n\to\infty} Y_n(t) \quad \text{in } L^2 \text{ for all } t. \tag{57}$$

Step 3. It remains to check that (Y, Z) satisfies (2). For any $\alpha \in R$, and $t \ge 0$, we have that for $n \ge t$,

$$\begin{aligned}e^{\alpha(t\wedge\tau)}Y_n(t\wedge\tau) = e^{\alpha\tau}\xi &+ \int_{t\wedge\tau}^{\tau} e^{\alpha s}[f(s, Y_n, Z_n) - \alpha Y_n]\,ds - \int_{t\wedge\tau}^{\tau} e^{\alpha s} Z_n\,dW\\&+\int_{n\wedge\tau}^{\tau} e^{\alpha s}[\lambda Y_n - f(s, Y_n, Z_n)]\,ds.\end{aligned}$$

We choose $\alpha < 0 \wedge \rho/2 \wedge \rho$, and take $\delta \equiv \rho - 2\alpha > 0$, with the result that

$$\begin{aligned}&\mu \equiv 1_{\{0\le t\le\tau\}}e^{\alpha t}\,dt \times dP \text{ is a finite measure on } [0,\infty)\times\Omega;\\&f(\cdot, 0, 0) \in M_\theta^2(0,\tau; R^k) \text{ for } \theta = 2\alpha+\delta \text{ and for } \theta = \alpha;\end{aligned}\tag{58}$$

$$(Y_n, Z_n) \to (Y, Z) \text{ in } M_\theta^2\big(0,\tau; R^k \times L(R^d; R^k)\big) \quad \text{for } \theta = 2\alpha+\delta \text{ and for } \theta = \alpha. \tag{59}$$

We shall combine (59) with Hölder's inequality to estimate $\int|Y_n - Y|\,d\mu$:

$$\begin{aligned}&\mathrm{E}\bigg[\int_0^\tau e^{\alpha s}|Y_n(s) - Y(s)|\,ds\bigg]\\&\quad\le \mathrm{E}\bigg[\bigg\{\int_0^\tau e^{(2\alpha+\delta)s}|Y_n(s)-Y(s)|^2\,ds\bigg\}^{1/2}\bigg\{\int_0^\tau e^{-\delta s}\,ds\bigg\}^{1/2}\bigg]\\&\quad\le \delta^{-1/2}\|Y_n - Y\|_{2\alpha+\delta}.\end{aligned}\tag{60}$$

The method used in the last calculation shows that

$$\int_{t\wedge\tau}^{\tau} e^{\alpha s}Y_n(s)\,ds \to \int_{t\wedge\tau}^{\tau} e^{\alpha s}Y(s)\,ds \quad \text{in } L^1(P) \text{ for all } t;$$

$$\int_{t\wedge\tau}^{\tau} e^{\alpha s}Z_n(s)\,dW \to \int_{t\wedge\tau}^{\tau} e^{\alpha s}Z(s)\,dW \quad \text{in } L^2(P) \text{ for all } t;$$

$$\int_{n\wedge\tau}^{\tau} e^{\alpha s}[\lambda Y_n(s) - f(s, Y_n, Z_n)]\,ds \to 0 \quad \text{in } L^1(P); \tag{61}$$

$$(62)\qquad \int_{t\wedge\tau}^{\tau} e^{\alpha s}\left| f(s, Y_n, Z_n) - f(s, Y_n, Z)\right| ds \to 0 \quad \text{in } L^1(P) \text{ for all } t.$$

Estimate (61) uses (22), (58) and (59); estimate (62) uses (5) and (59). To complete the proof, it suffices, in view of (62), to check that

$$(63)\qquad \int_{t\wedge\tau}^{\tau} e^{\alpha s}\left| f(s, Y_n, Z) - f(s, Y, Z)\right| ds \to 0 \quad \text{in } L^1(P) \text{ for all } t$$

or, equivalently, that $X_n \to 0$ in $L^1(\mu)$, where $X_n \equiv |f(\cdot, Y_n, Z) - f(\cdot, Y, Z)|$. For positive integers m and N, define

$$\Delta_N^m \equiv \left\{(y, \bar{y}) \in R^k \times R^k \colon |\bar{y}| \le N, |y - \bar{y}| \le 1/m\right\}.$$

Fix $\varepsilon > 0$. The continuity of f in y [condition (7)] implies that, for each ω, t, and N, there exists an integer $h(\omega, t, N)$ such that

$$(64)\qquad \begin{aligned} &m \ge h(\omega, t, N) \text{ and } (y, \bar{y}) \in \Delta_N^m \\ &\qquad \Rightarrow \left| f(\omega, t, y, Z(t)) - f(\omega, t, \bar{y}, Z(t))\right| \le \varepsilon. \end{aligned}$$

Next observe that the $\{X_n\}$ are uniformly μ-integrable, since μ is finite and

$$(65)\qquad \int X_n^2 \, d\mu = \mathrm{E}\left[\int_0^{\tau} e^{\alpha s}\left| f(s, Y_n, Z) - f(s, Y, Z)\right|^2 ds\right] \le C,$$

using (22), (58) and (59). Now by Fubini's theorem, for any $r > 0$,

$$\int |X_n| \, d\mu \le \int_0^{\infty} e^{\alpha t}\mathrm{E}\left[X_n(t) 1_{\{X_n(t) \le r\}}\right] dt + \frac{\int X_n^2 \, d\mu}{r}.$$

By (65), the second term on the right can be made arbitrarily small by choosing r large enough, and the first term goes to zero by dominated convergence (using the fact that $\alpha < 0$) provided we can prove that, for each fixed t, $X_n(t) \to 0$ in probability as $n \to \infty$. Now

$$\begin{aligned} &P\left(|X_n(t)| > \varepsilon\right) \\ &\qquad \le P\left(\left| f(t, Y_n(t), Z(t)) - f(t, Y(t), Z(t))\right| > \varepsilon, (Y_n, Y) \in \Delta_N^m\right) \\ &\qquad\quad + P\left(|Y(t)| > N\right) + P\left(|Y_n(t) - Y(t)| > 1/m\right) \\ &\qquad \le P(m < h(\omega, t, N)) + N^{-2}\mathrm{E}\left[|Y(t)|^2\right] + m^2\mathrm{E}\left[|Y_n(t) - Y(t)|^2\right]. \end{aligned}$$

Choosing N large enough, m large enough and n large enough, in that order, makes this arbitrarily small, using (64) and (57). Now we have proved that

$$e^{\alpha(t\wedge\tau)} Y(t \wedge \tau) = e^{\alpha\tau}\xi + \int_{t\wedge\tau}^{\tau} e^{\alpha s}\left[f(s, Y, Z) - \alpha Y\right] ds - \int_{t\wedge\tau}^{\tau} e^{\alpha s} Z \, dW$$

and so, by Itô's formula, (Y, Z) satisfies (2). □

6. Application to semilinear PDE's. For each $x \in R^d$, we may construct a Markov diffusion process with generator

$$L \equiv \beta \cdot \nabla + \left(\frac{1}{2}\right) \sum_{i,j=1}^{d} (\sigma\sigma^*)_{ij} \frac{\partial^2}{\partial x_i \, \partial x_j} \tag{66}$$

by solving the SDE

$$X_x(t) = x + \int_0^t \beta(X_x(s))\, ds + \int_0^t \sigma(X_x(s))\, dW(s), \qquad t \geq 0, \tag{67}$$

where the coefficients $\beta\colon R^d \to R^d$ and $\sigma\colon R^d \to L(R^d; R^d)$ satisfy $\beta \in C_b^1(U)$ and $\sigma \in C_b^2(U)$, for some open set U containing a bounded set D of the form

$$D \equiv \{x\colon \phi(x) > 0\},$$

for some $\phi \in C^2(R^d)$. We require that $|\nabla\phi(x)| \neq 0$ for all $x \notin \partial D \subseteq \{x\colon \phi(x) = 0\}$. These conditions ensure existence and uniqueness of a strong solution to (67) at least up to the stopping times

$$\tau_x \equiv \inf\{t \geq 0\colon X_x(t) \notin \overline{D}\}.$$

We assume that $P(\tau_x < \infty) = 1$ for all $x \in \overline{D}$, and that the set

$$\Gamma \equiv \{x \in \partial D\colon P(\tau_x > 0) = 0\} \tag{68}$$

is closed. Let $g \in C(R^d)$ (hence bounded on $\overline{D}$), and let $f \in C(R^d \times R \times L(R^d; R))$ be a function whose restriction to $\overline{D}$ satisfies

$$|f(x, y, z)| \leq |f(x, 0, z)| + \kappa(|y| + 1); \tag{69}$$

$$(y - \tilde{y})[f(x, y, z) - f(x, \tilde{y}, z)] \leq -a|y - \tilde{y}|^2; \tag{70}$$

$$|f(x, y, z) - f(x, y, \bar{z})| \leq b\|z - \bar{z}\|. \tag{71}$$

Also assume that, for some $\rho > b^2 - 2a$, we have

$$\sup_{x \in \overline{D}} \mathrm{E}[\exp(\rho\tau_x)] < \infty. \tag{72}$$

REMARKS. In view of the boundedness of g and $f(\cdot, 0, 0)$ on $\overline{D}$, condition (25) simplifies to (72). If for instance $D \subseteq \{(\sigma\sigma^*)_{11}(x) \geq \lambda > 0\}$, then there exists a ρ such that (72) holds. For more insight into the degenerate case, see Stroock and Varadhan (1972).

We now consider, for each $x \in \overline{D}$ the one-dimensional BSDE

$$\begin{aligned} Y_x(t) = g(X_x(\tau_x)) &+ \int_{t\wedge\tau_x}^{\tau_x} f(X_x(s), Y_x(s), Z_x(s))\, ds \\ &- \int_{t\wedge\tau_x}^{\tau_x} Z_x(s)\, dW(s), \end{aligned} \tag{73}$$

which has a unique solution in $M_\gamma^2(0, \tau; R \times L(R^d; R))$ by Theorem 3.4 and condition (72), where $\gamma \equiv b^2 - 2a$, and define

$$u(x) \equiv Y_x(0), \qquad x \in D. \tag{74}$$

LEMMA 6.1. *The function u is bounded on $\overline{D}$, and*

$$\sup_{x\in\overline{D}}\left(\mathrm{E}\left[\int_0^{\tau_x} e^{\rho s}\|Z_x(s)\|^2\,ds\right]\right) < \infty.$$

The proof is immediate from (72), (74) and Corollary 4.4.1 (taking t to be zero).

LEMMA 6.2. $Y_x(t) = u(X_x(t))$, $0 \le t \le \tau_x$ *a.s.; hence the processes* $\{Y_x(t)\}$ *are uniformly bounded.*

PROOF. The first result is a consequence of

$$\left(Y_{X_x(t\wedge\tau_x)}(s\wedge\tau_x), Z_{X_x(t\wedge\tau_x)}(s\wedge\tau_x)\right) = \left(Y_x((t+s)\wedge\tau_x), Z_x((t+s)\wedge\tau_x)\right)$$

for $s, t \ge 0$, which follows from uniqueness of the solution to the BSDE (73) on the time interval $[t\wedge\tau_x, \tau_x]$. The second now follows from Lemma 6.1. □

PROPOSITION 6.3. *The function u is continuous on $\overline{D}$.*

PROOF. The proof will be split into several steps, the first two consisting of the proof of the a.s. continuity of

$$x \to \tau_x \tag{75}$$

as $x' \to x$, which also proves the a.s. continuity of

$$x \to \left(\tau_x, X_x(\tau_x)\right), \tag{76}$$

using well-known spatial continuity properties of stochastic flows.

Step 1. First we shall prove that for any sequence $x(n) \to x$ in $\overline{D}$,

$$\limsup_{n\to\infty} \tau_{x(n)} \le \tau_x \quad \text{a.s.} \tag{77}$$

Suppose (77) is false. Then

$$P\left(\tau_x < \limsup_{n\to\infty} \tau_{x(n)}\right) > 0. \tag{78}$$

For each $\varepsilon > 0$, let

$$\tau_x^\varepsilon \equiv \inf\left\{t \ge 0: d\left(X_x(t), D\right) \ge \varepsilon\right\}.$$

If (78) holds, then there exist $\varepsilon > 0$ and T such that

$$P\left(\tau_x^\varepsilon < \limsup_{n\to\infty} \tau_{x(n)} \le T\right) > 0.$$

But since $X_{x(n)}(\cdot) \to X_x(\cdot)$ uniformly on $[0, T]$ a.s.,

$$P\left(\limsup_{n\to\infty} \tau_{x(n)}^{\varepsilon/2} \le \tau_x^\varepsilon < \limsup_{n\to\infty} \tau_{x(n)} \le T\right) > 0,$$

that is, for some n, $X_{x(n)}$ exits the $\varepsilon/2$-neighborhood of D before exiting D, on a set of positive probability; this is impossible. Hence (77) must be true.

Step 2. Secondly we shall prove that

$$\liminf_{n\to\infty} \tau_{x(n)} \ge \tau_x \quad \text{a.s.} \tag{79}$$

For this we need the assumption that Γ [see (68)] is closed—the result would clearly not be true otherwise. Let $\Omega_M \equiv \{\omega \in \Omega: \tau(x) \leq M\}$; since $\bigcup_{M>0} \Omega_M = \Omega$, it suffices to prove that (79) holds on each Ω_M, a.s.

From the result of Step 1, for almost all $\omega \in \Omega_M$ there exists $n(\omega)$ such that $n \geq n(\omega)$ implies $\tau_{x(n)}(\omega) \leq M + 1$. Since $X_{x(n)}(\cdot) \to X_x(\cdot)$ uniformly on $[0, M+1]$ a.s., on Ω_M $X_x(\cdot)$ reaches

$$\overline{\left\{X_{x(n)}(\tau_{x(n)}): n \in N\right\}} \subseteq \overline{\Gamma} = \Gamma$$

on the random interval $[0, \lim_{n\to\infty} \inf \tau_{x(n)}]$ a.s. But $\tau_x \leq \inf\{t: X_x(t) \in \Gamma\}$ a.s., hence

$$\tau_x \leq \liminf_{n\to\infty} \tau_{x(n)} \quad \text{a.s. on } \Omega_M.$$

Step 3. Let us fix x, and some $\theta \in (b^2 - 2a, \rho)$. According to Proposition 4.4, there exists a positive number δ such that

$$\begin{aligned} |u(x) - u(x')|^2 &\leq \mathrm{E}\left|\exp(\theta\tau_x/2)\, g(X_x(\tau_x)) - \exp(\theta\tau_{x'}/2)\, g(X_{x'}(\tau_{x'}))\right|^2 \\ &\quad + \delta^{-1}\mathrm{E}\left[\int_0^{\tau_x \vee \tau_{x'}} e^{\theta s}\left| f(X_x(s), Y_x(s), Z_x(s)) \right.\right. \\ &\qquad\qquad \left.\left. - f(X_{x'}(s), Y_x(s), Z_x(s))\right|^2 ds\right]. \end{aligned} \tag{80}$$

We shall first show $U_{x,x'} \equiv |\exp(\theta\tau_x'/2) g(X_x(\tau_x)) - \exp(\theta\tau_{x'}/2) g(X_{x'}(\tau_{x'}))|^2$ converges to 0 in $L^1(P)$ as $x' \to x$. It follows from the continuity of g and from (76) that $U_{x,x'} \to 0$ a.s. as $x' \to x$. By (72), $\sup_{x\in\overline{D}} \mathrm{E}[\exp(\rho\tau_x)] < \infty$, and therefore

$$\left\{\exp(\theta\tau_x),\ x \in \overline{D}\right\}$$

is uniformly integrable. Since $\{g(X_{x'}(\tau_{x'})),\ x' \in \overline{D}\}$ is bounded, an elementary calculation shows that the random variables $\{U_{x,x'},\ x' \in \overline{D}\}$ are uniformly integrable over $x' \in \overline{D}$, and hence $U_{x,x'} \to 0$ in $L^1(P)$ as $x' \to x$. This takes care of the first term on the right-hand side of (80).

Step 4. Let $V_{x,x'}(s) \equiv |f(X_x(s), Y_x(s), Z_x(s)) - f(X_{x'}(s), Y_x(s), Z_x(s))|^2$. We are going to show that

$$\mathrm{E}\left[\int_0^{\tau_x \wedge \tau_{x'}} e^{\theta s} V_{x,x'}(s)\, ds\right] \tag{81}$$

tends to zero as $x' \to x$. Lemma 6.1 shows that

$$\mathrm{E}\left[\int_0^{\tau_x} e^{\theta s} \|Z_x\|^2 1_{\{Z_x \notin K_n\}}\, ds\right] \to 0 \quad \text{as } n \to \infty \tag{82}$$

for compact sets $\{K_n\}$ increasing to $L(R^d; R)$ and let

$$C_n \equiv 2\sup\left\{|f(x, y, z)|^2: x \in \overline{D},\ y \in u(\overline{D}),\ z \in K_n\right\} < \infty. \tag{83}$$

Finiteness of C_n comes from the continuity of f in all variables, and the fact that u is a bounded function, proved in Lemma 6.1. By (69), (71) and Lemma

6.2, there is a constant $C > 0$ such that, omitting the time variable,

$$V_{x, x'} \le C\left(1 + \|Z_x\|^2\right). \tag{84}$$

Moreover, the continuity of f in the first argument, and compactness, show that for any $\varepsilon > 0$ there exists $\nu \equiv \nu(n, \varepsilon) > 0$ such that for $x, x' \in \overline{D}$, $y \in u(\overline{D})$, $z \in K_n$,

$$|x - x'| \le \nu \Rightarrow |f(x, y, z) - f(x', y, z)| \le \varepsilon.$$

We use the identity

$$V_{x, x'} = V_{x, x'} 1_{\{Z_x \in K_n\}}\left\{1_{\{|X_x - X_{x'}| \le \nu\}} + 1_{\{|X_x - X_{x'}| > \nu\}}\right\} + V_{x, x'} 1_{\{Z_x \notin K_n\}}$$

to deduce that the expression (81) is bounded above by

$$\mathrm{E}\left[\int_0^{\tau_x \wedge \tau_{x'}} e^{\theta s}\left\{\varepsilon + C_n 1_{\{|X_x - X_{x'}| > \nu\}} + C\left(1 + \|Z_x\|^2\right) 1_{\{Z_x \notin K_n\}}\right\} ds\right]. \tag{85}$$

This leads to a sum of three expectations; the third can be made arbitrarily small using (82) (this involves choice of n); the first can be made arbitrarily small by choice of ε, using the finiteness of $\mathrm{E}[\exp(\rho \tau_x)]$ [this involves a choice of $\nu(n, \varepsilon)$]; the second can be made arbitrarily small using well-known spatial continuity properties of stochastic flows, which imply that

$$P\left(\sup_s |X_x(s) - X_{x'}(s)| > \nu\right) \to 0 \quad \text{as } |x - x'| \to 0.$$

Thus as $|x - x'| \to 0$, the expression (81) tends to zero, as claimed.

Step 5. Finally we must check that

$$\mathrm{E}\left[\int_{\tau_x \wedge \tau_{x'}}^{\tau_x \vee \tau_{x'}} e^{\theta s} V_{x, x'}(s)\, ds\right] \to 0 \quad \text{as } |x - x'| \to 0. \tag{86}$$

Using (84) and the fact that $Z_x(s) = 0$ for $s > \tau_x$, we see that

$$\mathrm{E}\left[\int_{\tau_x \wedge \tau_{x'}}^{\tau_x \vee \tau_{x'}} e^{\theta s} V_{x, x'}(s)\, ds\right] \le C\,\mathrm{E}\left[\int_{\tau_x \wedge \tau_{x'}}^{\tau_x} e^{\theta s}\left(1 + \|Z_x(s)\|^2\right) ds\right]$$

and now (86) follows from (75) and Lemma 6.1. □

6.4. *Viscosity solutions of PDE*. For L as in (66) and f and g as in (73), we consider the following elliptic PDE:

$$Lu(x) + f(x, u(x), (\nabla u)\sigma(x)) = 0, \qquad x \in D;\ u|_{\delta D} = g. \tag{87}$$

Let us define what we mean by a viscosity solution of the equation (87). [For uniqueness results for viscosity solutions of such equations, see Barles and Murat (1995) and Barles and Burdeau (1995).]

DEFINITION 6.4.1. A continuous function u: $\overline{D} \to R$ is called a viscosity subsolution of (87) if, for any $\varphi \in C^2(\overline{D})$, and any local maximum point x of

$u - \varphi$, it is true that

$$-L\varphi(x) - f(x, u(x), (\nabla\varphi)\sigma(x)) \le 0 \quad \text{if } x \in D;$$

$$\min\{-L\varphi(x) - f(x, u(x), (\nabla\varphi)\sigma(x)), u(x) - g(x)\} \le 0 \quad \text{if } x \in \partial D.$$

The function u is called a viscosity supersolution of (87) if, for any $\varphi \in C^2(\overline{D})$ and any local minimum point x of $u - \varphi$, it is true that

$$-L\varphi(x) - f(x, u(x), (\nabla\varphi)\sigma(x)) \ge 0 \quad \text{if } x \in D;$$

$$\max\{-L\varphi(x) - f(x, u(x), (\nabla\varphi)\sigma(x)), u(x) - g(x)\} \ge 0 \quad \text{if } x \in \partial D.$$

A continuous function $u\colon \overline{D} \to R$ is said to be a viscosity solution of (87) if it is both a viscosity subsolution and a viscosity supersolution. Now we shall prove the main result of this section.

THEOREM 6.5. *The function $u\colon \overline{D} \to R$ given by* (74) *is a bounded, continuous viscosity solution of the elliptic equation* (87).

PROOF. We prove only that u is a viscosity subsolution, the proof of the other statement being similar. The boundedness comes from Lemma 6.1 and the continuity from Proposition 6.3.

We consider first the case where $u - \varphi$ achieves a local maximum (which we assume without loss of generality to be a global maximum) at $x \in D \cup (\partial D \cap \Gamma^c)$. We also assume that φ and its derivatives up to second order have at most polynomial growth at infinity. Since $x \in D \cup (\partial D \cap \Gamma^c)$, $\tau_x > 0$ a.s. We can and will assume that $u(x) = \varphi(x)$. Hence $u(\bar{x}) \le \varphi(\bar{x})$, $\bar{x} \in \overline{D}$.

For $0 \le s \le t$, Lemma 6.2 shows that

$$\begin{aligned} Y_x(s) = u(X_x(t \wedge \tau_x)) &+ \int_{s\wedge\tau_x}^{t\wedge\tau_x} f(X_x(r), Y_x(r), Z_x(r))\,dr \\ &- \int_{s\wedge\tau_x}^{t\wedge\tau_x} Z_x(r)\,dW(r). \end{aligned} \tag{88}$$

Let $(\overline{Y}_x, \overline{Z}_x)$ be the unique solution of the following BSDE:

$$\begin{aligned} \overline{Y}_x(s) = \varphi(X_x(t \wedge \tau_x)) &+ \int_{s\wedge\tau_x}^{t\wedge\tau_x} f(X_x(r), Y_x(r), \overline{Z}_x(r))\,dr \\ &- \int_{s\wedge\tau_x}^{t\wedge\tau_x} \overline{Z}_x(r)\,dW(r). \end{aligned} \tag{89}$$

The use of $Y_x(r)$ rather than $\overline{Y}_x(r)$ as the second argument of f is intentional. Note that, from Itô's formula,

$$\begin{aligned} \varphi(X_x(s)) = \varphi(X_x(t \wedge \tau_x)) &- \int_{s\wedge\tau_x}^{t\wedge\tau_x} L\varphi(X_x(r))\,dr \\ &- \int_{s\wedge\tau_x}^{t\wedge\tau_x} \nabla\varphi\sigma(X_x(r))\,dW(r). \end{aligned}$$

Define $\hat{Y}_x(s) \equiv \overline{Y}_x(s) - \varphi(X_x(s))$, $\hat{Z}_x(s) \equiv \overline{Z}_x(s) - \nabla\varphi\sigma(X_x(s))$. We have

$$\hat{Y}_x(s) = \int_{s\wedge\tau_x}^{t\wedge\tau_x} \Big[L\varphi(X_x) + f\big(X_x, Y_x, \hat{Z}_x + \nabla\varphi\sigma(X_x)\big)\Big]\, dr \\ - \int_{s\wedge\tau_x}^{t\wedge\tau_x} \hat{Z}_x\, dW, \tag{90}$$

omitting r in the integrands. Since $u(\bar{x}) \le \varphi(\bar{x})$, $\bar{x} \in \overline{D}$, we may apply Theorem 4.4.2 to (88) and (89) (think of the coefficient f as a random function of the z-argument only) to deduce that $u(x) = Y_x(0) \le \overline{Y}_x(0)$, and since $u(x) = \varphi(x)$, we see that $\hat{Y}_x(0) \ge 0$. Now (90) gives

$$\frac{1}{t}\mathrm{E}\bigg[\int_0^{t\wedge\tau_x} \Big[L\varphi(X_x) + f\big(X_x, Y_x, \hat{Z}_x + \nabla\varphi\sigma(X_x)\big)\Big]\, ds\bigg] = \frac{\hat{Y}_x(0)}{t} \ge 0. \tag{91}$$

Let us introduce the following lemma.

LEMMA 6.6.

$$\frac{1}{t}\mathrm{E}\bigg[\int_0^{t\wedge\tau_x} \big|\hat{Z}_x(s)\big|\, ds\bigg] \to 0 \quad \text{as } t \to 0.$$

From (91), Lemma 6.6, and (71) (Lipschitz continuity of f in z), we deduce

$$\frac{1}{t}\mathrm{E}\bigg[\int_0^{t\wedge\tau_x} \big[L\varphi(X_x) + f(X_x, Y_x, \nabla\varphi\sigma(X_x))\big]\, ds\bigg] + \frac{C}{t}\mathrm{E}\bigg[\int_0^{t\wedge\tau_x} \big|\hat{Z}_x(s)\big|\, ds\bigg] \ge 0.$$

We can take the limit as $t \to 0$ in the above inequality, to obtain, by dominated convergence,

$$L\varphi(x) + f(x, u(x), (\nabla\varphi)\sigma(x)) \ge 0. \tag{92}$$

Suppose now that $u - \varphi$ achieves a maximum at a point $x \in \Gamma$. Then $u(x) = g(x)$, so the condition for u to be a viscosity subsolution is satisfied. □

PROOF OF LEMMA 6.6. Recall that D is bounded, $\{X_x(t),\ 0 \le t \le \tau_x\}$ is bounded, and by Lemmas 6.1 and 6.2 $\{Y_x(t),\ 0 \le t \le \tau_x\}$ is also bounded. From Itô's formula and (90), we see that for $\theta > 0$ and $s < t$ (dropping the index x for brevity),

$$\mathrm{E}\Big[e^{\theta(s\wedge\tau)}\big|\hat{Y}(s\wedge\tau)\big|^2\Big] + \mathrm{E}\bigg[\int_{s\wedge\tau}^{t\wedge\tau} e^{\theta r}\Big[\theta\big|\hat{Y}(r)\big|^2 + \big\|\hat{Z}(r)\big\|^2\Big]\, dr\bigg] \\ = 2\mathrm{E}\bigg[\int_{s\wedge\tau}^{t\wedge\tau} e^{\theta r}\hat{Y}(r)\cdot\Big[L\varphi(X(r)) \\ + f\big(X(r), Y(r), \hat{Z}(r) + \nabla\varphi\sigma(X(r))\big)\Big]\, dr\bigg]. \tag{93}$$

Using (71) and the continuity of f, we find that the last expression is less than or equal to

$$(b^2+1)\mathrm{E}\int_{s\wedge\tau}^{t\wedge\tau} e^{\theta r}\left|\hat{Y}(r)\right|^2 dr + \mathrm{E}\int_{s\wedge\tau}^{t\wedge\tau} e^{\theta r}\left[\left\|\hat{Z}(r)\right\|^2 + c_1\right] dr, \tag{94}$$

using all the boundedness properties mentioned above. Taking $\theta = b^2+1$ proves that

$$\mathrm{E}\left[e^{(b^2+1)(s\wedge\tau)}\left|\hat{Y}(s\wedge\tau)\right|^2\right] \le C(t-s). \tag{95}$$

Working from (93) again with $\theta = 0$ and $s = 0$ gives that there exist c_i such that

$$\begin{aligned}
\mathrm{E}&\left[\int_0^{t\wedge\tau}\left\|\hat{Z}(r)\right\|^2 dr\right] \\
&\le 2\mathrm{E}\left[\int_0^{t\wedge\tau}\left|\hat{Y}(r)\right|\left(c_1 + b\left\|\hat{Z}(r)\right\|\right) dr\right] \\
&\ge \left(\tfrac{1}{2}\right)\mathrm{E}\left[\int_0^{t\wedge\tau}\left\|\hat{Z}(r)\right\|^2 dr\right] + c_2\mathrm{E}\left[\int_0^t\left(\left|\hat{Y}(r)\right| + \left|\hat{Y}(r)\right|^2\right) dr\right],
\end{aligned}$$

so we have, using (95), that this is

$$\begin{aligned}
\left(\tfrac{1}{2}\right)\mathrm{E}&\left[\int_0^{t\wedge\tau}\left\|\hat{Z}(r)\right\|^2 dr\right] \\
&\le c_2\left[t^{1/2}\left(\mathrm{E}\left[\int_0^t\left|\hat{Y}(r)\right|^2 dr\right]\right)^{1/2} + \mathrm{E}\left[\int_0^t\left|\hat{Y}(r)\right|^2 dr\right]\right] \\
&\le c_3\left[t^{1/2}\left(\int_0^t (t-r)\, dr\right)^{1/2} + \int_0^t (t-r)\, dr\right] \\
&\le c_3\left[t^{3/2} + t^2\right].
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{t}\mathrm{E}\left[\int_0^{t\wedge\tau(x)}\left|\hat{Z}_x(s)\right| ds\right] &\le t^{-1/2}\left\{\mathrm{E}\left[\int_0^{t\wedge\tau}\|\hat{Z}\|^2\, dr\right]\right\}^{1/2} \\
&\le c_3\frac{\left[t^{3/2}+t^2\right]^{1/2}}{t^{1/2}} \le c_3\left(t^{1/4} + t^{1/2}\right). \qquad \square
\end{aligned}$$

Acknowledgments. The first author thanks the Université de Provence, Laboratoire d'Analyse, Probabilités et Topologie–URA 225, and the Mathematics Research Centre, University of Warwick, for their hospitality during the conduct of this research. We are grateful for the corrections supplied by the referee and by an Associate Editor.

REFERENCES

BARLES, G., BUCKDAHN, R. and PARDOUX, E. (1997). Backward stochastic differential equations and integral-partial differential equations. *Stochastics Stochastics Rep.* **60** 57–83.

BARLES, G. and BURDEAU, J. (1995). The Dirichlet problem for semilinear second-order degenerate elliptic equations and applications to stochastic exit time control problems. *Comm. Partial Differential Equations* **20** 129–178.

BARLES, G. and MURAT, F. (1995). Uniqueness and the maximum principle for quasilinear elliptic equations with quadratic growth conditions. *Arch. Rational Mech. Anal.* **133** 77–101.

CRANDALL, M., ISHII, H. and LIONS, P. L. (1992). User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc.* **27** 1–67.

DARLING, R. W. R. (1995). Constructing gamma-martingales with prescribed limit using backwards SDE. *Ann. Probab.* **23** 1234–1261.

PARDOUX, E. and PENG, S. (1990). Adapted solution of a backward stochastic differential equation. *Systems Control Lett.* **14** 55–61.

PARDOUX, E. and PENG, S. (1994). Some backward stochastic differential equations with non-Lipschitz coefficients. Univ. Provence URA 225, Preprint 94-3.

PARDOUX, E., PRADEILLES, F. and RAO, Z. (1995). Probabilistic interpretation for a system of nonlinear parabolic partial differential equations. *Ann. Inst. H. Poincaré Probab. Statist.* To appear.

PENG, S. (1991). Probabilistic interpretation for systems of quasilinear parabolic partial differential equations. *Stochastics Stochastics Rep.* **37** 61–74.

REVUZ, D. and YOR, M. (1991). *Continuous Martingales and Brownian Motion*. Springer, Berlin.

STROOCK, D. W. and VARADHAN, S. R. S. (1972). On degenerate elliptic-parabolic operators of second order and their associated diffusions. *Comm. Pure Appl. Math.* **25** 651–713.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTH FLORIDA
TAMPA, FLORIDA 33620-5700
E-MAIL: darling@math.usf.edu

LATP-CMI
UNIVERSITÉ DE PROVENCE
39, RUE F. JOLIOT-CURIE
F-13453 MARSEILLE CEDEX
France
E-MAIL: pardoux@gyptis.univ-mrs.fr