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BROWNIAN MOTION ON A RANDOM RECURSIVE SIERPINSKI GASKET[1]

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We introduce a random recursive fractal based on the Sierpinski gasket and construct a diffusion upon the fractal via a Dirichlet form. This form and its symmetrizing measure are determined by the electrical resistance of the fractal. The effective resistance provides a metric with which to discuss the properties of the fractal and the diffusion. The main result is to obtain uniform upper and lower bounds for the transition density of the Brownian motion on the fractal in terms of this metric. The bounds are not tight as there are logarithmic corrections due to the randomness in the environment, and the behavior of the shortest paths in the effective resistance metric is not well understood. The results are deduced from the study of a suitable general branching process.

1. Introduction. The study of diffusion on fractals has concentrated on fractals with spatial symmetry. The randomization used in [14] preserved this property, which was crucial to the analysis. In this work we introduce a random recursive fractal with statistical self-similarity and extend the theory of diffusion on fractals to a simple family of Sierpinski gaskets from this broader class of fractals. The fractal is based on the Sierpinski gasket as this is a simple multiply connected fractal and provides an instructive example where the behavior of random recursive fractals can be seen. A more general class of random recursive fractals based on nested fractals, [23], or affine nested fractals [8], could be constructed but provides more work in the construction of a diffusion. Recently T. Hattori [15] has considered processes on scale irregular gaskets, another class of sets which are not exactly self-similar and include the homogeneous random fractals considered in [14].

The random recursive Sierpinski gasket is a random recursive construction in the sense of [25], [12] or net fractal in the terminology of [6]. A family of sets of contraction maps is used recursively to generate the fractal. We will consider sets from the family of Sierpinski gaskets defined in [14], examples of which are shown in Figure 1. The fractal $\mathrm{SG}(\nu)$ from the family has dimension $d_f = \log(\nu(\nu+1)/2)/\log \nu$ and can be described by a $\nu(\nu+1)/2$-ary tree in which the nth generation branches of the tree correspond to the sets in the fractal at the nth level of construction.

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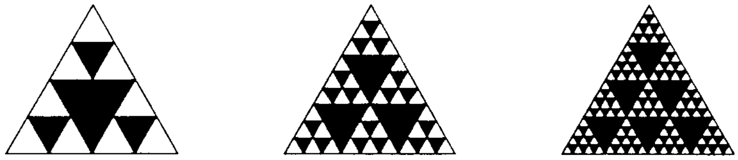

FIG. 1. *The first two stages in the construction of three fractals from the family of Sierpinski gaskets.*

The random recursive gasket can be described using a random tree. A branch is associated with each subset of the fractal of a given size and each splits into more branches, according to a simple Galton–Watson process, which describe the evolution of that component of the fractal. This tree will provide a means of discussing the geometry of the fractal. A version of the random recursive gasket formed from SG(2) and SG(3) is shown in Figure 2.

In this fractal the graph formed from the vertices and edges of the triangles after n iterations of the contractions has no well defined length scale. However, such a scale will be imposed using the electrical resistance of the fractal. The resistance of each edge in the graph approximation to the fractal is determined by the need to maintain a unit resistance across the graph of unit side length. The contraction maps are then iterated until the resistance of each edge is approximately e^{-n}. This provides a sequence of graphs G_n, convergent to the fractal, with the property that each edge has roughly the same resistance, and hence a random walker will move along any edge with roughly equal probability.

The construction of a diffusion process on the random recursive gasket will be accomplished using Dirichlet forms. The method uses the connection between reversible Markov chains and electrical networks. By our choice of resistance the graphs G_n form a compatible sequence of networks in the sense of [20]. The resistance also gives rise to a natural measure μ on the fractal which is equivalent to the Hausdorff measure in the effective resistance metric. From the associated sequence of Dirichlet forms and the corresponding Markov chains we can construct a natural diffusion process, X_t associated with the limiting Dirichlet form $(E, D(E))$ on $L^2(\mu)$ which will be called "Brownian motion" on the fractal G.

Let $(\Omega, F, \mathbb{P})$ be a probability space of infinite trees (specified in Section 2). For each tree there is a corresponding fractal, and a diffusion process can be constructed via its Dirichlet form.

THEOREM 1.1. *For each $\omega \in \Omega$ there exists a continuous symmetric strong Markov process $\{X_t\colon t \geq 0\}$ on the space $G(\omega)$.*

The properties of this process can be deduced with a further assumption, that there are only a finite number of possible gasket types. Then the results

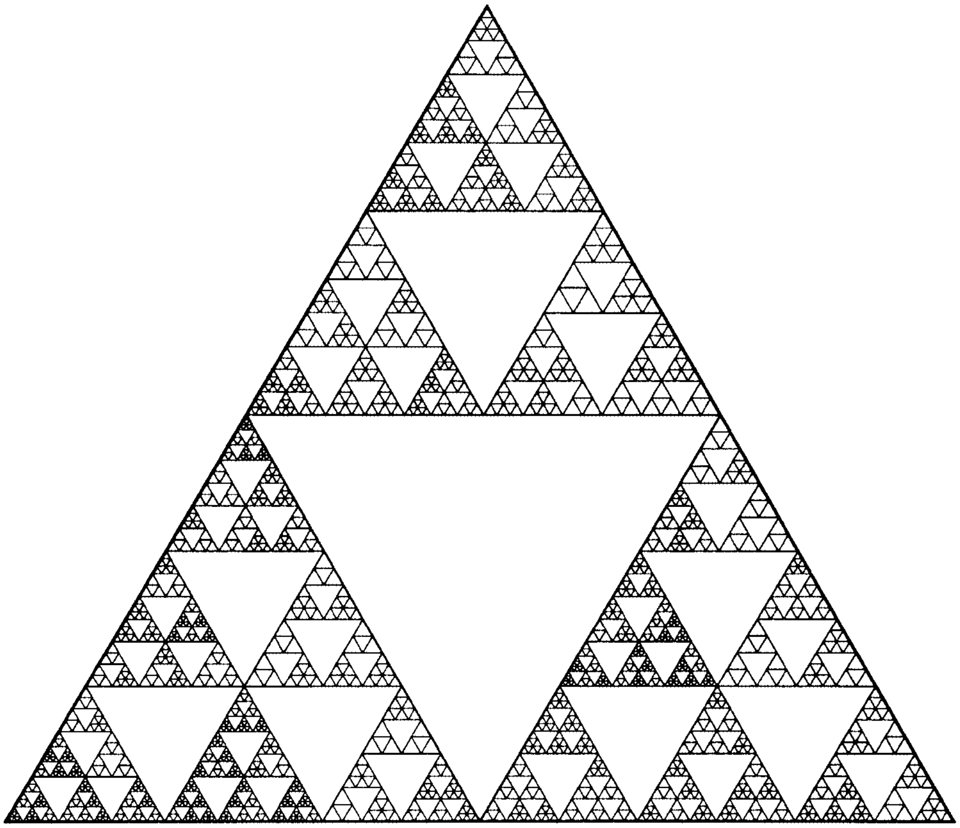

FIG. 2. *The graph formed from the first stages in the construction of a random recursive Sierpinski gasket.*

that we will state, concerning the properties of processes on random recursive gaskets, will be $\mathbb{P}$-almost sure results on the space of fractals Ω.

In order to determine the properties of the fractal we introduce a general branching process which describes the behavior of the sequence of approximating graphs. This will enable us to show that the box counting dimension is equal to the Hausdorff dimension. A feature of the geometry of the fractal is the existence of points in the fractal with widely differing ancestry, as the proportions of triangle types in each branch do not converge uniformly over the fractal. We also obtain uniform estimates on the measures of triangles in G_n.

The effective resistance between two points $x, y \in G$ can be defined in terms of the Dirichlet form as

$$d_r(x, y) = \left(\inf\{E(f, f): f \in D(E),\, f(x) = 0,\, f(y) = 1\}\right)^{-1}.$$

In [20] this was shown to be a metric, and it will be the appropriate metric with which to discuss the analytic properties of the fractal.

Recent work of [18] shows that the similarity dimension of the fractal provides a useful intrinsic notion of dimension. This is the Hausdorff dimension of the set in the effective resistance metric. The similarity dimension α is defined as the unique solution to the following equation:

$$\alpha = \left\{ s : \mathbb{E} \sum_{i=1}^{N} r_i^s = 1 \right\},$$

where r_i is the resistance of the ith component, of which there are N. For the case of the Sierpinski gasket based upon the two generators SG(2) and SG(3), chosen with probability p and q, respectively, the similarity dimension is the number s which satisfies

$$3p\left(\tfrac{3}{5}\right)^s + 6q\left(\tfrac{7}{15}\right)^s = 1.$$

From work in [19], where it is shown for P.C.F. self-similar sets, the similarity dimension can be written in terms of the spectral dimension as

$$\alpha = \frac{d_s}{2 - d_s}.$$

In [20] this relationship is conjectured to hold in much greater generality. We will denote by α_ν the corresponding parameter associated with SG(ν). The α_ν are seen to be increasing in ν, as for this family of gaskets the spectral dimension increases to 2 as $\nu \to \infty$ [27]. We write K for the maximum value of ν. In the case where $\nu = 2$, the similarity dimension of the Sierpinski gasket SG(2) is $\log 3/\log(5/3)$.

The effective resistance metric does not reflect the geometry of the fractal very well. In order to establish heat kernel estimates, we need to know how the shortest paths scale with effective resistance. The results in Section 7 show that there is a constant κ such that, if a_n denotes the number of steps in the shortest path on G_n required to cross the fractal G, then

$$(1.1) \qquad \kappa_2 = \exp\bigl(\log 2/\log(5/3)\bigr) \le \liminf_{n\to\infty} a_n^{1/n} \le \limsup_{n\to\infty} a_n^{1/n} \le \kappa.$$

Transition density estimates for Brownian motion on fractals were first obtained in [4]. The techniques were refined in [3], and applied to nested fractals in [21] and affine nested fractals in [8]. We will show how these techniques can be extended to incorporate the random recursive gaskets which are not spatially homogeneous. The spatial inhomogeneity in the structure leads to logarithmic corrections for the heat kernel. To express the results, we define the function $g_a(z) = (\log z \vee 1)^a$, $z > 0$, and the positive exponents

$$b_2 = \frac{\alpha}{\alpha_2} - 1, \qquad b_K = 1 - \frac{\alpha}{\alpha_K}, \qquad \xi = b_2 + b_K,$$

$$\eta = b_2 + 2\xi, \qquad \eta_1 = \eta\beta_2 + \xi, \qquad \hat{\eta} = \alpha\xi + b_K,$$

$$\beta_2 = \frac{\log \kappa_2}{\alpha + 1 - \log \kappa_2}, \qquad \beta_\varepsilon = \frac{\log(\kappa + \varepsilon)}{\alpha + 1 - \log(\kappa + \varepsilon)} \quad \text{for } \varepsilon > 0.$$

Combining the effective resistance metric and the similarity dimension, we will obtain the following transition density estimates.

THEOREM 1.2.

(i) *For each* $\omega \in \Omega$, *there exists a jointly continuous real-valued function* $p_t(x, y)$ *on* $(0,1] \times G(\omega) \times G(\omega)$.

(ii) *There exist constants* $c_{1.1}(\omega), c_{1.2}(\omega)$ *such that*

$$p_t(x, y) \le c_{1.1}(\omega) t^{-\alpha/(\alpha+1)} g_{b_2}(1/t)^{1/(\alpha+1)}$$
$$\times \exp\left(-c_{1.2}(\omega) D^*(x, y, t)^{\beta_2} \left(\log_+ D^*(x, y, t)\right)^{-\eta_1}\right),$$
$$0 < t \le t_0, \forall\, x, y \in G_R, \mathbb{P}\text{-}a.s.,$$

where $D^*(x, y, t) = d_r(x, y)^{\alpha+1}/t g_\eta(1/d_r(x, y))$.

For $\varepsilon > 0$ *there exist constants* $c_{1.3}(\omega), c_{1.4}(\omega)$ *such that*

$$p_t(x, y) \ge c_{1.3}(\omega) t^{-\alpha/(\alpha+1)} g_{\hat{\eta}}(1/t)^{-1/(\alpha+1)}$$
$$\times \exp\Bigg(-c_{1.4}(\omega) D_*(x, y, t)^{\beta_\varepsilon}$$
$$\times \left(\log_+(D_*(x, y, t))\right)^{\beta_\varepsilon \hat{\eta}} \left(\log_+\left(\frac{d_r(x, y)^{\log \kappa_\varepsilon - 1}}{t}\right)\right)^2\Bigg),$$
$$0 < t < t_0, \ x, y \in G_R, \mathbb{P}\text{-}a.s.,$$

where $D_*(x, y, t) = d_r(x, y)^{\alpha+1}/t g_{-\hat{\eta}}(1/t)$ *and* G_R *is a suitably small neighborhood in* $G(\omega)$ *and* $t_0 < 1$.

REMARKS. (i) The bounds are not tight. The fact that even the on-diagonal bounds are not tight is due to the local inhomogeneities in the structure. There will almost always exist neighborhoods in the fractal which have atypical behavior.

(ii) The exponents for the leading term in the exponential, $d_r(x, y)^{\alpha+1} t^{-1}$, differ in the bounds as it has not been possible to establish the existence of a shortest path exponent in the effective resistance metric for these fractals. In order to do this, we would need the existence of the limit in (1.1).

(iii) The estimates for the transition density on the homogeneous random Sierpinski gasket [14] differ as the nth term in the constructing sequence is equivalent to choosing about $e^{\alpha n}$ random variables in the random recursive model. The set of trees which are used to construct the homogeneous random gaskets form a null set in the space of all random recursive gaskets.

(iv) The transition density estimates for the Brownian motion on the Sierpinski gasket in the effective resistance metric can be recovered from the above. We set $\alpha = \alpha_2 = \alpha_K = \log 3/\log(5/3)$, so that all the correction exponents are 0. Also the shortest path exponent $\log \kappa = d_c = \log 2/\log(5/3)$ will control the path uniformly at each level, allowing us to remove the ε and the final term in the lower bound (from the proof) to obtain

$$p_t(x, y) \sim c_1 t^{-\alpha/(\alpha+1)} \exp\Big(-c_2\big(d_r(x, y)^{\alpha+1} t^{-1}\big)^{d_c/(\alpha+1-d_c)}\Big), \qquad \forall\, x, y \in G,\ 0 < t < 1,$$

where the constants c_1, c_2 differ for the upper and lower bounds.

2. A random recursive Sierpinski gasket. We define a class of random sets that have statistical self-similarity, following [11], [12], [25] or alternatively [6]. The fractal is formed recursively from the family of contractions described in [14]. At each stage a member of the set of possible contraction maps is chosen according to a probability distribution, for each copy of the original set. We begin by constructing a space of trees, which controls the number of components of the fractal and their position. We then form the random recursive gasket by considering the limit set obtained by associating triangles with the branches of the tree.

Let $I_n = \bigcup_{k=0}^{n} \mathbb{N}^k$ and $I = \bigcup_k I_k$ be the space of arbitrary length sequences. We write $\mathbf{i}, \mathbf{j}$ for concatenation of sequences. For a point $\mathbf{i} \in I \setminus I_n$ denote by $[\mathbf{i}]_n$ the sequence of length n such that $\mathbf{i} = [\mathbf{i}]_n, \mathbf{k}$ for a sequence $\mathbf{k}$. We write $\mathbf{j} \le \mathbf{i}$, if $\mathbf{i} = \mathbf{j}, \mathbf{k}$ for some $\mathbf{k}$, which provides a natural ordering on branches. Also denote by $|\mathbf{i}|$ the length of the sequence $\mathbf{i}$.

The infinite random tree, T, is a subset of the space I, defined as the sample path of a Galton–Watson process. Let the root be $T_0 = I_0$, the empty sequence. Let $U_{\mathbf{i}}$, $\mathbf{i} \in T$ be positive i.d. random variables, indicating the number of offspring of an individual, with probability distribution

$$P\big(U_{\mathbf{i}} = \nu(\nu+1)/2\big) = p_\nu, \qquad \nu = 2, 3, \dots .$$

Then $\mathbf{i} \in T$ if $[\mathbf{i}]_n \in T_n \subset I_n$ for each $n \le |\mathbf{i}|$, where $[\mathbf{i}]_n \in T_n$ if we have the following:

1. $[\mathbf{i}]_{n-1} \in T_{n-1}$.
2. There is a k: $1 \le k \le U_{[\mathbf{i}]_{n-1}}$ such that $[\mathbf{i}]_{n-1}, k = [\mathbf{i}]_n$.

There is also a labeling of the tree given by the associated random variables

$$V_{[\mathbf{i}]_n} = \nu \quad \text{if } U_{[\mathbf{i}]_n} = \nu(\nu+1)/2.$$

Then the pair $\{(U_{\mathbf{i}}, V_{\mathbf{i}});\ \mathbf{i} \in T\}$ denotes a labeled tree. There is a natural probability space associated with these trees given by $(\Omega, F, \mathbb{P})$, where the

σ-algebras are

$$F_n = \sigma\big(U_{\mathbf{i}}; \mathbf{i} \in T_{n-1}(\omega)\big), \qquad F = \bigcup_{n=1}^{\infty} F_n,$$

and the probability measure, $\mathbb{P}$, is determined by the Galton–Watson process with offspring distribution $\{p_\nu : \nu = 2, \dots\}$. Taking expectations with respect to this probability space of all constructing trees will be denoted by $\mathbb{E}$. For these random recursive fractals, the branching process will be supercritical with no possibility of extinction.

For example in the case of the $(2,3)$-gasket $G(\omega)$, shown in Figure 2, we have generating function for the offspring distribution $f(u) = pu^3 + qu^6$ and $V_{\mathbf{i}} = 2$ if $U_{\mathbf{i}} = 3$, $V_{\mathbf{i}} = 3$ if $U_{\mathbf{i}} = 6$.

We now define a sequence of sets to be attached to the branches and we drop the reference to the underlying probability space. Let $E = E_0$ be the unit equilateral triangle, and let G_0 denote the complete graph on the vertices of E_0. Let

$$\Psi^\nu = \left\{\psi_i^\nu,\ i = 1, 2, \dots, \frac{\nu(\nu+1)}{2}\right\},$$

denote the family of contraction maps of type ν, which involve dividing the side of the equilateral triangle by ν. Then set $E_{\mathbf{i}}$, $\mathbf{i} \in T_n$, geometrically similar to E, to be

$$E_{\mathbf{i}} = \psi_{\mathbf{i}}(E) = \psi_{[\mathbf{i}]_1}^{V_{[\mathbf{i}]_1}}\Big(\cdots\Big(\psi_{[\mathbf{i}]_n}^{V_{[\mathbf{i}]_n}}(E)\Big)\Big).$$

A random gasket can then be defined by

$$G(\omega) = \bigcap_{n=1}^{\infty} \bigcup_{\mathbf{i} \in T_n(\omega)} E_{\mathbf{i}}.$$

The Hausdorff dimension of the set $G(\omega)$ can be found by applying the results of [6], [25] and is given by,

$$d_f(G(\omega)) = \inf\left\{\alpha : \sum_{\nu=2}^{\infty} \frac{\nu(\nu+1)}{2} p_\nu \left(\frac{1}{\nu}\right)^{\alpha} = 1\right\} \quad \text{a.s.} \tag{2.1}$$

REMARK. The homogeneous random fractal of [14], generated by a set of trees which is a null set in Ω, has Hausdorff dimension

$$d_f = \frac{\sum_{\nu=2}^{\infty} p_\nu \log(\nu(\nu+1)/2)}{\sum_{\nu=2}^{\infty} p_\nu \log \nu}.$$

We can also use [12] to find the exact Hausdorff measure function for this random recursive construction.

LEMMA 2.1. *The exact Hausdorff measure function for $G(\omega)$ is given by*

$$h(t) = t^{d_f}(\log|\log t|)^{1-d_f/2}, \quad \mathbb{P}\text{-}a.s.$$

where $d_f = d_f(G(\omega))$.

PROOF. We apply [12], Theorem 5.2, which, by [12], Theorem 5.4 will follow if

$$E\left(\sum_{i=1}^{n} H_i^0\right) > 1,$$

where n is the maximum number of new components and the H_i, the ratio of the component sizes, take only finitely many values. Also a condition on nondegeneracy is needed:

$$P\left(\sum_{i=1}^{n} H_i^{d_f} \neq 1\right) > 0.$$

For the case of a finitely ramified fractal, the first condition is trivial and the second follows provided there is a p_ν with $0 < p_\nu < 1$. Thus the Hausdorff measure function is given by

$$h(t) = t^{d_f}(\log|\log t|)^{\theta},$$

where

$$\theta^{-1} = \sup\left\{a\colon a > 1, \frac{\nu(\nu+1)}{2}\left(\frac{1}{\nu}\right)^{d_f/(1-1/a)} \leq 1,\ \nu = 2, 3, \dots\right\}.$$

Evaluating this gives the result. □

The random recursive Sierpinski gasket G will be the set considered from now on. For a given realization of the constructing tree, let $\{G_n\}_{n \geq 0}$ be a sequence of graphs formed from the vertices of the triangles such that $G_n \subset G_{n+1}$. For each n, $G_n \subset G_\infty$, and the random fractal G can be recovered as $G = \mathrm{cl}(G_\infty)$. The corresponding set formed from the union of the triangles in G_n will be denoted by E_n. We will not explicitly refer to the underlying probability space when referring to the fractal or associated graph approximations. The results obtained will depend on this realization of the fractal and hence constants followed by a (ω) will denote a constant dependent on the environment. For $x \in G_n$, we will denote by $N_n(x)$ the points in the graph G_n directly connected to x by an edge. For $x \in G \setminus G_\infty$, let $\triangle_n(x)$ denote the n-level triangle containing x, and let $N_n(\triangle_n(x))$ denote the n-level triangles having a common vertex with $\triangle_n(x)$. Then let $D_n(x) = \triangle_n(x) \cup N_n(\triangle_n(x))$, the n-neighborhood of the n-triangle containing the point x. For a point $x \in G_n$ we define $D_n(x)$ to be the union of the n-triangles containing x. We may also write $\triangle_n(\mathbf{i})$ for $i \in T \setminus T_\infty$, the triangle associated with the branch $[\mathbf{i}]_n$ and then $E_n = \bigcup_{\mathbf{i} \in T_n} \triangle_n(\mathbf{i})$.

3. General branching processes. In order to discuss the random gasket, we introduce a description of the set via a general branching process. The random tree used to construct the fractal is just the realization of a Galton–Watson process. It contains information about the number of sets at each level but not about the size of the sets at a given level. This can be incorporated by using a general branching process. We begin by discussing a strictly supercritical general branching process without referring to the specific process needed in the discussion of fractals. The general branching process is described in [16] and [1]. The set-up is as follows.

We have a reproduction point process $\xi(t)$, which describes the birth events as well as a life-length τ and a function ϕ on $[0,\infty)$ called a random characteristic of the process. We make no assumptions about the joint distributions of these quantities. The individuals in the population are ordered according to their birth times σ_n. As we can have multiple births, this will not be a strictly increasing sequence. We denote the attributes of the nth individual by $(\xi_n, \tau_n, \phi_n, \dots)$. At time 0 we have an initial ancestor so that $\sigma_1 = 0$. The process that we wish to consider is written as

$$Z^{\phi}(t) = \sum_{n,\, \sigma_n \le t} \phi_n(t - \sigma_n).$$

An example of a random characteristic is

$$\phi(t) = I_{\{\tau > t\}},$$

so that $Z^{\phi}(t)$ is the total number of individuals alive at time t. This process, for the current population size, will be denoted by z_t. We also define the mean reproduction measure $m(t) = E\xi(t)$. We will assume that $m(0) = 0$ and in the supercritical case that it is Malthusian, so there exists an $\alpha > 0$ such that

$$\int_0^{\infty} e^{-\alpha t} m(dt) = 1.$$

We also assume that each individual has at least two offspring so there is no possibility of extinction and the process will grow rapidly. We will write

$$m_{\alpha}^{\phi}(t) = E\big(e^{-\alpha t} Z^{\phi}(t)\big),$$

for the discounted mean of the process with random characteristic ϕ. We now introduce a martingale, analogous to the standard branching process martingale, which will enable us to discuss the asymptotic growth of this process. This plays an important role in the discussion of random recursive fractals.

Let

$$A_n = \sigma\big((\xi_k, \tau_k, \phi_k^{(1)}, \dots): 1 \le k \le n\big).$$

Observe that the birth time of individuals is determined by their parents' reproduction process, so that the birth times σ_k are A_{k-1} measurable. Now define

$$R_n = \sum_{l=n+1}^{\infty} e^{-\alpha \sigma_l} I_{\{l \text{ is a child of } 1 \dots n\}}.$$

Then we have the following theorem.

THEOREM 3.1 ([1], Chapter VI, Theorem 4.1). *The quantity* $\{R_n\}_{n=1}^{\infty}$ *is a nonnegative martingale with respect to* A_n *and*

$$W = \lim_{n\to\infty} R_n \quad \textit{exists}.$$

Also $W > 0$ *if and only if*

$$E\left(\int_0^\infty e^{-\alpha t}\,\xi(dt)\right)\log\left(\int_0^\infty e^{-\alpha t}\,\xi(dt)\right) < \infty.$$

Otherwise $W = 0$, *a.s.*

We will also need to look at various characteristics of the process and so we will state here [1], Chapter VI, Theorem 5.1 and its Corollary 5.3.

THEOREM 3.2. *Let* ϕ_i, $i = 1, 2$ *be characteristics with sample functions that are right continuous and satisfying the following*:

(i) $E(\sup_{t\geq 0} e^{-at}\phi_i(t)) < \infty$ *for some* $0 < a < \alpha$, $i = 1, 2$;
(ii) $\alpha > 0, \lim_{t\to\infty} e^{-bt} m(t) < \infty$ *for some* $0 \leq b < \alpha$.

Then

$$\lim_{t\to\infty} \frac{Z^{\phi_1}(t)}{Z^{\phi_2}(t)} = \frac{m_\alpha^{\phi_1}(\infty)}{m_\alpha^{\phi_2}(\infty)} \quad a.s.,$$

and also

$$\lim_{t\to\infty} e^{-\alpha t} Z^{\phi_i}(t) = W m_\alpha^{\phi_i}(\infty) \quad a.s.\ i = 1, 2. \tag{3.1}$$

It will be useful for estimating properties of the process to have estimates on the tails of the random variable W. For the left tail upper bound the approach will be similar to that used in [2] and [13], where a loose estimate on the distribution function can be used with the branching structure to get an improved estimate.

LEMMA 3.3. *For a uniformly integrable family of nonnegative random variables* $\{X_a\colon a \in C\}$ *there exist constants* $c_{3.1} < 1$ *and* $c_{3.2}$ *such that*

$$P(X_a < sE(X_a)) \leq c_{3.1} + c_{3.2}s \qquad \forall\, s > 0, \forall\, a \in C.$$

PROOF. Let $Y_a = X_a/E(X_a)$ and observe that, by uniform integrability, for $\varepsilon > 0$ there exists a K such that $E(Y_a; Y_a > K) < \varepsilon$. Then

$$1 \leq \int_0^K \bigl(1 - P(Y_a \leq s)\bigr)\,ds + \varepsilon,$$

$$1 - \varepsilon \leq K\bigl(1 - P(Y_a \leq K)\bigr)$$

and thus

$$P(Y_a \leq s) \leq 1 - \frac{1-\varepsilon}{K} + \frac{1-\varepsilon}{K^2}s, \qquad s > 0,$$

as desired. □

We now use this estimate with a decomposition of the branching process martingale. Let $S_n = \inf\{m\colon \sigma_m \geq n\}$, the number of births when the first birth occurs after time n. Then there will be more than b^n offspring alive at this time, where b is the minimum family size born in a time unit (assumed strictly bigger than one). Thus the weighted sum of the offspring up to the stopping time S_n will be bigger than the sum of the weighted sums of the offspring of the first b^m individuals run for time $n - m$, giving

$$R_{S_n} \geq \sum_{i=1}^{b^m} \exp(-\alpha\sigma_{S_m}) R_{S_{n-m}, i}. \tag{3.2}$$

As R_n is a convergent martingale and $S_n \to \infty$ as $n \to \infty$, we can let $n \to \infty$ in (3.2) to get

$$W \geq \sum_{i=1}^{b^m} W_i e^{-\alpha m}. \tag{3.3}$$

Using this we have the following result on the tail of the random variable W.

LEMMA 3.4. *There exist constants $c_{3.3}, c_{3.4}$ such that if $\gamma = \log b$,*

$$P(W < \delta) \leq c_{3.3} \exp\left(-c_{3.4}\delta^{-\gamma/(\alpha-\gamma)}\right), \qquad \delta > 0.$$

PROOF. By Lemma 3.3 we have a loose estimate for W of the form

$$P(W < \delta E(W)) \leq c_{3.1} + c_{3.2}\delta.$$

We can write

$$P(We^{-\alpha m} < \delta) \leq c_{3.1} + ce^{\alpha m}\delta.$$

Now apply (3.3) and [2], Lemma 1.1, to obtain

$$\begin{aligned} \log P(W < \delta) &\leq -cb^m + C(b^m e^{\alpha m}\delta)^{1/2} \\ &= -cb^m\left(1 - \frac{C}{c}(e^{(\alpha-\gamma)m}\delta)^{1/2}\right). \end{aligned} \tag{3.4}$$

We can now choose an appropriate value of m to obtain the bound. That is, let $m_0 = \log(C^2/4c^2\delta)/(\alpha - \gamma)$. Then, for $m = [m_0]$,

$$e^{(\alpha-\gamma)m}\delta \leq \frac{1}{4}\left(\frac{C}{c}\right)^2.$$

Hence

$$1 - \frac{C}{c}(e^{(\alpha-\gamma)m}\delta)^{1/2} \geq \hat{c}$$

and

$$-cb^m \leq -c^*\delta^{-\gamma/(\alpha-\gamma)}.$$

We can now substitute this in (3.4) to obtain the result for small values of δ. By choice of $c_{3.3}$ we can extend the bound to all $\delta > 0$. □

In the case of the random recursive Sierpinski gasket generated from two types, the general branching process can be described explicitly. This is a Bellman–Harris or age dependent branching process as the offspring are born at the moment of death. In the case of this fractal, the life length is either $\log 3$, when six offspring are produced, or $\log 2$, when three appear. Thus from a single individual at time 0, there evolves a branching process, z_t, where individual x has life and offspring according to

$$(\lambda_x, \xi_x) = \begin{cases} \left(\log 2, 3 I_{\{t = \log 2\}}\right) & \text{with probability } p, \\ \left(\log 3, 6 I_{\{t = \log 3\}}\right) & \text{with probability } q. \end{cases}$$

Thus the number of individuals in the branching process z_n, at time n, is the number of sets of diameter at least e^{-n} in the fractal. Using the general branching process, results about the Hausdorff dimension and the upper bound for the Hausdorff measure function can be recovered.

However, we will approximate the fractal via graphs with the conductance determining the construction of the graph, as this determines the appropriate metric to use on the fractal. Let γ_ν denote the conductance of $\mathrm{SG}(\nu)$ (it is easy to calculate that $\gamma_2 = 5/3$, $\gamma_3 = 15/7$; see [14]). In this case the appropriate life length and reproduction point process are

$$(3.5) \qquad (\lambda_x, \xi_x) = \left(\log \gamma_\nu, \frac{\nu(\nu+1)}{2} I_{\{t = \log \gamma_\nu\}}\right) \quad \text{with probability } p_\nu.$$

Thus the fractal G is approximated by the graph G_n given by

$$G_n = \bigcup_{\mathbf{i} \in \tilde{T}_n} \psi_{\mathbf{i}}(G_0),$$

where $\tilde{T}$ is the family tree of the general branching process. This is a sequence of cut sets of the tree T. It is this approximate graph that will be considered from now on.

We will now assume that the number of components that can be chosen is finite. Thus the random variable $V_{\mathbf{i}} \in [2, K]$ where we let K denote the maximal family of maps. As a corollary to Lemma 3.4, we state the general branching process result for the process used here. Let α_ν denote the Malthusian parameter corresponding to the general branching process of only type ν, that is, the similarity dimension of the fractal $\mathrm{SG}(\nu)$, which is $\alpha_\nu = \log(\nu(\nu + 1)/2)/\log \gamma_\nu$. As noted before, $\alpha_\nu = d_s(\nu)/(2 - d_s(\nu))$ and as $d_s(\nu) \uparrow 2$ as $\nu \to \infty$, we see that α_ν is increasing in ν.

COROLLARY 3.5. *For the general branching process determined by* (3.5),

$$P(W < \delta) \le c_{3.3} \exp\left(-c_{3.4} \delta^{\alpha_2/(\alpha_2 - \alpha)}\right) \qquad \forall\, \delta > 0.$$

PROOF. This follows by calculating the minimum family size possible in a time unit. □

We can now provide an estimate on the right tail of W for the general branching process which is used here. First, observe that we can decompose W at the time of the first birth,

$$W = \sum_{i=1}^{U_1} \gamma_{V_1}^{-\alpha} W_i. \tag{3.6}$$

In order to use large deviations, we must estimate the moment generating function, $M(\theta) = E(\exp(\theta W))$. Observe that the number of offspring is bounded over a unit time interval and hence the moment generating function will exist. Using (3.6), this satisfies the following inequality:

$$\begin{aligned} M(\theta) &= E(\exp(\theta W)), \\ &= \sum_{\nu=2}^{K} p_\nu M(\theta\gamma_\nu^{-\alpha})^{\nu(\nu+1)/2} \\ &\leq \max_{2\leq\nu\leq K} M(\theta\gamma_\nu^{-\alpha})^{\nu(\nu+1)/2}. \end{aligned}$$

By iterating this inequality, we have

$$M(\theta) \leq \max_{2\leq\nu\leq K} M(\theta\gamma_\nu^{-\alpha n})^{(\nu(\nu+1)/2)^n}. \tag{3.7}$$

LEMMA 3.6. *There exist constants $c_{3.5}, c_{3.6}$ such that*

$$M(\theta) \leq c_{3.5} \exp\big(c_{3.6}\theta^{\alpha_K/\alpha}\big) \qquad \forall\, \theta \geq 0.$$

PROOF. Let $c_{3.6}$ be defined by

$$c_{3.6} = \sup_{\theta\in[1,\gamma_K]} \theta^{-\alpha_K/\alpha} \log M(\theta).$$

Assume that the result holds for $\theta \in [1, c]$ for some $c > \gamma_K$. Now consider the interval $[c, \gamma_2^{\alpha} c]$. Then, for all $\nu \in [2, K]$, $\gamma_\nu^{-\alpha}\theta \in [1, c]$. By (3.7) and the fact that the α_ν are increasing in ν, we have

$$\begin{aligned} M(\theta) &\leq \max_{2\leq\nu\leq K} M(\theta\gamma_\nu^{-\alpha})^{\nu(\nu+1)/2} \\ &\leq \max_{2\leq\nu\leq K} \exp\Big(c_{3.6}(\theta\gamma_\nu^{-\alpha})^{\alpha_K/\alpha}(\nu(\nu+1)/2)\Big) \\ &\leq \max_{2\leq\nu\leq K} \exp\big(c_{3.6}\theta^{\alpha_K/\alpha}\gamma_\nu^{\alpha_\nu-\alpha_K}\big) \\ &\leq \exp\big(c_{3.6}\theta^{\alpha_K/\alpha}\big). \end{aligned}$$

By repeatedly applying this procedure, we can extend the interval from $\theta \in [1, \gamma_K]$ to $\theta \geq 1$. For $\theta < 1$, we use the fact that $M(\theta)$ is increasing in θ and set $c_{3.5} = e^{c_{3.6}}$ to give the result. □

THEOREM 3.7. *There exist constants $c_{3.7}, c_{3.8}$ such that*

$$P(W > \delta) \leq c_{3.7} \exp\big(-c_{3.8}\delta^{\alpha_K/(\alpha_K-\alpha)}\big) \qquad \forall\, \delta > 0.$$

PROOF. Using Markov's inequality,

$$P(W > \delta) = P(\exp(\theta W) > \exp(\theta\delta))$$
$$\leq \exp(-\theta\delta) M(\theta) \leq c_{3.5} \exp\big(-\theta\delta + c_{3.6} \theta^{\alpha_K/\alpha}\big),$$

and now optimize over θ. □

4. The Dirichlet form and the diffusion. A natural method for constructing processes on finitely ramified fractals is via their Dirichlet forms (see [10], [17], [20]). We assign electrical conductances to each edge in the graph G_n and use this to determine the approximating Markov chains and invariant measure on the set.

We consider the sequence G_n of approximating graphs constructed from the general branching process. Let the corresponding family tree be denoted by $\tilde{T}_n$. The conductance assigned to the edges of the triangle $\triangle_n(\mathbf{i})$, $\mathbf{i} \in T \setminus \tilde{T}_{n-1}$ is the birth time of the individual corresponding to the triangle, that is,

$$\bar{\rho}_n(\mathbf{i}) = \prod_{\nu=2}^{\infty} (\gamma_\nu)^{k_n^{\mathbf{i}}(\nu)}, \qquad x, y \in G_n \cap \triangle_n(\mathbf{i}),$$

where

$$k_n^{\mathbf{i}}(\nu) = \sum_{j=1}^{n(\mathbf{i})-1} I_{\{V_{[\mathbf{i}]_j} = \nu\}},$$
$$n(\mathbf{i}) = \inf\left\{m: \prod_{j=1}^{m} \gamma_{V_{[\mathbf{i}]_j}} \geq e^n\right\}.$$

We will then write the conductance of edge (x, y) in the graph G_n as $\rho_n(x, y) = \bar{\rho}_n(\mathbf{i})$, if $x, y \in G_n \cap \triangle_n(\mathbf{i})$. We can also define the weight on each node $x \in G_n$ as $\rho_n(x) = \sum_{y \in N_n(x)} \rho_n(x, y)$ and the associated measure μ_n on the graph by

$$\mu_n(x) = \frac{\rho_n(x)}{\sum_{x \in G_n} \rho_n(x)}, \qquad x \in G_n.$$

By our construction of the graph G_n, the conductance $\rho_n(x, y)$ is bounded above by e^n and below by $c^* e^n$.

The random walk moving according to the conductance of each edge in the graph is the natural random walk on a graph. For a continuous time random walk, the time for it to take each step is also determined by the conductance, being exponential with rate $\rho_n(x)$. The natural Laplacian associated with the graph is determined by this conductance, and we write it as

$$\Delta_n f(x) = \sum_{y \in N_n(x)} (f(y) - f(x)) \rho_n(x, y).$$

The Laplacian on the fractal is obtained by taking the limit of this sequence as it is suitably normalized by our choice of conductance.

Now let $\tau_n = \sum_{x \in G_n} \rho_n(x)$ and rewrite

$$\Delta_n f(x) = \tau_n \sum_{y \in N_n(x)} (f(y) - f(x)) \frac{\rho_n(x, y)}{\rho_n(x)} \mu_n(x),$$

and then the Dirichlet form for the sequence of chains is defined as

$$\begin{aligned} E_n(f, g) &= -(f, \Delta_n g) \\ &= \frac{1}{2} \tau_n \sum (f(x) - f(y))(g(x) - g(y)) \frac{\rho_n(x, y)}{\rho_n(x)} \mu_n(x). \end{aligned}$$

Note that the Laplacian is determined when we fix the measure μ, and any finite measure with full support could be used. We choose this measure as it will turn out to be equivalent to the α-dimensional Hausdorff measure in the effective resistance metric.

The crucial property of this sequence of quadratic forms is that it is nested. This allows the construction of the process to be completed.

LEMMA 4.1. *Let the graph G_0 have edge conductivities $c_{xy} = c$ for all edges xy. If the graph G_1 is of type ν then, if the conductivity placed on each new edge is of the form $\gamma_\nu c$, the network conductivity is unchanged.*

PROOF. That there is a resistance which leaves the network conductivity unchanged follows as these are nested fractals [23, 26]. As they have three essential fixed points, there is a unique fixed point for the conductivity map; see [14] for the explicit case of SG(ν). □

A consequence of this result is that the sequence of Dirichlet forms $E_n(f, f)$ is monotone increasing. To see this write Q_{n+1} for the q-matrix, the generator of the Markov chain corresponding to the random walk on the graph G_{n+1}. Then the $(n+1)$-level Dirichlet form can be written as $E_{n+1}(f, g) = -f^t Q_{n+1} g$. If we write

$$Q_{n+1} = \begin{bmatrix} A_{n+1} & B_{n+1} \\ C_{n+1} & D_{n+1} \end{bmatrix},$$

where A_{n+1} determines transitions on G_n to itself (it is diagonal for nested fractals); B_{n+1} and $C_{n+1} = B_{n+1}^T$ give transitions from G_n to G_{n+1} and vice versa, and D_{n+1} denotes transitions from $G_{n+1} \setminus G_n$ to itself. Then, by the choice of γ_ν and hence $\rho_n(x)$, the generator on G_n satisfies

$$Q_n = A_{n+1} - B_{n+1}^T D_{n+1}^{-1} B_{n+1}. \tag{4.1}$$

This is the compatibility condition of [20] and ensures that the forms have the property that

$$E_{n-1}(g, g) = \inf\{E_n(f, f) : f|_{F_{n-1}} = g\} \quad \text{for } g \in C(F_{n-1}). \tag{4.2}$$

Thus the sequence of forms is monotone increasing and there exists a limiting bilinear form $(E, D(E))$ given by

$$D(E) = \left\{ f\colon \sup_n E_n(f,f) < \infty \right\},$$

$$E(f,f) = \lim_{n\to\infty} E_n(f,f), \qquad f \in D(E),$$

In order to prove we have a Dirichlet form, we need some further results. We extend the definition of $\rho_n(x)$ to all $x \in G$ by setting $\rho_n(x) = \bar{\rho}_n(\mathbf{i})$, if $x \in \operatorname{int}(\triangle_n(\mathbf{i}))$. Define an n-harmonic function h to be the function which satisfies $\triangle_n h = 0$. Following [17] and [22] it can be shown that this function is unique and can be extended uniquely to a function which is harmonic on the fractal G. For $f \in C(G)$ we write $H_n f$ for the function which takes the values of f on G_n and is harmonic everywhere else.

From [20] there is an effective resistance defined by the limiting Dirichlet form

$$d_r(x,y) = \left(\inf\{ E(f,f)\colon f(x) = 0,\ f(y) = 1\} \right)^{-1},$$

which is a metric on G.

LEMMA 4.2. *For $x, y \in \partial \triangle_n$ then there exist constants $c_{4.3}, c_{4.4}$ such that*

$$c_{4.3} e^{-n} \le d_r(x,y) \le c_{4.4} e^{-n}.$$

PROOF. This follows from the construction of the Dirichlet form. By definition we have for all f with $f(x) = 0$, $f(y) = 1$ with $y \in N_n(x)$,

$$ce^n \le \rho_n(x,y) \le E_n(f,f) \le E(f,f).$$

Thus there is a constant such that

$$\left(\inf\{ E(f,f)\colon f(x) = 0,\ f(y) = 1\} \right)^{-1} \le c_{4.4} e^{-n}.$$

For the lower bound, we take the n-harmonic function f_1 which is 1 at y and 0 at all other points of G_n. Then

$$\inf\{ E(f,f)\colon f(x) = 0,\ f(y) = 1\} \le E(f_1, f_1) = E_n(f_1, f_1) \le ce^n,$$

and we have the lower bound. □

Using the limiting form we have the following result from [20]:

$$|f(x) - f(y)|^2 \le d_r(x,y) E(f,f) \qquad \forall\, f \in D(E), \forall\, x, y \in G, \tag{4.3}$$

which allows us to control the continuity of functions in the domain.

LEMMA 4.3. *For any $\triangle_n(z)$ we have that for all $f \in D(E)$,*

$$\sup_{x, y \in \triangle_n(z)} |f(x) - f(y)| \le c_{4.2}\, \rho_n(z)^{-1/2} E(f,f)^{1/2}.$$

The proof follows by combining Lemma 4.2 and (4.3).

THEOREM 4.4. *The bilinear form* $(E, D(E))$ *is a local regular Dirichlet form on* $L^2(G(\omega), \mu)$.

An immediate corollary of this result is the existence of a diffusion process upon the fractal.

COROLLARY 4.5. *For each* $\omega \in \Omega$, *there exists a continuous, strong Markov diffusion process* X_t^{ω} *on the set* $G(\omega)$, *which is symmetric with respect to the measure* μ^{ω}.

The law of the process X_t^{ω} on $G(\omega)$ started from the point $x \in G(\omega)$ will be denoted by $P^{x,\omega}$, though we will usually suppress the ω.

REMARK. By Lemma 4.1, the addition of new nodes and edges with the given conductivities within a given triangle only affect the fractal through the vertices of that triangle. This decomposition into separate pieces allows this technique to be used for any sequence of cutsets of the constructing tree.

PROOF OF THEOREM 4.4. Fix a fractal $G(\omega)$. To prove that the limiting form is a Dirichlet form, observe that the Markov property is inherited from the Markov property for the Dirichlet forms in the approximating sequence and the closability can be proved as in [17], Theorem 7.2, or [22], Theorem 4.14.

Let $f \in C(G)$ and take $f_n = H_n f \in F \cap C(G)$. Then from Lemma 4.3 we see that $F \cap C(G)$ is dense in $C(G)$ in the sup norm and as $E_n(f, f) = E(H_n f, H_n f)$ we can prove the density in F. Thus the form is regular. The local property can be proved in the same manner as [17]. □

5. The geometry of the fractal. We begin by computing the other dimensional indices of the fractal with respect to the Euclidean metric. The Hausdorff dimension is given in (2.1). The box counting dimension of a set A is formed by covering the set with balls of fixed radius, ε, finding the minimum number $\overline{N}_{\varepsilon}(A)$, and setting

$$d_{ub} = \limsup_{\varepsilon \downarrow 0} \frac{\log \overline{N}_{\varepsilon}(A)}{-\log \varepsilon}.$$

LEMMA 5.1. *The box counting dimension of* G *is given by*

$$d_{ub} = d_f, \qquad \mathbb{P}\text{-}a.s.$$

PROOF. Let $\varepsilon = e^{-n}$. We regard the sequence of approximating graphs as those formed by constructing each set until its length is less than e^{-n}. Then each set in E_n is of side length at least e^{-n} and at most Ke^{-n}. By using the population size of the general branching process z_n, corresponding to the fractal in the Euclidean metric, the set G can certainly be covered by

$$N_{e^{-n}}(G) \le K^2 z_n \tag{5.1}$$

balls of size e^{-n}. (That this is in fact close enough to the minimum number of balls required will follow from the fact that value of the box counting dimension obtained is equal to the Hausdorff dimension.) Now

$$d_{ub} = \limsup_{n\to\infty} \log N_{e^{-n}}(G)^{1/n},$$

and by a simple application of (3.1) with characteristic given by $\phi(t) = I_{(\tau > t)}$ we see that

$$c_1(\omega)e^{sn} < z_n < c_2(w)e^{sn}, \qquad \mathbb{P}\text{-a.s.}$$

with s in the required form. Thus, letting $n \to \infty$, and using (5.1), gives

$$\begin{aligned} d_{ub} &= \limsup_{n\to\infty} \frac{1}{n}\log N_{e^{-n}}(G) \\ &\le \limsup_{n\to\infty} \frac{1}{n}\big(\log c_2(\omega)K^2 + sn\big), \qquad \mathbb{P}\text{-a.s.} \\ &= s = \inf\left\{u\colon \sum_{\nu=2}^{K} \frac{\nu(\nu+1)p_\nu}{2} e^{-u\log\nu} = 1\right\} = d_f, \qquad \mathbb{P}\text{-a.s.} \end{aligned}$$

as desired. □

Thus the geometric dimensions coincide and the set is fractal in the sense of Taylor [28]. The natural measure for the diffusion on this set is determined not by the Hausdorff measure in the Euclidean metric but by the Hausdorff measure in the effective resistance metric. The dimensions of the set can now be stated in terms of the effective resistance metric.

THEOREM 5.2. *The Hausdorff dimension of the set $G(w)$ in the effective resistance metric is given by*

$$\alpha = d_f^r = d_{ue}^r = \left\{s\colon \sum_{\nu=2}^{K} \frac{\nu(\nu+2)}{2} p_\nu \gamma_\nu^{-s} = 1\right\}, \qquad \mathbb{P}\text{-}a.s.$$

PROOF. Following Lemma 5.1, using Lemma 4.2 and the general branching process z_n in which the lifetimes are determined by the conductance, we have that $d_{ub} \le \alpha$.

For the lower bound we use the mass distribution given by the measure $\tilde{\mu}$ obtained by putting weight $1/z_n$ on each set in E_n. From the growth of the general branching process, $z_n \le c_1 W e^{\alpha n}$ for large n. Thus for $\varepsilon > 0$ there exists a constant c_2 such that $\tilde{\mu}(U) \le c_2 |U|_r^{\alpha-\varepsilon}$, for all sets U with diameter in the effective resistance metric $|U|_r < e^{-n}$. By [6], Theorem 4.2, the lower bound on the Hausdorff dimension in the effective resistance metric will be α as desired. □

The behavior of the measure μ introduced in Section 4 can be described.

LEMMA 5.3. *There exist constants $c_{5.1}(\omega)$, $c_{5.2}(\omega)$ such that*

$$c_{5.1}(\omega)e^{(\alpha+1)n} \le \sum_{x\in G_n} \rho_n(x) \le c_{5.2}(\omega)e^{(\alpha+1)n} \qquad \forall\ n \ge 0.$$

PROOF. To see this we use the general branching process with random characteristic given by

$$\phi(t) = e^{\sigma} I_{\{\tau < t\}}.$$

As the births in the general branching process are determined by the conductivity, each individual alive at time t corresponds to a subset of the fractal with conductance given by its birth time e^{σ}. Thus

$$Z_n^{\phi} = \sum_{x\in G_n} \rho_n(x).$$

By the assumption of boundedness of the components, there exists a constant c^* such that for all individuals alive at time t, their conductivity e^{σ_i} satisfies $c^* e^t \le e^{\sigma} \le e^t$. Now we can estimate the growth of the characteristic. For the upper bound we write z_n for the population size of the general branching process at time n; then there is a constant c such that

$$\begin{aligned} Z_n^{\phi} &= \sum_i e^{\sigma_i} I_{\{\tau_i > n - \sigma_i\}} \\ &\le e^n z_n \\ &\le cWe^{(\alpha+1)n}, \qquad \mathbb{P}\text{-a.s.} \end{aligned}$$

for large enough n. The lower bound is treated similarly. □

This allows us to prove the following result for the measure μ. Let $S_{n(y)}\omega = \omega_{n(y)+1}\omega_{n(y)+2}\dots$ be the shift on sequences y in the tree ω.

THEOREM 5.4. *The measures μ_n converge weakly to a measure μ equivalent to the α-dimensional Hausdorff measure in the effective resistance metric on the fractal G.*

PROOF. The weak convergence follows from Lemma 5.3 and the definition of μ_n. For any $\triangle_n(x) \subset G$ we have

$$\begin{aligned} \mu(\triangle_n(x)) &= \lim_{m\to\infty} \frac{\sum_{y\in \triangle_n(x)\cap G_m} \rho_m(y)}{\sum_{y\in G_m} \rho_m(y)} \\ &= \lim_{m\to\infty} \rho_n(x) \frac{\sum_{y\in G_{m-n}(S_{n(x)}\omega)} \rho_{m-n}(y)}{\sum_{y\in G_m} \rho_m(y)} \\ &\le ce^n \lim_{m\to\infty} \frac{W(S_{n(x)}\omega)e^{(\alpha+1)(m-n)}}{W(\omega)e^{(\alpha+1)m}} \end{aligned}$$

for some constant c. Thus

$$\mu(\triangle_n(x)) \le c_{5.3}(\omega) e^{-\alpha n}, \tag{5.2}$$

where the constant is given by $c_{3.5}(\omega) = c \sup_{\triangle_n} W(S_{n(x)}\omega) W^{-1}(\omega)$. We denote the corresponding lower bound constant by $c_{5.4}(\omega)$. Then for any $\varepsilon > 0$ there exist c_1, c_2 such that

$$c_1 \le \mu(\triangle_n(x)) e^{(\alpha+\varepsilon)n} \le \mu(\triangle_n(x)) e^{(\alpha-\varepsilon)n} \le c_2 \qquad \forall\, x \in G, \forall\, n \ge 0.$$

By using a density theorem, [7], Proposition 4.9, we see that the measure μ must be equivalent to the α-dimensional Hausdorff measure in the effective resistance metric on the fractal. □

We now obtain finer estimates on the measure of triangles $\triangle_n$. These will be crucial to the later estimates on the transition density for the Brownian motion.

THEOREM 5.5. *There exist constants $c_{5.5}(\omega), c_{5.6}(\omega)$ such that for all $n > 0$ and $x \in G(\omega)$,*

$$\begin{aligned} c_{5.5}(\omega)(\log n)^{-b_2} e^{-\alpha n} &\le \mu(\triangle_n(x)) \\ &\le c_{5.6}(\omega)(\log n)^{b_K} e^{-\alpha n}, \qquad \mathbb{P}\text{-}a.s. \end{aligned} \tag{5.3}$$

There exist constants $c_{5.7}(\omega), c_{5.8}(\omega)$ such that for all $n > 0$,

$$\begin{aligned} c_{5.7}(\omega) n^{-b_2} e^{-\alpha n} &\le \min_{\triangle_n} \mu(\triangle_n) \le \max_{\triangle_n} \mu(\triangle_n) \\ &\le c_{5.8}(\omega) n^{b_K} e^{-\alpha n}, \qquad \mathbb{P}\text{-}a.s. \end{aligned} \tag{5.4}$$

PROOF. For (5.3) consider (5.2) in which the constants $c_{5.3}(\omega)$ and $c_{5.4}(\omega)$ can be written in terms of W. For the lower bound, consider a given $x \in G$, then for all n,

$$\mu(\triangle_n(x)) \ge c \frac{W(S_{n(x)}\omega)}{W(\omega)} e^{-\alpha n}. \tag{5.5}$$

As a function of n the random variable W is a constant. Thus all we need is an estimate on the size of the fluctuations in $W(S_{n(x)}\omega)$, each of which has the same distribution as W.

For the lower bound, we use the estimates on the tail of W obtained in Corollary 3.5 and the first Borel–Cantelli lemma. As

$$\mathbb{P}(W(S_n\omega) \le \delta) \le c_{3.3} \exp(-c_{3.4} \delta^{\alpha_2/(\alpha_2-\alpha)}) \qquad \forall\, n,$$

if we set $\delta = c(\log n)^{(\alpha_2-\alpha)/\alpha_2}$, then

$$\mathbb{P}\big(W(S_n\omega) \le c(\log n)^{-b_2}\big) \le c_{3.3} n^{-c_{3.4}c}.$$

Hence, as

$$\mathbb{P}\left(\frac{W(S_n\omega)}{W} \le \frac{c}{W}(\log n)^{-b_2} \right) = \mathbb{P}\left(W(S_n\omega) \le c(\log n)^{-b_2} \right),$$

we see that

$$\sum_n \mathbb{P}\left(\frac{W(S_n\omega)}{W} \le \frac{c}{W}(\log n)^{-b_2} \right) \le c_{3.3} \sum_n n^{-c_{3.4}c} < \infty,$$

by the choice of c. Thus

$$\frac{W(S_n\omega)}{W} \ge c(\omega)(\log n)^{-b_2}, \qquad \mathbb{P}\text{-a.s.}$$

For the upper bound we follow the same approach and use Theorem 3.7 to obtain the upper bound on the fluctuations in $W(S_n\omega)$.

For (5.4) we need to estimate the maximum and minimum of $W(S_{n(x)}\omega)$ over level n. As the number of possible triangles on level n is bounded above by $e^{\alpha_K n}$ and each $W(S_{n(x)}\omega)$ is independent and identically distributed we can use the following:

$$\begin{aligned} P\left(\max_{\triangle_n} W(S_n\omega) > \delta \right) &\le 1 - \left(1 - P\left(W(S_n\omega) > \delta\right)\right)^{e^{\alpha_K n}} \\ &\le e^{\alpha_K n} P\left(W(S_n\omega) > \delta\right) \\ &\le c^n \quad \text{with } c < 1, \end{aligned}$$

if $\delta \le c_1 n^{b_K}$, by choice of c_1 in Theorem 3.7. In which case

$$\max_{\triangle_n} W(S_n\omega) \le c_{5.8} n^{b_K}, \qquad \mathbb{P}\text{-a.s.}$$

as desired. As above we use the first Borel–Cantelli lemma to obtain the result. Similar methods yield for the lower bound. □

We conclude with another indication of the type of bad behavior that will occur within the fractal. Define the ratio of types of map at level n for a branch by

$$\eta_n(\mathbf{i}) = \frac{1}{n} \sum_{j=0}^{n} I_{\{V_{[\mathbf{i}]_j} = 2\}}, \qquad \mathbf{i} \in T \setminus T_\infty.$$

For each $\mathbf{i} \in T \setminus T_\infty$, $\eta_n(\mathbf{i}) \to p_2$, pointwise; however, the convergence is not uniform.

LEMMA 5.6. *If $p_\nu > 2/\nu(\nu+1)$ for $\nu \ge 3$, then*

$$\lim_{n\to\infty} \inf_{\mathbf{i} \in T\setminus T_\infty} \eta_n(\mathbf{i}) = 0 \quad a.s.$$

If $p_2 > 1/3$, then

$$\lim_{n\to\infty} \sup_{\mathbf{i} \in T\setminus T_\infty} \eta_n(\mathbf{i}) = 1 \quad a.s.$$

PROOF. We consider the case $p_3 > 1/6$ as the other case follows in the same way. The process which generates type 3 triangles, $\hat{Z}_n$, has generating function $f(u) = 1 - p_3 + p_3 u^6$ and is supercritical for $p_3 > 1/6$. Hence it has an extinction probability $p_e = f(p_e) < 1$.

Let $\varepsilon > 0$ and consider

$$P\left(\inf_{\mathbf{i}\in T\setminus T_\infty} \eta_n(\mathbf{i}) > \varepsilon\right) = P\left(\sum_{j=1}^{n} I_{\{V_{[\mathbf{i}]_j}=2\}} > n\varepsilon, \forall\, \mathbf{i} \in T \setminus T_\infty\right).$$

For this event to occur, we require that, for each of the individuals alive at generation $[n\varepsilon]$, all their $\hat{Z}_n$ offspring must become extinct. As the minimum number of individuals at this generation is $3^{[n\varepsilon]}$,

$$P\left(\inf_{\mathbf{i}\in T\setminus T_\infty} \eta_n(\mathbf{i}) > \varepsilon\right) \le p_e^{3^{[n\varepsilon]}} \le p_e^{c^n},$$

for $c > 1$. By the Borel–Cantelli lemma, as

$$\sum_n P\left(\inf_{\mathbf{i}\in T\setminus T_\infty} \eta_n(\mathbf{i}) > \varepsilon\right) < \infty,$$

then

$$P\left(\inf_{\mathbf{i}\in T\setminus T_\infty} \eta_n(\mathbf{i}) > \varepsilon \text{ i.o.}\right) = 0.$$

This holds for each ε, which gives the result. $\square$

6. Resolvent and local time for the process. The Dirichlet form has an associated resolvent which is approximated by the sequence of resolvents associated with the sequence of forms. In this section we will approximate the process by the sequence of Markov chains associated with the resolvents. Let X_t^n denote the Markov chain on G_n moving according to the conductivities assigned to the edges of G_n. By construction this is the process on G watched only when it is in G_n (see [14]).

Let the local time at x for the chain X^n be denoted by

$$L_t^x(X^n) = \int_0^t I_{\{X_s^n = x\}}\, ds.$$

We normalize the local time by writing

$$\Lambda_t^x(X^n) = \frac{L_t^x(X^n)}{\mu_n(x)} \qquad \forall\, x \in G_n.$$

We will write $T_A^n = \inf\{t\colon\ X_t^n \in A^c\}$ for the exit time from a set A. The normalized resolvent densities for the Markov chain killed upon leaving an open connected set A are defined as

$$r_A^n(x, y) = E^x \int_0^{T_A^n} \frac{I_{\{X_s^n = y\}}}{\mu_n(y)}\, ds = E^x \Lambda_{T_A^n}^y(X^n) \qquad \forall\, x, y \in G_n.$$

We begin by proving some elementary results for the sequence of resolvents.

LEMMA 6.1.

$$(6.1)(\mathrm{i}) \qquad r_A^n(x,y) \le \frac{\delta(x,y)}{\rho_m(x)} + \sum_{z \in N_m(x)} \frac{\rho_m(x,z)}{\rho_m(x)} r_A^n(z,y), \qquad x, y \in G_m;$$

$$(6.2)(\mathrm{ii}) \qquad r_A^n(x,y) = r_A^k(x,y) \qquad \forall\, k \ge n,\ x, y \in G_n;$$

$$(6.3)(\mathrm{iii}) \qquad r_A^n(x,y) = r_A^n(y,x) \qquad \forall\, x, y \in G_n.$$

The proof is essentially the same as [14], Lemma 7.1.

From Lemma 6.1(ii) we can work with the limiting resolvent. Let

$$r_A(x,y) = \lim_{n\to\infty} r_A^n(x,y) \qquad \forall x, y \in G_m, \forall\, m.$$

For each n, by the strong Markov property of the process, we can write

$$r_A^n(x,y) = \psi_A^n(x,y) r_A^n(y,y),$$

where $\psi_A^n(x,y) = P^x(T_y^n < T_A^n)$ with $T_y^n = \inf\{t\colon\ X_t^n = y\}$. Thus continuity properties of the resolvent can be recovered from these first passage time probabilities. The existence of the limit of the sequence of first passage time probabilities follows from the convergence of the resolvents, which is a consequence of the convergence of the Dirichlet forms; hence

$$(6.4) \qquad \psi_A(x,y) = \lim_{n\to\infty} \psi_A^n(x,y) = \frac{r_A(x,y)}{r_A(y,y)}.$$

In order to get bounds on the passage time probabilities, we need bounds on ratios of resolvents. The methods to be used will be the same as those in [4].

First, we bound the resolvent on the diagonal.

LEMMA 6.2. *There are constants $c_{6.1}, c_{6.2}$ such that*

$$(6.5) \qquad r_A(x,x) \ge \frac{c_{6.1}}{\rho_m(x)}, \qquad x \in G_\infty \cap A,\ \triangle_m(x) \subset A,$$

$$(6.6) \qquad r_A(x,x) \le \frac{c_{6.2}}{\rho_m(x)} + \max_{z \in \partial \triangle_m(x)} r_A(z,z) \qquad \forall\, x \in G_\infty.$$

PROOF. For (6.5), let $A = D_m(x)$; then, doing some simple calculations on the interior of one possible neighborhood, shown in Figure 3,

$$r_A(a,a) = (\rho_m(a))^{-1} + \frac{2\rho_m(a,b)}{\rho_m(a)} r_A(b,a),$$

$$r_A(b,a) = \tfrac{1}{4}(r_A(a,a) + r_A(b,a)),$$

$$r_A(c,a) = r_A(d,a) = r_A(e,a) = 0.$$

Solving this gives

$$r_A(a,a) = \frac{3}{3\rho_m(a) - 2\rho_m(a,b)} = c(\rho_m(a))^{-1},$$

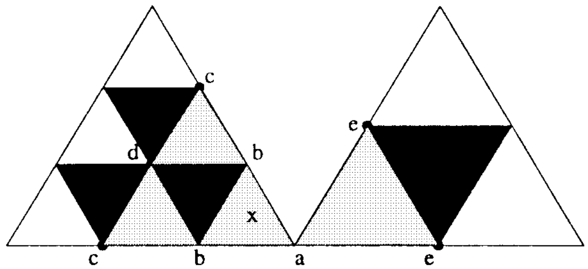

FIG. 3. *A neighborhood of x for the random recursive gasket.*

and over all the interior points, there are constants $C, \hat{C}$ such that

$$C(\rho_m(a))^{-1} \le r_A(\cdot,\cdot) \le \hat{C}(\rho_m(a))^{-1}.$$

A similar calculation can be done for each possible neighborhood. For points in $G_m \setminus G_{m-1}$ the equations will be the same as for a triangle of that type, the only complications occur at the higher level vertices, such as a in Figure 2, where we introduce the terms dependent upon the environment. As there are only a finite number of neighborhoods, though there may be a large number, we can always find a constant $c_{6.1} = \min\{c\colon r_A(\cdot,\cdot) \ge c(\rho_m(a))^{-1}\}$, which gives the bound for $a \in G_n$. Then $r_A(x,x) \ge r_A(z,x) = r_A(x,z)$, by (6.4) and symmetry, for $z \in \partial \triangle_m(x)$. So

$$\begin{aligned} r_A(x,z) &= \sum_{a \in \partial \triangle_m(x)} P^x\big(X_{T_{\triangle_m}} = a\big) r_A(a,z) \\ &\ge \min_{a \in \partial \triangle_m(x)} r_A(a,z) \qquad (6.7)\\ &\ge c_{6.1}(\rho_m(a))^{-1}, \qquad a \in D_m(x) \cap G_m \setminus G_{m-1}. \end{aligned}$$

For (6.6), we follow [4], Lemma 5.4, in a similar fashion to the lower bound. □

Let

$$\phi_A(x,y) = 1 - \psi_A(x,y) = \frac{r_A(y,y) - r_A(x,y)}{r_A(y,y)}. \qquad (6.8)$$

LEMMA 6.3. *There exist constants $c_{6.3}, c_{6.4}$ such that if $\triangle_m \subset A$, then*

$$\phi_A(x,z) r_A(z,z) \le c_{6.3}\, \rho_m(z)^{-1}, \qquad x \in \triangle_m,\ z \in \partial \triangle_m(x), \qquad (6.9)$$

$$r_A(y,y) - r_A(z,z) \le c_{6.4}\, \rho_m(z)^{-1}, \qquad y \in \triangle_m,\ z \in \partial \triangle_m(y). \qquad (6.10)$$

PROOF. For (6.9), we recall (6.1) and rearrange to get

$$\sum_{b \in N_m(z)} \frac{\rho_m(z,b)}{\rho_m(z)} \big(r_A(z,z) - r_A(b,z) \big) \le \rho_m(z)^{-1}, \qquad z \in G_m .$$

Thus for $a \in N_m(z)$, we have

$$\begin{aligned} \frac{\rho_m(z,a)}{\rho_m(z)} \big(r_A(z,z) - r_A(a,z) \big) &\le \sum_{b \in N_m(z)} \frac{\rho_m(z,b)}{\rho_m(z)} \big(r_A(z,z) - r_A(b,z) \big) \\ &\le \rho_m(z)^{-1}, \qquad a, z \in G_m \cap \triangle_m(x). \end{aligned}$$

Rearranging gives, for $a, z \in G_m \cap \triangle_m(x)$,

$$\begin{aligned} \phi_A(a,z) r_A(z,z) &= r_A(z,z) - r_A(a,z) \\ &\le \rho_m(z,a)^{-1} \le c_1 \rho_m(z)^{-1}. \end{aligned} \tag{6.11}$$

To extend this to the interior of the triangle we use the fact that $r_A(\cdot, z)$ is harmonic within $\triangle_m$ and thus attains its extreme values on the boundary. So for $x \in \triangle_m$ and $z \in \partial \triangle_m(x)$ the result holds.

For (6.10), use (6.6),

$$r_A(y,y) \le c_{6.2}\, \rho_m(z)^{-1} + \max_{z \in \partial \triangle_m(y)} r_A(z,z) \qquad \forall\, y \in G_\infty .$$

Then using (6.11) and $r_A(a,a) \ge r_A(z,a) = r_A(a,z)$, we get

$$r_A(y,y) - r_A(z,z) \le c_2\, \rho_m(z)^{-1},$$

and we have the result. □

We now have a fixed upper bound on the resolvent diagonal, when the set $A = D_m(x)$.

COROLLARY 6.4. *There exist constants $c_{6.5}, c_{6.6}$ such that for $x \in G_\infty$,*

$$c_{6.5}\, \rho_m(x)^{-1} \le r_{D_m(x)}(x,x) \le c_{6.6}\, \rho_m(x)^{-1}. \tag{6.12}$$

For the proof, the lower bound is given in Lemma 6.2. The upper is similar to [4], Lemma 5.8.

Now we obtain a loose bound on the denominator in (6.8).

LEMMA 6.5. *There exists a constant, $c_{6.7}$, such that,*

$$|r_A(y,y) - r_A(x,y)| \le c_{6.7}\, d_r(x,y) \qquad \forall\, x, y \in G_\infty .$$

PROOF. Let m be such that $e^{-m-1} \le d_r(x,y) \le e^{-m}$. By Lemma 4.2 and our choice of graph, $\triangle_m(x) \cap \triangle_m(y) \ne \emptyset$, and we can choose $z \in \partial \triangle_m(x) \cap$

$\partial \triangle_m(y)$. Then

$$\begin{aligned} r_A(y,y) - r_A(x,y) &= \phi_A(x,y) r_A(y,y) \\ &\le (\phi_A(x,z) + \phi_A(z,y)) r_A(y,y) \\ &= \phi_A(x,z) r_A(z,z) \\ &\quad + (1+\phi_A(x,z))(r_A(y,y) - r_A(z,z)) \\ &\quad + \phi_A(z,y) r_A(y,y) - r_A(y,y) + r_A(z,z). \end{aligned}$$

Using $\phi_A(z,y) r_A(y,y) - r_A(y,y) + r_A(z,z) = \phi_A(y,z) r_A(z,z)$ we get that

$$\begin{aligned} r_A(y,y) - r_A(x,y) \le 2\phi_A(x,z) r_A(z,z) \\ + 2(r_A(y,y) - r_A(z,z)). \end{aligned} \tag{6.13}$$

By Lemma 6.3 we have bounds on both terms in this expression,

$$\begin{aligned} r_A(y,y) - r_A(x,y) &\le 2c_{6.3}(\rho_m(z))^{-1} + 2c_{6.4}\rho_m(z)^{-1} \\ &= C\rho_m(z)^{-1}. \end{aligned}$$

Now by the choice of m, we get $\rho_m(z)^{-1} \le ce^{-m} \le c_{6.7} d_r(x,y)$, the required upper bound. □

Thus the resolvent is uniformly continuous in both variables and can be extended uniquely to a continuous function on $G \times G$. The estimates obtained above will hold for all $x, y \in G$ and we then have the following corollary.

COROLLARY 6.6. (i) *The process X_t hits points, $P^x(T_x = 0) = 1$.*
(ii) *The fine topology on G is the ordinary topology.*

For the proof, these results can be obtained from the above estimates in the same way as [4], Corollary 5.14.

We end this section with some estimates on exit times from neighborhoods.

LEMMA 6.7. *There exist constants $c_{6.8}(\omega), c_{6.9}(\omega)$, such that $\mathbb{P}$-a.s. for each $x \in G$ and all $m > 0$,*

$$\begin{aligned} c_{6.8}(\omega)(\log m)^{-b_2} e^{-(\alpha+1)m} &\le E^x T_{D_m(x)} \\ &\le c_{6.9}(\omega)(\log m)^{b_K} e^{-(\alpha+1)m}, \end{aligned} \tag{6.14}$$

and constants $c_{6.10}(\omega), c_{6.11}(\omega)$ such that $\mathbb{P}$-a.s. for all $m > 0$,

$$\begin{aligned} c_{6.10}(\omega) m^{-b_2} e^{-(\alpha+1)m} &\le \inf_{x\in G} E^x T_{\triangle_m(x)} \\ &\le \sup_{x \in G} E^x T_{\triangle_m(x)} \le c_{6.11}(\omega) m^{b_K} e^{-(\alpha+1)m}. \end{aligned} \tag{6.15}$$

PROOF. The exit time from an m-level triangle can be written

$$E^x T_{D_m(x)} = E^x \int_{D_m(x)} L^y_{T_{D_m(x)}} \mu(dy) = \int_{D_{m(x)}} r_{D_m(x)}(x, y) \mu(dy).$$

By Corollary 6.4, $r_{D_m(x)}(x, y) = r_{D_m(x)}(y, x) \le r_{D_m(x)}(x, x) \le c_{6.7} \rho_m(x)^{-1}$. Then

$$E^x T_{D_m(x)} \le c_{6.6} \rho_m(x)^{-1} \mu(D_m(x)) \le c_{6.9}(\omega)(\log m)^{b_K} e^{-(\alpha+1)m},$$

by Theorem 5.5 (5.3).

For the lower bound we use the Hölder continuity of the resolvent. If we write

$$r_{D_m(x)}(x, y) = r_{D_m(x)}(y, y) - \left(r_{D_m(x)}(y, y) - r_{D_m(x)}(x, y)\right),$$

then, applying Corollary 6.4 and Lemma 6.5, we see that there are c_1, k such that for $y \in \triangle_{m+k}(x)$,

$$r_{D_m(x)}(x, y) \ge c_1 \rho_m(x)^{-1}.$$

Then

$$E^x T_{\triangle_m(x)} \ge c_1 \rho_m(x)^{-1} \mu(\triangle_{m+k}(x)),$$

and applying Theorem 5.5 gives the desired result.

For the second result we apply the same approach but use (5.4) from Theorem 5.5. □

From this result we can obtain the following results on the oscillations in the mean exit times.

COROLLARY 6.8. *There exist constants* $c_{6.12}, c_{6.13}, c_{6.14}, c_{6.15}$, *dependent on* ω *such that*

$$\begin{aligned} c_{6.12}(\omega) &\le \liminf_{m\to\infty} \frac{E^0 T_{\triangle_m(0)}}{e^{-(\alpha+1)m}(\log m)^{-b_2}} \\ &\le \limsup_{m\to\infty} \frac{E^0 T_{\triangle_m(0)}}{e^{-(\alpha+1)m}(\log m)^{b_K}} \le c_{6.13}(\omega), \qquad \mathbb{P}\text{-}a.s. \end{aligned}$$

and

$$\begin{aligned} c_{6.14}(\omega) &\le \liminf_{r\downarrow 0} \frac{E^0 T_{B_r(0)}}{r^{\alpha+1}(\log\log(1/r))^{-b_2}} \\ &\le \limsup_{r\downarrow 0} \frac{E^0 T_{B_r(0)}}{r^{\alpha+1}(\log\log(1/r))^{b_K}} \le c_{6.15}(\omega), \qquad \mathbb{P}\text{-}a.s. \end{aligned}$$

PROOF. The first result is a simple consequence of (6.14). For the second result observe that $B_{c_{4.3}r}(0) \subset \triangle_m(0) \subset B_{c_{4.4}r}(0)$ if $r = e^{-m}$. □

7. Shortest paths and crossing times. Now that we have some preliminary resolvent estimates, we can obtain an estimate on the crossing time of a triangle. This will then be used to obtain some more resolvent estimates and then estimates on the transition density using the techniques developed in [3] and [8]. We use [2], Lemma 1.1 and Lemma 4.3, to show how the mean crossing times determine the tail behavior of the actual crossing time.

Let

$$T_0^n(\omega) = \inf\{t \geq 0 : X_t \in G_n\},$$

$$T_{i+1}^n(\omega) = \inf\left\{t > T_i^n(\omega) : X_t \in G_n(\omega) \setminus \left\{X_{T_i^n(\omega)}\right\}\right\},$$

$$\tilde{T}_i^n(\omega) = T_i^n(\omega) - T_{i-1}^n(\omega),$$

denote the sequence of crossing times for the process on G_n.

A simple extension of Lemma 6.7 leads to the following lemma.

LEMMA 7.1. *There exist constants $c_{7.1}(\omega), c_{7.2}(\omega)$, such that $\mathbb{P}$-a.s. for all $m > 0$,*

$$\begin{aligned} c_{7.1}(\omega)\, m^{-b_2} e^{-(\alpha+1)m} \leq \inf_{x \in G} E^x T_1^m \leq \sup_{x \in G} E^x T_1^m \\ \leq c_{7.2}(\omega)\, m^{b_\kappa} e^{-(\alpha+1)m}. \end{aligned} \tag{7.1}$$

We now give an elementary technical lemma which will be useful.

LEMMA 7.2. *If, for $x, y > e$, there exist positive constants c_1, c_2 such that for some a, b,*

$$c_1 y \leq x^a (\log x)^b \leq c_2 y,$$

then there exist positive constants c_3, c_4 such that

$$c_3 y^{1/a} (\log y)^{-b/a} \leq x \leq c_4 y^{1/a} (\log y)^{-b/a} \qquad \forall\, x, y > e.$$

The effective resistance metric does not take into account the geometry of the shortest paths on the fractal. We determine bounds on the growth rate in n of the number of steps in the shortest path on G_n across the unit fractal. For $x, y \in G_n$, $m > n$, let $\Pi_{n,m}(x, y)$ denote the set of all paths from x to y on G_m. Let $|\pi|$ denote the number of steps in the path π and let

$$a_{n,m}(x, y) = \inf\{|\pi| : \pi \in \Pi_{n,m}(x, y)\}$$

be the number of steps in the shortest path on G_m between $x, y \in G_n$.

LEMMA 7.3. *There exists a constant κ such that, for each pair $x, y \in G_0$,*

$$\begin{aligned} \kappa_2 = \exp(\log 2/\log(5/3)) \leq \liminf_{m \to \infty} a_{0,m}(x, y)^{1/m} \\ \leq \limsup_{m \to \infty} a_{0,m}(x, y)^{1/m} \leq \kappa, \qquad \mathbb{P}\text{-a.s.} \end{aligned}$$

PROOF. The lower bound is just the trivial worst case bound. The upper bound uses a branching argument. The following subbranching inequality holds:

$$a_{0,m+n}(x,y) \le \sum_{i=1}^{a_{0,m}(x,y)} a_{m,m+n}(x_{i-1},x_i) \qquad \forall\, m,n > 0, \tag{7.2}$$

where $a_{m,m+n}(x_{i-1},x_i) = a_{m,m+n} = a_{0,n}$ in distribution (and we suppress the reference to the points x, y). We then follow the proof of a similar result in [5]. First, taking expectations and using the independence, we have a submultiplicative sequence and hence there exists κ such that

$$\lim_{n\to\infty} \mathbb{E}(a_{0,n})^{1/n} = \inf_{n>0} \mathbb{E}(a_{0,n})^{1/n} = \kappa.$$

By inequality (7.2) we have

$$a_{0,(n+1)k} \le \sum_{i=1}^{a_{0,nk}} a_{0,k}(i).$$

Then consider a branching process Z_n with offspring distribution given by $a_{0,k}$, so that $a_{0,(n+1)k} \le Z_{n+1}$. Now let $u > \kappa$ and consider

$$\mathbb{P}\big(a_{0,nk} \ge u^{nk}\big) \le \mathbb{P}\big(Z_n \ge u^{nk}\big) \le \mathbb{P}\big(Z_n \ge c^n\big),$$

where $c = u^k \ge m = \mathbb{E}(a_{0,k}) \ge \kappa^k$. From the boundedness of $a_{0,k}$, using the ideas of Lemma 3.6 and Theorem 3.7, we have an exponential tail for the limiting random variable of the standard branching process martingale, $W = \lim_{n\to\infty} Z_n/m^n$. Thus there are constants c_1, c_2 such that

$$\mathbb{P}(Z_n > \delta m^n) \le \hat{c}\mathbb{P}(W > \delta) \le c_1 \exp(-c_2\delta) \qquad \forall\,\delta > 0. \tag{7.3}$$

Hence

$$\mathbb{P}(Z_n > c^n) \le c_1 \exp\big(-c_2(c/m)^n\big),$$

and thus

$$\sum_n \mathbb{P}\big(a_{0,nk} \ge u^{nk}\big) < \infty \qquad \forall\, u > \kappa.$$

We can approximate the general case using, for all $v > 0$,

$$\mathbb{P}\big(a_{0,nk+j} \ge v^{nk+j}\big) \le \mathbb{P}\big(a_{0,nk+j} \ge v^{nk}\big), \qquad j = 1,\dots,k$$

and, for all $c > 0$,

$$\mathbb{P}\big(a_{0,nk+j} \ge c\big) \le \mathbb{P}\big(a_{0,(n+1)k} \ge c\big), \qquad j = 1,\dots,k.$$

Thus we have

$$\begin{aligned}\sum_n \mathbb{P}(a_{0,n} \ge v^n) &= \sum_n \sum_{j=1}^k \mathbb{P}\big(a_{0,nk+j} \ge v^{nk+j}\big)\\ &\le \sum_j \sum_n \mathbb{P}\big(a_{0,nk+j} \ge v^{nk}\big)\\ &\le k\sum_n \mathbb{P}\big(a_{0,(n+1)k} \ge v^{nk}\big) < \infty,\end{aligned}$$

for all $v > \kappa$. Hence $\limsup_{n\to\infty} a_{0,n}^{1/n} \le \kappa$, $\mathbb{P}$-a.s. □

The shortest paths on G_m between points on G_0 will not necessarily be straight. Thus we can find an upper bound on κ using a general branching process which describes the growth of the number of steps in the straight path. In this case the appropriate life length and reproduction point process are

$$(\lambda_x, \xi_x) = \left(\log \gamma_\nu, \nu I_{\{t = \log \gamma_\nu\}}\right) \quad \text{with probability } p_\nu.$$

The parameter $\kappa \le \hat{\kappa}$, the Malthusian parameter for the branching process, is given by

$$\hat{\kappa} = \left\{ s \colon \sum_{\nu=2}^{K} \nu \gamma_\nu^{-s} p_\nu = 1 \right\}.$$

We need a uniform bound on the length of the shortest path between two points in the fractal, where the step size is determined by the resistance metric.

LEMMA 7.4. *For $\varepsilon > 0$ there exists a constant $c_{7.3}$ such that for all $i, n \ge 0$, if $x, y \in G_i$ with $y \in N_i(x)$, then*

$$a_{i, n+i}(x, y) \le c_{7.3}(i+1)(\kappa + \varepsilon)^n, \qquad \mathbb{P}\text{-}a.s.$$

PROOF. By the previous lemma we know that $a_{0,n}^{1/n}(x, y)$ converges and we wish to determine how the convergence depends on x, y. We begin by considering the behavior in n:

$$(7.4) \quad \mathbb{P}\left(\sup_n \frac{a_{i, n+i}(x, y)}{(\kappa + \varepsilon)^n} > \delta \right) \le \sum_{n=0}^{\infty} \mathbb{P}\left(a_{i, n+i}(x, y) > \delta (\kappa + \varepsilon)^n \right).$$

We know that $a_{i, n+i}(x, y)$, for $x, y \in G_i$ with $x \in N_i(y)$, is a bounded discrete random variable equal in distribution to $a_{0,n}(x_0, y_0)$, where $x_0, y_0 \in G_0$, and that

$$\kappa = \lim_{k \to \infty} \mathbb{E}(a_{0,k})^{1/k}.$$

Hence for $\varepsilon > 0$ there exists a k_0 such that $\mathbb{E}(a_{0,k}) < (\kappa + \varepsilon)^k$ for all $k > k_0$. Fix such a k and let Z_n be the branching process with offspring distribution $a_{0,k}$. Then $a_{0,(n+1)k} \le Z_{n+1}$ and, as in (7.3), we have

$$\mathbb{P}\left(a_{0,nk} > \delta (\mathbb{E} a_{0,k})^n \right) \le \mathbb{P}(Z_n > \delta m^n) \le c_1 \exp(-c_2 \delta).$$

From this we can deduce that for any l, which we can write $l = nk + j$, we have

$$\begin{aligned}
\mathbb{P}\Big(a_{0,l} > \delta \mathbb{E}(a_{0,k})^{l/k}\Big) &= \mathbb{P}\Big(a_{0,nk+j} > \delta \mathbb{E}(a_{0,k})^{n+j/k}\Big) \\
&\le \mathbb{P}\Big(a_{0,(n+1)k} > \delta \big(\mathbb{E}(a_{0,k})\big)^{n+1} \big(\mathbb{E}(a_{0,k})\big)^{j/k-1}\Big) \\
&\le c_1 \exp\Big(-c_2 \big(\mathbb{E}(a_{0,k})\big)^{j/k-1} \delta\Big) \\
&\le c_1 \exp(-c_3 \delta).
\end{aligned}$$

Using this in (7.4) we have

$$\begin{aligned}
\mathbb{P}\left(\sup_n \frac{a_{0,n}}{(\kappa+\varepsilon)^n} > \delta\right) &\le \sum_{n=0}^{\infty} \mathbb{P}\left(\frac{a_{0,n}}{(\kappa+\varepsilon)^n} > \delta\right) \\
&\le \sum_{n=0}^{\infty} \mathbb{P}\left(\frac{a_{0,n}}{\big(\mathbb{E}(a_{0,k})\big)^{n/k}} > \delta \frac{(\kappa+\varepsilon)^n}{\big(\mathbb{E}(a_{0,k})\big)^{n/k}}\right) \\
&\le \sum_{n=0}^{\infty} c_1 \exp\left(-c_2 \delta \left(\frac{\kappa+\varepsilon}{\big(\mathbb{E}(a_{0,k})\big)^{1/k}}\right)^n\right) \\
&\le c_3 \exp(-c_4 \delta),
\end{aligned}$$

by choice of $k > k_0$.

Finally,

$$\begin{aligned}
\mathbb{P}\left(\max_{x, y \in G_i} \sup_n \frac{a_{i,n+i}}{(\kappa+\varepsilon)^n} > \delta\right) &\le \sum_{x, y \in G_i} \mathbb{P}\left(\sup_n \frac{a_{i,n+i}}{(\kappa+\varepsilon)^n} > \delta\right) \\
&\le c_5 \exp(2\alpha i) \exp(-c_4 \delta).
\end{aligned}$$

Hence there exists a constant c_6 such that

$$\mathbb{P}\left(\max_{x, y \in G_i} \sup_n \frac{a_{i,n+i}}{(\kappa+\varepsilon)^n} > c_6 i\right) \le \exp(-c_7 i),$$

and Borel–Cantelli gives the result. $\square$

From Lemma 7.4 we can control the number of steps in the shortest path. First, we extend the definition of $\Pi_{n,m}(x, y)$ to all $x, y \in G$ by setting $\Pi_m(x, y) = \Pi_{m,m}(x_m, y_m)$, where $x_m, y_m \in G_m$ are given by the lower left corners of $\triangle_m(x)$ and $\triangle_m(y)$, respectively. Hence we can define

$$a_m(x, y) = \inf\{|\pi| : \pi \in \Pi_m(x, y)\}.$$

THEOREM 7.5. *For $\varepsilon > 0$, there exists a constant $c_{7.4}$ such that for all $x, y \in G$ with $d_r(x, y) \le e^{-k}$ and for each m there exists a path $\pi = \{x_0, \dots, x_{|\pi|}\} \in \Pi_{k+m}(x, y)$ such that*

$$|\pi| \le c_{7.4}(k+1)(\kappa+\varepsilon)^m, \qquad \mathbb{P}\text{-}a.s.$$

PROOF. For any points $x, y \in G$ with $d_r(x, y) \le e^{-k}$, we can find a sequence of points $\{\tilde{x}_i, \tilde{y}_i\}_{i=k}^{k+m}$ where $\tilde{x}_i, \tilde{y}_i \in G_i$ and $\tilde{x}_i \in \triangle_i(\tilde{x}_{i+1})$, $\tilde{x}_{k+m} \in \triangle_{k+m}(x)$, and similarly for $\tilde{y}$ and $\tilde{x}_k \in N_k(\tilde{y}_k)$. The path π between x, y on G_{k+m} can then be constructed as the concatenation of the shortest paths on G_{k+m} linking the points $\tilde{x}_{k+m}, \tilde{x}_{k+m-1}, \dots, \tilde{x}_k, \tilde{y}_k, \dots, \tilde{y}_{k+m}$. The length of this path is then bounded by

$$|\pi| \le \sum_{i=k+1}^{k+m} a_{k+m}(\tilde{x}_i, \tilde{x}_{i-1}) + a_{k+m}(\tilde{x}_k, \tilde{y}_k) + \sum_{i=k+1}^{k+m} a_{k+m}(\tilde{y}_i, \tilde{y}_{i-1}).$$

By Lemma 7.4 we have a uniform bound on the length of each piece of path and, if M is an upper bound on the number of steps across G_0 on G_1, then $\mathbb{P}$-a.s. we have

$$\begin{aligned} |\pi| &\le 2c_{7.3} M \sum_{i=k+1}^{k+m} (i+1)(\kappa+\varepsilon)^{k+m-i} + c_{7.3}(k+1)(\kappa+\varepsilon)^m \\ &\le c_1 \sum_{i=0}^{m} (i+k+1)(\kappa+\varepsilon)^{m-i} \\ &\le c_2(k+1)(\kappa+\varepsilon)^m, \end{aligned}$$

as desired. $\square$

We now turn our attention to the tail of the crossing time distribution. We follow the same idea as [3] but, as the mean crossing time for a cell is a random variable, we must first establish a weak *uniform* estimate on the crossing time distribution.

LEMMA 7.6. *There exist constants $c_{7.5}(\omega), c_{7.6}(\omega)$ such that for all $n \ge 0$, $i \ge 1$ and all $x \in G$, $\mathbb{P}$-a.s.,*

$$P^x(\tilde{T}_i^n < s) \le 1 - c_{7.5}(\omega) n^{-\xi} + s c_{7.6}(\omega) n^{b_2} e^{(\alpha+1)n}, \qquad s > 0. \tag{7.5}$$

PROOF. We use [2], Lemma 4.3. Consider T_1^n and condition on the position of the process at T_0^n to get

$$P^x(\tilde{T}_1^n < s) = \sum_{z \in \partial \triangle_n(x)} P^z(\tilde{T}_1^m < s) P^x(X_{T_0^m} = z).$$

For the process started at a point $z \in G_n$ we can write

$$\tilde{T}_1^n \le s + (\tilde{T}_1^n - s) I(\tilde{T}_1^n > s).$$

Taking expectations, using the Markov property and the fact that starting from z, T_1^m is a first hitting time, gives

$$E^z(\tilde{T}_1^n) \le s + P^z(\tilde{T}_1^n < s) E^{X_s}(T_{D_n(z)}).$$

Taking suprema using $D_n(z) \subset D_n(x)$ for $z \in \partial \triangle_n(x)$ and then rearranging, we have

$$P^z\left(\tilde{T}_1^n < s\right) \le 1 - \frac{E^z\left(\tilde{T}_1^n\right)}{\sup_{x \in G} E^x\left(T_{D_n(x)}\right)} + \frac{s}{\sup_{x \in G} E^x\left(T_{D_n(x)}\right)} \quad \forall\ n, i \ge 0,\ s > 0.$$

Thus

$$P^x\left(\tilde{T}_i^n < s\right) \le 1 - \frac{E^x\left(\tilde{T}_i^n\right)}{\sup_{x \in G} E^x\left(T_{D_n(x)}\right)} + \frac{s}{\sup_{x \in G} E^x\left(T_{D_n(x)}\right)} \quad \forall\ n, i \ge 0,\ s > 0.$$

From an application of both the upper and lower bounds from (6.15) and the lower bound from (7.1), we obtain the result. $\square$

This leads to the general estimate on the crossing time distribution. Let $\theta_n = c_{7.1}(\omega) n^{\eta} e^{(\alpha+1)n} \vee 1$, so that $(E^x T_1^n)^{-1} \le n^{-2\xi} \theta_n$ for all $x \in G$ by (7.1).

LEMMA 7.7. *There exist constants $c_{7.7}(\omega), c_{7.8}(\omega) > 0$ such that for each $n \ge 0$ and each $x \in G$,*

$$P^x(T_1^n < t) \le c_{7.7}(\omega) \exp\left(-c_{7.8}(\omega)(\theta_n t)^{-\beta_2} \left(\log_+ (1/\theta_n t)\right)^{-\eta_1}\right) \quad \forall\ t > 0,\ \mathbb{P}\text{-}a.s. \tag{7.6}$$

PROOF. The crossing time random variable T_1^n satisfies the following inequality:

$$T_1^n \ge \sum_{i=1}^{\kappa_2^m} \tilde{T}_i^{n+m}, \tag{7.7}$$

as κ_2^m is a uniform bound on the minimum number of steps on G_{n+m} required to cross the cell of G_n. This differs from the usual case as the $\tilde{T}_i^k$ are not identically distributed.

In order to estimate the tail of the probability distribution for the crossing time we will use [2], Lemma 1.1 and thus we must estimate $P^x(\tilde{T}_i^n < \delta | \tilde{T}_j^n, 1 \le j < i)$. By Lemma 7.6 and the strong Markov property, we have that

$$P^x\left(\tilde{T}_i^n < \delta | \tilde{T}_j^n, 1 \le j < i\right) \le 1 - c_{7.5} n^{-\xi} + \delta c_{7.6} n^{b_2} e^{(\alpha+1)n}.$$

This estimate, with our branching inequality (7.7), allows us to use [2], Lemma 1.1, to deduce that there are constants c_1, c_2, c_3, c_4 such that for all $m > 0$,

$$\begin{aligned}
\log P^x(T_1^n < t) &\le c_1 \kappa_2^m \log\Big(1 - c_{7.1}(n+m)^{-\xi}\Big) \\
&\quad + c_2\Big(\kappa_2^m (n+m)^{b_2} \exp((\alpha+1)(n+m))t\Big)^{1/2}, \\
&\le -c_3 (n+m)^{-\xi}\kappa_2^m \\
&\quad \times\Big(1 - c_4\big((n+m)^{\eta}\exp((\alpha+1-\log\kappa_2)m)\exp((\alpha+1)n)t\big)^{1/2}\Big).
\end{aligned}$$

Using the fact that $mn \ge (m+n)/2$ for $m, n \ge 1$ we have

$$\begin{aligned}
\log P^x(T_1^n < t) \le -c_3(nm)^{-\xi}\kappa_2^m\Big(1 - c_4\big((nm)^{\eta}\exp((\alpha+1-\log\kappa_2)m) \\
\exp((\alpha+1)n)t\big)^{1/2}\Big).
\end{aligned}$$

Hence there is a t_1 such that for $0 < t < t_1$ we can choose an $m \ge 1$ such that

$$c_4^2 m^{\eta}\exp((\alpha+1-\log\kappa_2)m) \le \left(\frac{c_3}{2}\right)^2 (n^{\eta}\exp((\alpha+1)n)t)^{-1}, \tag{7.8}$$

and then

$$\log P^x(T_1^n(\omega) < t) \le -\frac{c_3}{2} n^{-\xi} m^{-\xi}\kappa_2^m, \qquad 0 < t < t_1,\ \mathbb{P}\text{-a.s.} \tag{7.9}$$

From (7.8), using Lemma 7.2, there are constants c_5, c_6 such that

$$e^m \le c_5(\theta_n t)^{1/(\alpha+1-\log\kappa_2)}(\log(1/\theta_n t))^{-\eta/(\alpha+1-\log\kappa_2)},$$

and $m \ge c_6\log(1/\theta_n t)$. Using this in (7.9), we obtain

$$P^x(T_1^n(\omega) < t) \le \exp\Big(-c_{7.8}(\theta_n t)^{-\beta_2}\Big(\log_+\big((\theta_n t)^{-1}\big)\Big)^{-\eta_1}\Big) \qquad \forall\, 0 < t < t_1,\ \mathbb{P}\text{-a.s.}$$

By setting the constant $c_{7.7}(\omega) = \exp(c_{7.8}(\omega)t_1^{-\beta_2}\log(1/t_1)^{\eta_1})$, we obtain the result for all $t > 0$. □

THEOREM 7.8. *There exist constants $c_{7.9}(\omega), c_{7.10}(\omega) > 0$ such that for all $0 < \delta < 1$, $0 < t < 1$, $x \in G$, $\mathbb{P}$-a.s.,*

$$\begin{aligned}
&P^x\Big(\sup_{s\le t} d_r(X_s, X_0) > \delta\Big) \\
&\quad \le c_{7.9}(\omega)\exp\left(-c_{7.10}(\omega)\left(\frac{\delta^{\alpha+1}}{tg_\eta(1/\delta)}\right)^{\beta_2}\left(\log_+\left(\frac{\delta^{\alpha+1}}{tg_\eta(1/\delta)}\right)\right)^{-\eta_1}\right),
\end{aligned} \tag{7.10}$$

PROOF. To prove the assertion we choose n such that $D_n(x) \subset B(x, \delta)$, so that $e^{-n} < \delta$. Then

$$P^x\Big(\sup_{s \le t} d_r(X_s, X_0) > \delta\Big) \le P^x(T_1^n < t),$$

and using (7.6), with Lemma 7.2 we obtain the result. □

The λ-resolvent can now be defined for the process X_t and is given by

$$R_\lambda f(x) = E^x \int_0^\infty e^{-\lambda t} f(X_t)\, dt.$$

The existence of resolvent densities $\{r_\lambda(x, y);\ \lambda > 0,\ (x, y) \in G \times G\}$ follows as in [4], Section 5, and then we can write

$$R_\lambda f(x) = \int_G r_\lambda(x, y) f(y) \mu(dy).$$

We need to estimate the probability of exiting a large triangle before a random exponential time. Let ζ_λ denote a random variable which has the exponential distribution with parameter λ.

LEMMA 7.9. *There exist constants $c_{7.11}(\omega), c_{7.12}(\omega)$, such that $\mathbb{P}$-a.s. for all $m > 0$,*

$$\sup_{x \in G} P^x\big(T_{D_m(x)} > \zeta_\lambda\big) \le \tfrac{1}{2} \quad \text{for } \lambda \le c_{7.10}(\omega)\, m^{-b_K} e^{(\alpha+1)m}, \tag{7.11}$$

$$\sup_{x \in G} P^x\big(T_{D_m(x)} \le \zeta_\lambda\big) \le \tfrac{1}{2} \quad \text{for } \lambda \ge c_{7.11}(\omega)\, m^{b_2} e^{(\alpha+1)m}. \tag{7.12}$$

PROOF. This follows the proof of [3], Corollary 3.5. For (7.11), we use Lemma 7.1 to bound the tail of the probability distribution of the hitting time

$$P^x\big(T_{D_m(x)} > t\big) \le t^{-1} E^x T_{D_m(x)} \le \big(c_{7.2}(\omega)\, m^{b_K} e^{-(\alpha+1)m} t^{-1}\big) \wedge 1 \qquad \forall\, x \in G.$$

Then integrating this estimate against $\lambda e^{-\lambda t}$ and observing that the integral decreases as $\lambda c_{7.2}(\omega) m^{b_K} e^{-(\alpha+1)m}$ decreases, gives the result.

For (7.12) we use (7.6) and that $P^x(T_{D_m(x)} \le t) \le P(T_1^m \le t)$, so that

$$P^x\big(T_{D_m(x)} \le t\big) \le c_{7.9}(\omega) \exp\Big(-c_{7.10}(\omega)(\theta_m t)^{-\beta_2}\big(\log_+(1/\theta_m t)\big)^{-\eta_1}\Big).$$

Integrating then gives

$$\begin{aligned}
&P^x\big(T_{D_m(x)} \le \zeta_\lambda\big) \\
&\qquad \le c_{7.9}(\omega) \int_0^\infty \lambda \exp(-\lambda t) \exp\Big(-c_{7.10}(\omega)(\theta_m t)^{-\beta_2}\big(\log_+(1/\theta_m t)\big)^{-\eta_1}\Big)\, dt \\
&\qquad = c_{7.9}(\omega) \int_0^\infty \exp(-u) \exp\left(-c_{7.10}(\omega)\left(\frac{u\theta_m}{\lambda}\right)^{-\beta_2}\left(\log_+\left(\frac{\lambda}{u\theta_m}\right)\right)^{-\eta_1}\right) du \\
&\qquad = I(v(x), \omega),
\end{aligned}$$

where $v(x) = \theta_m/\lambda$. Thus $I(v) \to 0$ as $v \to 0$. By choosing $c_{7.12}(\omega)$ such that $I(c_{7.12}(\omega)) \le 1/2$, and using the form of θ_m, we have the result. □

LEMMA 7.10. *There exist constants $c_{7.13}(\omega)$, $c_{7.14}(\omega)$ such that for $\lambda > 0$,*

$$\begin{aligned} c_{7.13}(\omega)\lambda^{1/(\alpha+1)} g_{b_K}(\lambda)^{-1/(\alpha+1)} &\le \inf_{x\in G} r_\lambda(x,x) \\ &\le \sup_{x\in G} r_\lambda(x,x) \\ &\le c_{7.14}(\omega)\lambda^{1/(\alpha+1)} g_{b_2}(\lambda)^{1/(\alpha+1)}. \end{aligned} \tag{7.13}$$

PROOF. As in [3], Lemma 4.3, conditioning on the exit time we have, for all $x \in G$,

$$\begin{aligned} r_A(x,x) = r_\lambda(x,x) &+ E^x\Big(I_{\{T_A \ge \zeta_\lambda\}} r_A\big(X_{\zeta_\lambda}, x\big)\Big) \\ &- E^x\Big(I_{\{T_A < \zeta_\lambda\}} r_\lambda\big(X_{T_A}, x\big)\Big). \end{aligned} \tag{7.14}$$

Rearranging gives

$$r_\lambda(x,x) \le P^x(T_A \ge \zeta_\lambda)^{-1} r_A(x,x).$$

For $A = D_m(x)$, by Corollary 6.4 and Lemma 7.9 we have

$$r_\lambda(x,x) \le 2c_{6.6}e^{-m} \quad \text{if } \lambda \ge c_{7.12}(\omega) m^{b_2} e^{(\alpha+1)m}.$$

Using Lemma 7.2 we obtain the result.

For the lower bound we use (7.14) again with $A = D_m(x)$, so that

$$r_{D_m(x)}(x,x) \le r_\lambda(x,x) + P\big(\zeta_\lambda \le T_{D_m(x)}\big) r_{D_m(x)}(x,x).$$

Rearranging and applying Lemma 7.9 and Corollary 6.4 again we have

$$r_\lambda(x,x) \ge \frac{c_{6.5}}{2} e^{-m} \quad \text{if } \lambda \le c_{7.11}(\omega) m^{-b_K} e^{(\alpha+1)m}.$$

Once again Lemma 7.2 gives the result. □

This demonstrates that the spectral dimension of the fractal is $d_s = 2\alpha/(\alpha+1)$, which agrees with the conjecture made in [20].

The estimates derived from the potential theory also allow us to bound the first passage time Laplace transform.

THEOREM 7.11. *There exists a constant $c_{7.15}(\omega)$ such that*

$$1 - \psi_\lambda(x,y) \le c_{7.15}(\omega)\big(\lambda g_{b_K}(\lambda)\big)^{-1/(\alpha+1)} d_r(x,y).$$

PROOF. Using the techniques in [14], estimates can be obtained for the λ-resolvent. Substituting these estimates into

$$1 - \psi_\lambda(x,y) = \frac{r_\lambda(y,y) - r_\lambda(x,y)}{r_\lambda(y,y)},$$

gives the result. □

The continuity of the local time comes from the Hölder continuity of the Laplace transform of the first passage time as given in Theorem 7.11.

THEOREM 7.12. *The Brownian motion X_t on $G(\omega)$ has a jointly continuous local time Λ_t^x, for $x \in G$, $0 < t < 1$, which is the density of occupation for the process with respect to the measure μ.*

PROOF. The local time continuity will follow from a bound on $1 - \psi_1(x, y)$. The estimate obtained in Theorem 7.11 shows the metric $\hat{d}_1(x, y) = (r_1(x, x) + r_1(y, y) - 2r_1(x, y))^{1/2}$ is bounded by a function $\hat{d}_1(x, y) = \sigma(x - y)$ where $\sigma(u) = Cu$, which is of the correct form to apply the results of [24], Theorem 1, Theorem 8.4. The density of occupation result is standard as in [4], Theorem 1.11. □

8. Transition density estimates. Now that we have a process and resolvent estimates, we can obtain estimates on the transition density using the techniques developed in [3], [8]. The fundamental crossing time estimate, Lemma 7.7, will be the key result for the off-diagonal upper bound. The bounds will be uniform for $x, y \in G(\omega)$ in a small region and for short times $0 < t < t_0 < e^{-1}$ for $\mathbb{P}$-almost every $\omega \in \Omega$.

In the manner of [3], we can use the Mercer expansion theorem to obtain an eigenvalue expansion of the resolvent for the process killed on hitting the boundary, the vertices of G_0. Thus there are strictly positive eigenvalues γ_j and eigenfunctions ϕ_j such that

$$r_{\triangle_0}(x, y) = \sum_{j=1}^{\infty} \gamma_j \phi_j(x) \phi_j(y) \qquad \forall\, x, y \in G(\omega).$$

Then the transition density for this process can be written as

$$\bar{p}_t(x, y) = \sum_{j=1}^{\infty} e^{-\gamma_j t} \phi_j(x) \phi_j(y) \qquad \forall\, x, y \in G(\omega).$$

As in [3], Section 8.5, it is easy to convert results about the killed process to those for the reflected process. From now on we will use $p_t(x, y)$ for the transition density of either process in a small region G_R away from the boundary and t_0 will denote a small time. The on-diagonal upper bound is a consequence of our resolvent estimates in the previous section.

LEMMA 8.1. (a) *$p_t(x, y)$ is nonincreasing in t for $0 < t \le t_0$, for all $x, y \in G \setminus G_0$.*

(b) *$p_t(x, y) \le p_t(x, x)^{1/2} p_t(y, y)^{1/2}$ for $0 < t \le t_0$, for all $x, y \in G \setminus G_0$.*

(c) *There exists a constant $c_{8.1}(\omega)$ such that*

$$\sup_{x \in G_R} p_t(x, x) \le c_{8.1}(\omega) t^{-\alpha/(\alpha+1)} g_{b_2}(1/t)^{1/(\alpha+1)}, \qquad 0 < t < t_0, \mathbb{P}\text{-}a.s.$$

PROOF. Parts (a) and (b) follow from Mercer's theorem and Cauchy–Schwarz. For (c), the diagonal resolvent estimate (7.13) is used as in [3], Lemma 5.2. □

We are now ready to derive global upper bounds on the transition density.

THEOREM 8.2. *The heat kernel $p_t(x, y)$ is continuous in t, x, y. For $\varepsilon > 0$ there exist constants $c_{8.2}(\omega), c_{8.3}(\omega)$ such that*

$$\begin{aligned} p_t(x, y) \le c_{8.2}(\omega) t^{-\alpha/(\alpha+1)} g_{b_2}(1/t)^{1/(\alpha+1)} \\ \times \exp\left(- c_{8.3}(\omega) \left(\frac{d_r(x, y)^{\alpha+1}}{t g_\eta(1/d_r(x, y))} \right)^{\beta_2} \right. \\ \left. \times \left(\log_+ \left(\frac{d_r(x, y)^{\alpha+1}}{t g_\eta(1/d_r(x, y))} \right) \right)^{-\eta_1} \right) \\ \forall\, x, y \in G_R, 0 < t < t_0, \mathbb{P}\text{-}a.s. \end{aligned}$$

PROOF. For the continuity we can use the diagonal upper bound and the general theory as indicated in [8].

For the upper bound we follow [3], Theorem 6.2. Fix $x \ne y \in G_R$ and t, and let $\varepsilon < 6 d_r(x, y)$, $C_x = B(x, \varepsilon) \cap G$, $C_y = B(y, \varepsilon) \cap G$ and $\mu^z = \mu|_{C_z}$, $z = x, y$. Let $A_1 = \{z\colon d_r(x, z) \le \frac{1}{2} d_r(z, y)\} \cap G$ and $A_2 = A_1^c \cap G$. Then

$$\begin{aligned} P^{\mu^x}(X_t \in C_y) &= P^{\mu^x}(X_t \in C_y, X_{t/2} \in A_1) + P^{\mu^x}(X_t \in C_y, X_{t/2} \in A_2) \\ &= I_1 + I_2. \end{aligned}$$

We begin by considering I_2. For $z \in C_x$, use Lemma 7.7. Let n be the smallest such that $D_{n-1}(x) \subset A_1$, then by (7.10),

$$\begin{aligned} P^z(X_{t/2} \in A_2) &\le P^z(T_1^n < t/2) \\ &\le c_{7.9}(\omega) \exp\left(-c_{7.10}(\omega) \left(\frac{d_r(x, y)^{\alpha+1}}{t g_\eta(1/d_r(x, y))} \right)^{\beta_2} \right. \\ &\qquad \left. \times \left(\log_+ \left(\frac{d_r(x, y)^{\alpha+1}}{t g_\eta(1/d_r(x, y))} \right) \right)^{-\eta_1} \right). \end{aligned}$$

Now let $q(z) = P(X_t \in C_y \mid X_{t/2} = z)$ so that by the uniform diagonal upper bound, Lemma 8.1(c), we have

$$q(z) = \int_{C_y} p_{t/2}(z, u) \mu(du) \le c(\omega) t^{-\alpha/(\alpha+1)} g_{b_2}(1/t)^{1/(\alpha+1)} \mu(C_y).$$

Thus we estimate I_2 as

$$\begin{aligned} I_2 &= E^{\mu^x}\big(q(X_{t/2}); X_{t/2} \in A_2\big) \\ &\le C(\omega)\mu(C_x)\mu(C_y)t^{-\alpha/(\alpha+1)}g_{b_2}(1/t)^{1/(\alpha+1)} \\ &\quad \times \exp\left(-c_{7.10}(\omega)\left(\frac{d_r(x,y)^{\alpha+1}}{tg_\eta(1/d_r(x,y))}\right)^{\beta_2}\left(\log_+\left(\frac{d_r(x,y)^{\alpha+1}}{tg_\eta(1/d_r(x,y))}\right)\right)^{-\eta_1}\right). \end{aligned}$$

We proceed similarly for I_1, observing that from the symmetry of the process

$$P^{\mu^x}\big(X_t \in C_y,\, X_{t/2} \in A_1\big) = P^{\mu^y}\big(X_t \in C_x,\, X_{t/2} \in A_1\big),$$

and applying the same approach as for I_2. Now divide by $\mu(C_x)\mu(C_y)$, let $\varepsilon \to 0$ and with the continuity of $p_t(x,y)$, we obtain the result. □

The final task is to prove a lower bound. Again we use the standard probabilistic approach of obtaining an on-diagonal bound, using the Hölder continuity of the transition density and then a chaining argument.

LEMMA 8.3. *There exists a constant $c_{8.4}(\omega)$ such that*

$$\inf_{x\in G_R} p_t(x,x) \ge c_{8.4}(\omega)t^{-\alpha/(\alpha+1)}g_{\hat\eta}(1/t)^{-1/(\alpha+1)}, \qquad 0 < t < t_0,\ \mathbb{P}\text{-}a.s.$$

PROOF. Follow [3], Lemma 7.1. Fix t and let a be a constant such that

$$c_{7.7}(\omega)\exp\big(-c_{7.8}a^{-\beta_2}(\log_+(1/a))^{-\eta_1}\big) \le 1/2.$$

Then using (7.6),

$$P^x\big(X_t \in \triangle_n(x)\big) \ge P^x\big(T_1^n > t\big) \ge 1/2 \quad \text{if } \theta_n t \le a.$$

Let n be such that $\theta_n \le a/t$ so that

$$c_{7.1}(\omega)^{-1}n^{b_2}e^{(\alpha+1)n} \le a/t. \tag{8.1}$$

Thus, as in [3], Lemma 7.1,

$$\tfrac{1}{4} \le P^x\big(X_t \in \triangle_n(x)\big)^2 \le \mu\big(\triangle_n(x)\big)p_{2t}(x,x).$$

Once we have this result, we can appeal to Lemma 7.2 and (8.1) to get

$$\mu\big(\triangle_n(x)\big) \le cn^{b_K}e^{\alpha n} \le c_2\big(tg_{b_2}(1/t)\big)^{\alpha/(\alpha+1)}g_{b_K}(1/t),$$

and we have the result by definition of $\hat\eta$. □

The extension to a ball about the diagonal follows from an estimate on the Hölder continuity of the heat kernel.

LEMMA 8.4. *There exists a constant* $c_{8.5}$ *such that for all* $y \in G(\omega)$,

$$\sup_{x, x' \in \triangle_n} |p_t(x, y) - p_t(x', y)| \le c_{8.5}\big(\rho_n(x)^{-1} t^{-1} p_t(y, y)\big)^{1/2} \qquad \forall\, n \ge 0,\ 0 < t < t_0.$$

PROOF. As P_t: $L^2(\mu) \to D(E)$, we see that $p_t(\cdot, y) = P_{t/2}\, p_{t/2}(\cdot, y)(\cdot) \in D(E)$ if $p_{t/2}(\cdot, y) \in L^2(\mu)$. From the upper bound, we know

$$\| p_{t/2}(\cdot, y)\|_2^2 = \int_G p_{t/2}(z, y)^2 \mu(dz) = p_t(y, y) < \infty,$$

and hence $p_t(\cdot, y) \in D(E)$. Now we can use Lemma 4.3,

$$\sup_{x, x' \in \triangle_n} |p_t(x, y) - p_t(x', y)| \le c_{4.1}\, \rho_n(x)^{-1/2} E(p_t(\cdot, y), p_t(\cdot, y))^{1/2}.$$

Let $u(x) = p_{t/2}(x, y)$ and use [9], Lemma 1.3.3(i),

$$E\big(P_{t/2}u, P_{t/2}u\big) \le \frac{1}{t}\big(\|u\|_2^2 - \|P_{t/2}u\|_2^2\big) \le t^{-1} p_t(y, y)$$

as desired. □

We write $\hat{\eta}_1 = \hat{\eta}/(\alpha + 1)$.

LEMMA 8.5. *There exists a constant* $c_{8.6}(\omega)$ *such that for all* $x \in G_R$,

$$p_t(x, y) \ge \frac{c_{8.4}(\omega)}{2} t^{-\alpha/(\alpha+1)} g_{\hat{\eta}_1}(1/t)^{-1}$$

for $y \in \big\{z \in G(\omega): d_r(x, z) < c_{8.6}(\omega) t^{1/(\alpha+1)} g_{\hat{\eta}}(1/t)^{-1},\big\}, \qquad 0 < t < t_0.$

PROOF. We use the above estimate on the Hölder continuity as follows:

$$\begin{aligned} p_t(x, y) &\ge p_t(x, x) - |p_t(x, x) - p_t(x, y)| \\ &\ge p_t(x, x)\left(1 - c_{8.5}\sqrt{\frac{\rho_n(x)^{-1}}{t p_t(x, x)}}\right) \end{aligned}$$

if $y \in \triangle_n(x)$. Now we choose n such that

$$1 - \sqrt{\frac{\rho_n(x)^{-1}}{t \inf_{x \in G} p_t(x, x)}} \ge \frac{1}{2},$$

which gives the result. □

We can now obtain the lower bound on the transition probability using the chaining argument. The proof can be used to give a slightly sharper bound than the one stated, but this provides more complication than insight. Recall that $D_*(x, y, t) = d_r(x, y)^{\alpha+1}/t g_{\hat{\eta}}(1/t)^{-1}$.

THEOREM 8.6. *For $\varepsilon > 0$ there exist constants $c_{8.7}(\omega), c_{8.8}(\omega)$ such that*

$$\begin{aligned} p_t(x, y) \geq c_{8.7}(\omega) t^{-\alpha/(\alpha+1)} g_{\hat{\eta}}(1/t)^{-1/(\alpha+1)} \\ \times \exp\Bigg(-c_{8.8}(\omega) D_*(x, y, t)^{\beta_\varepsilon} (\log_+(D_*(x, y, t)))^{\beta_\varepsilon \hat{\eta}} \\ \times \left(\log_+\left(\frac{d_r(x, y)^{\log \kappa_\varepsilon - 1}}{t}\right)\right)^2\Bigg), \\ 0 < t < t_0,\ x, y \in G_R,\ \mathbb{P}\text{-}a.s. \end{aligned}$$

PROOF. The chaining argument can be used with the shortest path linking the points along which we can apply the lemmas. Fix x, y and t. The theorem follows immediately for $d_r(x, y) \leq c_{8.6} t^{1/(\alpha+1)} g_{\hat{\eta}_1}(1/t)^{-1}$ by Lemma 8.5. Let $D_*(x, y, t) > c_{8.6}$, and write κ_ε for $\kappa + \varepsilon$. We can find a constant c_1 such that the following statement holds. For an m such that

$$\begin{aligned} &\exp((\alpha + 1 - \log \kappa_\varepsilon) m)(\log(\kappa_\varepsilon^m))^{-\hat{\eta}} \\ &\quad \leq c_1 D_*(x, y, t) \\ &\quad \leq \exp((\alpha + 1 - \log \kappa_\varepsilon)(m + 1))\big(\log\big(\kappa_\varepsilon^{(m+1)}\big)\big)^{-\hat{\eta}}, \end{aligned} \tag{8.2}$$

we have

$$\frac{d_r(x, y)}{e^m} \leq \frac{c_{8.6}}{3ec_{4.4}} \left(\frac{t}{\kappa_\varepsilon^m} (\log(1/t))^{-\hat{\eta}} (\log(\kappa_\varepsilon^m))^{-\hat{\eta}}\right)^{1/(\alpha+1)}.$$

As the following relationship holds for N, $1/t > 2$,

$$2(\log N \log(1/t))^{1/2} \leq \log N/t \leq \log N \log(1/t), \tag{8.3}$$

we have

$$\frac{d_r(x, y)}{e^m} \leq \frac{c_{8.6}}{3ec_{4.4}} \left(\frac{t}{\kappa_\varepsilon^m} (\log(\kappa_\varepsilon^m/t))^{-\hat{\eta}}\right)^{1/(\alpha+1)}.$$

Now choose k such that

$$e^{-k-1} \leq d_r(x, y) \leq e^{-k}.$$

By Theorem 7.5 there is a path π on G_{k+m} given by $x_0 = x$, $x_N = y$, $x_i \in G_{k+m}$, $0 < i < N$ and $x_i, x_{i+1} \in \triangle_{k+m}(x_i)$, $0 \leq i < N$ where $N = |\pi| \leq c_{7.4}(k + 1)\kappa_\varepsilon^m$. Then, by our choice of G_{k+m},

$$d_r(x_i, x_{i+1}) \leq c_{4.4} e^{-k-m} \leq \frac{ec_{4.4} d_r(x, y)}{e^m} \leq \frac{c_{8.6}}{3} \left(\frac{t}{\kappa_\varepsilon^m} (\log(\kappa_\varepsilon^m/t))^{-\hat{\eta}}\right)^{1/(\alpha+1)}.$$

Using the same approach, with (8.2) and the lower bound in (8.3), we have a lower bound on the distance between points in the path. There is a constant

c_2 such that

$$c_2\Big(\big(t/\kappa_\varepsilon^m\big)\big(\log(\kappa_\varepsilon^m/t)\big)^{-\hat\eta/2}\Big)^{1/(\alpha+1)} \le d_r(x_i, x_{i+1}). \tag{8.4}$$

Let B_i denote the ball of radius ε about x_i, where $\varepsilon = c_{4.4}e^{-k-m}$. Then for $z_i \in B_i$, $z_{i+1} \in B_{i+1}$,

$$d_r(z_i, z_{i+1}) \le 2\varepsilon + c_{4.4}e^{-m-k} \le c_{8.6}\left(\frac{t}{\kappa_\varepsilon^m}\big(\log(\kappa_\varepsilon^m/t)\big)^{-\hat\eta}\right)^{1/(\alpha+1)}. \tag{8.5}$$

Now we can apply the chaining argument:

$$\begin{aligned} p_t(x,y) \ge \int_{B_1}\cdots\int_{B_{N-1}} p_{t/N}(x,y_1)\cdots p_{t/N}(y_{N-1},y)\,\mu(dy_1) \\ \cdots\,\mu(dy_{N-1}). \end{aligned} \tag{8.6}$$

As we have (8.5) we can use Lemma 8.5 to bound the terms in (8.6) below to get

$$\begin{aligned} p_t(x,y) &\ge \prod_{i=0}^{N-1}\frac{c_{8.4}}{2}(t/\kappa_\varepsilon^m)^{-\alpha/(\alpha+1)}\big(\log(\kappa_\varepsilon^m/t)\big)^{-\hat\eta_1}\prod_{i=1}^{N-1}\mu(B_i) \\ &= (t/\kappa_\varepsilon^m)^{-\alpha/(\alpha+1)}\big(\log(\kappa_\varepsilon^m/t)\big)^{-\hat\eta_1} \\ &\quad\times\prod_{i=1}^{N-1}\frac{c_{8.4}}{2}(t/\kappa_\varepsilon^m)^{-\alpha/(\alpha+1)}\big(\log(\kappa_\varepsilon^m/t)\big)^{-\hat\eta_1}\mu(B_i). \end{aligned} \tag{8.7}$$

Now by (5.4) we have

$$\mu(B_i) \ge \mu(\triangle_{k+m}(x_i)) \ge c_3(k+m)^{-b_2}e^{-\alpha(k+m)} \qquad \forall\, x_i \in G_{k+m}.$$

By the choice of G_{k+m} we have, using (8.4), that

$$\mu(B_i) \ge c_4(t/\kappa_\varepsilon^m)^{\alpha/(\alpha+1)}\big(\log(\kappa_\varepsilon^m/t)\big)^{-\hat\eta_2},$$

where $\hat\eta_2 = \hat\eta\alpha/2 + b_2 > 0$. Hence, substituting this into (8.7), we get

$$\begin{aligned} p_t(x,y) &\ge (t/\kappa_\varepsilon^m)^{-\alpha/(\alpha+1)}\big(\log(\kappa_\varepsilon^m/t)\big)^{-\hat\eta_1} \\ &\quad\times\Big(c_5\big(\log(\kappa_\varepsilon^m/t)\big)^{-\hat\eta_1}\big(\log(\kappa_\varepsilon^m/t)\big)^{-\hat\eta_2}\Big)^N, \\ &\ge c_6(\omega)t^{-\alpha/(\alpha+1)}\big(\log(1/t)\big)^{-\hat\eta_1} \\ &\quad\times\exp\Big(-c_{7.4}(k+1)\kappa_\varepsilon^m\big(c_7+(\hat\eta_1+\hat\eta_2)\log\log(\kappa_\varepsilon^m/t)\big)\Big), \end{aligned} \tag{8.8}$$

Using Lemma 7.2 on (8.2) we have

$$\kappa_\varepsilon^m \le c_{11}D_*^{\beta_\varepsilon}(\log D_*)^{\beta_\varepsilon\hat\eta}. \tag{8.9}$$

Observe that, with (8.9),

$$1 + c_{12}\log\log(\kappa_\varepsilon^m/t) \le c_{13}\Big(\log\big(d_r(x,y)^{\log\kappa_\varepsilon}/t\big)\Big)$$

for $m > 0$, $t < t_0$. As $k \le -\log d_r(x, y)$, we have, by (8.3), that

$$\begin{aligned}(k+1)\big(c_7 + (\hat{\eta}_1 + \hat{\eta}_2)\log\log(\kappa_\varepsilon^m/t)\big) \\ \le c_{14}\Big(\log\big(d_r(x,y)^{\log\kappa_\varepsilon - 1}/t\big)\Big)^2.\end{aligned} \tag{8.10}$$

Putting estimates (8.9), (8.10) into (8.8) gives the result. □

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