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SPECIAL INVITED PAPER

SOME BROWNIAN FUNCTIONALS AND THEIR LAWS

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We develop some topics about Brownian motion with a particular emphasis on the study of principal values of Brownian local times. We show some links between principal values and Doob's h-transforms of Brownian motion, for nonpositive harmonic functions h. We also give a survey and complement some martingale approaches to Ray–Knight theorems for local times.

Contents

0. Introduction. The contents of this paper were presented by the second author, in a much less advanced form, at the IMS Conference in Montreal (July 1995). Our aim is to develop a number of topics found in [65] and in [66] (throughout this paper, Chapter 1 in [65] will be referred to as Chapter 1*, Chapter 12 in [66] as Chapter 12*, and so on). However, to a large extent we do not assume that the reader is familiar with [65] and [66].

The main objective of [65] was to derive explicitly the laws of more and more complicated Brownian functionals, essentially using stochastic calculus and excursion theory. Here, although our aim is somewhat similar, we took a different direction; in fact, we drifted strongly, as we shall now explain, towards the study of principal values of Brownian local times.

In Section 1, we recall some quite well-known links between space–time harmonic functions h and the set of laws of processes which admit the same bridges as Brownian motion, that is, Doob's h-transforms of Brownian motion.

More generally, one may also look for some definition of h-transforms, when h is not necessarily positive. This topic was first suggested by P. A. Meyer to Ruiz de Chavez [52], in order to obtain a more general martingale characterization of Brownian motion, possibly involving signed measures. This may

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seem somewhat exotic but, in fact, appears to be quite natural in the context of Girsanov's theorem and in some studies in mathematical finance.

Indeed, an interpretation of Girsanov's theorem is that it gives a formula for a bounded variation process A such that, if $D_t = dQ/dP|_{\mathscr{F}_t}$ is the Radon–Nikodym derivative of Q with respect to P, P and Q being two probabilities on a filtered space $(\Omega, \mathscr{F}, (\mathscr{F}_t))$, and if (M_t) is a $(P, (\mathscr{F}_t))$ local martingale, then

$$(M_t - A_t)D_t \text{ is also a } (P, (\mathscr{F}_t)) \text{ local martingale.}$$

With this formulation, the probability Q does not appear any more, and one could start with any pair (D_t, M_t) of $(P, (\mathscr{F}_t))$ local martingales.

Formally, Girsanov's formula is $dA_t = d\langle M, D\rangle_t / D_t$, which, in case (D_t) is not necessarily positive, leads naturally to the study of principal values, that is, limits of

$$\int_0^t \frac{d\langle M, D\rangle_s}{D_s} 1_{(|D_s| \geq \varepsilon)} \quad \text{as } \varepsilon \to 0.$$

In the original study [7] of such quantities for $D_t = B_t$, and particularly for $M_t = D_t = B_t$, in which case one obtains

$$A_t = p.v. \int_0^t \frac{ds}{B_s},$$

it was found that the study of this additive functional is closely related to that of $\int_0^t ds\, B_s^2$.

This brings us naturally to the presentation in Section 2 below of some developments of Chapter 2*, which was devoted to the study of the laws of some quadratic functionals of Brownian motion, and more generally, Bessel processes.

In Section 3, we survey and develop martingale approaches for a number of Ray–Knight theorems about local times, which, as is now well known, may be described in terms of squares of Bessel processes.

In Section 4, we get back to principal values of Brownian local times to discuss a striking result of Alili [1, 2, 3] concerning the law of

$$p.v. \int_0^u ds \coth(\lambda B_s), \qquad u \geq 0,$$

taken at the inverse local time of B.

In Section 5, we consider one of the consequences of Alili's result, namely:

$$\int_0^{T_1(R_2)} \frac{ds}{1 - (R_2(s))^2} \overset{\text{(law)}}{=} T_{\pi/2}(R_3),$$

where R_δ denotes the δ-dimensional Bessel process starting from 0 and $T_a(R_\delta)$ is its first hitting time of $a > 0$. We discuss a number of extensions of this result, leading us in particular to the porous medium equation [4].

In Section 6, we study exponential functionals of the normalized Brownian excursion.

1. Around the Gaussian space of Brownian motion. Chapter 1* contains a description of all probability measures on $(C(\mathbb{R}_+,\mathbb{R}),\mathscr{F}_\infty)$, where $X_t=\omega(t)$, $t\geq 0$, denotes the canonical process, and $\mathscr{F}_t=\sigma(X_s,\ s\leq t)$ $(t\leq\infty)$ such that, for each finite t, $(X_u-(u/t)X_t,\ u\leq t)$ is a standard Brownian bridge (of duration t), independent of X_t. This topic led naturally to various computations involving the first Wiener chaos, that is, the Gaussian space of Brownian motion.

1.1. *Relationship with space–time harmonic functions.* Let W denote the Wiener measure on $(C(\mathbb{R}_+,\mathbb{R}),\mathscr{F}_\infty)$. We give a description of $\mathscr{I}$, the set of all probabilities W^h on Ω such that

$$W^h\big|_{\mathscr{F}_t}=h(t,X_t)W\big|_{\mathscr{F}_t}$$

for a positive function h, which is therefore a space–time harmonic function with $h(0,0)=1$. Assuming the integral representation of positive space–time harmonic functions (see, e.g., [59]), there exists a unique probability measure ν on $\mathbb{R}$ such that

$$h(x,t)=\int\nu(dy)\exp(xy-y^2t/2). \tag{1.1}$$

We have the following characterization of $\mathscr{I}$.

PROPOSITION 1.1. *Under W^h, the process $(X_t,\ t\geq 0)$ satisfies*

$$X_t=B_t+tY,\qquad t\geq 0, \tag{1.2}$$

where $Y:=\lim_{t\to\infty}(X_t/t)$, and (B_t) is a Brownian motion independent of Y. Moreover, the law of Y is ν, h and ν being related by (1.1).

PROOF. This is elementary and relies mainly on Fubini's theorem. Let $T>0$, and $\varphi\in L^2([0,T])$; then,

$$\begin{aligned}
&W^h\left[\exp i\int_0^T\varphi(s)\,dX_s\right]\\
&\quad=W\left[h(T,X_T)\exp i\int_0^T\varphi(s)\,dX_s\right]\\
&\quad=\int\nu(dy)W\left[\exp\left(i\int_0^T\varphi(s)\,dX_s+yX_T\right)\right]\exp\left(-\frac{y^2}{2}T\right)\\
&\quad=\int\nu(dy)\exp\left(-\frac{y^2}{2}T\right)\exp\left(\frac{1}{2}E\left[\left(i\int_0^T\varphi(s)\,dB_s+yB_T\right)^2\right]\right)\\
&\quad=\int\nu(dy)\exp\left(iy\int_0^T\varphi(s)\,ds-\frac{1}{2}\int_0^T\varphi^2(s)\,ds\right)\\
&\quad=E\left[\exp i\int_0^T\varphi(s)d(B_s+sY)\right].
\end{aligned}$$

□

REMARKS. (i) Equation (1.2) is a noncanonical representation of the diffusion process with infinitesimal generator

$$G^h = \frac{1}{2}\frac{d^2}{dx^2} + \frac{h'_x(x,t)}{h(x,t)}\frac{d}{dx}.$$

(ii) Note that, in general, G^h is inhomogeneous; that is, $h'_x(x,t)/h(x,t)$ depends on t; however, for the space–time harmonic function $h_\lambda(t,x) = \cosh(\lambda x)\exp(-\lambda^2 t/2)$ ($\lambda \neq 0$), $h'_x(x,t)/h(x,t)$ does not depend on t and the generator G^h takes the form

$$G^h = \frac{1}{2}\frac{d^2}{dx^2} + \lambda \tanh(\lambda x)\frac{d}{dx}.$$

More generally, Benjamini and Lee [5] remark that for the two parameter family of drifts $\mu(x) = k\tanh(kx + c)$, the bridge of the diffusion with drift μ is a Brownian bridge.

(iii) In Theorem 1.3*, a more complete description of the set $\mathscr{I}$ is given and in particular, the integral representation (1.1) is not assumed, but, in fact, is obtained as the consequence of the proof of the theorem.

(iv) These results admit some partial generalizations to infinite dimensions, that is, when the original Brownian motion is replaced by a Brownian sheet (see [10], [19]).

1.2. *Extensions to Gaussian–Markov and other processes.*

EXTENSION a. Proposition 1.1 is still valid when we replace Brownian motion $(B_t,\ t \geq 0)$ by a continuous Gaussian–Markov process U_t. It is well known [41] that U can be expressed in terms of B by $U_t = u(t)B_{v(t)}$, where u, v are continuous, strictly positive and v is nondecreasing. In this case, Proposition 1.1 generalizes as follows (and the proof is quite analogous).

PROPOSITION 1.2. *We denote by $\mathscr{J}$ the set of probabilities $P^{U,k}$ on Ω such that*

$$P^{U,k}\big|_{\mathscr{F}_t} = k(t, X_t)P^U\big|_{\mathscr{F}_t},$$

where P^U is the law of the Gaussian process U on Ω and k is U-harmonic in the sense that $k(t, U_t)$ is a martingale with mean 1.

Under $P^{U,k}$, $X_t = U_t + u(t)v(t)Y$, where Y is a random variable with law ν, independent of $(U_t)_{t\geq 0}$ and ν and k are linked by the formula

$$k(x,t) = \int \nu(dy)\exp\left(\frac{xy}{u(t)} - \frac{y^2}{2}v(t)\right). \tag{1.3}$$

In particular, when U is the standard Brownian bridge $b(t)$ on $[0,1]$, then under P^k, $X_t = b(t) + tY$; when U is the Ornstein–Uhlenbeck process U_t^λ of parameter λ (i.e., the solution of $dU_t = dB_t + \lambda U_t dt$), $X_t = U_t^\lambda + (\sinh(\lambda t)/\lambda)Y$.

For the Brownian bridge b, $u(t) = 1 - t$; $v(t) = t/(1-t)$, but the statement of Proposition 1.2 must be slightly modified with Ω changed into $C([0,1[,\mathbb{R})$; details are left to the reader.

For the Ornstein–Uhlenbeck process U^λ, $u(t) = \exp(\lambda t)$; $v(t) = (1 - \exp(-2\lambda t))/2\lambda$.

Applications. (1) From the absolute continuity relation

$$P^b\big|_{\mathscr{F}_t} = \frac{1}{\sqrt{1-t}} \exp\left(\frac{-X_t^2}{2(1-t)}\right) W\big|_{\mathscr{F}_t}, \qquad t < 1,$$

we recover that the function $k(x,t) = \sqrt{1-t}\,\exp(x^2/(2(1-t)))$ is b-harmonic and the corresponding measure ν associated by (1.3) is the Gaussian standard density.

(2) Let us denote by X the Ornstein–Uhlenbeck process of parameter $-\frac{1}{2}$, starting from 0. Breiman's formula [9],

$$E[\exp(-\alpha T_c)] = \frac{1}{\varphi_\alpha(c)}, \tag{1.4}$$

where T_c is the first hitting time of level c by $|X|$ and

$$\varphi_\alpha(x) = \frac{1}{\Gamma(\alpha/2)2^{\alpha/2-1}} \int_0^\infty dz\, z^{\alpha-1} \exp\left(-\frac{z^2}{2}\right) \cosh(xz),$$

may be recovered from Proposition 1.2, by checking that

$$k(x,t) = \varphi_\alpha(x)\exp(-\alpha t), \qquad \alpha > 0$$

is a particular solution of (1.3).

EXTENSION b. We now present a generalization of Proposition 1.1 to signed measures (or to nonpositive space–time harmonic functions). Let ν be a signed measure on $\mathbb{R}$ satisfying $\int |\nu|(dy) < \infty$, and define $Q_\nu = W \otimes \nu$ a signed measure on $\tilde{\Omega} = C(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R}$. For $(\omega, y) \in \tilde{\Omega}$, let $Y_t(\omega, y) = \omega(t)$ and $Y(\omega, y) = y$, and define $Z_t = Y_t + tY$.

PROPOSITION 1.3. *The law of the process Z under Q_ν is W^h, the law of the process of coordinates X on Ω, where W^h is the signed measure on Ω defined by*

$$W^h\big|_{\mathscr{F}_t} = h(t, X_t) W\big|_{\mathscr{F}_t}$$

and

$$h(x,t) = \int \nu(dy)\exp(xy - y^2 t/2) \tag{1.5}$$

is a space–time harmonic function (not necessarily positive).

The proof of Proposition 1.3 is the same as that of Proposition 1.1.

EXAMPLE 1. Let $h_\lambda(x,t) = \sinh(\lambda x)\exp(-(\lambda^2/2)t)$ be the space–time harmonic function corresponding to the measure $\nu = \frac{1}{2}(\delta_\lambda - \delta_{-\lambda})$ where δ_α denotes the Dirac measure at α. In this case, the function h'_x/h does not depend on t and under W^{h_λ},

$$\hat{X}_t := X_t - \lambda \int_0^t \coth(\lambda X_s)\,ds$$

is a local martingale; that is, $\hat{X}_t h_\lambda(t, X_t)$ is a W local martingale. The study of $H_t^\lambda := \text{v.p.} \int_0^t ds \coth(\lambda B_s)$ has been made by Alili [1] in his thesis; we shall study the process H^λ in Section 4.

EXAMPLE 2. Let h_0 be the harmonic function $h_0(x,t) = x$. Under W^{h_0},

$$\hat{X}_t = X_t - \int_0^t \frac{ds}{X_s}$$

is a local martingale. We refer to Biane and Yor [7] for the study of $p.v. \int_0^t (ds/B_s)$. Note that h_0 is not of the form (1.5) but

$$h_0(x,t) = \lim_{\lambda \to 0} \frac{h_\lambda(x,t)}{\lambda}.$$

EXTENSION c. We now give an analogue of Proposition 1.1 when we replace Brownian motion by a particular Lévy process, namely the gamma process Γ, which is the subordinator (that is, a Lévy process valued in $\mathbb{R}_+$) with Lévy measure $\mu(dx) = ((\exp -x)/x)dx$ on $\mathbb{R}_+$. The Laplace exponent of Γ, Ψ, defined by

$$E[\exp(-\lambda \Gamma_t)] = \exp -t\Psi(\lambda), \qquad \lambda \ge 0$$

is $\Psi(\lambda) = \ln(\lambda + 1)$.

We set P^* the law of Γ on $\mathbb{D} = \mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$ the space of positive cadlag functions. Let ν be a probability measure on $\mathbb{R}_+$ and $Q_\nu^* = P^* \otimes \nu$ on $\hat{\Omega} = \mathbb{D}(\mathbb{R}_+, \mathbb{R}_+) \times \mathbb{R}_+$. For $(\omega, y) \in \hat{\Omega}$, let $Y_t(\omega, y) = \omega(t)$ and $Y(\omega, y) = y$ and define $Z_t = YY_t$.

PROPOSITION 1.4 ([58]). *The law of Z on $\hat{\Omega}$, under Q_ν^* is the law of the canonical process X on $\mathbb{D}$ under $P^{*,h}$, the probability measure on $\mathbb{D}$ defined by*

$$P^{*,h}\big|_{\mathscr{F}_t} = h(t, X_t) P^*\big|_{\mathscr{F}_t}$$

with

$$h(x,t) = \int \nu(dy) \exp\left(-\left(\frac{1}{y} - 1\right)x - t \ln y\right).$$

1.3. *Loss of information.* We now discuss a relationship between space–time harmonic functions and the phenomenon of loss of information for Brownian motion.

1. Let $h: \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$ be a space–time harmonic function such that $h(0,0)=0$ and $h'_x(x,t) \neq 0$ for every t and x (this is the case in Examples 1 and 2). Then we can define the process $(B_t^h,\ t\geq 0)$ (see Example 5.a in Chapter 17*) by

$$B_t^h = B_t - p.v.\int_0^t ds \frac{h'_x(B_s,s)}{h(B_s,s)}.$$

We denote by $\mathscr{F}_t^h$ (resp., $\mathscr{F}_t$) the natural filtration of B^h (resp., B); then we have $\mathscr{F}_t^h \not\subseteq \mathscr{F}_t$ (see [7], Appendix A and [66]).

More generally, let (D_t) denote a Brownian martingale such that $D_0=0$, $(D_t,\ t\geq 0)\not\equiv 0$ and such that

$$\int_0^t \frac{d\langle B,D\rangle_s}{D_s} \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0}\int_0^t 1_{(|B_s|\geq\varepsilon)}\frac{d\langle B,D\rangle_s}{D_s}$$

exists. Then if we denote $B_t^D = B_t - \int_0^t (d\langle B,D\rangle_s)/D_s$, Ruiz de Chavez [52] proves that

$$E\big(D_t/B_s^D,\ s\leq t\big)=0,$$

implying that the natural filtration of B^D is strictly smaller than that of B.

2. We recall another much simpler example of loss of information taken from [30] (see also Theorem 1.1* and Nagasawa and Domenig [40] for recent developments).

PROPOSITION 1.5. (i) *The process* $\beta_t = B_t - \int_0^t (ds/s)B_s$ *is a one-dimensional Brownian motion.*

(ii) *For every* $t>0$, $\mathscr{F}_t^\beta = \sigma\{B_u - (u/t)B_t,\ u\leq t\} \not\subseteq \mathscr{F}_t$; *in particular, for fixed* $t>0$, B_t *is independent of* $\mathscr{F}_t^\beta$.

We note that, in this case,

$$\beta_t = B_t - \int_0^t ds \frac{h'_x(B_s,s)}{h(B_s,s)}$$

for the positive space–time harmonic function defined on $\mathbb{R}\times\mathbb{R}_+^*$ by

$$h(x,t) = \frac{1}{\sqrt{t}}\exp\left(\frac{x^2}{2t}\right).$$

We refer to Chapter 17* for other examples of loss of information, in particular Tsirel'son's equation (see also [64]).

To conclude Section 1.3, we ask the following.

Question. Under the hypothesis made in point 1 above, is the filtration $\mathscr{B}_t^D \equiv \sigma\{B_s^D,\ s\leq t\}$ generated by a Brownian motion?

The answer to this question, even for Example 2 above, and a fortiori for Example 1, is unknown. One does not even know whether all $(\mathscr{B}_t^D)$ martingales are continuous.

2. Quadratic functionals of Brownian motion.

We refer to [21] and [22] for studies of Brownian quadratic functionals in the recent literature.

2.1. *Lévy's formula.*

We consider $(B_t,\ t \geq 0)$ a δ-dimensional Brownian motion, starting from 0. Then,

$$E\left(\exp\left(-\alpha|B_t|^2 - \frac{b^2}{2}\int_0^t ds|B_s|^2\right)\right) = \left(\cosh(bt) + 2\frac{\alpha}{b}\sinh(bt)\right)^{-\delta/2}. \tag{2.1}$$

A well-known method to prove this formula is to consider a new probability $P^{(b)}$ defined by

$$P^{(b)}\big|_{\mathscr{F}_t} = \exp\left(-\frac{b}{2}(|B_t|^2 - \delta t) - \frac{b^2}{2}\int_0^t ds|B_s|^2\right)P\big|_{\mathscr{F}_t},$$

in order to "take care" of the quadratic functional integral and reduce the problem to the computation of the mean and variance of the Gaussian process $(B_t,\ t \geq 0)$ under $P^{(b)}$, where it becomes a Ornstein–Uhlenbeck process.

From formula (2.1), we deduce the conditional formula

$$\begin{aligned} &E\left(\exp\left(-\frac{b^2}{2}\int_0^t ds|B_s|^2\right)\bigg| B_t = a\right) \\ &\qquad = \left(\frac{bt}{\sinh(bt)}\right)^{\delta/2}\exp\left(-\frac{|a|^2}{2t}(bt\coth(bt) - 1)\right), \end{aligned} \tag{2.2}$$

which, in the case $a = 0$, gives

$$E\left(\exp\left(-\frac{b^2}{2}\int_0^t ds|B_s|^2\right)\bigg| B_t = 0\right) = \left(\frac{bt}{\sinh(bt)}\right)^{\delta/2}. \tag{2.3}$$

Lévy's formula ($\delta = 2$) for the stochastic area of $B_s = (X_s, Y_s)$, $s \leq t$,

$$\begin{aligned} &E\left(\exp\left(ib\int_0^t (X_s dY_s - Y_s dX_s)\right)\bigg|(X_t, Y_t) = a\right) \\ &\qquad = \left(\frac{bt}{\sinh(bt)}\right)\exp\left(-\frac{|a|^2}{2t}(bt\coth(bt) - 1)\right), \end{aligned}$$

follows from (2.2), using the rotational invariance of Brownian motion.

2.2. *Integrating out: some consequences of a path decomposition of Brownian motion.*

We first recall an elementary path decomposition of Brownian motion up to time 1. For a process X and two random times a, b, with $0 \leq a < b$, we denote

$$X^{[a,b]} = \left(\frac{1}{\sqrt{b-a}}X_{a+t(b-a)};\ 0 \leq t \leq 1\right).$$

Then, if g_1 denotes the last zero of B before time 1 (g_1 is arc-sine distributed), we have (see [6], [7], [50]) the following:

1. $b := B^{[0,\, g_1]}$ is a Brownian bridge;
2. $m := |B|^{[g_1,\, 1]}$ is a Brownian meander;
3. g_1, b and m are independent.

Our aim here is to check partially and to look at some consequences of the following identity in law:

4. $m^2 \overset{\text{(law)}}{=} \tilde{b}^2 + R^2$, where $\tilde{b}$ is a Brownian bridge independent of a two-dimensional Bessel process R (see Corollary 3.9.1*).

Now, putting together 1 through 4, we obtain

$$\int_0^1 du\, B_u^2 \overset{\text{(law)}}{=} g_1^2 \int_0^1 du\, b_u^2 + (1-g_1)^2 \int_0^1 du(\tilde{b}_u^2 + R_u^2) \tag{2.4}$$

with b, $\tilde{b}$, R, g_1 independent. Since the Laplace transforms of $\int_0^1 X_s^2\, ds$ for X a Brownian bridge and for $X = R$ are given in particular by (2.3) and (2.1), the identity in law in (2.4) implies the following relation:

$$\frac{1}{(\cosh\mu)^{1/2}} = \frac{1}{\pi}\int_0^\mu \frac{da}{(\sinh a)^{1/2}(\sinh(\mu-a))^{1/2}(\cosh(\mu-a))}. \tag{2.5}$$

More generally, from the identity

$$E\left(\exp\left(-\frac{\mu^2}{2}\int_0^1 du\, B_u^2\right) f(B_1^2)\right)$$
$$= E\left(\exp\left(-\frac{\mu^2}{2}\left\{g_1^2\int_0^1 du b_u^2 + (1-g_1)^2\int_0^1 du[\tilde{b}_u^2 + R_u^2]\right\}\right) f((1-g_1)R_1^2)\right),$$

we obtain

$$\begin{aligned}&\frac{1}{\sqrt{z}}\frac{1}{(\sinh\mu)^{1/2}}\exp\left(-\frac{z}{2}\coth\mu\right)\\ &\quad= \frac{1}{\sqrt{2\pi}}\int_0^\mu \frac{da}{(\sinh a)^{1/2}(\sinh(\mu-a))^{3/2}}\exp\left(-\frac{z}{2}\coth(\mu-a)\right).\end{aligned} \tag{2.6}$$

We note that (2.5) is obtained from (2.6) by integrating with respect to dz. Taking the Laplace transform in λ (with respect to z) in the above identity, we obtain

$$\begin{aligned}&\frac{1}{(\cosh\mu + \lambda\sinh\mu)^{1/2}}\\ &\quad= \frac{1}{\pi}\int_0^\mu \frac{da}{(\sinh a)^{1/2}(\sinh(\mu-a))^{1/2}(\cosh(\mu-a)+\lambda\sinh(\mu-a))}.\end{aligned} \tag{2.7}$$

We briefly indicate an elementary proof of this identity. We denote by L (resp., R) the left-hand side (resp., right-hand side) of (2.7). We set $a = \sinh(\mu)$; then,

$$L = \frac{1}{(\sqrt{1+a^2}+\lambda a)^{1/2}};$$

$$\pi R = \int_0^\mu dx \frac{(\sinh x)^{-1/2}(a\cosh x - \sqrt{1+a^2}\sinh x)^{-1/2}}{[(\sqrt{1+a^2}+\lambda a)\cosh x - (a+\lambda\sqrt{1+a^2})\sinh x]}$$

$$= \int_0^a dy \frac{(1+y^2)^{-1/2}(ay\sqrt{1+y^2} - \sqrt{1+a^2}\,y^2)^{-1/2}}{[(\sqrt{1+a^2}+\lambda a)\sqrt{1+y^2} - (a+\lambda\sqrt{1+a^2})y]}$$

$$(\text{taking } y = \sinh(x))$$

$$= \int_0^b \frac{dv}{(av - \sqrt{1+a^2}v^2)^{1/2}[(\sqrt{1+a^2}+\lambda a) - (a+\lambda\sqrt{1+a^2})v]}$$

$$\left(\text{taking } v = \frac{y}{\sqrt{1+y^2}} \text{ and } b = \frac{a}{\sqrt{1+a^2}}\right)$$

$$= 2(1+a^2)^{1/4}\int_0^\pi \frac{d\varphi}{(2+a^2+\lambda a\sqrt{1+a^2}) - (a^2+\lambda a\sqrt{1+a^2})\cos\varphi}$$

$$\left(\text{taking } v - \frac{a}{2\sqrt{1+a^2}} = -\frac{a}{2\sqrt{1+a^2}}\cos\varphi\right)$$

$$= 2(1+a^2)^{1/4}\int_0^\infty \frac{dt}{1+(1+a^2+\lambda a\sqrt{1+a^2})t^2} \quad \left(\text{taking } t = \tan\left(\frac{\varphi}{2}\right)\right)$$

$$= \frac{1}{(\sqrt{1+a^2}+\lambda a)^{1/2}}\pi = \pi L$$

as desired.

2.3. *Identities in law between two quadratic functionals.* We present a simple proof of the identity in law:

$$\int_0^1 dt(B_t - G)^2 \overset{\text{(law)}}{=} \int_0^1 dt\, b_t^2, \tag{2.8}$$

where B is a standard Brownian motion, $G = \int_0^1 B_s\,ds$ and b is a Brownian bridge on $[0,1]$. The identity in law (2.8) has been obtained independently by Chiang, Chow and Lee [14] and Chan, Dean, Jansons and Rogers [12], using a diagonalization procedure; in [15], we prove this identity as a consequence of a Fubini type theorem.

Here, we shall give a new variant, involving a Brownian sheet, of our proof of (2.8). First, we introduce the following notation. Let X and Y denote two r.v.'s; we write $X \sim_{L\to F} Y$, and also $Y \sim_{F\to L} X$ if

$$E(\exp(i\lambda Y)) = E\left(\exp\left(-\frac{\lambda^2}{2}X\right)\right) \qquad \forall\ \lambda \in \mathbb{R}.$$

Now we note that the left-hand side of (2.8) equals $\int_0^1 du \int_0^u ds(B_u - B_s)^2$; moreover,

$$\int_0^1 du \int_0^u ds(B_u - B_s)^2 \underset{L\to F}{\sim} \int_0^1\int_0^1 dW_{s,u}(B_u - B_s)1_{(s<u)},$$

where $\{W_{s,u};\ s,u \le 1\}$ denotes a Brownian sheet independent of B. Using Fubini's theorem,

$$\int_0^1 \int_0^1 dW_{s,u}(B_u - B_s) 1_{(s<u)} = \int_0^1 dB_t \int_0^1 \int_0^1 dW_{s,u} 1_{(s<t<u)}.$$

Below, we use the notation $W(A)$ for $\int_A dW_{s,u}$ and we recall that $E[W(A)W(C)] = m(A \cap C)$, where m is Lebesgue measure on $\mathbb{R}_+^2$, and A and C are two Borel sets.

Now, consider the rectangles $T_t = \{(s,u);\ 0 \le s < t < u \le 1\}$, $0 \le t \le 1$. The process $\{W(T_t);\ t \le 1\}$ is a centered Gaussian process with covariance

$$E(W(T_t)W(T_{t_0})) = m(T_t \cap T_{t_0}) = t_0(1-t) \quad \text{for } t_0 \le t.$$

That is, $(W(T_t),\ t \le 1)$ is a Brownian bridge. Thus,

$$\int_0^1 dt(B_t - G)^2 \underset{L \to F}{\sim} \int_0^1 dB_t W(T_t) \underset{F \to L}{\sim} \int_0^1 dt(W(T_t))^2$$

proving (2.8).

Note. Clearly, this type of argument may be applied to more general quadratic functionals of Brownian motion; for example, the reader may give a proof of Exercise 2.4* along the preceding lines.

We give a second example which originates from Chan and Jansons [13]. The authors are interested in the law of

$$\int_0^1 (X_t - \bar{X})^2\, dt,$$

where

$$\begin{cases} dX_t = \alpha U_t dt, \\ dU_t = dB_t - \alpha U_t dt, \end{cases} \qquad \alpha > 0$$

and $\bar{X} = \int_0^1 X_s\, ds$; U is an Ornstein–Uhlenbeck process. Then,

$$\begin{aligned} \int_0^1 (X_t - \bar{X})^2\, dt &= \int_0^1 ds \int_0^s du\, (X_u - X_s)^2 \\ &\underset{L \to F}{\sim} \int_0^1 dW_{s,u} \left(\int_s^u dX_t \right) 1_{(s<u)} \\ &= \int_0^1 dX_t\, W(T_t) = \alpha \int_0^1 dt\, U_t\, W(T_t), \end{aligned}$$

where $b_t := W(T_t)$ is a Brownian bridge independent of U;

$$E\left(\exp\left(-\frac{\lambda^2}{2} \int_0^1 (X_t - \bar{X})^2\, dt \right) \right) = E_b\left(\exp -\frac{\lambda^2}{2} \alpha^2 E_U\left(\left\{ \int_0^1 dt U_t\, b_t \right\}^2 \right) \right),$$

where the second expectation is taken with respect to U, b being fixed.

We assume that $U_0 \sim N(0, 1/2\alpha)$ so that U_t is stationary and $E(U_t U_s) = (\exp(-\alpha|t-s|))/2\alpha$. Finally, we obtain

$$E\left(\exp\left(-\frac{\lambda^2}{2}\int_0^1 (X_t - \bar{X})^2\,dt\right)\right)$$
$$= E\left(\exp\left(-\frac{\lambda^2}{2}\alpha\int_0^1 dt\int_0^t ds\exp(-\alpha(t-s))b_s b_t\right)\right)$$

proving the identity in law:

$$\int_0^1 (X_t - \bar{X})^2\,dt \overset{\text{(law)}}{=} \alpha\int_0^1 dt\int_0^t ds\ \exp(-\alpha(t-s))b_s b_t. \tag{2.9}$$

3. A martingale approach to Ray–Knight theorems. The results of this section are a continuation of Chapter 3*. On one hand, we prove again Theorems 3.3* and 3.4* without using excursion theory. On the other hand, we give some complements to Theorem 3.3* in studying the two parameter process $(L_\infty^a(|B| + (2/\delta)l);\ a \geq 0,\ \delta > 0)$, where $(l_t;\ t \geq 0)$ denotes the local time at 0 of B. We also include well-known Ray–Knight theorems for Bessel processes.

To prove these results, we use stochastic calculus; this approach has already been used by McGill [38, 39], Jeulin [27, 28] and by Jeulin and Yor [29] to study some functionals of Brownian motion. In Chapter 3*, Ray–Knight theorems are proved with the tools of excursion theory and Lévy–Khintchine representation of squares of Bessel processes.

We refer to [51], [17], [56], [36], [45], [54] for other Ray–Knight theorems and other approaches (among which are excursion theory and Dynkin's isomorphism theorem). From the classical Ray–Knight theorems (at time T_a and τ_t), one can obtain the Ray–Knight theorem for local times taken at an independent exponential time (see [8]). For recent simplifications of the Ray–Knight theorem at a fixed time [44], [28], we refer to [37], [57].

First, let us introduce some notation. We set $\Sigma_\Delta(t) = |B_t| + \Delta^{-1}(2l_t)$ where $\Delta: \mathbb{R}_+ \to \mathbb{R}_+$ is a C^1 function, strictly increasing, with $\Delta(0) = 0$ and $\Delta(\infty) = \infty$.

We define the squared Bessel process with generalized dimension Δ' as a $\mathbb{R}_+$-valued process, which is solution of the SDE

$$X_t = x + 2\int_0^t \sqrt{X_s}\,d\beta_s + \Delta(t), \tag{3.1}$$

and $Q_x^{\Delta'}$ denotes the law on $C(\mathbb{R}_+, \mathbb{R}_+)$ of a process which satisfies (3.1). The law of the squared Bessel process of dimension δ ($\delta \geq 0$) corresponding to $\Delta(t) = \delta t$ is Q_x^δ.

We now introduce some notation about the solutions of the Sturm–Liouville equation

$$\Phi'' = f\Phi \tag{SL}$$

associated with a measurable locally integrable function $f: \mathbb{R}_+ \to \mathbb{R}_+$. Φ_f and

Ψ_f are the solutions of (SL) which satisfy the following:

(i) $\Phi_f(0) = 1,\ \Phi_f \geq 0$ and Φ_f is decreasing;

(ii) $\Psi_f(0) = 0,\ \Psi'_f(0) = 1.$

The functions Φ_f and Ψ_f are linked by

$$\Psi_f(t) = \Phi_f(t) \int_0^t \frac{ds}{\Phi_f^2(s)}. \tag{3.2}$$

We recall that the Laplace transform of the law Q_x^δ can be expressed in terms of solutions of the Sturm–Liouville equation (SL).

LEMMA 3.1. *Let $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function with compact support. Then*

$$Q_x^{\Delta'}\left(\exp -\frac{1}{2}\int_0^\infty X_t f(t)\,dt\right) = \exp\left(\frac{1}{2}\int_0^\infty \frac{\Phi'_f(t)}{\Phi_f(t)}\Delta'(t)\,dt\right)\exp\left(\frac{x}{2}\Phi'_f(0)\right), \tag{3.3}$$

which, in the case $\Delta'(t) = \delta$, becomes

$$Q_x^\delta\left(\exp -\frac{1}{2}\int_0^\infty X_t f(t)\,dt\right) = (\Phi_f(\infty))^{\delta/2}\exp\left(\frac{x}{2}\Phi'_f(0)\right). \tag{3.4}$$

3.1. *The Ray–Knight theorem for the process Σ_Δ.* We can now state the Ray–Knight theorem for the family of local times of the process Σ_Δ.

THEOREM 3.1 ([35], Theorem 3.3*). *The law of the process $(L_\infty^a(\Sigma_\Delta))$, the family of local times of the process Σ_Δ, is $Q_0^{\Delta'}$. In particular, the law of the family of local times of $(\Sigma_\delta(t) \equiv |B_t| + (2/\delta)l_t;\ t \geq 0)$ is Q_0^δ.*

The proof of this theorem given in [35], Chapter 3*, is based on excursion theory. See also [43] for some extensions. Doney, Warren and Yor [16] show that Q_0^δ is also the law of the family of local times of $R_{3,\alpha}$, where $R_{3,\alpha}$ is an α-perturbed Bessel process of dimension 3 with $\delta = 2(1-\alpha)$ and, relying partly on [34], they explain the relationships between the two processes Σ_δ and $R_{3,\alpha}$.

We shall obtain Theorem 3.1 as a consequence of the following general result.

PROPOSITION 3.1. *We denote $X_t = |B_t| + l_t$. Let f be a measurable function on $\{(x,l);\ x \geq l \geq 0\}$, positive with compact support. Then,*

$$E\left(\exp -\tfrac{1}{2}\int_0^\infty f(X_s, l_s)\,ds\right) = \exp\left(\int_0^\infty F'_x(u,u)\,du\right), \tag{3.5}$$

where

$$F(x,l) = \ln(\Phi_{f(\cdot,l)}(x)), \qquad x \geq l \tag{3.6}$$

and $\Phi_{f(\cdot,l)}$ is the positive decreasing solution of the (SL) equation associated with the function $x \mapsto f(x,l),\ x \geq l$, satisfying $\Phi_{f(\cdot,l)}(l) = 1$.

PROOF. Let F satisfy (3.6); then F (or rather F'_x) solves the Riccati differential equation

$$F''_{x^2}(x,l)+(F'_x(x,l))^2=f(x,l).$$

Define

$$M_t^f=\exp\left\{F(X_t,l_t)-\int_0^{l_t}F'_x(u,u)\,du-\tfrac{1}{2}\int_0^t f(X_s,l_s)\,ds\right\}. \tag{3.7}$$

Using the decomposition $X_t=\beta_t+2l_t$, with the help of Itô's formula, one can easily prove that (M_t^f) is a $\mathscr{G}_t:=\sigma(|B_s|,\ s\le t)$ local martingale. Moreover, since f has compact support in $\{(x,l);\ x\ge l\ge 0\}$, this martingale is bounded (F'_x has compact support). Equality (3.5) then follows from the optional stopping theorem. □

PROOF OF THEOREM 3.1. We apply (3.5) to the function $f(x,l)=g(x-l+\Delta^{-1}(2l))$. Then $F(x,l)=\ln\Phi_g(x-l+\Delta^{-1}(2l))-\ln\Phi_g(\Delta^{-1}(2l))$ and (3.5) becomes

$$E\left(\exp-\frac{1}{2}\int_0^\infty g(\Sigma_\Delta(s))\,ds\right)=\exp\left(\frac{1}{2}\int_0^\infty\frac{\Phi'_g(t)}{\Phi_g(t)}\Delta'(t)\,dt\right).$$

That is,

$$E\left(\exp-\tfrac{1}{2}\int_0^\infty g(a)L_\infty^a(\Sigma_\Delta)\,da\right)=Q_0^{\Delta'}\left(\exp-\tfrac{1}{2}\int_0^\infty X_t\,g(t)\,dt\right)$$

by (3.3). Note that, in the case $\Delta'(t)=\delta$, the martingale M^f takes the form

$$M^g(t)=\exp\left\{G_g(\Sigma_\delta(t))-(1+(\delta/2))G_g\left(\frac{2}{\delta}l_t\right)-\frac{1}{2}\int_0^t g(\Sigma_\delta(s))\,ds\right\}, \tag{3.8}$$

where $G_g(x)=\ln\Phi_g(x)$. □

SOME COMMENTS ON PROPOSITION 3.1. (1) According to Pitman's representation of Bes(3), we have the following equality in law:

$$(X_t,l_t;\ t\ge 0)\overset{\text{(law)}}{=}(R_t,J_t;\ t\ge 0), \tag{3.9}$$

where $(R_t,t\ge 0)$ is a Bes(3) process and $J_t=\inf_{s\ge t}R_s$. Thus, (3.5) is equivalent to

$$E\left(\exp-\tfrac{1}{2}\int_0^\infty f(R_s,J_s)\,ds\right)=\exp\left(\int_0^\infty F'_x(u,u)\,du\right), \tag{3.10}$$

where F satisfies (3.6).

(2) We now consider the process $\Sigma_\delta(t)$ in the case $\delta=2$, that is, for local times of a Bessel process of dimension 3 (beware: not 2!). We note $R=\Sigma_2$; the martingale M_t^f defined by (3.8) takes the form

$$M^f(t)=\exp\left\{G_f(R(t))-2G_f(l_t)-\tfrac{1}{2}\int_0^t f(R(s))\,ds\right\}, \tag{3.11}$$

where $G_f(x) = \ln \Phi_f(x)$; M^f is a martingale with respect to the natural filtration $\mathscr{G}_t$ of $|B|$. The natural filtration $\mathscr{R}_t$ of the process $R_t = |B_t| + l_t$ is strictly smaller than $\mathscr{G}_t$ (see [18]). A natural question arises: what is the projection of the martingale M^f on the filtration $\mathscr{R}_t$? Here is the solution:

We set $N_t^f = E[M_t^f / \mathscr{R}_t]$. To calculate N_t^f, we use the following fact: for fixed t, the conditional law of l_t is the uniform law on $[0, R_t]$; thus

$$\begin{aligned} N_t^f &= \exp\left\{G_f(R(t)) - \frac{1}{2}\int_0^t f(R(s))\,ds\right\}\frac{1}{R_t}\int_0^{R_t}\exp(-2G_f(u))\,du \\ &= \frac{1}{R_t}\Phi_f(R_t)\int_0^{R_t}\frac{du}{\Phi_f^2(u)}\ \exp\left\{-\frac{1}{2}\int_0^t f(R(s))\,ds\right\} \qquad (3.12)\\ &= \frac{1}{R_t}\Psi_f(R_t)\exp\left\{-\frac{1}{2}\int_0^t f(R(s))\,ds\right\}, \end{aligned}$$

using (3.2). Formula (3.12) yields another proof of the Ray–Knight theorem for the family of local times of Bes(3). Indeed, we can apply the optional stopping theorem to the bounded martingale N^f and then use

$$\frac{\Psi_f(x)}{x} \underset{x\to\infty}{\rightarrow} \frac{1}{\Phi_f(\infty)}.$$

We can also obtain the Ray–Knight theorem for the family of local times at time T_1, the hitting time of 1 by R. The optional stopping theorem applied to (3.12) at time T_1 gives

$$\begin{aligned} E\left[\exp -\frac{1}{2}\int_0^{T_1} f(R(s))\,ds\right] &= \frac{1}{\Psi_f(1)} \\ &= Q^2_{0\to 0}\left[\exp -\frac{1}{2}\int_0^1 f(x)X_x\,dx\right] \end{aligned}$$

using a result of Pitman and Yor [47], where $Q^2_{0\to 0}$ is the law of the squared two-dimensional Bessel bridge. In other words,

$$(L^a_{T_1}(R),\ 0 \le a \le 1) \overset{\text{(law)}}{=} (|\tilde{Z}_t|^2,\ 0 \le t \le 1), \qquad (3.13)$$

where $\tilde{Z}$ is a Brownian bridge in $\mathbb{R}^2$.

3.2. *A second Ray–Knight theorem for the process* Σ_δ. The martingale M^f introduced in the proof of Proposition 3.1 enables us to obtain the law of the family of local times up to τ_x, the inverse local time.

THEOREM 3.2 (Theorem 3.4*). *Let $x > 0$ and consider $\tau_x = \inf\{t \ge 0,\ l_t > x\}$. The process $(L^a_{\tau_x}(\Sigma_\delta),\ a \ge 0)$ is an inhomogeneous Markov process, starting at 0, which is the square of a δ-dimensional Bessel process for $a \le 2x/\delta$, and the square of a 0-dimensional Bessel process for $a \ge 2x/\delta$.*

PROOF. The optional stopping theorem $E[M_0^f] = E[M_{\tau_x}^f]$ shows

$$E\left[\exp -\frac{1}{2}\int_0^\infty f(a)L_{\tau_x}^a \, da\right] = E\left[\exp -\frac{1}{2}\int_0^{\tau_x} f(\Sigma_\delta(s))\, ds\right]$$
$$= \left\{\Phi_f\left(\frac{2x}{\delta}\right)\right\}^{\delta/2}.$$

Let $(Y_a,\ a \geq 0)$ be an inhomogeneous Markov process as specified in the above theorem. Then,

$$E\left[\exp -\frac{1}{2}\int_0^\infty f(a)Y_a \, da\right]$$
$$= E\left[\exp\left(-\frac{1}{2}\int_0^b f(a)Y_a \, da\right)\exp -\frac{1}{2}\int_0^\infty f_b(a)Y_{a+b}\, da\right]$$
$$\text{with } b := \frac{2x}{\delta} \text{ and } f_b(a) = f(a+b)$$
$$= Q_0^\delta\left[\exp\left(-\frac{1}{2}\int_0^b f(a)X_a \, da\right)Q_{X_b}^0\left(\exp\left(-\frac{1}{2}\int_0^\infty f_b(a)X_a\, da\right)\right)\right]$$

from the Markov property. Now, by (3.3),

$$Q_{X_b}^0\left(\exp\left(-\frac{1}{2}\int_0^\infty f_b(a)X_a \, da\right)\right) = \exp\left\{\frac{X_b}{2}\Phi'_{f_b}(0)\right\},$$

$\Phi_{f_b}(x) = \Phi(x+b)/\Phi(b)$ and $\Phi'_{f_b}(0) = \Phi'(b)/\Phi(b)$. Then,

$$E\left[\exp -\frac{1}{2}\int_0^\infty f(a)Y_a \, da\right]$$
$$= Q_0^\delta\left[\exp\left(-\frac{1}{2}\int_0^b f(a)X_a \, da + \frac{1}{2}\frac{\Phi'(b)}{\Phi(b)}X_b\right)\right]$$
$$= \left\{\frac{1}{\Psi'_f(b) - (\Phi'(b)/\Phi(b))\Psi_f(b)}\right\}^{\delta/2} \quad \text{(by [47], Equation (1.h))}$$
$$= (\Phi_f(b))^{\delta/2} \quad \text{using (3.2).}$$

This proves the equality in law between the processes $(L_{\tau_x}^a(\Sigma_\delta),\ a \geq 0)$ and $(Y_a,\ a \geq 0)$.

3.3. *Relation between the local times* $(L_\infty^a(\Sigma_\delta),\ a \geq 0)$. For $\delta > 0$, we denote C_a^δ the local time at level a and at time ∞ for the process $\Sigma_\delta(t) = |B_t| + (2/\delta)l_t$. One can prove [35] that there exists a jointly continuous version of the process $(C_a^\delta;\ a \geq 0,\ \delta > 0)$. In Section 3.1 above, we have obtained the law of $C^\delta = (C_a^\delta;\ a \geq 0)$, namely Q_0^δ. We are now interested in the joint law of the processes $(C^\delta;\ \delta > 0)$ and we show the following theorem.

THEOREM 3.3. *Define* $M_a^\delta = \frac{1}{2}(C_a^\delta - a\delta)$*; then* M_a^δ *is a two-parameter martingale with respect to the filtration* $\mathscr{C}_a^\delta = \sigma\{C_b^\gamma;\ b \leq a,\ 0 < \gamma \leq \delta\}$*; its* *increasing process is given by*

$$\langle M\rangle_a^\delta = \int_0^a dx\, C_x^\delta. \tag{3.14}$$

Moreover, $\langle M\rangle$ *is* C^1 *and*

$$\begin{aligned}\frac{\partial}{\partial\delta}\langle M\rangle_a^\delta &= \frac{2}{\delta^2}\int_0^\infty du\big(C_a^\delta - C_a^\delta(\tau_u)\big)\\ &= \frac{2}{\delta^2}\int_0^\infty dl_s\big(C_a^\delta - C_a^\delta(s)\big),\end{aligned} \tag{3.15}$$

where τ *is the inverse of the local time* l.

PROOF. From Tanaka's formula,

$$\begin{aligned}\frac{1}{2}(C_a^\delta(t) - a\delta) &= \int_0^t 1_{(|B_s|+(2/\delta)l_s<a)}\,\mathrm{sgn}(B_s)dB_s\\ &\quad + \left(|B_t| + \frac{2}{\delta}l_t - a\right)^- - \left(1+\frac{2}{\delta}\right)\left(l_t - \frac{\delta}{2}a\right)^-,\end{aligned} \tag{3.16}$$

where $C_a^\delta(t)$ is the local time at level a at time t of the process Σ_δ. For $t=\infty$, (3.16) simplifies to

$$\begin{aligned}\tfrac{1}{2}(C_a^\delta - a\delta) &= \int_0^\infty 1_{(|B_s|+(2/\delta)l_s<a)}\,\mathrm{sgn}(B_s)\,dB_s\\ &\triangleq M_a^\delta.\end{aligned} \tag{3.17}$$

Now, we shall use a representation of the square integrable random variables measurable with respect to $\mathscr{C}_a^\delta$.

LEMMA 3.2. *Any r.v.* H *of* $L^2(\mathscr{C}_a^\delta)$ *may be written*

$$H = E(H) + \int_0^\infty h_s 1_{(|B_s|+(2/\delta)l_s<a)}\,dB_s, \tag{3.18}$$

where h *is predictable w.r.t the filtration of* B *and*

$$E\left[\int_0^\infty h_s^2 1_{(|B_s|+(2/\delta)l_s<a)}\,ds\right] < \infty. \tag{3.19}$$

PROOF. It is enough to prove (3.18) for a r.v. H of the form

$$\begin{aligned}H &= \exp\left\{-\sum_{i=1}^n \int_0^a g_i(b)C_b^{\gamma_i}\,db\right\}\\ &= \exp\left\{-\sum_{i=1}^n \int_0^\infty g_i(\Sigma_i(s))\,ds\right\}\\ &= \exp\left\{-\sum_{i=1}^n \int_0^\infty g_i(\Sigma_i(s))1_{(l_s\le(a\delta/2))}\,ds\right\}\end{aligned}$$

with $g_i \ge 0$ with support in $[0,a]$, $\gamma_i \le \delta$ and $\Sigma_i(s) := |B_s| + (2/\gamma_i)l_s$.

Clearly, H can be written as $\exp\{-\int_0^\infty f(X_t, l_t)\,dt\}$ with $X_t = |B_t| + l_t$ and

$$f(x,l) = \sum_{i=1}^{n} g_i\left(x - l + \frac{2}{\gamma_i} l\right) 1_{(l \le (a\delta/2))}.$$

Let us consider the martingale M^f associated with the function f introduced in the proof of Proposition 3.1. Then

$$M_t^f = 1 + \int_0^t M_s^f F'_x(X_s, l_s)\,\mathrm{sgn}(B_s)\,dB_s,$$

where F is defined by (3.6). Since supp $g_i \in [0,a]$ and $\gamma_i \le \delta$, $f(x,l) = 0$ for $x \ge a + l - (2/\delta)l$. Now, we easily verify that supp $f(\cdot, l) \subset [l, a + l - (2/\delta)l]$, $(l \le (a\delta/2))$ implies supp $F'_x(\cdot, l) \subset [l, a + l - (2/\delta)l]$ and therefore

$$\begin{aligned} M_t^f &= 1 + \int_0^t M_s^f F'_x(X_s, l_s) 1_{(X_s \le a + l_s - (2/\delta) l_s)}\,\mathrm{sgn}(B_s)\,dB_s \\ &= 1 + \int_0^t M_s^f F'_x(X_s, l_s) 1_{(|B_s| + (2/\delta) l_s \le a)}\,\mathrm{sgn}(B_s)\,dB_s. \end{aligned}$$

Since $H = M_\infty^f \exp\{\int_0^\infty F'_x(u,u)\,du\}$, H can be written

$$H = C + \int_0^\infty h(s) 1_{(|B_s| + (2/\delta) l_s \le a)}\,dB_s. \qquad \square$$

It is now easy to prove that M_a^δ is a $\mathscr{C}_a^\delta$ martingale. By (3.16), we can see that $E(C_a^\delta) = a\delta$ when we take $t \to \infty$, which implies $E(M_a^\delta) = 0$. Let H be a $\mathscr{C}_b^\gamma$ measurable r.v. with $\gamma \le \delta$ and $b \le a$. By the lemma, we can write

$$H = E(H) + \int_0^\infty h(s) 1_{(|B_s| + (2/\gamma) l_s \le b)}\,dB_s$$

and

$$\begin{aligned} E(M_a^\delta H) &= E\left(\int_0^\infty h(s) 1_{(|B_s| + (2/\gamma) l_s \le b)} 1_{(|B_s| + (2/\delta) l_s \le a)}\,\mathrm{sgn}(B_s)\,ds\right) \\ &\qquad \text{(using (3.17))}, \\ &= E(M_b^\gamma H), \end{aligned}$$

proving the martingale property of M_a^δ.

Following Wong and Zakai [61] (see also [41]), we say that a two-parameter continuous martingale $(M_z, \mathscr{F}_z,\ z \in \mathbb{R}_+^2)$ is path-independent if for every continuous increasing path γ from $[0,1]$ to $\mathbb{R}_+^2$, $M_{\gamma(t)}$ is a one-parameter martingale with increasing process A_t such that A_1 is the same for all increasing paths γ having the same endpoints $\gamma(0)$ and $\gamma(1)$. For a path-independent martingale, one can define a function $\langle M, M\rangle_z$, called the increasing process, as the increasing function A_1 for all paths γ connecting $\gamma(0)$ and $\gamma(1) = z$.

We shall verify that M_a^δ is path-independent. Let γ be an increasing path connecting $\gamma(0) = 0$ and $\gamma(1) = (a, \delta)$ and Δ_n a sequence of subdivisions (t_i) of

$[0,1]$ such that $|\Delta_n| \to 0$. We set $(a_i, \delta_i) = \gamma(t_i)$. Let A_t denote the increasing process of the martingale $M_{\gamma(t)}$. Then,

$$A_1 = \lim_{n\to\infty} \sum_{\Delta_n} \big(M_{a_{i+1}}^{\delta_{i+1}} - M_{a_i}^{\delta_i}\big)^2.$$

Set $M_a^\delta(t) = \int_0^t 1_{(|B_s|+(2/\delta)l_s \le a)} \operatorname{sgn}(B_s)\, dB_s$ and apply Itô's formula to $(M_{a_{i+1}}^{\delta_{i+1}}(t) - M_{a_i}^{\delta_i}(t))^2$ to obtain, by letting $t \to \infty$,

$$\begin{aligned}\big(M_{a_{i+1}}^{\delta_{i+1}} - M_{a_i}^{\delta_i}\big)^2 = 2\int_0^\infty &\big(M_{a_{i+1}}^{\delta_{i+1}}(s) - M_{a_i}^{\delta_i}(s)\big) \\ &\times \big(1_{(|B_s|+(2/\delta_{i+1})l_s \le a_{i+1})} - 1_{(|B_s|+(2/\delta_i)l_s \le a_i)}\big) \operatorname{sgn}(B_s)\, dB_s \\ &+ \int_0^\infty \big(1_{(|B_s|,\, l_s)\in\Gamma_{i+1}} - 1_{(|B_s|,\, l_s)\in\Gamma_i}\big)^2\, ds,\end{aligned}$$

where $\Gamma_i = \{(b,l) \in (\mathbb{R}_+)^2;\ b + (2/\delta_i)l \le a_i\}$. We note that $\Gamma_i \subset \Gamma_{i+1}$ since $a_i \le a_{i+1}$ and $\delta_i \le \delta_{i+1}$. Hence

$$\begin{aligned}\sum_{\Delta_n} \big(M_{a_{i+1}}^{\delta_{i+1}} - M_{a_i}^{\delta_i}\big)^2 = \sum_{\Delta_n} 2\int_0^\infty &\big(M_{a_{i+1}}^{\delta_{i+1}}(s) - M_{a_i}^{\delta_i}(s)\big) 1_{\Gamma_{i+1}\setminus\Gamma_i}(|B_s|, l_s) \operatorname{sgn}(B_s)\, dB_s \\ &+ \int_0^\infty 1_{(|B_s|,\, l_s)\in\cup(\Gamma_{i+1}\setminus\Gamma_i)}\, ds.\end{aligned}$$

Now,

$$\begin{aligned}\int_0^\infty 1_{(|B_s|, l_s)\in\cup(\Gamma_{i+1}\setminus\Gamma_i)}\, ds &= \int_0^\infty 1_{(|B_s|+(2/\delta)l_s\le a)}\, ds \\ &= \int_0^a C_b^\delta\, db.\end{aligned}$$

It remains to prove that $\lim_{n\to\infty} M_n(\infty) = 0$ in probability, where

$$M_n(t) = \sum_{\Delta_n} \int_0^t \big(M_{a_{i+1}}^{\delta_{i+1}}(s) - M_{a_i}^{\delta_i}(s)\big) 1_{\Gamma_{i+1}\setminus\Gamma_i}(|B_s|, l_s) \operatorname{sgn}(B_s)\, dB_s.$$

This follows from

$$\begin{aligned}\langle M_n\rangle_\infty &= \sum_{\Delta_n} \int_0^\infty \big(M_{a_{i+1}}^{\delta_{i+1}}(s) - M_{a_i}^{\delta_i}(s)\big)^2 1_{\Gamma_{i+1}\setminus\Gamma_i}(|B_s|, l_s)\, ds \\ &\le \int_0^\infty \sup_i \big(M_{a_{i+1}}^{\delta_{i+1}}(s) - M_{a_i}^{\delta_i}(s)\big)^2 1_{(|B_s|+(2/\delta)l_s\le a)}\, ds \\ &\le \sup_{s,\,i} \big(M_{a_{i+1}}^{\delta_{i+1}}(s) - M_{a_i}^{\delta_i}(s)\big)^2 \int_0^\infty 1_{(|B_s|+(2/\delta)l_s\le a)}\, ds \\ &\underset{n\to\infty}{\longrightarrow} 0,\end{aligned}$$

using the continuity of the map $(a,\delta) \mapsto M_a^\delta(\cdot)$, whose proof will be given in Section 3.5. This proves that $A_1 = \int_0^a C_b^\delta\, db$, which is independent of the path

γ with endpoint (a,δ). Thus,

$$\langle M\rangle_a^\delta = \int_0^a C_b^\delta\, db.$$

To prove (3.15), we write

$$\begin{aligned}\langle M\rangle_a^\delta &= \int_0^a dx\, C_x^\delta \\ &= \int_0^a dx\left(x\delta + 2\int_0^\infty 1_{(|B_s|+(2/\delta)l_s\le x)}\operatorname{sgn}(B_s)\,dB_s\right) \\ &= \delta\frac{a^2}{2} + 2\int_0^\infty \left(a - |B_s| - \frac{2}{\delta}l_s\right)^+ \operatorname{sgn}(B_s)\,dB_s.\end{aligned}$$

Then,

$$\frac{\partial}{\partial\delta}\langle M\rangle_a^\delta = \frac{a^2}{2} + 2\int_0^\infty 1_{(|B_s|+(2/\delta)l_s\le a)} l_s \operatorname{sgn}(B_s)\,dB_s,$$

and we easily verify that this expression is the same as (3.15). □

SOME COMMENTS ON THEOREM 3.3.

1. The two-parameter filtration $\mathscr{C}_a^\delta$ does not satisfy the important (F.4) property for two-parameter processes (see [11]); that is, $\mathscr{C}_a^\infty = \vee_{\gamma>0}\mathscr{C}_a^\gamma$ and $\mathscr{C}_\infty^\delta = \vee_{b\ge 0}\mathscr{C}_b^\delta$ are not conditionally independent given $\mathscr{C}_a^\delta$.

   Indeed, let $a_0 \le a$ and $\delta_0 \le \delta$ and compute $E(M_{a_0}^\delta M_a^{\delta_0})$. By (3.17),

$$\begin{aligned}E(M_{a_0}^\delta M_a^{\delta_0}) &= E\left(\int_0^\infty 1_{(|B_s|+(2/\delta_0)l_s<a)} 1_{(|B_s|+(2/\delta)l_s<a_0)}\, ds\right) \\ &= E\left[\min\left(\frac{a^2}{(|B_1|+(2/\delta_0)l_1)^2}, \frac{a_0^2}{(|B_1|+(2/\delta)l_1)^2}\right)\right] \quad \text{(by scaling)} \\ &= E\left[\min\left(\frac{a^2}{(U+(2/\delta_0)(1-U))^2}, \frac{a_0^2}{(U+(2/\delta)(1-U))^2}\right)\right],\end{aligned}$$

   where U is an uniform r.v. on $[0,1]$ and the last equality follows from the identity in law $(|B_1|, l_1) \overset{\text{(law)}}{=} R_1(U, 1-U)$ with R_1, a Bes(3) at time 1, independent of U. For example, in the case $(a_0/\delta_0) - (a/\delta) < 0$ and $(a_0/\delta_0) - (a/\delta) + a - a_0 > 0$, we find $E(M_{a_0}^\delta M_a^{\delta_0}) = \delta(a_0^2/2)$.

   On the other hand, if (F.4) were satisfied, then $E(M_{a_0}^\delta M_a^{\delta_0})$ would be equal to

$$E((M_{a_0}^{\delta_0})^2) = E\left(\int_0^{a_0} C_b^{\delta_0}\, db\right) = \int_0^{a_0} (\delta_0 b)\, db = \delta_0\frac{a_0^2}{2}.$$

2. Another way to find the increasing process $\langle M\rangle$ would be to prove directly that

$$(M_a^\delta)^2 - \langle M\rangle_a^\delta \quad \text{is a } \mathscr{C}_a^\delta \text{ martingale.}$$

We can see that

$$(M_a^\delta)^2 - \int_0^a C_b^\delta db = 2\int_0^\infty M_a^\delta(s) 1_{(|B_s|+(2/\delta)l_s<a)} \operatorname{sgn}(B_s)\, dB_s,$$

but we do not know how to prove directly that the right hand side is a $\mathscr{C}_a^\delta$ martingale.

Infinite divisibility of the law of the process C. We recall another property of the law of the process $C = (C_a^\delta;\ a \geq 0,\ \delta > 0)$ proved by Le Gall and Yor [35]: if $\tilde{C}$ is an independent copy of C, then for $p > 0,\ q > 0$,

$$\big(C_a^{p\delta} + \tilde{C}_a^{q\delta};\ a \geq 0,\ \delta > 0\big) \overset{(\text{law})}{=} \big(C_a^{(p+q)\delta};\ a \geq 0,\ \delta > 0\big), \tag{3.20}$$

which proves the infinite divisibility of the law (on $C(\mathbb{R}_+ \times \mathbb{R}_+^*, \mathbb{R}_+)$) of C. We can describe the Lévy–Khintchine representation of the law of C as follows:

$$\begin{aligned} &E\left[\exp\left\{-\sum_{i=1}^n \int_0^\infty f_i(a) C_a^{\delta_i}\, da\right\}\right] \\ &\quad = \exp\left\{-\int \nu(d\tilde{\omega})\left(1 - \exp - \sum_{i=1}^n \int_0^\infty f_i(a)\tilde{\omega}(a, \delta_i)\, da\right)\right\}, \end{aligned} \tag{3.21}$$

where ν is the image of the measure $M(d\omega) \otimes ds$ under the mapping

$$\begin{aligned} C(\mathbb{R}_+, \mathbb{R}_+) \times \mathbb{R}_+ &\to C(\mathbb{R}_+ \times \mathbb{R}_+^*, \mathbb{R}_+) \\ (\omega,\quad s) &\mapsto (\tilde{\omega}(a, \delta) = \omega((a - s/\delta)^+),\ a \geq 0,\ \delta > 0) \end{aligned}$$

and M is the image of the Itô measure n^+ of positive excursions under the mapping which associates with an excursion e its family of local times $(l_V^a(e), a \geq 0)$ (see [35], [65] and another description of M in [45]). Equation (3.21) is a consequence of master formulas of excursion theory ([50], Chapter XII).

3.4. *Some extensions to Bessel processes.* In this subsection, we shall extend Proposition 3.1, or more precisely equation (3.10), to a Bessel process of dimension d for $d > 2$. Let R denote a d-dimensional Bessel process and $J_t = \inf_{s \geq t} R_s$. We denote by $\mathscr{R}_t = \sigma(R_s, s \leq t)$ and $\tilde{\mathscr{R}}_t = \mathscr{R}_t \vee \sigma(J_t)$. The decomposition of the semimartingale R in the filtration $\tilde{\mathscr{R}}_t$ is the following (Corollary 12.7.1*, as well as [53], [55], [49]):

$$R_t = r + \beta_t + 2J_t - \frac{d-3}{2}\int_0^t \frac{du}{R_u}, \tag{3.22}$$

where β is an $(\tilde{\mathscr{R}}_t)$ Brownian motion. We can now state the theorem.

Theorem 3.4. *Let f be a measurable function on $\{(x, l);\ x \geq l \geq 0\}$, positive with compact support. Then*

$$E\left(\exp - \tfrac{1}{2}\int_0^\infty f(R_s, J_s)\, ds\right) = \exp\left(\int_0^\infty G_x'(u, u)\, du\right), \tag{3.23}$$

where G satisfies

$$G(x,l) = \ln \Phi_{g^{(\alpha)}(\cdot,l)}(x^\alpha) \tag{3.24}$$

and the function $g^{(\alpha)}$ is defined on the set $\{(x,l);\ x \geq l^\alpha \geq 0\}$ by

$$g^{(\alpha)}(x,l) = \frac{1}{\alpha^2} x^{(2/\alpha)-2} f(x^{1/\alpha}, l) \tag{3.25}$$

with $\alpha := d-2$ and as before $\Phi_{g^\alpha(\cdot,l)}$ is the positive, decreasing solution of the (SL) equation with initial value $\Phi_{g^\alpha(\cdot,l)}(l^\alpha) = 1$.

PROOF. As in the proof of Proposition 3.1, we are looking for an $\tilde{\mathscr{R}}_t$ martingale of the form

$$M_t^f = \exp\left\{G(R_t, J_t) + \varphi(J_t) - \tfrac{1}{2}\int_0^t f(R_s, J_s)\,ds\right\}. \tag{3.26}$$

Using the decomposition (3.22) and Itô's formula, one sees that G and φ should satisfy

$$\begin{cases} f(x,l) = G''_{x^2}(x,l) - \dfrac{d-3}{x} G'_x(x,l) + (G'_x(x,l))^2, \\ \varphi(l) = -G(l,l) - \displaystyle\int_0^l G'_x(u,u)\,du. \end{cases} \tag{3.27}$$

Now, to solve the differential equation

$$y'' - \frac{\alpha-1}{x} y' + (y')^2 = f(x),$$

we use the change of variable $z(x) = y(x^{1/\alpha})$; then z satisfies a standard Ricatti equation

$$z'' + (z')^2 = g^{(\alpha)}(x)$$

with

$$g^{(\alpha)}(x) = \frac{1}{\alpha^2} x^{(2/\alpha)-2} f(x^{1/\alpha}). \tag{3.28}$$

Now $z(x) = \ln \Phi_{g^{(\alpha)}}(x)$ solves the preceding Ricatti equation, and $G(x,l) = \ln \Phi_{g^{(\alpha)}(\cdot,l)}(x^\alpha)$ is a solution of (3.27) such that the martingale defined by (3.26) is bounded. Then the end of the proof is the same as in Proposition 3.1. □

Note that when $f(x,l)$ does not depend on l, the martingale M^f is

$$M_t^f = \exp\left\{G(R_t) - 2G(J_t) - \tfrac{1}{2}\int_0^t f(R_s)\,ds\right\},$$

where $G(x) = \ln \Phi_{g^{(\alpha)}}(x^\alpha)$ with $g^{(\alpha)}$ satisfying (3.28) and $\Phi_{g^{(\alpha)}}(0) = 1$.

As in Section 2.1, we can compute the projection of the martingale M^f on the filtration $\mathscr{R}_t$. To do this, we need the conditional law of J_t given $\mathscr{R}_t$. Using the Markov property of R and the local martingale $(1/R_t^\alpha)$ (see, e.g., Lemma 12.1*), one can prove that

$$P\big(J_t \le y \big| \mathscr{R}_t\big) = \frac{y^\alpha}{R_t^\alpha}, \qquad y \le R_t.$$

Then,

$$\begin{aligned}
\hat{M}_l^f &\overset{\text{def}}{=} E\big(M_t^f \big| \mathscr{R}_t\big) \\
&= \exp\left\{G(R_t) - \frac{1}{2}\int_0^t f(R_s)\,ds\right\} \frac{1}{R_t^\alpha}\int_0^{R_t} \exp(-2G(u))\alpha u^{\alpha-1}\,du \\
&= \Phi_{g^{(\alpha)}}(R_t^\alpha)\exp\left\{-\frac{1}{2}\int_0^t f(R_s)\,ds\right\}\frac{1}{R_t^\alpha}\int_0^{R_t}\frac{1}{\Phi^2_{g^{(\alpha)}}(u^\alpha)}\alpha u^{\alpha-1}\,du \qquad (3.29)\\
&= \frac{1}{R_t^\alpha}\Phi_{g^{(\alpha)}}(R_t^\alpha)\int_0^{R_t^\alpha}\frac{1}{\Phi^2_{g^{(\alpha)}}(v)}dv\ \exp\left\{-\frac{1}{2}\int_0^t f(R_s)\,ds\right\} \\
&= \frac{1}{R_t^\alpha}\Psi_{g^{(\alpha)}}(R_t^\alpha)\exp\left\{-\frac{1}{2}\int_0^t f(R_s)\,ds\right\}.
\end{aligned}$$

COROLLARY 3.4.1 (Ray–Knight theorem for local times of Bessel processes for $d > 2$). *Let $d > 2$ and $\alpha = d-2$, we denote by Z (resp., $\tilde{Z}$) a two-dimensional Brownian motion (resp., Brownian bridge). Then*:

$$\text{(i)} \qquad (L_\infty^a(R_d);\ a \ge 0) \overset{\text{(law)}}{=} \left(\frac{1}{\alpha a^{\alpha-1}}|Z_{a^\alpha}|^2;\ a \ge 0\right),$$

$$\text{(ii)} \qquad (L_{T_1}^a(R_d);\ 0 < a \le 1) \overset{\text{(law)}}{=} \left(\frac{1}{\alpha a^{\alpha-1}}|\tilde{Z}_{a^\alpha}|^2;\ 0 < a \le 1\right),$$

where R_d denotes a d-dimensional Bessel process and T_1 is the hitting time of 1 by R_d.

The assertions of the corollary are well known: they are obtained from the classical Ray–Knight theorem for Bes(3) by time change (see [34], [63]). Here, we give another proof, using the expression (3.29) of the martingale $\hat{M}^f$. We apply the optional stopping theorem to the bounded martingale $\hat{M}^f$:

$$\begin{aligned}
E\left[\exp\left\{-\frac{1}{2}\int_0^\infty f(R_d(s))\,ds\right\}\right] &= \lim_{x\to\infty}\frac{x}{\Psi_{g^{(\alpha)}}(x)} = \Phi_{g^{(\alpha)}}(\infty) \\
&= Q_0^2\left[\exp\left\{-\frac{1}{2}\int_0^\infty g^{(\alpha)}(x)X_x\,dx\right\}\right] \\
&= E\left[\exp\left\{-\frac{1}{2}\int_0^\infty g^{(\alpha)}(x)|Z_x|^2\,dx\right\}\right] \\
&= E\left[\exp\left\{-\frac{1}{2}\int_0^\infty f(x)\frac{1}{\alpha x^{\alpha-1}}|Z_{x^\alpha}|^2\,dx\right\}\right].
\end{aligned}$$

Part (ii) is obtained in the same way, using

$$\frac{1}{\Psi_{g^{(\alpha)}}(1)} = Q^2_{0\to 0}\left[\exp\left\{-\frac{1}{2}\int_0^1 g^{(\alpha)}(x)X_x\,dx\right\}\right]$$
$$= E\left[\exp\left\{-\frac{1}{2}\int_0^1 g^{(\alpha)}(x)|\tilde{Z}_x|^2\,dx\right\}\right].$$

COROLLARY 3.4.2 (Ray–Knight theorem for local times of Bessel processes for $d < 2$). *Let $d < 2$ and $\beta = 2 - d$. Assume R_d is a d-dimensional Bessel process with $R_d(0) = r > 0$ and T_0 is the hitting time of 0 by R_d. Then*

$$(L^a_{T_0}(R_d);\ a \ge 0) \overset{\text{(law)}}{=} \left(\frac{1}{\beta a^{\beta-1}} Y_{a^\beta};\ a \ge 0\right), \tag{3.30}$$

where $(Y_a, a \ge 0)$ is an inhomogeneous Markov process, starting at 0, which is the square of a two-dimensional Bessel process for $a \le r^\beta$, and the square of a 0-dimensional Bessel process for $a \ge r^\beta$.

The identity (3.30) is obtained in a similar way as above. Note that there exists a version of (3.22) for a Bessel process of dimension $d < 2$; that is,

$$R_t = r + \gamma_t + 2K_t - \frac{d-3}{2}\int_0^t \frac{du}{R_u}, \qquad t \le T_0, \tag{3.31}$$

where $K_t = \sup_{t\le u\le T_0} R_u$, and γ is an $(\hat{\mathscr{R}}_t)$ Brownian motion, $(\hat{\mathscr{R}}_t)$ being the filtration $\mathscr{R}_t \vee \sigma(K_t)$.

The decomposition (3.31) leads, as might be expected, to the same result as above:

$$\frac{1}{R_t^\alpha}\Psi_{g^{(\alpha)}}(R_t^\alpha)\exp\left\{-\frac{1}{2}\int_0^t f(R_s)\,ds\right\}$$

is an $(\mathscr{R}_t)$ martingale, where $\alpha = d - 2 = -\beta$ and $g^{(\alpha)}$ is defined by (3.28).

Note also that by time reversal, we can partially recover Corollary 3.4.2 from (i) of Corollary 3.4.1.

3.5. *Proof of Lemma* 3.3. As announced, we shall now prove the following lemma.

LEMMA 3.3. *There exists a continuous version of the map $(a, \delta) \to M_a^\delta(\cdot)$.*

PROOF. We assume $a \le N$ and $\delta \le D$, for arbitrary but fixed N and D. To use Kolmogorov's criterion, we must prove that there exist $p > 0$, $\alpha > 2$ and $C > 0$ such that, for $a, b \le N$, and $\delta, \gamma \le D$:

$$E\Big(\sup_{s\ge 0}|M_a^\delta(s) - M_b^\gamma(s)|^p\Big) \le C(|a-b| + |\delta-\gamma|)^\alpha. \tag{3.32}$$

Let $p > 0$ be an even integer; then

$$E\Big(\sup_{s\geq 0}|M_a^\delta(s) - M_b^\gamma(s)|^p\Big) \leq C_p E\left(\left\{\int_0^\infty (1_\Gamma(|B_s|, l_s) - 1_{\tilde\Gamma}(|B_s|, l_s))^2\, ds\right\}^{p/2}\right),$$

where

$$\Gamma = \left\{(x, l);\ x + \frac{2}{\delta} l \leq a\right\}, \qquad \tilde\Gamma = \left\{(x, l);\ x + \frac{2}{\gamma} l \leq b\right\}.$$

Let $a > 0$, $\delta > 0$ and $0 < h \leq 1$; then

$$\begin{aligned} E\Big(\sup_{s\geq 0}|M_{a+h}^\delta(s) - M_a^\delta(s)|^p\Big) &\leq C_p E\left(\left\{\int_0^\infty (1_{a<|B_s|+(2/\delta)l_s\leq a+h}\, ds\right\}^{p/2}\right) \\ &= C_p E\left(\left\{\int_a^{a+h} C_x^\delta\, dx\right\}^{p/2}\right) \\ &= C_p h^{p/2} \sup_{x\in[a,\, a+1]} E\big[(C_x^\delta)^{p/2}\big] \\ &\leq C_p h^{p/2} E\big[(C_{N+1}^\delta)^{p/2}\big]. \end{aligned}$$

On the other hand, the estimates for increments involving the dimension parameter are more involved; first,

$$E\Big(\sup_{s\geq 0}|M_a^{\delta+h}(s) - M_a^\delta(s)|^p\Big) \leq C_p E\left(\left\{\int_0^\infty 1_{\Gamma\setminus\tilde\Gamma}(|B_s|, l_s)\, ds\right\}^{p/2}\right),$$

where

$$\Gamma = \left\{(x, l);\ x + \frac{2}{\delta + h} l \leq a\right\}; \qquad \tilde\Gamma = \left\{(x, l);\ x + \frac{2}{\delta} l \leq a\right\}.$$

We now write

$$\begin{aligned} \int_0^\infty 1_{\Gamma\setminus\tilde\Gamma}(|B_s|, l_s)\, ds &= \int_0^\infty 1_{(|B_s|+(2/\delta+h)l_s\leq a<|B_s|+(2/\delta)l_s)}\, ds \\ &= \int_0^\infty 1_{((a-(2/\delta)l_s)^+<|B_s|\leq(a-(2/\delta+h)l_s)^+)}\, ds \\ &= \int_0^a dy \int_0^\infty 1_{((a-(2/\delta)l_s)^+<y\leq(a-(2/\delta+h)l_s)^+)}\, d_s L_s^y, \end{aligned}$$

where $(L_s^y,\ y \geq 0)$ denotes the family of local times of $|B|$. Now, for $y \in [0, a]$,

$$\begin{aligned} \int_0^\infty 1_{((a-(2/\delta)l_s)^+<y\leq(a-(2/\delta+h)l_s)^+)}\, d_s L_s^y &= \int_0^\infty 1_{((\delta/2)(a-y)<l_s\leq(\delta+h/2)(a-y))}\, d_s L_s^y \\ &= \int_0^\infty 1_{(\tau_{\alpha(y)}<s\leq\tau_{\beta(y)})}\, d_s L_s^y \\ &= L_{\tau_{\beta(y)}}^y - L_{\tau_{\alpha(y)}}^y \end{aligned}$$

where $\tau(u) = \inf\{t > 0,\ l_t > u\}$ and

$$\alpha(y) = \frac{\delta}{2}(a - y), \tag{3.33}$$

$$\beta(y) = \frac{\delta + h}{2}(a - y). \tag{3.34}$$

Therefore,

$$Z := \int_0^\infty 1_{\Gamma\setminus\tilde{\Gamma}}(|B_s|, l_s)\, ds = \int_0^a dy\big(L^y_{\tau_{\beta(y)}} - L^y_{\tau_{\alpha(y)}}\big). \tag{3.35}$$

Now, we must estimate the moment of order $p/2$ of the r.v. Z. Let $k \in \mathbb{N}^*$,

$$E(Z^k) = k! \int_{[0,a]^k} 1_{(x_1 < x_2 < \cdots < x_k)} \prod_{i=1}^k dx_i\ E\bigg(\prod_{i=1}^k \big(L^{x_i}_{\tau_{\beta(x_i)}} - L^{x_i}_{\tau_{\alpha(x_i)}}\big)\bigg).$$

We note that the functions α and β defined by (3.33), (3.34) are decreasing. We want to compute the function:

$$A(x_1, \dots, x_k) = E\bigg(\prod_{i=1}^k \big(L^{x_i}_{\tau_{\beta(x_i)}} - L^{x_i}_{\tau_{\alpha(x_i)}}\big)\bigg).$$

To do this, we use the Markov property of B and the additive functional identity:

$$L^x_{\tau_C} - L^x_{\tau_A} = L^x_{\tau_{C-A}} \circ \theta_{\tau_A} \quad \text{for } A < C,$$

where θ denotes the translation operator on the Wiener space.

The computation of $A(x)$ depends on the position between $\alpha(x_i)$ and $\beta(x_j)$ for $i < j$. Let us first study the easier case, that is, let $x_1 < x_2 < \cdots < x_k$ such that $\alpha(x_i) \geq \beta(x_{i+1})$ (recall that α and β are decreasing). Then

$$\begin{aligned}
A(x) &= E\bigg(\prod_{i=2}^k \big(L^{x_i}_{\tau_{\beta(x_i)}} - L^{x_i}_{\tau_{\alpha(x_i)}}\big) E_{B_{\tau_{\alpha(x_1)}}}\big(L^{x_1}_{\tau_{\beta(x_1)-\alpha(x_1)}}\big)\bigg) \\
&= E\bigg(\prod_{i=2}^k \big(L^{x_i}_{\tau_{\beta(x_i)}} - L^{x_i}_{\tau_{\alpha(x_i)}}\big)\bigg) E_0\big(L^{x_1}_{\tau_{\beta(x_1)-\alpha(x_1)}}\big) \\
&= \prod_{i=1}^k E\big(L^{x_i}_{\tau_{\beta(x_i)-\alpha(x_i)}}\big) \\
&= 2^k \prod_{i=1}^k (\beta(x_i) - \alpha(x_i)) \\
&= 2^k h^k \prod_{i=1}^k (a - x_i).
\end{aligned}$$

The equality $E(L^x_{\tau_A}) = 2A$ is a consequence of the Ray–Knight theorem for the local times of Brownian motion at time τ_A; that is,

$$(l^x_{\tau_A},\ x \ge 0) \overset{\text{(law)}}{=} Q^0_A,$$

and $L^x_{\tau_A} = l^x_{\tau_A} + l^{-x}_{\tau_A}$. We have thus obtained

$$\int 1_{(x_1<x_2<\cdots<x_k)} 1_{\alpha(x_i)\ge\beta(x_{i+1})} A(x)\,dx \le Ch^k.$$

It remains to study the case where $\alpha(x_i) \le \beta(x_{i+1})$ for some i. We shall study the case $k = 3$ (which is enough to prove the continuity). Let $x < y < z$ and suppose that $\alpha(x) > \beta(y)$, $\alpha(y) < \beta(z)$. The last condition implies that z varies in an interval $I(y)$ of amplitude less than Ch. Now, we can prove that in this case,

$$A(x, y, z) \le Ch^2$$

and

$$\int_{I(y)} A(x, y, z)\,dz \le Ch^3.$$

The other cases are similar and finally, we have obtained that

$$E(Z^3) \le Ch^3,$$

which proves the lemma. □

4. On some principal values of Brownian local times.

4.1. *Distributions of principal values of Brownian local times, taken at the inverse local time.* In Section 1, we introduced the two processes

$$H_t = p.v.\int_0^t \frac{ds}{B_s} \quad \text{and} \quad H^\lambda_t = p.v.\int_0^t ds\,\coth(\lambda B_s),$$

corresponding to the space–time harmonic functions $h_0(x, t) = x$ and $h_\lambda(x, t) = \sinh(\lambda x)\exp(-\lambda^2 t/2)$. The first process has been studied by Biane and Yor [7], the second one by Alili [1, 2].

More generally, let f be an odd function, locally integrable on $\mathbb{R}^*$, satisfying $f(x) \sim_0 (C/x)$. We can define

$$\begin{aligned} A^f_t &= p.v.\int_0^t f(B_s)\,ds := \lim_{\varepsilon\to 0}\int_0^t f(B_s)1_{(|B_s|\ge\varepsilon)}\,ds \\ &= \int_0^\infty f(a)(l^a_t - l^{-a}_t)\,da. \end{aligned}$$

The existence of A^f_t follows from the Hölder continuity of Brownian local times. For a general survey of principal values of local times, see [62].

Let τ denote the inverse local time of B at 0. We present two approaches to study the law of the r.v. $A^f_{\tau_t}$. The first one is based on excursion theory and gives the Lévy measure of the Lévy process $(A^f_{\tau_t},\ t\ge 0)$ in terms of n the Itô measure of excursions. This approach has been used by Biane–Yor [7] to describe the law of (H_{τ_t},τ_t).

PROPOSITION 4.1. *The characteristic function of $A^f_{\tau_t}$ is given by*

$$E\big(\exp(i\xi A^f_{\tau_t})\big)$$

$$= \exp\left\{-t\int \mathrm{n}(de)\left(1-\exp\left(i\xi\int_0^{V(e)} f(e(u))\,du\right)\right)\right\} \tag{4.1}$$

$$= \exp\left\{-t\int_0^\infty \frac{dv}{\sqrt{2\pi v^3}}\Pi_v\left(1-\cos\left(\xi\int_0^v f(r(u))\,du\right)\right)\right\}. \tag{4.2}$$

The identity (4.2) is a consequence of the following description of n^+, the Itô measure of positive excursions (see [50], Chapter 12):

(D)
(1) under n^+, the law of the lifetime V of the excursion is $dv/2\sqrt{2\pi v^3}$;
(2) conditionally on $V=v$, the law of $(e(s), s\le V)$ is the law, denoted by Π_v, of a three-dimensional Bessel bridge over $[0,v]$.

A second method (used by Alili) to obtain the law of $A^f_{\tau_t}$ uses the Ray–Knight theorem for $(l^x_{\tau_t},\ x\in\mathbb{R})$ and the description of squared Bessel processes in terms of Sturm–Liouville equations (see Section 3). Let us first introduce the processes

$$A_t^{f,+} = \int_0^1 f(a)(l_t^a - l_t^0)\,da + \int_1^\infty f(a) l_t^a\,da,$$

$$A_t^{f,-} = \int_0^1 f(a)(l_t^{-a} - l_t^0)\,da + \int_1^\infty f(a) l_t^{-a}\,da.$$

REMARK. If $\int^\infty f(a)\,da<\infty$, we can set another definition for the processes $A^{f,+}$, $A^{f,-}$; that is,

$$A_t^{f,+} = \int_0^\infty f(a)(l_t^a - l_t^0)\,da, \qquad A_t^{f,-} = \int_0^\infty f(a)(l_t^{-a} - l_t^0)\,da.$$

Alili in [1] and [2] chooses this renormalization.

We assume that $f_{/\mathbb{R}^+}\ge 0$; as in Section 3, Φ_f denotes the decreasing solution of the (SL) equation associated with $f_{/\mathbb{R}^+}$.

PROPOSITION 4.2. *Let $\theta>0$; then*

$$E\left(\exp\left(-\frac{\theta}{2}A^{f,+}_{\tau_t}\right)\right) = \exp\left\{\frac{t}{2}\lim_{\varepsilon\to 0}\left(\frac{\Phi'_{\theta f}(\varepsilon)}{\Phi_{\theta f}(\varepsilon)-\varepsilon\Phi'_{\theta f}(\varepsilon)}+\theta\int_\varepsilon^1 f(a)\,da\right)\right\} \tag{4.3}$$

PROOF.

$$A^{f,+}_{\tau_t} = \lim_{\varepsilon\to 0}\left(\int_\varepsilon^\infty f(a) l^a_{\tau_t}\, da - t\int_\varepsilon^1 f(a)\, da\right).$$

Set $f_\varepsilon(x) = f(x)1_{(x\geq\varepsilon)}$ and denote by Φ_{f_ε} the solution of (SL) associated with f_ε. Then, it is easy to see that

$$\Phi'_{\theta f_\varepsilon}(0_+) = \frac{\Phi'_{\theta f}(\varepsilon)}{\Phi_{\theta f}(\varepsilon) - \varepsilon\Phi'_{\theta f}(\varepsilon)}.$$

Formula (4.3) follows then from the Ray–Knight theorem and (3.4). □

Denote by $I(\theta)$ the Laplace exponent of $A^{f,+}_{\tau_t}$ which is given in (4.3); then

$$E\left(\exp\left(i\frac{\xi}{2}A^f_{\tau_t}\right)\right) = \exp\left\{\frac{t}{2}(I(i\xi) + I(-i\xi))\right\}. \tag{4.4}$$

Applying these results to $f_0(x) = (1/x)$ and $f_\lambda(x) = \coth(\lambda x)$, one obtains the theorem. □

THEOREM 4.1 ([2], Theorem 3). *Let $\xi \in \mathbb{R}$ and $\mu \in \mathbb{R}^+$; then we have*

$$E\big(\exp(i\xi H^\lambda_{\tau_t} - \mu\tau_t)\big) = \exp\left(\frac{-\pi t|\xi|}{\lambda}\coth\left(\frac{\pi}{\lambda}\sqrt{(\mu^2+\xi^2)^{1/2} - \mu}\right)\right). \tag{4.5}$$

In particular, for $\mu = 0$,

$$E\big(\exp(i\xi H^\lambda_{\tau_t})\big) = \exp\left(\frac{-\pi t|\xi|}{\lambda}\coth\left(\frac{\pi}{\lambda}\sqrt{|\xi|}\right)\right). \tag{4.6}$$

THEOREM 4.2 ([7]). *Let $\xi \in \mathbb{R}$ and $\mu \in \mathbb{R}^+$; then we have*

$$E\big(\exp(i\xi H_{\tau_t} - \mu\tau_t)\big) = \exp\left(-\pi t|\xi|\coth\left(\pi\frac{|\xi|}{\sqrt{2\mu}}\right)\right). \tag{4.7}$$

The comparison of these two results implies the following puzzling identity in law:

$$\lambda H^\lambda_{\tau_t} \overset{\text{(law)}}{=} H_{\tau_t} + \frac{\lambda}{2}C_{\tau_t}, \tag{4.8}$$

where $(C_u,\ u \geq 0)$ denotes a standard Cauchy process, independent of B.

4.2. *An identity in law for the normalized excursion.* If we rewrite (4.5) and (4.7) using excursion theory as in Proposition 4.1, we can see (cf. [2], Theorem 9) that (4.5) and (4.7) imply the following surprising fact: for $\nu \in \mathbb{R}^*$, the law of

$$\theta_\nu := \nu^2\left\{\left(\int_0^1 ds \coth(\nu r_s)\right)^2 - 1\right\} \tag{4.9}$$

does not depend on ν, where $(r_s,\ s \le 1)$ is a three-dimensional Bessel bridge of length 1. Now, we give a partial explanation of (4.9). We refer to [3] for a more complete discussion of this identity in law.

First verification. We verify that $\theta_0 := \lim_{\nu \to 0} \theta_\nu$ and $\theta_\infty := \lim_{\nu \to \infty} \theta_\nu$ have the same law. Letting ν go to 0, we find that

$$\theta_0 = \left(\int_0^1 \frac{ds}{r_s} \right)^2. \tag{4.10}$$

Now, it is well known ([7], (2.g) and (5.d)) that

$$\theta_0 \overset{\text{(law)}}{=} T_\pi^{(3)} + \hat{T}_\pi^{(3)},$$

where $T_\pi^{(3)}$ and $\hat{T}_\pi^{(3)}$ are two independent copies of the first hitting time of π by a three-dimensional Bessel process, so that

$$E\left(\exp\left(-\frac{\lambda^2}{2} \theta_0 \right) \right) = \left(\frac{\lambda \pi}{\sinh(\lambda \pi)} \right)^2.$$

On the other hand, when $\nu \to \infty$,

$$\theta_\nu \sim 2\nu^2 \int_0^1 ds\, f_*(\nu r_s),$$

where

$$f_*(x) = \coth x - 1 = \frac{2}{\exp(2x) - 1}.$$

We now apply the following result (see [20], [26]): for f a bounded function, with compact support (and in fact for a larger class of functions),

$$\nu^2 \int_0^1 ds\, f(\nu r_s) \underset{\nu \to \infty}{\overset{\text{(law)}}{\to}} \int_0^\infty ds\, f(R_s) + \int_0^\infty ds\, f(\hat{R}_s), \tag{4.11}$$

where R and $\hat{R}$ are two independent three-dimensional Bessel processes. We apply this result to f_* even though this function does not have a compact support. Thus, to prove the identity in law between θ_0 and θ_∞, we must show that

$$\int_0^\infty \frac{ds}{\exp(R_s) - 1} \overset{\text{(law)}}{=} T_\pi^{(3)}.$$

This identity is a consequence of

$$\int_0^\infty \frac{ds}{\exp(2R_s) - 1} \overset{\text{(law)}}{=}_{\text{(i)}} \int_0^{T_1^{(2)}} \frac{ds}{1 - (R_2(s))^2} \overset{\text{(law)}}{=}_{\text{(ii)}} T_{\pi/2}^{(3)}, \tag{4.12}$$

where R_2 is a two-dimensional Bessel process. This identity in law (4.12) will be proved and generalized in Section 5.

Second verification via the scaling property. If we assume (4.9), the function $\Phi_\nu(\lambda) = E(\exp(-\lambda\theta_\nu))$ does not depend on ν, hence

$$\frac{d}{d\nu}\Phi_\nu(\lambda) = 0.$$

This implies

$$E\left(\frac{dC_\nu}{d\nu}\bigg| C_\nu\right) = -\frac{1}{\nu}\left(C_\nu - \frac{1}{C_\nu}\right), \tag{4.13}$$

where $C_\nu = \int_0^1 ds \coth(\nu r_s)$. We multiply both sides of (4.13) by ν^3; then, we can write (4.13) as

$$E\left(\nu^2 \int_0^\infty ds\, g_*(\nu r_s)|\theta_\nu\right) = \frac{\theta_\nu}{C_\nu},$$

where $g_*(x) = x/(\sinh x)^2$. Let ν go to ∞ and use (4.11) to obtain from the above equation

$$\begin{aligned} E\bigg(\int_0^\infty ds\, g_*(R_s) + \int_0^\infty ds\, g_*(\hat{R}_s)\bigg| \int_0^\infty ds\, f_*(R_s) + \int_0^\infty ds\, f_*(\hat{R}_s)\bigg) \\ = 2\bigg(\int_0^\infty ds\, f_*(R_s) + \int_0^\infty ds\, f_*(\hat{R}_s)\bigg). \end{aligned}$$

Thus, we must show that

$$E\left(\int_0^\infty ds\, g_*(R_s)\bigg| \int_0^\infty ds\, f_*(R_s)\right) = 2\int_0^\infty ds\, f_*(R_s). \tag{4.14}$$

The identity (4.14) is just a consequence of the scaling property of Bes(3). More generally, for any C^1 function f, and $t \ge 0$,

$$E\left(tf(R_t) - \tfrac{1}{2}\int_0^t ds\, R_s f'(R_s)\bigg| \int_0^t ds\, f(R_s)\right) = \int_0^t ds\, f(R_s). \tag{4.15}$$

PROOF OF (4.15). For $\lambda \ge 0$, we can write, using the scaling property,

$$E\left(\exp\left(-\lambda\mu^2 \int_0^t ds\, f(\mu R_s)\right)\right) = E\left(\exp\left(-\lambda \int_0^{\mu^2 t} ds\, f(R_s)\right)\right).$$

We then differentiate the two sides of the above equality with respect to μ and take $\mu = 1$ in the resulting formula. This yields (4.15). □

4.3. *An explicit computation.* We come back to Section 4.1. We shall compute the Laplace transform of the functional $A_{\tau_t}^{f_1,+}$ for the function $f_1(x) = 1/(\exp(2x) - 1)$. The choice of this function is motivated by the identity in law (4.12). The computation of Φ_{f_1}, the solution of the (SL) equation associated with f_1, gives both the Laplace transform of $A_{\tau_t}^{f,+}$ [see (4.3)] and the

Laplace transform of $\int_0^\infty (ds/\exp(2R_s)-1)$, proving directly the identity in law between this functional and $T^{(3)}_{\pi/2}$. Indeed,

$$E\left(\exp\left(-\frac{\theta^2}{2}\int_0^\infty \frac{ds}{\exp(2R_s)-1}\right)\right) = Q_0^2\left(\exp\left(-\frac{\theta^2}{2}\int_0^\infty \frac{dt}{\exp(2t)-1}X_t\right)\right)$$
$$= \Phi_{\theta^2 f_1}(\infty)$$

where $\Phi_{\theta^2 f_1}$ is the positive, decreasing solution of

$$\Phi''(x) = \frac{\theta^2}{\exp(2x)-1}\Phi(x), \qquad \Phi(0)=1. \tag{4.16}$$

To solve (4.16), we set $\Psi(x)=\Phi(\ln(1/x))$, $0<x<1$; Ψ satisfies the differential equation

$$\Psi''(x)+\frac{\Psi'(x)}{x} = \frac{\theta^2}{1-x^2}\Psi(x).$$

A solution of this equation is given by

$$\Psi(x) = {}_2F_1\left(\frac{i\theta}{2}, \frac{-i\theta}{2}; 1; x^2\right),$$

where ${}_2F_1(\alpha,\beta;\gamma;z)$ denotes the hypergeometric function, defined for $|z|<1$ by the series

$${}_2F_1(\alpha,\beta;\gamma;z) = \sum_{n=0}^\infty \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} z^n$$

with $(\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n-1)$.

Clearly, Ψ is increasing on $[0,1]$. Thus, the solution $\Phi_{\theta^2 f_1}$ is given by

$$\Phi_{\theta^2 f_1}(x) = \left|\Gamma\left(1+\frac{i\theta}{2}\right)\right|^2 {}_2F_1\left(\frac{i\theta}{2}, \frac{-i\theta}{2}; 1;\ \exp(-2x)\right) \tag{4.17}$$

and

$$\Phi_{\theta^2 f_1}(\infty) = \left|\Gamma\left(1+\frac{i\theta}{2}\right)\right|^2 = \frac{\pi(\theta/2)}{\sinh(\pi(\theta/2))} = E\left(\exp\left(-\frac{\theta^2}{2}T^{(3)}_{\pi/2}\right)\right), \tag{4.18}$$

which proves the identity in law between the two extreme terms of (4.12).

We now proceed to the study of the law of $A^{f_1,+}_{\tau_t}$ or more generally to $K^\lambda_{\tau_t}$ where we denote

$$K_t^\lambda := p.v.\int_0^t \frac{ds}{\exp(2\lambda B_s)-1} 1_{(B_s\ge 0)}, \qquad \lambda>0,$$
$$= \int_0^\infty \frac{da}{\exp(2\lambda a)-1}(l_t^a - l_t^0).$$

THEOREM 4.3. *Let $\lambda > 0$; then,*

$$E\left(\exp\left(-\frac{\theta^2}{2}K^{\lambda}_{\tau_t}\right)\right) = \exp\left(-\frac{t\theta^2}{4\lambda}\left\{-2\gamma - \Psi\left(1+\frac{i\theta}{2}\right) - \Psi\left(1-\frac{i\theta}{2}\right)\right\}\right), \tag{4.19}$$

where γ denotes Euler's constant and $\Psi(x) = \Gamma'(x)/\Gamma(x)$.

PROOF. We apply (4.3) with the function $f_\lambda(x) = 1/(\exp(2\lambda x) - 1)$. Now,

$$\begin{aligned}\Phi_{\theta^2 f_\lambda}(x) &= \Phi_{(\theta/\lambda)^2 f_1}(\lambda x)\\ &= \left|\Gamma\left(1+\frac{i\theta}{2\lambda}\right)\right|^2 {}_2F_1\left(\frac{i\theta}{2\lambda}, \frac{-i\theta}{2\lambda}; 1;\ \exp(-2\lambda x)\right) \quad \text{using (4.17).}\end{aligned}$$

According to Lebedev ([33], (9.2.2)),

$$\Phi'_{\theta^2 f_\lambda}(x) = -\left|\Gamma\left(1+\frac{i\theta}{2\lambda}\right)\right|^2 \frac{\theta^2}{2\lambda}\exp(-2\lambda x)\, {}_2F_1\left(\frac{i\theta}{2\lambda}+1, \frac{-i\theta}{2\lambda}+1; 2;\ \exp(-2\lambda x)\right).$$

By (4.3), we must find $\lim_{\varepsilon\to 0} I_\varepsilon(\theta,\lambda)$, where

$$\begin{aligned}I_\varepsilon(\theta,\lambda) &= \frac{\Phi'_{\theta^2 f_\lambda}(\varepsilon)}{\Phi_{\theta^2 f_\lambda}(\varepsilon) - \varepsilon\Phi'_{\theta^2 f_\lambda}(\varepsilon)} + \theta^2\int_\varepsilon^\infty f_\lambda(a)\,da\\ &= \frac{\Phi'_{\theta^2 f_\lambda}(\varepsilon)}{\Phi_{\theta^2 f_\lambda}(\varepsilon) - \varepsilon\Phi'_{\theta^2 f_\lambda}(\varepsilon)} + \frac{\theta^2}{2\lambda}\ln\left(\frac{1}{2\lambda\varepsilon}\right).\end{aligned}$$

To study $\lim_{\varepsilon\to 0} I_\varepsilon(\theta,\lambda)$, we need the following approximation near 1 of the function ${}_2F_1(a,b;c;x)$ for $c = a+b$:

$$\begin{aligned}&{}_2F_1(a,b;a+b;x)\\ &\quad = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}(2\Psi(1) - \Psi(a) - \Psi(b) - \ln(1-x)) + O((1-x)\ln(1-x)),\end{aligned}$$

where $\Psi(x) = \Gamma'(x)/\Gamma(x)$. Then, we see that $\lim_{\varepsilon\to 0}\varepsilon\Phi'_{\theta^2 f_\lambda}(\varepsilon) = 0$ and

$$\lim_{\varepsilon\to 0} I_\varepsilon(\theta,\lambda) = \lim_{\varepsilon\to 0}\left(\Phi'_{\theta^2 f_\lambda}(\varepsilon) + \frac{\theta^2}{2\lambda}\ln\left(\frac{1}{2\lambda\varepsilon}\right)\right).$$

Now,

$$\begin{aligned}\Phi'_{\theta^2 f_\lambda}(\varepsilon) = -\frac{\theta^2}{2\lambda}\exp(-2\lambda\varepsilon)\biggl(&2\Psi(1) - \Psi\left(1+\frac{i\theta}{2\lambda}\right) - \Psi\left(1-\frac{i\theta}{2\lambda}\right)\\ &- \ln(1-\exp(-2\lambda\varepsilon))\biggr)\\ &+ O(\varepsilon\ln(\varepsilon)),\end{aligned}$$

and therefore,

$$\lim_{\varepsilon\to 0} I_\varepsilon(\theta,\lambda) = -\frac{\theta^2}{2\lambda}\left(2\Psi(1) - \Psi\left(1+\frac{i\theta}{2\lambda}\right) - \Psi\left(1-\frac{i\theta}{2\lambda}\right)\right). \qquad \square$$

REMARK. We have obtained the Laplace transform of

$$p.v.\int_0^\infty (da/\exp(2\lambda a)-1)l^a_{\tau_x}.$$

Since $2/(\exp(2a)-1)=\coth(a)-1$,

$$E\left(\exp\left(-\frac{\theta^2}{2}p.v.\int_0^\infty \frac{da}{\exp(2\lambda a)-1}l^a_{\tau_x}\right)\right)=E\left(\exp\left(-\frac{\theta^2}{4}H^{\lambda,+}_{\tau_x}+\frac{\theta^2}{4}\tau_x^+\right)\right),$$

where $H^{\lambda,+}_{\tau_x}=\text{p.v.}\int_0^{\tau_x} ds\coth(\lambda B_s)1_{(B_s\ge 0)}$ and $\tau_x^+=\int_0^\infty l^a_{\tau_x}\,da$.

Alili [1, 2] obtained the joint law of $(H^{\lambda,+}_{\tau_x},\tau_x^+)$; taking formally $u^2=\theta^2/2$ and $\mu^2=-(\theta^2/2)$ in the formula obtained by Alili, we recover (4.19) up to a coefficient, that is, $\ln(2\lambda)$, which does not appear in our (4.19). This comes from the fact that the "normalization" $\ln(2\lambda\varepsilon)$ in our definition of *p.v.* is not the same as that of Alili, who uses $\ln(\varepsilon)$.

5. Generalized Bessel processes and the porous medium equation. In the preceding section, we presented the following identities in law:

$$\int_0^\infty \frac{ds}{\exp(2R_3(s))-1} \overset{\text{(law)}}{=}_{\text{(i)}} \int_0^{T_1^{(2)}} \frac{ds}{1-(R_2(s))^2} \overset{\text{(law)}}{=}_{\text{(ii)}} T^{(3)}_{\pi/2}, \tag{5.1}$$

where R_d denotes a d-dimensional Bessel process starting from 0 and $T_a^{(d)}=\inf\{t>0,\ R_d(t)=a\}$. The first, (i), is an easy consequence of the following representation (see [34]):

$$\begin{aligned}\ln\left(\frac{1}{R_2(t)}\right)&=R_3\left(\sup\left\{u,\int_u^\infty \exp(-2R_3(s))\,ds>t\right\}\right)\\&=R_3\left(\int_t^{T_1^{(2)}}\frac{ds}{R_2^2(s)}\right),\qquad 0<t\le T_1^{(2)}.\end{aligned}\tag{5.2}$$

The second identity in law, (ii), will follow from the study of the functional

$$A_a(R)=\int_0^{T_a(R)}\frac{ds}{1-(R(s))^2},\qquad a\le 1,$$

where R is a positive process which belongs to a certain class of diffusions (to be defined in the next subsection), including the Bessel processes.

5.1. *Generalized Bessel processes and hypergeometric functions.* We consider a family $\mathscr{L}^{\delta,c}$ of diffusions $(R_t,\ t\ge 0)$ with values in $[0,1]$, depending on two parameters $\delta>0$ and c, whose infinitesimal generator is given by

$$L^{\delta,c}f(x)=\frac{1}{2}f''(x)+\frac{\delta-1}{2x}f'(x)+\frac{2cx}{1-x^2}f'(x),\qquad f\in C^2. \tag{5.3}$$

For $\delta<2$, 0 is an instantaneously reflecting regular boundary (for $\delta\ge 2$, 0 is not reached). The process R, starting from $x\in[0,1[$, is stopped at time T_1. One can verify by computing the scale function of R that $T_1<\infty$ p.s. if and only if $c > -\frac{1}{2}$. We denote by $P_x^{\delta, c}$ the law of the process R of generator $L^{\delta, c}$, starting from x. Then, we have the following proposition.

PROPOSITION 5.1. *Let $\delta = 2(\nu+1) > 0$ and $c \in \mathbb{R}$.*

(i) *For $a \in [0, 1[$ and $x \le a$,*

$$E_x^{\delta, c}\left[\exp\left(-\frac{k^2}{2}\int_0^{T_a(R)} \frac{ds}{1-(R(s))^2}\right)\right] = \frac{{}_2F_1(\alpha, \beta; \nu+1; x^2)}{{}_2F_1(\alpha, \beta; \nu+1; a^2)}, \tag{5.4}$$

where ${}_2F_1$ denotes the hypergeometric function (see [33]);

$$\alpha := \frac{\nu - 2c + i\theta}{2}; \qquad \beta := \frac{\nu - 2c - i\theta}{2} \quad \text{if } k^2 \ge (\nu - 2c)^2,$$

$$\alpha := \frac{\nu - 2c + \theta}{2}; \qquad \beta := \frac{\nu - 2c - \theta}{2} \quad \text{if } k^2 < (\nu - 2c)^2,$$

and $\theta = \sqrt{|k^2 - (\nu - 2c)^2|}$.

(ii) *When $c > -\frac{1}{2}$ ($T_1 < \infty$ a.s.),*

$$\begin{aligned} &E_0^{\delta, c}\left[\exp\left(-\frac{k^2}{2}\int_0^{T_1(R)} \frac{ds}{1-(R(s))^2}\right)\right] \\ &\quad = \frac{|\Gamma(\nu/2 + (1+c) + (i\theta/2))|^2}{\Gamma(\nu+1)\Gamma(1+2c)} \quad \text{if } k^2 \ge (\nu-2c)^2, \end{aligned} \tag{5.5}$$

$$\begin{aligned} &E_0^{\delta, c}\left[\exp\left(-\frac{k^2}{2}\int_0^{T_1(R)} \frac{ds}{1-(R(s))^2}\right)\right] \\ &\quad = \frac{\Gamma((\nu/2) + (1+c) + (\theta/2))\Gamma((\nu/2) + (1+c) - (\theta/2))}{\Gamma(\nu+1)\Gamma(1+2c)} \quad \text{if } k^2 < (\nu-2c)^2. \end{aligned} \tag{5.6}$$

PROOF. It is well known that if φ is the bounded solution on $[0, a]$ of

$$L^{\delta, c}\varphi(r) = \frac{k^2}{2(1-r^2)}\varphi(r)$$

with $\varphi(0) = 1$, then

$$E_x^{\delta, c}\left[\exp\left(-\frac{k^2}{2}\int_0^{T_a(R)} \frac{ds}{1-(R(s))^2}\right)\right] = \frac{\varphi(x)}{\varphi(a)}, \qquad x \le a.$$

Thus φ is the solution of

$$\varphi''(x) + \frac{\delta - 1}{x}\varphi'(x) + \frac{4cx}{1-x^2}\varphi'(x) = \frac{k^2}{(1-x^2)}\varphi(x). \tag{5.7}$$

Now, according to Lebedev ([33], page 164), the equation

$$\varphi''(x) + \frac{2[(\gamma - \frac{1}{2}) - (\alpha + \beta + \frac{1}{2})x^2]}{x(1-x^2)}\varphi'(x) = \frac{4\alpha\beta}{(1-x^2)}\varphi(x)$$

admits ${}_2F_1(\alpha, \beta; \gamma; x^2)$ as a solution. Now, we compute the coefficients α, β, γ so that the above equation is precisely (5.7). Therefore, we set

$$\begin{cases} 2\gamma - 1 = \delta - 1, \\ 2\alpha + 2\beta + 1 = \delta - 1 - 4c, \\ 4\alpha\beta = k^2, \end{cases}$$

or

$$\begin{cases} \gamma = \dfrac{\delta}{2} = \nu + 1, \\ \alpha + \beta = \nu - 2c, \\ 4\alpha\beta = k^2. \end{cases}$$

We assume that $k^2 \geq (\nu - 2c)^2$ so that we can set $k^2 = (\nu - 2c)^2 + \theta^2$. The above system gives

$$\begin{cases} \gamma = \nu + 1, \\ \alpha = \dfrac{\nu - 2c + i\theta}{2}, \\ \beta = \dfrac{\nu - 2c - i\theta}{2}. \end{cases}$$

The function $\varphi(x) = {}_2F_1((\nu - 2c + i\theta)/2, (\nu - 2c - i\theta)/2; \nu + 1; x^2)$ is then a solution of (5.7) bounded on $[0, a]$ and $\varphi(0) = 1$. This proves the first part of Proposition 5.1; (5.5) is a consequence of the following relation ([33], (9.3.4)):

$$ {}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - (a + b))}{\Gamma(c - a)\Gamma(c - b)} \quad \text{for } \mathrm{Re}(c - a - b) > 0.$$

The case $k^2 \leq (\nu - 2c)^2$ is similar. □

COROLLARY 5.1.1. *We have*

$$\int_0^{T_1^{(2)}} \frac{ds}{1 - (R_2(s))^2} \overset{\text{(law)}}{=} T^{(3)}_{\pi/2}. \tag{5.8}$$

PROOF. We apply (5.4) in the particular case $\delta = 2$, $c = 0$, that is, for the two-dimensional Bessel process

$$\begin{aligned} &E\left[\exp\left(-\frac{\theta^2}{2}\int_0^{T_1^{(2)}} \frac{ds}{1 - (R_2(s))^2}\right)\right] \\ &\quad = \left|\Gamma\left(1 + \frac{i\theta}{2}\right)\right|^2 = \frac{\pi\theta/2}{\sinh(\pi\theta/2)} = E\left[\exp\left(-\frac{\theta^2}{2} T^{(3)}_{\pi/2}\right)\right]. \end{aligned}$$

□

5.2. *A particular case:* $c = \nu/2$. When $c = (\nu/2)$, (5.5) becomes

$$E_0^{\delta,\nu/2}\left[\exp\left(-\frac{\theta^2}{2}\int_0^{T_1(R)}\frac{ds}{1-(R(s))^2}\right)\right] = \frac{|\Gamma(1+\nu+(i\theta/2))|^2}{(\Gamma(\nu+1))^2}. \tag{5.9}$$

Note that in this case, the infinitesimal generator $L^{\delta,c}$ is

$$L^{\delta,\nu/2}f(x) = \frac{1}{2}f''(x) + \frac{2\nu+1-x^2}{2x(1-x^2)}f'(x). \tag{5.10}$$

In the special case $\nu = -1/2$ that is $\delta = 1$, we obtain the following result:

$$E_0\left[\exp\left(-\frac{\theta^2}{2}\int_0^{T_1(R)}\frac{ds}{1-(R(s))^2}\right)\right] = \frac{1}{\cosh(\theta\pi/2)}, \tag{5.11}$$

where R is the (reflected) diffusion with infinitesimal generator

$$Lf(x) = \frac{1}{2}f''(x) - \frac{x}{2(1-x^2)}f'(x), \qquad x \in \mathbb{R}^+. \tag{5.12}$$

Let X be the diffusion, with values in $[-1, 1]$, which solves the SDE

$$\begin{cases} dX_t = d\beta_t - \dfrac{X_t}{2(1-X_t^2)}dt, & t < T_1 \wedge T_{-1}, \\ X_0 = 0, \end{cases} \tag{5.13}$$

where β is a real valued Brownian motion. The scale function of X is $s(x) = \arcsin x$, so that

$$\arcsin X_t = B\left(\int_0^t \frac{ds}{1-X_s^2}\right), \qquad t < T_1^* = \inf\{s, X_s = 1 \text{ or } -1\} \tag{5.14}$$

for a real valued Brownian motion B, starting from 0.

The representation (5.14) gives an explanation for the Laplace transform obtained in (5.11). In fact, by (5.14),

$$\int_0^{T_1(X)\wedge T_{-1}(X)}\frac{ds}{1-X^2(s)} \overset{\text{(law)}}{=} T_{\pi/2}(|B|). \tag{5.15}$$

This obviously agrees with (5.11), since it is well known that

$$E\left[\exp -\frac{\theta^2}{2}T_a(|B|)\right] = \frac{1}{\cosh(\theta a)}, \qquad a > 0.$$

Given the identities in law (5.8) and (5.15), which exhibit $T_{\pi/2}^{(3)}$ and $T_{\pi/2}^{(1)}$, and the well-known probabilistic interpretation of the factorization

$$\frac{1}{\cosh(\theta a)} = \left(\frac{\tanh(\theta a)}{\theta a}\right)\left(\frac{\theta a}{\sinh(\theta a)}\right)$$

as the expression of the Laplace transform (in $\theta^2/2$) for

$$T_a^{(1)} = g_{T_a^{(1)}} + (T_a^{(1)} - g_{T_a^{(1)}}), \tag{5.16}$$

one obtains the following:

$$\int_0^{g_{T_1^*}} \frac{ds}{1-X_s^2} \overset{\text{(law)}}{=} g_{T^{(1)}_{\pi/2}}(B); \qquad \int_{g_{T_1^*}}^{T_1^*} \frac{ds}{1-X_s^2} \overset{\text{(law)}}{=} T^{(3)}_{\pi/2}. \tag{5.17}$$

In the above equations, we denote

$$g_{T_a^{(1)}} = \sup\{t < T_a^{(1)},\ R_1(t) = 0\}$$

and

$$g_{T_1^*} = \sup\{t < T_1^*,\ X(t) = 0\}.$$

The Laplace transform of the right-hand side of (5.16) follows from the path decomposition of Brownian motion at time g_{T_a} (see [60], [66]), which states that $(B(g_{T_a}+t);\ t \le T_a - g_{T_a})$ is a three-dimensional Bessel process, independent of $(B(t);\ t \le g_{T_a})$. Concerning the last identity in law in (5.17), it is now tempting, in view of Corollary 5.1, to think that

$$\big(|X(g_{T_1^*}+u)|;\ u \le T_1^* - g_{T_1^*}\big) \underset{?}{\overset{\text{(law)}}{=}} \big(R_2(u);\ u \le T_1^{(2)}\big), \tag{5.18}$$

which would give a nice explanation of Corollary 5.1.

However, the identity in law (5.18) does not hold, as will be shown, with the description of the diffusion $(\tilde{X}(u);\ u \le \tilde{T}_1)$ on the left-hand side of (5.18).

To show this, we may use the enlargement formula for $L \equiv g_{T_1^*}$, that is, a formula which gives the decomposition of the Brownian motion $(\beta_u,\ u \ge 0)$ driving X:

$$X_t = \beta_t + \int_0^t ds\, b_0(X_s) \quad \text{where } b_0(x) = -\frac{x}{2(1-x^2)},$$

as a semimartingale in the enlarged filtration; that is, in terms of X,

$$\tilde{X}_t = X_{L+t} = \tilde{\beta}_t + \int_0^t ds\, b(\tilde{X}_s), \tag{5.19}$$

where $(\tilde{\beta}_t)$ is a Brownian motion in the enlarged filtration, and

$$b(x) = \frac{\varphi'(x)}{\varphi(x)} + b_0(x) \quad \text{and} \quad \varphi(X(u \wedge T_1^*)) \equiv 1 - Z_u^L,$$

$(Z_u^L,\ u \ge 0)$ being the customary notation for Azéma's supermartingale associated with L (see Chapter 12*, and [26], Chapter 5). Since

$$\arcsin(X_t) = B\left(\int_0^t \frac{ds}{1-X_s^2}\right), \qquad Z_t^{g_{T_a^*}(B)} = 1 - \frac{1}{a}|B(t \wedge T_a^*(B))|,$$

it is not difficult to show that $1 - Z_t^L = (2/\pi)\arcsin(|X(t \wedge T_1^*)|)$, which implies

$$\frac{\varphi'(x)}{\varphi(x)} = \frac{1}{\sqrt{1-x^2}(\arcsin(x))};$$

hence,

$$b(x) = \frac{1}{\sqrt{1-x^2}(\arcsin(x))} - \frac{x}{2(1-x^2)},$$

so that the process $(|\tilde{X}_t|,\ t \le \tilde{T}_1)$, where $\tilde{X}$ satisfies (5.19), is definitely not a two-dimensional Bessel process. However, both $\tilde{X}$ and R_2 satisfy

$$\int_0^{T_1(|\tilde{X}|)} \frac{ds}{1-\tilde{X}^2(s)} \overset{\text{(law)}}{=} \int_0^{T_1(R_2)} \frac{ds}{1-R_2^2(s)}. \tag{5.20}$$

5.3. *The porous medium equation.* The porous medium equation is the partial differential equation

$$(E)_\mu \qquad \begin{cases} u_t = \frac{1}{2}(u^{2m+1})_{xx}, & m > 0, \\ u(0,\cdot) = \mu, \end{cases}$$

where u: $\mathbb{R}_+^* \times \mathbb{R} \to \mathbb{R}$. If μ is a probability measure, the positive solution of (E_μ) is a probability density and it is natural to associate to (E_μ) a random process X, such that $P(X_t \in dx) = u(t,x)\,dx$. More precisely, we consider the solution of

(S_μ) $\quad X_t = X_0 + \int_0^t u^m(s, X_s)\,dB_s,$ and the density of the law of X_t is $u(t,\cdot)$.

The systems (E_μ) and (S_μ) have been studied by Benachour, Chassaing, Roynette and Vallois [4] for a large class of measures μ. The purpose of this paragraph is to link the generalized Bessel processes introduced in Section 5.1 and the solution of (S_μ), for $\mu = \delta_0$ the Dirac measure at 0.

If X is the solution of (S_{δ_0}), then the process $Z_t := \exp(-t\beta)X_{\exp(t)}$ is the unique stationary process, with invariant measure $\varphi(x)\,dx$, which solves

$$Z_t = Z_s + \int_s^t \varphi^m(Z_r)\,dB_r - \beta\int_s^t Z_r\,dr, \qquad t \ge s, \tag{5.21}$$

where

$$\begin{cases} \beta = \dfrac{1}{2m+2}, \\ a_m = \left\{\dfrac{m^{1/2}}{[(2m+1)(m+1)]^{1/2} B((2m+1)/2m, 1/2)}\right\}^{m/m+1} \\ \qquad\qquad\qquad \text{where } B(a,b) = \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \\ \gamma_m = a_m\left(\dfrac{(2m+1)(m+1)}{m}\right)^{1/2}, \\ \varphi(x) = a_m^{1/m}\left(1-\left(\dfrac{x}{\gamma_m}\right)^2\right)_+^{1/2m}. \end{cases} \tag{5.22}$$

Equation (5.21) admits a unique weak solution with values in $]-\gamma_m, \gamma_m[$ (see [4]). We denote by $(Z_t^y,\ t \geq 0)$ the solution of (5.21) satisfying $Z_0^y = y$ and

$$S_{x,y} = \inf\{t > 0,\ Z_t^x = y\};\ x, y \in]-\gamma_m, \gamma_m[.$$

In [4], the authors obtain the Laplace transform of the hitting time $S_{x,y}$, the result being expressed in terms of hypergeometric functions. We shall establish an identity in law between the time $S_{x,y} \wedge S_{x,-y}$ and the functional $A_y(R)$, already introduced at the beginning of Section 5, for a diffusion R of the family $\mathscr{L}^{\delta,c}$, which enables us to recover some results from [4]. We consider $Y_t^y = (Z_t^{y\gamma_m})/\gamma_m$ with values in $]-1,1[$, where Y_t^y solves

$$Y_t^y = y + \int_0^t \sigma^m(Y_r^y)\, dB_r - \beta \int_0^t Y_r^y\, dr$$

with

$$\sigma(x) = \delta_m^{1/m}(1-x^2)_+^{1/2m}, \qquad \delta_m = \left(\frac{m}{(2m+1)(m+1)}\right)^{1/2}.$$

Let $T_{x,y} = \inf\{t > 0,\ Y_t^x = y\} = S_{x\gamma_m, y\gamma_m}$.

By a change of time, there exists a process ξ^y such that

$$Y_t^y = \xi^y\left(\int_0^t \sigma^{2m}(Y_s^y)\, ds\right)$$

and ξ^y is the solution of

$$\xi_t^y = y + \gamma_t - \beta \int_0^t \frac{\xi_s^y}{\sigma^{2m}(\xi_s^y)} ds$$

for a Brownian motion γ:

$$\xi_t^y = y + \gamma_t - \beta \int_0^t \frac{\xi_s^y}{\delta_m^2(1-(\xi_s^y)^2)} ds \quad \text{for } y \in]-1,1[,\ t \leq T_1(|\gamma|). \tag{5.23}$$

The generator of ξ, the solution of (5.23), belongs to the class $\mathscr{L}^{\delta,c}$ with $\delta = 1$ and $c = -(\beta/2\delta_m^2) < -(1/2)$. The diffusion ξ is symmetric; that is,

$$(\xi_t^y,\ t \geq 0) \overset{\text{(law)}}{=} (-\xi_t^{-y},\ t \geq 0).$$

Thus $(|\xi_t^y|,\ t \geq 0) \overset{\text{(law)}}{=} (R_t, t \geq 0)$ under $P_{|y|}^{1,-(\beta/2\delta_m^2)}$. We denote by α_u the inverse of the increasing functional

$$t \to \int_0^t \sigma^{2m}(Y_s)\, ds,$$

so that $\xi_u = Y_{\alpha_u}$. One easily verifies that

$$\alpha_u = \int_0^u \frac{ds}{\sigma^{2m}(\xi_s)} = \int_0^u \frac{ds}{\delta_m^2(1-\xi_s^2)}. \tag{5.24}$$

It is now easy to obtain the law of $T_{x,y}$ with the help of Proposition 5.1.

PROPOSITION 5.2. *Let* $0 \le x \le y \le 1$,

$$E\big[\exp(-\lambda(T_{x,y} \wedge T_{x,-y}))\big] \overset{\text{(i)}}{=} E\left[\exp\left(-\lambda \int_0^{T_y(|\xi|)} \frac{ds}{\delta_m^2(1-\xi_s^2)}\right)\right] \overset{\text{(ii)}}{=} \frac{\varphi_\lambda(x)}{\varphi_\lambda(y)} \tag{5.25}$$

with

$$\begin{cases} \varphi_\lambda(x) = {}_2F_1\left(\dfrac{\xi_m}{2}, \dfrac{b_m}{2}; \dfrac{1}{2}; x^2\right), \\ \xi_m = \dfrac{m+1}{2m} + i\theta, \\ b_m = \dfrac{m+1}{2m} - i\theta, \\ \theta = \sqrt{\dfrac{2(2m+1)(m+1)}{m}}\sqrt{\lambda - \lambda_m} \quad \text{with } \lambda_m = \dfrac{m+1}{8m(2m+1)}, \end{cases} \tag{5.26}$$

(5.25) being given for $\lambda \ge \lambda_m$.

PROOF. The identity (i) follows from the relation $\xi_u = Y_{\alpha_u}$ and the expression of α_u. Identity (ii) is a consequence of Proposition 5.1, since the law of $|\xi|$ is $P^{\delta,c}$with $\delta = 1$ ($\nu = -1/2$), $c = -(\beta/2\delta_m^2)$. In this case, $\theta^2 = (2\lambda/\delta_m^2) - (\nu - 2c)^2$. □

REMARKS. (i) In [4], Corollaire II.6, the authors obtain the following:

$$E\big[\exp(-\lambda(T_{x,y} \wedge T_{x,-y}))\big] = \frac{k_\lambda(x) + k_\lambda(-x)}{k_\lambda(y) + k_\lambda(-y)} \tag{5.27}$$

with

$$k_\lambda(x) = {}_2F_1\left(\xi_m, b_m; \frac{2m+1}{2}; \frac{1+x}{2}\right) = {}_2F_1\left(\xi_m, b_m; \xi_m + b_m + \frac{1}{2}; \frac{1+x}{2}\right).$$

It follows from Lebedev ([33], (9.6.11)) that

$$k_\lambda(x) + k_\lambda(-x) = C\varphi_\lambda(x).$$

Formulas (5.25) and (5.27) are then equivalent.

(ii) Theorem II.2 in [4] gives the Laplace transform of the law of $T_{x,y}$. We could obtain this result with the family of diffusions introduced in Section 5.1, but we must choose symmetric diffusions with values in $[-1, 1]$ instead of reflected diffusions. For example, in order to compute the Laplace transform of $T_{x,y}$ for $x \le y$, we look for a function φ solution of (5.7) with $\delta = 1$, $c = -(\beta/2\delta_m^2)$, $k = \sqrt{2\lambda}$, which is bounded on $[-1, y]$. This is exactly the function $k_\lambda(x)$.

TABLE 1
Laplace transforms of some variables

X	$E(\exp(-(\theta^2/2)X))$
$\int_0^\infty \frac{ds}{\exp(2R_3(s)) - a^2}, \quad a \in]0,1]$	$\frac{1}{{}_2F_1((-i\theta)/2a, i\theta/2a; 1; a^2)}$
$\int_0^\infty \frac{ds}{\exp(2R_3(s)) - 1} \overset{\text{(law)}}{=} T^{(3)}_{\pi/2}$	$\frac{\pi\theta/2}{\sinh(\pi\theta/2)}$
$\int_0^\infty \frac{ds}{\exp(2\Sigma_\delta(s)) - 1}$	$\left(\frac{\pi\theta/2}{\sinh(\pi\theta/2)}\right)^{\delta/2}$
$\int_0^\infty \exp(-2R_3(s))\, ds \overset{\text{(law)}}{=} T^{(2)}_1$	$\frac{1}{I_0(\theta)}$
$\int_0^\infty \exp(-2\Sigma_\delta(s))\, ds$	$\left(\frac{1}{I_0(\theta)}\right)^{\delta/2}$

6. Complements. Let I_ν and K_ν denote the modified Bessel functions and let J_ν, Y_ν stand for the Bessel functions of the first and second kind (see [33], Chapter 5).

6.1. *Table of formulas.* In Table 1, we give the Laplace transforms of some variables encountered in the preceding sections (and some generalizations). We denote by R_δ a δ-dimensional Bessel process and by $T_x^{(\delta)}$ its first hitting time of x; B is a Brownian motion and l_t its local time at 0 and at time t; $\Sigma_\delta(t) = |B_t| + (2/\delta)l_t$.

SOME COMMENTS. (i) The first line follows from the identity in law

$$\int_0^\infty \frac{ds}{\exp(2R_3(s)) - a^2} \overset{\text{(law)}}{=} \frac{1}{a^2}\int_0^{T_a^{(2)}} \frac{du}{1 - (R_2(u))^2},$$

and Proposition 5.1.

(ii) The fourth identity in law follows from the representation (5.2). Moreover, we verify that

$$\lim_{a\to 0} \frac{1}{{}_2F_1(-i\theta/2a, i\theta/2a; 1; a^2)} = \frac{1}{I_0(\theta)}.$$

(iii) The third and fifth lines are consequences of Theorem 3.1 and of the additivity of squared Bessel processes.

6.2. *Some functionals of the three-dimensional Bessel bridge.* In Section 4, we have seen that Alili obtained the law of $\int_0^1 \coth(\lambda r_s)\, ds$ from the study of the law of $(H^\lambda_{\tau_t}, \tau_t)$, where r denotes a three-dimensional Bessel bridge (i.e., a normalized excursion). In view of Table 1, we are interested in the laws of $\int_0^1 (ds/(\exp(2r_s) - a^2))$ $(0 < a < 1)$ and of $\int_0^1 (ds/(\exp(2r_s)))$. This study relies on the following proposition.

PROPOSITION 6.1 ([25]). *Let f be a locally bounded function and define*

$$u(k;x) = E_x\left(\exp -\left(\frac{k^2}{2} T_0 + \int_0^{T_0} f(B_s)\,ds\right)\right).$$

Then,

$$\frac{d}{dx}\bigg|_{x=0^+} u(k;x) = \int_0^\infty \frac{dt}{\sqrt{2\pi t^3}}\left(1 - \exp\left(-\frac{k^2}{2}t\right)K_f(t)\right)$$
$$= k + \int_0^\infty \frac{dt}{\sqrt{2\pi t^3}} \exp\left(-\frac{k^2}{2}t\right)(1 - K_f(t)),$$

where

$$K_f(t) = E\left(\exp - \int_0^t du\, f(R_3(u)) | R_3(t) = 0\right). \tag{6.1}$$

Applying these results to $f(x) = \theta^2/(2(\exp(2x) - a^2))$ $(a \in [0,1[)$, we obtain the following proposition.

PROPOSITION 6.2. (i) *Let $0 < a < 1$; then*

$$\begin{aligned} &\int_0^\infty \frac{dt}{\sqrt{2\pi t^3}} \exp\left(-\frac{k^2}{2}t\right) \\ &\quad\times \left(1 - E\left(\exp - \frac{\theta^2}{2}\int_0^t \frac{du}{\exp(2R_3(u)) - a^2}\bigg| R_3(t) = 0\right)\right) \\ &= \int_0^\infty \frac{dt}{\sqrt{2\pi t^3}} \exp\left(-\frac{k^2}{2}t\right) \\ &\quad\times \left(1 - E\left(\exp - \frac{\theta^2}{2} t \int_0^1 \frac{du}{\exp(2t^{1/2} r(u)) - a^2}\right)\right) \\ &= \frac{\theta^2}{2(k+1)} \frac{{}_2F_1(\alpha+1, \beta+1; k+2; a^2)}{{}_2F_1(\alpha, \beta; k+1; a^2)}, \end{aligned} \tag{6.2}$$

where

$$\alpha = \frac{1}{2}\left(k + i\sqrt{\frac{\theta^2}{a^2} - k^2}\right), \quad \beta = \frac{1}{2}\left(k - i\sqrt{\frac{\theta^2}{a^2} - k^2}\right), \qquad k^2 < \frac{\theta^2}{a^2}.$$

(ii) *For* $a = 0$,

$$\begin{aligned}
&\int_0^\infty \frac{dt}{\sqrt{2\pi t^3}} \exp\left(-\frac{k^2}{2}t\right) \\
&\quad \times \left(1 - E\left(\exp - \frac{\theta^2}{2}\int_0^t du\, \exp(-2R_3(u)) \,\middle|\, R_3(t) = 0\right)\right) \\
&\qquad = \int_0^\infty \frac{dt}{\sqrt{2\pi t^3}} \exp\left(-\frac{k^2}{2}t\right) \\
&\qquad\quad \times \left(1 - E\left(\exp - \frac{\theta^2}{2} t \int_0^1 du\, \exp(-2t^{1/2} r(u))\right)\right) \\
&\qquad = \theta \frac{I_{k+1}(\theta)}{I_k(\theta)}.
\end{aligned} \tag{6.3}$$

Formula (6.3) follows from [25] or from (6.2), letting $a \to 0$. We denote by $X_t = t\int_0^1 du\, \exp(-2t^{1/2} r(u))$. From (6.3),

$$\int_0^\infty \frac{dt}{\sqrt{2\pi t^3}} \exp\left(-\frac{k^2}{2}t\right) E\left(\int_0^{X_t} du\, \exp\left(-\frac{\theta^2}{2}u\right)\right) = \frac{2}{\theta}\frac{I_{k+1}(\theta)}{I_k(\theta)}.$$

Now (see [24]),

$$\frac{2}{\theta}\frac{I_{k+1}(\theta)}{I_k(\theta)} = 4\int_0^\infty dz\, \exp(-\theta^2 z) G_k(z),$$

where

$$G_k(z) = \sum_{n=1}^\infty \exp(-j_{k,n}^2 z), \tag{6.4}$$

$(j_{k,n})_n$ being the increasing sequence of positive zeros of the Bessel function J_k. The function G_k is a generalized theta function.

Then, by inverting the Laplace transform in $\theta^2/2$, one obtains

$$\int_0^\infty \frac{dt}{\sqrt{2\pi t^3}} \exp\left(-\frac{k^2}{2}t\right) P(X_t > z) = 2G_k\left(\frac{z}{2}\right). \tag{6.5}$$

From (6.5), we can compute the moments of X_t (via a Laplace transform). Let $p \geq 1$,

$$\int_0^\infty \frac{dt}{\sqrt{2\pi t^3}} \exp\left(-\frac{k^2}{2}t\right) E((X_t)^p) = 2^{p+1} p! \sum_{n=1}^\infty \frac{1}{j_{k,n}^{2p}}.$$

For $k = \frac{1}{2}$, $J_{1/2}(z) = (2/\pi z)^{1/2} \sin(z)$; therefore, we have

$$j_{1/2,n} = n\pi.$$

Then, the above equation becomes

$$\int_0^\infty \frac{dt}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{8}t\right) E((X_t)^p) = 2^{p+1} p! \frac{1}{\pi^{2p}} \zeta(2p),$$

where $\zeta(s) = \sum_{n=1}^\infty (1/n^s)$ denotes the Riemann zeta function.

We can obtain an analogous result for the random variable $Y_t = t\int_0^1 du\,\exp(+2t^{1/2}r(u))$. According to [25],

$$\int_0^\infty \frac{dt}{\sqrt{2\pi t^3}}\exp\left(-\frac{k^2}{2}t\right)E\left(\int_0^{Y_t} du\ \exp\left(-\frac{\theta^2}{2}u\right)\right) = \frac{2}{\theta}\frac{K_{k-1}(\theta)}{K_k(\theta)}.$$

Now, according to Ismail [23],

$$\frac{2}{\theta}\frac{K_{k-1}(\theta)}{K_k(\theta)} = 2\int_0^\infty dz\ \exp(-\theta^2 z)H_{k-1}(z),$$

with

$$H_{k-1}(z) = \frac{2}{\pi^2}\int_0^\infty dt\,t^{-1}\exp(-tz)(J_k^2+Y_k^2)^{-1}(t^{1/2}). \tag{6.6}$$

Thus,

$$\begin{aligned}&\int_0^\infty \frac{dt}{\sqrt{2\pi t^3}}\exp\left(-\frac{k^2}{2}t\right)P(Y_t > z)\\ &\quad = H_{k-1}\left(\frac{z}{2}\right) = \frac{2}{\pi^2}\int_0^\infty dt\,t^{-1}\exp\left(-\frac{tz}{2}\right)(J_k^2+Y_k^2)^{-1}(t^{1/2}).\end{aligned} \tag{6.7}$$

Let $p \geq 1$; then

$$\int_0^\infty \frac{dt}{\sqrt{2\pi t^3}}\exp\left(-\frac{k^2}{2}t\right)E((Y_t)^p) = \frac{2^{p+1}}{\pi^2}\,p!\int_0^\infty \frac{dt}{t^{p+1}}\frac{1}{J_k^2(\sqrt{t})+Y_k^2(\sqrt{t})},$$

the right-hand side being finite for $p < k$, using the behavior of the Bessel functions J_k and Y_k near 0. (See [33], Section 5.16.)

REMARKS. (i) From (6.7) and (6.5), we see that the functions $\nu \to H_{\sqrt{\nu}}(z)$ and $\nu \to G_{\sqrt{\nu}}(z)$ are completely monotonic, a well-known fact proved by Ismail [23] and Ismail and Kelker [24] by analytic means and interpreted probabilistically by Pitman and Yor [46].

(ii) Using Krein's theory, Kotani and Watanabe [32] and Knight [31] described the Lévy measure Π_f of the increasing Lévy process $A^f_{\tau_t} = \int_0^{\tau_t} f(B_s)\,ds$ for a positive function f. Now, Π_f can be expressed in terms of the Itô measure n and using Williams's description of n [see (D) following Proposition 4.1], in terms of r, one obtains that if $Z_t = t\int_0^1 \varphi(\sqrt{t}r_s)\,ds$ for a positive function φ, then

$$\int_0^\infty \frac{dt}{\sqrt{2\pi t^3}}\exp\left(-\frac{k^2}{2}t\right)P(Z_t \in dz) = h_k(z)dz,$$

where h_k is the Laplace transform of a positive measure. In the particular cases when $Z_t = X_t$ or $Z_t = Y_t$, (6.5) and (6.7) provide us with two particularly explicit and rich examples.

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