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THE FLUCTUATION RESULT FOR THE MULTIPLE POINT RANGE OF TWO DIMENSIONAL RECURRENT RANDOM WALKS

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Dedicated to Professor Hiroshi Kunita on his 60th birthday

We study the fluctuation problem for the multiple point range of random walks in the two dimensional integer lattice with mean 0 and finite variance. The p-multiple point range means the number of distinct sites with multiplicity p of random walk paths before time n. The suitably normalized multiple point range is proved to converge to a constant, which is independent of the multiplicity, multiple of the renormalized self-intersection local time of a planar Brownian motion.

1. Introduction.

In the present article, we will treat the fluctuation problem for the number of distinct lattice points with multiplicity p of random walk paths in the first n steps. A random walk in the d dimensional integer lattice $\mathbb{Z}^d$, denoted by $\{S_n\}_{n=0}^{\infty}$, means a sequence of random variables defined by

$$S_0 = 0, \qquad S_n = \sum_{k=1}^{n} X_k,$$

where $\{X_n\}_{n=1}^{\infty}$ is a sequence of independent identically distributed random variables with values in $\mathbb{Z}^d$.

We assume for convenience that the random walk is adapted, which implies that no proper subgroup of $\mathbb{Z}^d$ contains the support of X_1. In particular, the random walk is genuinely d dimensional. The p-multiple point range of a random walk, denoted by $Q_n^{(p)}$, means the number of distinct sites visited exactly p times by the random walk in the first n steps. Pitt [19] proved that for all transient random walks and each $p \geq 1$, $Q_n^{(p)}/n$ converges to $\mu^2(1-\mu)^{p-1}$ almost surely as $n \to \infty$, where μ is the probability that the random walk never returns to the origin. In the two dimensional case, Flatto [4] showed that $(\log n)^2 Q_n^{(p)}/n$ converges to π^2 almost surely as n tends to infinity for the simple random walk.

The first work on the fluctuation problem for $Q_n^{(p)}$ was by Hamana [5]. He proved that if $d \geq 5$ and $\mu < 1$, $\mathrm{Cov}(Q_n^{(k)}, Q_n^{(l)})/n$ converges to some constant $\sigma^{k,l}$ for each $k, l \geq 1$, and for fixed integer $K \geq 1$, the K dimensional random vector $\Psi_n = (Q_n^{(1)}, Q_n^{(2)}, \dots, Q_n^{(K)})$ obeys the central limit theorem; that is, the law of $(\Psi_n - E\Psi_n)/\sqrt{n}$ is asymptotically equal to the K dimensional normal distribution with mean 0 and the covariance matrix Σ of which the (k, l)-

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component is $\sigma^{k,l}$. Moreover, if $d \geq 4$ and $\mu < 1$, there exists a positive constant σ^2 such that Var $Q_n^{(1)} \sim \sigma^2 n$ and $[Q_n^{(1)} - \mu^2 n]/\sigma\sqrt{n}$ converges to the standard normal variable in the distribution sense. If $\mu = 1$, it is not interesting since $Q_n^{(1)} = n$ and $Q_n^{(p)} = 0$ for any $p \geq 2$. Hamana [8] also proved that if $d = 3$ and $\mu < 1$, one has that Var $Q_n^{(1)} \sim n\psi(n)$ for some nondecreasing slowly varying function ψ and $[Q_n^{(1)} - \mu^2 n]/\sqrt{n\psi(n)}$ tends to the normal with mean 0 and variance 1 in law. In the two dimensional case, he showed in the same paper that if $EX_1 = 0$ and $E|X_1|^2 < \infty$, there exists a positive constant L such that Var $Q_n^{(1)} \sim Ln^2/(\log n)^6$ and the distribution of $(\log n)^3[Q_n^{(1)} - EQ_n^{(1)}]/n$ is asymptotically equal to that of a constant multiple of the renormalized self-intersection local time of a two dimensional Brownian motion. However, one needs a more restrictive assumption—aperiodicity—to prove this result. Aperiodicity means that for every $x \in \mathbb{Z}^d$ there exists an $n \geq 1$ such that $P(S_m = x) > 0$ whenever $m \geq n$.

For general p, we shall consider the fluctuation problem for $Q_n^{(p)}$ when the random walk moves on $\mathbb{Z}^2$ and has zero mean and finite variance. Section 3 is devoted to the study of the variance of $Q_n^{(p)}$ and the asymptotic behavior of the distribution of $Q_n^{(p)} - EQ_n^{(p)}$. We will show that Var $Q_n^{(p)} \leq Cn^2/(\log n)^6$ for some positive constant C and the law of $(\log n)^3[Q_n^{(p)} - EQ_n^{(p)}]/n$ converges to that of a constant, which is independent of p, times the renormalized intersection local time of a planar Brownian motion. Section 4 is devoted to giving several lemmas which are useful to estimate the probabilities of various quantities of random walks. In Sections 5 and 6, we prove lemmas used in Section 3 by making the most of lemmas in Section 4.

We will now offer an intuitive explanation for the fact that $Q_n^{(p)}$ behaves like $Q_n^{(1)}$. The event that the lattice point x is a p-multiple point of the random walk path before time n is the intersection of the following three events. The first is the event that the random walk first reaches x at some time, the second is the event that the random walk returns $(p-1)$ times to x for some steps and the third is the event that the random walk never returns to x in the remaining steps. The second event can also be described in terms of intersections of $(p-1)$ events in which the random walk returns to x in the first time for some steps. Note that the probability that the random walk returns to the origin up to time n converges to 1 as $n \to \infty$ in the two dimensional recurrent case, and so it seems intuitively clear that the condition that the random walk eventually returns to its starting point $(p-1)$ times will be asymptotically negligible. On the other hand, the condition that it never returns again to a given point after a certain time should play an important role. Thus, when the time n is very large, the number of times which the random walk is required to return to a point which has already been reached should not be significant. Only the fact that the random walk reaches the point is important.

Let $R_n^{(p)}$ be the number of distinct points visited at least p times by a random walk in the first n steps. For $p = 1$, various results were shown (cf. [2, 10, 11, 12, 13, 14, 16, 18]). The results about $Q_n^{(p)}$ are refinements of these results. We can also study $R_n^{(p)}$ for $p \geq 1$. In the transient case, the law of large numbers was established in [19] and the central limit theorem was partially proved in [5]. For the two dimensional random walk with $EX_1 = 0$ and $E|X_1|^2 < \infty$, we can also derive the limiting behaviors of Var $R_n^{(p)}$ and the law of $[R_n^{(p)} - ER_n^{(p)}]/[\text{Var}\, R_n^{(p)}]^{1/2}$ for each $p \geq 2$, and we can conclude that they are not different asymptotically from the case of $p = 1$.

2. Notation and preliminaries. We assume a random walk is adapted. In terms of the characteristic function $\varphi(\xi)$, the adaptation means that, for $\xi \in (-\pi, \pi]^d$, $\varphi(\xi)$ is equal to 1 if and only if $\xi = 0$. However this property is not restrictive. If it is not satisfied, we may consider the smallest subgroup G of $\mathbb{Z}^d$ on which the random walk takes place and then can find a linear isomorphism from G to $\mathbb{Z}^m$ for some $m \leq d$. Under this situation, the random walk translated by this isomorphism is adapted and moves on $\mathbb{Z}^m$. Therefore we will investigate the adapted random walk in $\mathbb{Z}^d$ throughout this paper. There may exist nonzero values of ξ satisfying $|\varphi(\xi)| = 1$. Let ρ be the number of such ξ. We call ρ the period of the random walk. Aperiodicity is equivalent to the condition $\rho = 1$.

In this section we will give some notation and basic lemmas. For $x \in \mathbb{Z}^d$, the notation $P_x(\cdot)$ will be used to denote the probability measures of events related to the random walk starting at x. When $x = 0$, we will simply use $P(\cdot)$ instead of $P_0(\cdot)$. For $n \geq 0$ and $x, y \in \mathbb{Z}^d$, the notation $p^n(x, y)$ means $P_x(S_n = y)$. Note that $p^n(x, y) = p^n(0, y - x)$. For $x \in \mathbb{Z}^d$, τ_x will denote the first hitting time of x; that is,

$$\tau_x = \inf\{n \geq 1;\ S_n = x\}.$$

If there are no positive integers with $S_n = x$, then $\tau_x = \infty$. The taboo probabilities are defined by

$$p_z^n(x, y) = P_x(S_n = y,\ \tau_z \geq n),$$
$$p_{zw}^n(x, y) = P_x(S_n = y,\ \tau_z \geq n,\ \tau_w \geq n).$$

The following lemma is very important.

LEMMA 2.1 ([11], [20]). *If $\mu < 1$, there is a positive constant A such that*

$$p^n(0, x) \leq A n^{-d/2}$$

for all $x \in \mathbb{Z}^d$ and $n \geq 1$.

Another standard result is that for $n \geq 1$ and $x \in \mathbb{Z}^d$,

$$p_0^n(0, x) = p_x^n(0, x). \tag{2.1}$$

This can be checked easily by considering the reversed random walk.

We will use r_n for $P(\tau_0 > n)$, f_n for $p_0^n(0, 0)$ and u_n for $p^n(0, 0)$. When $d = 2$, Kesten and Spitzer [15] proved that r_n is slowly varying. Here the meaning of slowly varying is that for any positive real number c, $r_{[cn]}/r_n \to 1$

as n tends to infinity, where $[x]$ is the integer part of a real number x. If the two dimensional random walk is transient, the result of Kesten and Spitzer is trivial since $r_n \to \mu$ as $n \to \infty$. However, their result has an important meaning in the recurrent case. In order to obtain the asymptotic behavior of r_n, we will need a simple observation about slowly varying functions.

LEMMA 2.2 ([13]). *Let $\{\alpha(n)\}_{n=1}^{\infty}$ be a sequence of nonincreasing and slowly varying functions. Then there is a positive constant B such that $j\alpha(j) \le Bn\alpha(n)$ for all $j \le n$ and $n \ge 1$. In particular, this implies that there is a constant C such that $jr_j^{\beta} \le Cnr_n^{\beta}$ for $j \le n$ and $\beta \ge 1$.*

In this paper, we will use the following convenient notation. If $\{a_n\}$ and $\{b_n\}$ $(b_n > 0)$ are sequences of real numbers, then $a_n = o(b_n)$ means $a_n b_n^{-1} \to 0$; $a_n = O(b_n)$ means $a_n b_n^{-1}$ remains bounded; $a_n \sim b_n$ means $a_n b_n^{-1} \to 1$, as $n \to \infty$. Let $C_1, C_2, \dots, C_{36}$ denote suitable positive real constants. Throughout this paper, $\sum_{i=1}^{0} a_i$ and $\prod_{i=1}^{0} a_i$ imply 0 and 1, respectively.

Let Ξ be the symmetric matrix satisfying $E(\theta, X_1)^2 = (\theta, \Xi^2\theta)$ for any $\theta \in \mathbb{R}^d$, where $(\cdot,\cdot)$ is the standard inner product on $\mathbb{R}^d$. If the random walk is adapted with $EX_1 = 0$ and $E|X_1|^2 < \infty$, it is known that Ξ is strictly positive definite (cf. Spitzer [20]).

From now on, we consider the adapted random walk in $\mathbb{Z}^2$ satisfying $EX_1 = 0$ and $E|X_1|^2 < \infty$. We can derive the asymptotic behaviors of r_n and f_n.

LEMMA 2.3. *We have that*

$$r_n \sim \frac{c}{\log n}, \tag{2.2}$$

where $c = 2\pi(\det \Xi)$.

PROOF. Let ρ be the period of the random walk. By Proposition 2.4 in [18], we have that

$$u_{\rho n} = \frac{1}{cn} + o\left(\frac{1}{n}\right)$$

and then

$$\sum_{k=0}^{n} u_{\rho k} = \frac{\log n}{c} + o(\log n).$$

Note that $u_m = 0$ if m is not a multiple of ρ. By Lemma 2.3 in [10],

$$r_{\rho n} \sim \left(\sum_{k=0}^{n} u_{\rho n}\right)^{-1} \sim \frac{c}{\log n}.$$

This implies (2.2) since r_n is slowly varying. $\square$

Observing the proof of Theorem 3 in [15], we can easily check

$$f_n = O\left\{\frac{1}{n(\log n)^2}\right\} \tag{2.3}$$

by Lemma 2.1, (2.1) and Lemma 2.3. If, in addition, the random walk is aperiodic, Jain and Pruitt [13] showed that

$$f_n \sim \frac{c}{n(\log n)^2}.$$

Employing this asymptotic behavior, they also derived that

$$\sum_{k=1}^{m} f_k(r_{m-k}^{\gamma} - r_m^{\gamma}) = O(r_m^{\gamma+2}) \tag{2.4}$$

for any integer $\gamma \geq 1$. By Lemma 2.3 and (2.3), we can improve (2.4) for the adapted random walk. Moreover, we can sharpen this estimate for $\gamma = 1$.

Lemma 2.4.

$$\sum_{k=1}^{m} f_k(r_{m-k} - r_m) = O(r_m^4).$$

This lemma can be proved in the same fashion as Lemma 5.8 in [8].

3. The fluctuation of $Q_n^{(p)}$. We are given an adapted random walk moving on $\mathbb{Z}^2$ with $EX_1 = 0$ and $E|X_1|^2 < \infty$. Our goal in this section is to establish the fluctuation theorem for $Q_n^{(p)}$ under this situation. If, in addition, the random walk is aperiodic, Hamana [8] showed that

$$\operatorname{Var} Q_n^{(1)} \sim \frac{Ln^2}{(\log n)^6}$$

for some positive constant L and that

$$\lim_{n\to\infty} \frac{(\log n)^3}{n}\Big[Q_n^{(1)} - EQ_n^{(1)}\Big] = -16\pi^3(\det \Xi)^2\gamma(\mathscr{C})$$

in the distribution sense, where $\mathscr{C} = \{(s,t) \in \mathbb{R}^2;\ 0 \leq s < t \leq 1\}$ and $\gamma(\mathscr{C})$ is the renormalized self-intersection local time of a planar Brownian motion $\{W_t\}_{t\geq 0}$, which is expressed formally by

$$\iint_{\mathscr{C}} \delta_0(W_t - W_s)\,ds\,dt - E\Big[\iint_{\mathscr{C}} \delta_0(W_t - W_s)\,ds\,dt\Big]$$

(cf. [16, 17]). To consider the asymptotic behavior of the fluctuation of $Q_n^{(p)}$ around its expectation for general p, we need to improve his observations and introduce some notation. For $0 \leq a < b$, let $S(a,b) = \{S_k;\ a < k \leq b\}$ and $S^p(a,b)$ be the set of distinct sites visited exactly p times by a random walk between time $a+1$ and time b. Let

$$Q_n^{(p)}(i;h) = \left|S^p((i-1)2^{-h}n, i2^{-h}n)\right|$$

for $h \ge 0$ and $i \ge 1$, and

$$I_n^{k,j} = \left|S((2j-2)2^{-k}n, (2j-1)2^{-k}n) \cap S((2j-1)2^{-k}n, 2j2^{-k}n)\right|,$$
$$L_n^{k,j}(p) = \left|S((2j-2)2^{-k}n, (2j-1)2^{-k}n) \cap S^p((2j-1)2^{-k}n, 2j2^{-k}n)\right|,$$
$$M_n^{k,j}(p) = \left|S^p((2j-2)2^{-k}n, (2j-1)2^{-k}n) \cap S((2j-1)2^{-k}n, 2j2^{-k}n)\right|,$$
$$N_n^{k,j}(p,q) = \left|S^p((2j-2)2^{-k}n, (2j-1)2^{-k}n) \cap S^q((2j-1)2^{-k}n, 2j2^{-k}n)\right|$$

for $k \ge 0$ and $j \ge 1$, where $|A|$ denotes the number of elements which belong to a set A. It is clear that the distributions of these random variables are equal to those of $I_{2^{-k}n}^{0,1}$, $L_{2^{-k}n}^{0,1}(p)$, $M_{2^{-k}n}^{0,1}(p)$ and $N_{2^{-k}n}^{0,1}(p,q)$, respectively. Note that

$$\begin{aligned} Q_{2n}^{(p)} &= \left|S^p(0,n)\right| - \left|S^p(0,n) \cap S(n,2n)\right| + \left|S^p(n,2n)\right| \\ &\quad - \left|S(0,n) \cap S^p(n,2n)\right| + \sum_{l=1}^{p-1} \left|S^l(0,n) \cap S^{p-l}(n,2n)\right|. \end{aligned} \tag{3.1}$$

Employing the same observation as we used to derive (3.1), we have that for each integer $h \ge 1$,

$$Q_n^{(p)} = \sum_{i=1}^{2^h} Q_n^{(p)}(i;h) - \sum_{k=1}^{h} \sum_{j=1}^{2^{k-1}} \left\{ L_n^{k,j}(p) + M_n^{k,j}(p) - \sum_{l=1}^{p-1} N_n^{k,j}(l, p-l) \right\}.$$

By applying Le Gall's argument used to obtain Theorem 6.1 in [16], we can prove the fluctuation results for $Q_n^{(p)}$ if we succeed in showing that the terms involving N's are negligible and that $\sum_{k,j} L_n^{k,j}(p)$ and $\sum_{k,j} M_n^{k,j}(p)$ are independent of p asymptotically. Le Gall [16] proved that $n^{-2}r_n^{-4}E|I_n^{0,1}|^2$ converges to some constant. In particular, $E|I_n^{0,1}|^2 = O(n^2 r_n^4)$ (see Theorem 5.1). This plays an essential role to establish the fluctuation result for $R_n^{(1)}$. To observe the fluctuation of $Q_n^{(p)}$, we need $r_n I_n^{0,1}$ instead of $I_n^{0,1}$, which will be used later in the proof of Theorem 3.5. The following lemma implies that $r_n I_n^{0,1}$ is the dominant part comparing with $N_n^{0,1}(p,q)$. We defer its proof to Section 5.

LEMMA 3.1. *For* $p, q \ge 1$,

$$E\left|N_n^{0,1}(p,q)\right|^2 = O(n^2 r_n^7). \tag{3.2}$$

The following lemma implies that the difference between $L_n^{0,1}(p)$ and $L_n^{0,1}(p+1)$ is small compared with $r_n I_n^{0,1}$.

LEMMA 3.2. *For* $p \ge 1$,

$$E\left|L_n^{0,1}(p) - L_n^{0,1}(p+1)\right|^2 = O(n^2 r_n^7), \tag{3.3}$$

$$E\left|M_n^{0,1}(p) - M_n^{0,1}(p+1)\right|^2 = O(n^2 r_n^7). \tag{3.4}$$

We also defer the proof of this lemma to Section 6. Consequently Lemma 3.2 indicates that $L_n^{0,1}(p)$ is not much different from $r_n I_n^{0,1}$ asymptotically for each $p \geq 1$. Namely, we can obtain the following corollary.

COROLLARY 3.3. *For an arbitrary fixed integer $p \geq 1$,*

$$E\left|L_n^{0,1}(p) - r_n I_n^{0,1}\right|^2 = O(n^2 r_n^7), \tag{3.5}$$

$$E\left|M_n^{0,1}(p) - r_n I_n^{0,1}\right|^2 = O(n^2 r_n^7). \tag{3.6}$$

In particular, we have that both $E\left|L_n^{0,1}(p)\right|^2$ and $E\left|M_n^{0,1}(p)\right|^2$ are of order $n^2 r_n^6$.

PROOF. For $p = 1$, the assertions were proved in Lemma 6.2 in [8] if, in addition, the random walk is aperiodic. However, we can extend the results to the adapted case along the same line by applying Lemma 2.3 and (2.3). Thus we need to prove (3.5) and (3.6) when $p \geq 2$. For $p \geq 2$, we have that

$$L_n^{0,1}(p) - r_n I_n^{0,1} = \sum_{l=2}^{p}\left\{L_n^{0,1}(l) - L_n^{0,1}(l-1)\right\} + L_n^{0,1}(1) - r_n I_n^{0,1}.$$

Using Minkowski's inequality, we have that

$$\begin{aligned}\left\{E\left|L_n^{0,1}(p) - r_n I_n^{0,1}\right|^2\right\}^{1/2} &\leq \sum_{l=2}^{p}\left\{E\left|L_n^{0,1}(l) - L_n^{0,1}(l-1)\right|^2\right\}^{1/2} \\ &\quad + \left\{E\left|L_n^{0,1}(1) - r_n I_n^{0,1}\right|^2\right\}^{1/2}.\end{aligned}$$

The first term of the right-hand side is of order $n r_n^{7/2}$ by applying Lemma 3.2, and we mentioned that the second term is of order $n r_n^{7/2}$ in the beginning of this proof. Therefore we conclude (3.5).

The method of obtaining (3.6) is similar to (3.5), and then we obtain (3.6). □

Now we are ready to give a bound of the variance of $Q_n^{(p)}$ and to establish the fluctuation result for $Q_n^{(p)}$.

PROPOSITION 3.4. *Let p be an arbitrary fixed positive integer. There exists a constant C such that, for $n \geq 2$,*

$$\operatorname{Var} Q_n^{(p)} \leq \frac{Cn^2}{(\log n)^6}.$$

PROOF. We will prove this proposition along the same line as Lemma 6.2 in [16]. Recall that (3.1) is equivalent to the equality

$$Q_{2n}^{(p)} = \left|S^p(0,n)\right| + \left|S^p(n,2n)\right| - L_n^{0,1}(p) - M_n^{0,1}(p) + \sum_{l=1}^{p-1} N_n^{0,1}(l,\, p-l).$$

Noting that $\left|S^p(0,n)\right|$ is independent of $\left|S^p(n,2n)\right|$ and that both $\left|S^p(0,n)\right|$ and $\left|S^p(n,2n)\right|$ have the same distribution as $Q_n^{(p)}$, we obtain that

$$\begin{aligned}\left[\mathrm{Var}\, Q_{2n}^{(p)}\right]^{1/2} \le \left[2\mathrm{Var}\, Q_n^{(p)}\right]^{1/2} + \left[E\left|L_n^{0,1}(p)\right|^2\right]^{1/2} + \left[E\left|M_n^{0,1}(p)\right|^2\right]^{1/2} \\ + \sum_{l=1}^{p-1}\left[E\left|N_n^{0,1}(l,\, p-l)\right|^2\right]^{1/2}.\end{aligned}$$

By Lemma 3.1 and Corollary 3.3,

$$\left[\mathrm{Var}\, Q_{2n}^{(p)}\right]^{1/2} \le \left[2\mathrm{Var}\, Q_n^{(p)}\right]^{1/2} + C_1 n r_n^3 + C_2 n r_n^{7/2}.$$

For $k \ge 1$, let

$$a_k^{(p)} = \sup\left\{\left[\mathrm{Var}\, Q_n^{(p)}\right]^{1/2};\ 2^k < n \le 2^{k+1}\right\}.$$

Since $n r_n^3 \le C_3 k^{-3} 2^k$ for $2^k < n \le 2^{k+1}$, we have

$$a_{k+1}^{(p)} \le \sqrt{2} a_k^{(p)} + C_4 k^{-3} 2^k.$$

Put $b_k^{(p)} = k^3 2^{-k} a_k^{(p)}$. For a given $\alpha \in (1/\sqrt{2}, 1)$, there is some constant k_0 such that, for any $k > k_0$,

$$b_{k+1}^{(p)} \le \alpha b_k^{(p)} + C_4.$$

This means that the sequence $\{b_k^{(p)}\}$ is bounded. Therefore we can conclude the assertion of this proposition. □

THEOREM 3.5. *For a two dimensional adapted random walk with mean* 0 *and finite variance,*

$$\lim_{n\to\infty} \frac{(\log n)^3}{n}\left[Q_n^{(p)} - EQ_n^{(p)}\right] = -16\pi^3(\det \Xi)^2\gamma(\mathscr{C})$$

in the distribution sense.

PROOF. Let h be a given positive integer. Recall that $Q_n^{(p)}$ is equal to

$$\begin{aligned}\sum_{i=1}^{2^h} Q_n^{(p)}(i;h) - \sum_{k=1}^{h}\sum_{j=1}^{2^{k-1}}\left\{L_n^{k,j}(p) + M_n^{k,j}(p)\right\} \\ + \sum_{k=1}^{h}\sum_{j=1}^{2^{k-1}}\sum_{l=1}^{p-1} N_n^{k,j}(l,\, p-l)\end{aligned} \tag{3.7}$$

and that the laws of $L_n^{k,j}(p)$, $M_n^{k,j}(p)$, and $N_n^{k,j}(l, p-l)$ coincide with those of $L_{2^{-k}n}^{0,1}(p)$, $M_{2^{-k}n}^{0,1}(p)$, and $N_{2^{-k}n}^{0,1}(l, p-l)$, respectively. By Lemma 3.1 and Minkowski's inequality,

$$\begin{aligned} E\left[\left|\frac{(\log n)^3}{n}\sum_{k=1}^{h}\sum_{j=1}^{2^{k-1}}\sum_{l=1}^{p-1} N_n^{k,j}(l, p-l)\right|^2\right] &\le \frac{(\log n)^6}{n^2}\left[\sum_{k=1}^{h}\sum_{j=1}^{2^{k-1}}\sum_{l=1}^{p-1}\left\{E\left|N_n^{k,j}(l, p-l)\right|^2\right\}^{1/2}\right]^2 \\ &\le \frac{C_5 p^2 h^2 (\log n)^6}{(\log 2^{-h} n)^7} \to 0 \end{aligned}$$

as $n \to \infty$. Then we can find that the third part of (3.7) is the negligible part. By Corollary 3.3, we have that

$$\begin{aligned} &E\left[\left|\frac{(\log n)^3}{n}\sum_{k=1}^{h}\sum_{j=1}^{2^{k-1}}\left\{L_n^{k,j}(p) + M_n^{k,j}(p) - 2r_{2^{-k}n} I_n^{k,j}\right\}\right|^2\right] \\ &\quad\le \frac{(\log n)^6}{n^2}\left[\sum_{k=1}^{h}\sum_{j=1}^{2^{k-1}}\left\{E\left|L_n^{k,j}(p) + M_n^{k,j}(p) - 2r_{2^{-k}n} I_n^{k,j}\right|^2\right\}^{1/2}\right]^2 \\ &\quad\le \frac{C_6 h^2 (\log n)^6}{(\log 2^{-h} n)^7} \to 0 \end{aligned}$$

as $n \to \infty$. This implies that we can regard both $L_n^{k,j}(p)$ and $M_n^{k,j}(p)$ as $r_{2^{-k}n} I_n^{k,j}$ asymptotically. Moreover, the fact that

$$0 \le r_{2^{-k}n} - r_n \le \frac{C_7 k}{(\log n)(\log 2^{-k} n)},$$

which is obtained by Lemma 2.3, allows us to exchange $r_{2^{-k}n}$ for r_n. Indeed, using that $E\left|I_n^{0,1}\right|^2 = O(n^2 r_n^4)$ and the fact that the distribution of $I_n^{k,j}$ is equal to that of $I_{2^{-k}n}^{0,1}$, we obtain that

$$E\left[\left|\frac{(\log n)^3}{n}\sum_{k=1}^{h}\sum_{j=1}^{2^{k-1}}(r_{2^{-k}n} - r_n) I_n^{k,j}\right|^2\right] \le \frac{C_8 h^4 (\log n)^4}{(\log 2^{-h} n)^6},$$

which converges to 0 as n tends to infinity. Therefore it is sufficient to consider

$$\frac{(\log n)^3}{n}\sum_{i=1}^{2^h}\left\langle Q_n^{(p)}(i; h)\right\rangle - \frac{2r_n(\log n)^3}{n}\sum_{k=1}^{h}\sum_{j=1}^{2^{k-1}}\left\langle I_n^{k,j}\right\rangle. \tag{3.8}$$

Here the notation $\langle\cdot\rangle$ means that $\langle X\rangle = X - EX$ for any random variable X. Le Gall [16] showed that

$$\lim_{n\to\infty}\frac{(\log n)^2}{n}\sum_{k=1}^{h}\sum_{j=1}^{2^{k-1}}\left\langle I_n^{k,j}\right\rangle = 4\pi^2(\det \Xi)\sum_{k=1}^{h}\sum_{j=1}^{2^{k-1}}\left\langle\iint_{A_j^k}\delta_0(W_t - W_s)\,ds\,dt\right\rangle$$

in law, where $A_j^k = [(2j-2)2^{-k}, (2j-1)2^{-k}) \times ((2j-1)2^{-k}, 2j2^{-k}] \in \mathbb{R}^2$ (see Proposition 6.3), and by the definition of the renormalized self-intersection local time, we have

$$\lim_{h\to\infty} \sum_{k=1}^{h} \sum_{j=1}^{2^{k-1}} \left\langle \iint_{A_j^k} \delta_0(W_t - W_s)\, ds\, dt \right\rangle = \gamma(\mathscr{C})$$

in $L^2(\Omega, \mathfrak{B}, P)$. Then, by Lemma 2.3, we can see that the second part of (3.8) converges to $-16\pi^3(\det \Xi)^2\gamma(\mathscr{C})$ in the distribution sense. It remains to prove that the first part of (3.8) is negligible. Note that $\{Q_n^{(p)}(i;h);\ 1 \le i \le 2^h\}$ is a sequence of independent identically distributed random variables and that the distribution of $Q_n^{(p)}(i;h)$ coincides with that of $Q_{2^{-h}n}^{(p)}$. Then, by Proposition 3.4, we have that

$$\begin{aligned} E\left[\frac{(\log n)^3}{n} \sum_{i=1}^{2^h} \langle Q_n^{(p)}(i;h)\rangle\right]^2 &= \frac{(\log n)^6}{n^2} \sum_{i=1}^{2^h} \operatorname{Var}\left[Q_n^{(p)}(i;h)\right] \\ &\le \frac{C_9 2^{-h} (\log n)^6}{(\log 2^{-h} n)^6} \\ &\le C_{10} 2^{-h} \end{aligned}$$

as n is sufficiently large. Hence we can conclude that

$$\lim_{n\to\infty} \frac{(\log n)^3}{n} \left\langle Q_n^{(p)} \right\rangle = -16\pi^3 (\det \Xi)^2 \gamma(\mathscr{C})$$

in the distribution sense by choosing h sufficiently large in the beginning. □

On the other hand, for $p \ge 1$, let $R_n^{(p)}$ be the number of distinct points entered at least p times by a random walk in the first n steps. If the random walk is adapted with mean 0 and finite variance, Jain and Pruitt [13] showed that

$$R_n^{(1)} \sim \frac{Kn^2}{(\log n)^4}, \tag{3.9}$$

where $K = 8\pi^2 K_1 (\det \Xi)^2$ and

$$K_1 = -\int_0^1 \frac{\log x}{1 - x + x^2}\, dx + \frac{1}{2} - \frac{1}{12}\pi^2.$$

Moreover, Le Gall [16] proved that

$$\lim_{n\to\infty} \frac{(\log n)^2}{n} \left[R_n^{(1)} - ER_n^{(1)}\right] = -4\pi^2 (\det \Xi) \gamma(\mathscr{C})$$

in the distribution sense. Applying Proposition 3.4 and observing the proof of Theorem 3.5, we can improve these results to $R_n^{(p)}$ for general p.

THEOREM 3.6. *Let p be an arbitrary given positive integer. For a two dimensional adapted random walk with mean* 0 *and finite variance,*

$$R_n^{(p)} \sim \frac{Kn^2}{(\log n)^4}, \tag{3.10}$$

where K is the same as we used in (3.9), *and*

$$\lim_{n\to\infty} \frac{(\log n)^2}{n}\left[R_n^{(p)} - ER_n^{(p)}\right] = -4\pi^2(\det \Xi)\gamma(\mathscr{C}) \tag{3.11}$$

in the distribution sense.

PROOF. First we prove (3.10) by induction on p. If $p = 1$, the assertion is (3.9), which was proved by Jain and Pruitt. Note that $R_n^{(p+1)} = R_n^{(p)} - Q_n^{(p)}$ for $p \geq 1$. By Schwarz's inequality, we have that

$$\begin{aligned}\left|\operatorname{Var} R_n^{(p+1)} - \operatorname{Var} R_n^{(p)}\right| &\leq \left|\operatorname{Cov}(R_n^{(p)}, Q_n^{(p)})\right| + \operatorname{Var} Q_n^{(p)} \\ &\leq \sqrt{\operatorname{Var} R_n^{(p)} \cdot \operatorname{Var} Q_n^{(p)}} + \operatorname{Var} Q_n^{(p)}.\end{aligned}$$

By the induction assumption and Proposition 3.4,

$$\left|\operatorname{Var} R_n^{(p+1)} - \operatorname{Var} R_n^{(p)}\right| \leq \frac{C_{11}n^2}{(\log n)^5}.$$

Therefore we obtain that

$$\operatorname{Var} R_n^{(p+1)} \sim \frac{Kn^2}{(\log n)^4}.$$

This completes the proof of (3.10).

We next prove (3.11) for $p \geq 2$. Let $R_n^{(p)}(i;h)$ be the number of points visited at least p times between time $(i-1)2^{-h}n + 1$ and time $i2^{-h}n$. It is clear that

$$R_n^{(p)}(i;h) = R_n^{(1)}(i;h) - \sum_{m=1}^{p-1} Q_n^{(m)}(i;h)$$

for $h \geq 0$, $i \geq 1$ and $p \geq 2$. In particular,

$$R_n^{(p)} = R_n^{(1)} - \sum_{m=1}^{p-1} Q_n^{(m)}.$$

Moreover we have that for $h \geq 1$,

$$R_n^{(1)} = \sum_{i=1}^{2^h} R_n^{(1)}(i;h) - \sum_{k=1}^{h} \sum_{j=1}^{2^{k-1}} I_n^{k,j}.$$

Then, by using (3.7), we can easily obtain that

$$R_n^{(p)} = \sum_{i=1}^{2^h} R_n^{(p)}(i;h) - \sum_{k=1}^{h} \sum_{j=1}^{2^{k-1}} I_n^{k,j}$$
$$+ \sum_{k=1}^{h} \sum_{j=1}^{2^{k-1}} \sum_{m=1}^{p-1} \left\{ L_n^{k,j}(m) + M_n^{k,j}(m) - \sum_{l=1}^{m-1} N_n^{k,j}(l, m-l) \right\}.$$

Combining Lemma 3.1 and Corollary 3.3, we immediately see that the third part is negligible, and so it is enough to study

$$\sum_{i=1}^{2^h} \langle R_n^{(p)}(i;h) \rangle - \sum_{k=1}^{h} \sum_{j=1}^{2^{k-1}} \langle I_n^{k,j} \rangle.$$

Considering analogously with the proof of Theorem 3.5, we easily conclude (3.11). □

4. Some lemmas. In this section, we supply several estimates of functions of the transition probabilities of random walks which will be used in Sections 5 and 6.

LEMMA 4.1 ([8]). *Let $x \neq 0$ and $m \geq 1$. Then*

$$P_0(\tau_x < \tau_0 \leq m) = \sum_{j=1}^{m} \sum_{i=1}^{m-j} p_0^j(x,0) p_x^i(0,x)$$
$$\times \left\{ r_{m-j-i} + P_0(\tau_x < \tau_0 \leq m-j-i) \right\},$$
$$P_x(\tau_0 \leq m < \tau_x) = \sum_{k=1}^{m} p_x^k(x,0) P_0(\tau_x > m-k,\ \tau_0 > m-k).$$

In particular, we have that

$$P_0(\tau_x < \tau_0 \leq m) \leq \sum_{j=1}^{m} \sum_{i=1}^{m-j} p_0^j(x,0) p_x^i(0,x),$$
$$P_x(\tau_0 \leq m < \tau_x) \leq \sum_{k=1}^{m} p_x^k(x,0) r_{m-k}.$$

The following two lemmas can be obtained by simple calculations.

LEMMA 4.2 ([13]). *For $x \neq 0$, $m \geq 1$ and $\gamma \geq 0$,*

$$\sum_{k=1}^{m} p_0^k(0,x) r_{m-k}^{\gamma} = \sum_{k=1}^{m} p^k(0,x) r_{m-k}^{\gamma+1} - \sum_{k=1}^{m} p^k(0,x) \sum_{j=1}^{m-k} f_j (r_{m-k-j}^{\gamma} - r_{m-k}^{\gamma})$$
$$\leq \sum_{k=1}^{m} p^k(0,x) r_{m-k}^{\gamma+1}.$$

LEMMA 4.3 ([8]). *For $x \neq 0$, $m \geq 1$ and $\gamma \geq 0$, we have that*

$$\begin{aligned}\sum_{k=1}^{m} p_{0x}^{k}(0,x) r_{m-k}^{\gamma} = \sum_{k=1}^{m} p_{x}^{k}(0,x) r_{m-k}^{\gamma+1} \\ + \sum_{k=1}^{m} p_{x}^{k}(0,x) P_0(\tau_x < \tau_0 \leq m-k) r_{m-k}^{\gamma} \\ - \sum_{k=1}^{m} \sum_{j=1}^{m-k} p_{x}^{k}(0,x) p_{0x}^{j}(0,0)(r_{m-k-j}^{\gamma} - r_{m-k}^{\gamma}).\end{aligned}$$

We need some refinements of the argument which Jain and Pruitt used in [13] to estimate the negligible parts in proving the convergence of $(\log n)^4 \operatorname{Var} R_n^{(1)}/n^2$. For $n \geq 2$ and $h, i \geq 1$, let

$$T_{h,i}(n) = \sum_{\substack{1 \leq a_1, \ldots, a_h, \\ b_1, \ldots, b_i \leq n \\ x \in \mathbb{Z}^2}} \prod_{\mu=1}^{h} p^{a_\mu}(0,x) \prod_{\rho=1}^{i} p^{b_\rho}(x,0),$$

and for $n, h, i, j, k, l, \alpha, \beta \geq 0$ satisfying $h+i+j+k+l < n$ and $h+i \geq 1$, let

$$\begin{aligned}T_{h,i,j,k,l}^{\alpha,\beta}(n) \\ = \sum_{\substack{1 \leq f + a_1 + \cdots + a_h \\ + b_1 + \cdots + b_i \\ + c_1 + \cdots + c_j \\ + d_1 + \cdots + d_k \\ + e_1 + \cdots + e_l \leq n \\ x \neq 0}} \prod_{\mu=1}^{h} p_x^{a_\mu}(0,x) \prod_{\rho=1}^{i} p_0^{b_\rho}(x,0) \prod_{\lambda=1}^{j} p^{c_\lambda}(0,x) \prod_{\zeta=1}^{k} p^{d_\zeta}(x,0) \\ \times \prod_{\gamma=1}^{l} \Gamma_{e_\gamma}^{(\gamma)} r_f^{\alpha} r^{\beta}_{n - \sum_{\xi=1}^{h} a_\xi - \sum_{\eta=1}^{i} b_\eta - \sum_{\sigma=1}^{j} c_\sigma - \sum_{\nu=1}^{k} d_\nu - \sum_{\kappa=1}^{l} e_\kappa - f}.\end{aligned}$$

Here $\Gamma_j^{(i)}$ are arbitrary nonnegative real numbers with $\sum_{j=1}^{\infty} \Gamma_j^{(i)} \leq 1$ for each $i \geq 1$.

LEMMA 4.4. *We have that*

$$T_{h,i}(n) = O(n),$$
$$T_{h,i,j,k,l}^{\alpha,\beta}(n) = O(n^2 r_n^{\alpha+\beta+h+i}).$$

PROOF. The idea of the proof is the same as Lemma 5.6 in [8]. First we estimate $T_{h,i}(n)$. By Lemma 2.1, since it is symmetric in $a_1, \ldots, a_h$ and in

$b_1, \dots, b_i$, $T_{h,i}(n)$ is bounded by

$$\begin{aligned} A^{h+i-2}h!i! &\sum_{\substack{1\le a_1\le\cdots\le a_h\le n\\ 1\le b_1\le\cdots\le b_i\le n}} \sum_{x\in\mathbb{Z}^2} p^{a_1}(0,x)p^{b_1}(x,0)\prod_{\mu=2}^{h} a_\mu^{-1}\prod_{\rho=2}^{i} b_\rho^{-1} \\ &\le C_{12}\sum_{a=1}^{n}\sum_{b=1}^{n} p^{a+b}(0,0)\left(\log\frac{en}{a}\right)^{h-1}\left(\log\frac{en}{b}\right)^{i-1} \\ &\le C_{13}\sum_{m=1}^{2n}\sum_{b=1}^{m-1} m^{-1}\left(\log\frac{en}{m-b}\right)^{h-1}\left(\log\frac{en}{b}\right)^{i-1}. \end{aligned}$$

The bound of the double sum is

$$\sum_{m=1}^{2n}\left(\log\frac{e^2n}{m}\right)^{h+i-2} \le C_{14}n,$$

which means that $T_{h,i}(n) = O(n)$. We will next estimate the order of $T^{\alpha,\beta}_{h,i,j,k,l}(n)$. By Lemma 2.3,

$$\sum_{q=1}^{m} r_q^\gamma r_{m-q}^\delta \le C_{15} m r_m^{\gamma+\delta} \tag{4.1}$$

for $\gamma, \delta \ge 0$ since r_n is nonincreasing and slowly varying. Then we have

$$T^{\alpha,\beta}_{h,i,j,k,l}(n) \le C_{16} n r_n^{\alpha+\beta+h+i} T_{h+j,i+k}(n),$$

where Lemma 4.2 has been used $(h+i)$ times at first; second, the summations on f and $e_1, \dots, e_l$ have been taken in this order; and last, Lemma 2.2 has been applied. Therefore we obtain that $T^{\alpha,\beta}_{h,i,j,k,l}(n)$ is of order $n^2 r_n^{\alpha+\beta+h+i}$. □

We need another estimate more complicated than $T^{\alpha,\beta}_{h,i,j,k,l}(n)$. For $n, h, i, j, k, l, m, s, \alpha, \beta \ge 0$ with $h+i+j+k+l+m+s < n$ and $h+i \ge 1$, let

$$\begin{aligned} &T^{\alpha,\beta}_{h,i,j,k,l,m,s}(n) \\ &= \sum_{\substack{1\le q+a_1+\cdots+a_h+b_1+\cdots+b_i\\ +c_1+\cdots+c_j+d_1+\cdots+d_k\\ +e_1+\cdots+e_l+f_1+\cdots+f_m\\ +g_1+\cdots+g_s\le n\\ x\ne 0}} \prod_{\mu=1}^{h} p_{0x}^{a_\mu}(0,x)\prod_{\rho=1}^{i} p_{0x}^{b_\rho}(x,0)\prod_{\lambda=1}^{j} p_x^{c_\lambda}(0,x) \\ &\qquad\times\prod_{\zeta=1}^{k} p_0^{d_\zeta}(x,0)\prod_{\gamma=1}^{l} p^{e_\gamma}(0,x)\prod_{\delta=1}^{m} p^{f_\delta}(x,0)\prod_{\chi=1}^{s} \Gamma_{g_\chi}^{(\chi)} \\ &\qquad\times r_q^\alpha r^\beta_{n-\sum_1^h a_\xi-\sum_1^i b_\eta-\sum_1^j c_\sigma-\sum_1^k d_\nu-\sum_1^l e_\kappa-\sum_1^m f_\pi-\sum_1^s g_\theta-q}. \end{aligned}$$

The following lemma plays an important role in the proof of the main theorem of this paper, where it is used to show that many terms are negligible.

LEMMA 4.5. *We have that*

$$T^{\alpha,\beta}_{h,i,j,k,l,m,s}(n) = O(n^2 r_n^{\alpha+\beta+2h+2i+j+k}). \tag{4.2}$$

PROOF. First we show (4.2) when either h or i is 0; however, it is enough to prove it when $h \geq 1$ and $i = 0$ since

$$T^{\alpha,\beta}_{h,i,j,k,l,m,s}(n) = T^{\alpha,\beta}_{i,h,k,j,m,l,s}(n),$$

which is obtained by making the substitution $y = -x$ in the summation on x. We try to prove

$$T^{\alpha,\beta}_{h,0,j,k,l,m,s}(n) = O(n^2 r_n^{\alpha+\beta+2h+j+k}) \tag{4.3}$$

by induction with respect to h. Applying Lemma 4.3 and noting that r_n is nonincreasing, we have that

$$\begin{aligned} T^{\alpha,\beta}_{1,0,j,k,l,m,s}(n) &\leq T^{\alpha,\beta+1}_{j+1,k,l,m,s}(n) + T^{\alpha,\beta}_{j+2,k+1,l,m,s}(n) \\ &= O(n^2 r_n^{\alpha+\beta+j+k+2}). \end{aligned}$$

The last estimate was obtained by Lemma 4.4. We assume (4.3) and apply Lemma 4.3 to $\sum_{a_{h+1}} p_{0x}^{a_{h+1}}(0,x)$. Using Lemma 4.4 again, we obtain that

$$\begin{aligned} T^{\alpha,\beta}_{h+1,0,j,k,l,m,s}(n) &\leq T^{\alpha,\beta+1}_{h,0,j+1,k,l,m,s}(n) + T^{\alpha,\beta}_{h,0,j+2,k+1,l,m,s}(n) \\ &= O(n^2 r_n^{\alpha+\beta+2h+j+k+2}). \end{aligned}$$

Hence we have (4.2) when $h \geq 1$ and $i = 0$.

We next observe (4.2) when $h, i \geq 1$. In this case, we show it by induction on i. By Lemma 4.3,

$$\begin{aligned} T^{\alpha,\beta}_{h,1,j,k,l,m,s}(n) &\leq T^{\alpha,\beta+1}_{h,0,j,k+1,l,m,s}(n) + T^{\alpha,\beta}_{h,0,j+1,k+2,l,m,s}(n) \\ &= O(n^2 r_n^{\alpha+\beta+2h+j+k+2}). \end{aligned}$$

Assuming (4.2) for $i \geq 1$ and applying Lemma 4.3 to $\sum_{b_{i+1}} p_{0x}^{b_{i+1}}(x,0)$, we have that

$$\begin{aligned} T^{\alpha,\beta}_{h,i+1,j,k,l,m,s}(n) &\leq T^{\alpha,\beta+1}_{h,i,j,k+1,l,m,s}(n) + T^{\alpha,\beta}_{h,i,j+1,k+2,l,m,s}(n) \\ &= O(n^2 r_n^{\alpha+\beta+2h+2i+j+k+2} + n^2 r_n^{\alpha+\beta+2h+2i+j+k+3}), \end{aligned}$$

which is of order $n^2 r_n^{\alpha+\beta+2h+2(i+1)+j+k}$.

Therefore we can conclude (4.2). □

5. The proof of Lemma 3.1. For simplicity, we put $N_n(p,q) = N_n^{0,1}(p,q) = \left|S^p(0,n)\cap S^q(n,2n)\right|$. We can express $N_n(p,q)$ by the summation of several sequences of indicator random variables. For $0\le i<j$, let

$$Z_i^j = \begin{cases} 1, & \text{if } S_i \neq S_\alpha \text{ for } i<\alpha\le j,\\ 0, & \text{otherwise,}\end{cases}$$

$$Y_j^i = \begin{cases} 1, & \text{if } S_j \neq S_\alpha \text{ for } i\le\alpha< j,\\ 0, & \text{otherwise.}\end{cases}$$

For $0\le i<j$ and $p\ge 1$, let

$$W_i^j(p) = \begin{cases} 1, & \text{if there are exactly } (p-1) \text{ indices } \alpha \text{ in}\\ & \quad \{i+1,\dots,j-1\} \text{ such that } S_\alpha = S_i \text{ and } S_i = S_j,\\ 0, & \text{otherwise.}\end{cases}$$

Using these indicators, we have that for $p,q\ge 2$,

$$N_n(1,1) = \sum_{j=n+1}^{2n}\sum_{i=1}^{n}\sum_{l=j+1}^{2n} Y_i^0 Z_i^n Y_j^n Z_j^{2n}\chi(S_i = S_j),$$

$$N_n(1,q) = \sum_{j=n+1}^{2n}\sum_{i=1}^{n}\sum_{l=j+1}^{2n} Y_i^0 Z_i^n Y_j^n W_j^l(q-1) Z_l^{2n}\chi(S_i = S_j),$$

$$N_n(p,q) = \sum_{j=n+1}^{2n}\sum_{i=1}^{n}\sum_{h=j+1}^{2n}\sum_{l=i+1}^{n} Y_i^0 W_i^l(p-1) Z_l^n \times Y_j^n W_j^h(q-1) Z_h^{2n}\chi(S_i = S_j),$$

where $\chi(A)$ means the indicator function of a set A. To estimate these random variables, we need to introduce some notation of taboo probabilities. For $x\in\mathbb{Z}^2$ and $\alpha\ge 1$, $\tau_x^{(\alpha)}$ will denote the time of the αth entrance into x, so that $\tau_x^{(1)} = \tau_x$. For the sake of convenience, we put $\tau_x^{(0)} = 0$ for each $x\in\mathbb{Z}^2$. For $x, y\in\mathbb{Z}^2$ and $\alpha,\beta\ge 0$, let

$$\begin{aligned} q_y^n(x;\alpha) &= P_x(\tau_x^{(\alpha)} = n,\ \tau_y^{(1)} > n),\\ q^n(x,y;\alpha) &= P_x(\tau_y^{(1)} = n,\ \tau_x^{(\alpha)} < n,\ \tau_x^{(\alpha+1)} \ge n),\\ q^n(x,y;\alpha,\beta) &= P_x(\tau_y^{(\beta)} = n,\ \tau_x^{(\alpha)} < n,\ \tau_x^{(\alpha+1)} \ge n),\\ q_y^n(x;\alpha,\beta) &= P_x(\tau_x^{(\alpha)} = n,\ \tau_y^{(\beta)} < n,\ \tau_y^{(\beta+1)} \ge n)\end{aligned}$$

and $f_n^{(\alpha)} = P_x(\tau_x^{(\alpha)} = n)$. For $x\neq y$, we obtain the estimates

$$\left.\begin{aligned} q_y^n(x;\alpha)\\ q_y^n(x;\alpha,\beta)\end{aligned}\right\} \le f_n^{(\alpha)}, \tag{5.1}$$

$$q^n(x,y;\alpha) \le \begin{cases} \sum_{k=1}^{n-1} f_{n-k}^{(\alpha)} p_{xy}^k(x,y), & \text{if } \alpha \ge 1, \\ p_{xy}^n(x,y), & \text{if } \alpha = 0, \end{cases} \tag{5.2}$$

$$q^n(x,y;\alpha,\beta) \le \begin{cases} p^n(x,y), & \text{if } \alpha \ge 1 \text{ and } \beta \ge 1, \\ p_x^n(x,y), & \text{if } \alpha = 0 \text{ and } \beta \ge 1. \end{cases} \tag{5.3}$$

From now on, we prove Lemma 3.1 and first show (3.2) when $p, q \ge 2$. In this case, it is equivalent to the estimate that $E\left|N_n(\bar{p}+1, \bar{q}+1)\right|^2 = O(n^2 r_n^7)$ for $\bar{p}, \bar{q} \ge 1$. After this, for simplicity, we adopt p and q instead of $\bar{p}$ and $\bar{q}$, respectively. For $p, q \ge 1$, let

$$\Sigma_{i,j}^n(p,q) = \sum_{h=j+1}^{2n} \sum_{l=i+1}^{n} \Theta_{i,l,j,h}^n(p,q),$$

where $\Theta_{i,l,j,h}^n(p,q) = Y_i^0 W_i^l(p) Z_l^n Y_j^n W_j^h(q) Z_h^{2n} \chi(S_i = S_j)$. Then we have

$$N_n(p+1, q+1) = \sum_{j=n+1}^{2n} \sum_{i=1}^{n} \Sigma_{i,j}^n(p,q)$$

and also have that

$$\begin{aligned} E\left|N_n(p+1,q+1)\right|^2 &= \sum_{j=n+1}^{2n} \sum_{i=1}^{n} E\{\Sigma_{i,j}^n(p,q)\}^2 \\ &\quad + 2 \sum_{\substack{n<h<j\le 2n \\ 0<k<i\le n}} E\left[\Sigma_{i,j}^n(p,q)\Sigma_{k,h}^n(p,q)\right] \\ &\quad + 2 \sum_{\substack{n<h<j\le 2n \\ 0<i<k\le n}} E\left[\Sigma_{i,j}^n(p,q)\Sigma_{k,h}^n(p,q)\right] \\ &=: \mathrm{I} + 2\mathrm{II} + 2\mathrm{III}. \end{aligned}$$

Our goal is to show that $\mathrm{I} = O(n)$ and both II and III are of order $n^2 r_n^7$. The method of estimating I is very easy. Indeed, noting that $\Sigma_{i,j}^n(p,q)$ is also an indicator random variable for $i < j$,

$$\mathrm{I} = \sum_{j=n+1}^{2n} \sum_{i=1}^{n} \sum_{h=j+1}^{2n} \sum_{l=i+1}^{n} r_i f_{l-i}^{(p)} f_{j-l} f_{h-j}^{(q)} r_{2n-h}.$$

Dominating r_i and r_{2n-h} by 1, we have that $\mathrm{I} \le n$.

We now try to estimate II, which is equal to

$$\sum_{\substack{n<h<j\le 2n \\ 0<k<i\le n}} \sum_{\substack{h<r\le 2n \\ j<m\le 2n \\ i<l\le n \\ k<s\le n}} E\left[\Theta_{i,l,j,m}^n(p,q)\Theta_{k,s,h,r}^n(p,q)\right].$$

To estimate this summation, we should divide it into the following nine parts by the order of subscript letters of Θ:

(1) $1 \le k < s < i < l \le n < h < r < j < m \le 2n$;
(2) $1 \le k < i < s < l \le n < h < r < j < m \le 2n$;
(3) $1 \le k < i < l < s \le n < h < r < j < m \le 2n$;
(4) $1 \le k < s < i < l \le n < h < j < r < m \le 2n$;
(5) $1 \le k < i < s < l \le n < h < j < r < m \le 2n$;
(6) $1 \le k < i < l < s \le n < h < j < r < m \le 2n$;
(7) $1 \le k < s < i < l \le n < h < j < m < r \le 2n$;
(8) $1 \le k < i < s < l \le n < h < j < m < r \le 2n$;
(9) $1 \le k < i < l < s \le n < h < j < m < r \le 2n$.

Now we introduce the notation $\mathrm{II}(u)$ for $u = 1, 2, \dots, 9$, where $\mathrm{II}(u)$ means the summation of $E[\Theta^n_{i,l,j,m}(p,q)\Theta^n_{k,s,h,r}(p,q)]$ on h, i, j, k, l, m, s, r in the case (u).

In calculating the order of each $\mathrm{II}(u)$, we express the summands by taboo probabilities by using the Markov property, and next we apply lemmas in Section 4, especially Lemma 4.5. II(1) is equal to

$$\sum_{\substack{n<h<r<j<m\le 2n\\ 1\le k<s<i<l\le n\\ x\neq 0}} q_x^{s-k}(0;p)p_{0x}^{i-s}(0,x)q_0^{l-i}(x;p)p_{0x}^{h-l}(x,0) \times q_x^{r-h}(0;q)p_{0x}^{j-r}(0,x)q_0^{m-j}(x;q) \times P_x(\tau_0 > 2n-m,\ \tau_x > 2n-m)P_x(\tau_0 > k,\ \tau_x > k).$$

Neglecting the events $\{\tau_0 > 2n-m\}$ and $\{\tau_0 > k\}$ and employing (5.1), we have that II(1) is bounded by

$$\begin{aligned}&\sum_{\substack{1\le k<s<i<l<h\\ <r<j<m\le 2n\\ x\neq 0}} f_{s-k}^{(p)}p_{0x}^{i-s}(0,x)f_{l-i}^{(p)}p_{0x}^{h-l}(x,0)f_{r-h}^{(q)}p_{0x}^{j-r}(0,x)f_{m-j}^{(q)}r_{2n-m}r_k\\ &\le \sum_{\substack{1\le k+s+i+l+h\\ +r+j+m\le 2n\\ x\neq 0}} f_s^{(p)}p_{0x}^{i}(0,x)f_l^{(p)}p_{0x}^{h}(x,0)f_r^{(q)} \times p_{0x}^{j}(0,x)f_m^{(q)}r_{2n-m-j-r-h-l-i-s-k}r_k\\ &= T^{1,1}_{2,1,0,0,0,0,3}(2n).\end{aligned}$$

By Lemma 4.5, this is of order $n^2 r_n^8$. In the case (2), we may consider the estimate of

$$\sum_{\substack{n<h<r<j<m\le 2n\\ 1\le k<i<s<l\le n\\ x\neq 0\\ 0\le\alpha,\,\beta<p}} q^{i-k}(0,x;\alpha)q^{s-i}(x,0;\beta,\,p-\alpha)q^{l-s}(0,x;0,\,p-\beta) \times p_{0x}^{h-l}(x,0)q_x^{r-h}(0;q)p_{0x}^{j-r}(0,x)q_0^{m-j}(x;q)r_{2n-m}r_k. \tag{5.4}$$

It is clear that $q^n(0,x;\alpha) \le p^n(0,x)$ and $q^n(x,0;\beta,\,p-\alpha)$ is not more than $p^n(x,0)$, and then (5.4) is bounded by

$$p^2 \sum_{\substack{1\le k+i+s+l+h\\ +r+j+m\le 2n\\ x\ne 0}} p^i(0,x)p^s(x,0)p_0^l(0,x)p_{0x}^h(x,0)f_r^{(q)} \times p_{0x}^j(0,x)f_m^{(q)} r_{2n-m-j-r-h-l-s-i-k} r_k.$$

By (2.1), we can exchange $p_0^l(0,x)$ for $p_x^l(x,0)$, and we have that this summation is $p^2 T^{1,1}_{1,1,1,0,1,1,2}(2n)$. Therefore II(2) is of order $n^2 r_n^7$ by Lemma 4.5. The term II(3) is not larger than

$$\sum_{\substack{n<h<r<j<m\le 2n\\ 1\le k<i<l<s\le n\\ x\ne 0\\ 0\le\alpha+\beta<p}} q^{i-k}(0,x;\alpha)q_0^{l-i}(x;\,p,\beta)q^{s-l}(x,0;0,\,p-\alpha-\beta) \times p_{0x}^{h-s}(0,0)q_x^{r-h}(0;q)p_{0x}^{j-r}(0,x)q_0^{m-j}(x;q)r_{2n-m}r_k.$$

Applying (5.1) and (5.3), we obtain that II(3) is bounded by

$$p \sum_{\substack{1\le k+i+l+s+h\\ +r+j+m\le 2n\\ x\ne 0\\ 0\le\alpha<p}} q^i(0,x;\alpha)f_l^{(p)}p_x^s(x,0)f_h f_r^{(q)} \times p_{0x}^j(0,x)f_m^{(q)} r_{2n-m-j-r-h-l-s-i-k} r_k. \tag{5.5}$$

In estimating II(2), we used the rough inequality that $q^i(0,x;\alpha) \le p^i(0,x)$ without regard to the value of α. However, in this case, we must estimate (5.5) more carefully and split the summation into the two cases—$\alpha = 0$ and $\alpha \ne 0$—to derive the order of II(3). Using (5.2), we have that a bound of II(3) is

$$p \sum_{\substack{1\le k+i+l+s+h\\ +r+j+m\le 2n\\ x\ne 0}} \left(\sum_{\alpha=1}^{p-1} \sum_{u=1}^{i-1} f_{i-u}^{(\alpha)} p_{0x}^u(0,x) + p_{0x}^i(0,x) \right) f_l^{(p)} p_x^s(x,0) \times f_h f_r^{(q)} p_{0x}^j(0,x) f_m^{(q)} r_{2n-m-j-r-h-l-s-i-k} r_k.$$

The order of the first summation is $O\{T^{1,1}_{2,0,0,1,0,0,5}(2n)\}$ and the second part is $O\{T^{1,1}_{2,0,0,1,0,0,4}(2n)\}$. Hence we obtain that II(3) is of order $n^2 r_n^7$ by Lemma 4.5. In the case (4), it is sufficient to calculate

$$\sum_{\substack{n<h<j<r<m\le 2n\\ 1\le k<s<i<l\le n\\ x\ne 0\\ 0\le\alpha,\,\beta<q}} q_x^{s-k}(0;\,p)p_{0x}^{i-s}(0,x)q_0^{l-i}(x;\,p)p_{0x}^{h-l}(x,0)q^{j-h}(0,x;\alpha) \times q^{r-j}(x,0;\beta,\,q-\alpha)q^{m-r}(0,x;0,\,q-\beta)r_{2n-m}r_k.$$

There is no necessity of applying the careful argument used in estimating (5.5), and it is enough to use the rough estimate that $q^{j-h}(0, x; \alpha) \leq p^{j-h}(0, x)$. Using (5.1) and (5.3), II(4) is dominated by

$$p^2 \sum_{\substack{1 \leq k+s+i+l+h \\ +j+r+m \leq 2n \\ x \neq 0}} f_s^{(p)} p_{0x}^i(0, x) f_l^{(p)} p_{0x}^h(x, 0) p^j(0, x)$$

$$\times\, p^r(x, 0) p_0^m(0, x) r_{2n-m-r-j-h-l-i-s-k} r_k$$

$$= p^2 T^{1,1}_{1,1,1,0,0,0,2}(2n) = O(n^2 r_n^7).$$

A bound of II(5) is

$$\sum_{\substack{n<h<j<r<m \leq 2n \\ 1 \leq k<i<s<l \leq n \\ x \neq 0}} \sum_{\substack{0 \leq \alpha,\, \beta < p \\ 0 \leq \gamma,\, \delta < q}} q^{i-k}(0, x; \alpha) q^{s-i}(x, 0; \beta,\, p-\alpha)$$

$$\times\, q^{l-s}(0, x; 0,\, p-\beta) p_{0x}^{h-l}(x, 0) q^{j-h}(0, x; \gamma)$$

$$\times\, q^{r-j}(x, 0; \delta,\, q-\gamma) q^{m-r}(0, x; 0,\, q-\delta) r_{2n-m} r_k.$$

In the summands, there are two parts to which we can apply (5.2), and they are $q^{i-k}(0, x; \alpha)$ and $q^{j-r}(0, x; \gamma)$. However, we need not apply (5.2) to both parts, and so we adopt the estimate that $q^{j-h}(0, x; \gamma) \leq p^{j-h}(0, x)$. Then, by (5.3), we have that II(5) is bounded by

$$pq^2 \sum_{\substack{1 \leq k+i+s+l+h \\ +j+r+m \leq 2n \\ x \neq 0 \\ 0 \leq \alpha < p}} q^i(0, x; \alpha) p^s(x, 0) p_0^l(0, x) p_{0x}^h(x, 0) p^j(0, x)$$

$$\times\, p^r(x, 0) p_0^m(0, x) r_{2n-m-j-r-h-l-s-i-k} r_k. \tag{5.6}$$

We can estimate (5.6) in a way similar to (5.5). Indeed, calculating the part $\alpha = 0$ and the part $1 \leq \alpha < p$ separately, we obtain that (5.6) is $O\{T^{1,1}_{1,1,2,0,1,2,1}(2n) + T^{1,1}_{1,1,2,0,1,2,0}(2n)\}$, which is of order $n^2 r_n^8$ by Lemma 4.5. In the case (6), we may calculate

$$\sum_{\substack{n<h<j<r<m \leq 2n \\ 1 \leq k<i<l<s \leq n \\ x \neq 0}} \sum_{\substack{0 \leq \alpha+\beta < p \\ 0 \leq \gamma,\, \delta < q}} q^{i-k}(0, x; \alpha) q_0^{l-i}(x; p, \beta)$$

$$\times\, q^{s-l}(x, 0; 0,\, p-\alpha-\beta) p_{0x}^{h-s}(0, 0) q^{j-h}(0, x; \gamma)$$

$$\times\, q^{r-j}(x, 0; \delta,\, q-\gamma) q^{m-r}(0, x; 0,\, q-\delta) r_{2n-m} r_k.$$

This is not larger than

$$pq \sum_{\substack{1\le k+i+l+s+h\\ +j+r+m\le 2n\\ x\neq 0\\ 0\le\alpha<p\\ 0\le\gamma<q}} q^i(0,x;\alpha) f_l^{(p)} p_x^s(x,0) f_h q^j(0,x;\gamma) \times p^r(x,0) p_0^m(0,x) r_{2n-m-j-r-h-s-l-i-k} r_k. \tag{5.7}$$

We must use neither $q^i(0,x;\alpha) \le p^i(0,x)$ nor $q^j(0,x;\gamma) \le p^j(0,x)$, and so have to apply (5.2) to both $q^i(0,x;\alpha)$ and $q^j(0,x;\gamma)$. Then we must split the sum on α and γ into four parts: (i) $\alpha,\gamma \neq 0$; (ii) $\alpha \neq 0$, $\gamma = 0$; (iii) $\alpha = 0$, $\gamma \neq 0$; (vi) $\alpha,\gamma = 0$. The first part is $O\{T^{1,1}_{2,0,1,1,0,1,4}(2n)\}$; the second and the third parts are of order $T^{1,1}_{2,0,1,1,0,1,3}(2n)$; and the fourth part is of order $T^{1,1}_{2,0,1,1,0,1,2}(2n)$. Hence II(6) is $O(n^2 r_n^8)$ by Lemma 4.5. The term II(7) can be estimated by calculating

$$\sum_{\substack{n<h<j<m<r\le 2n\\ 1\le k<s<i<l\le n\\ x\neq 0\\ 0\le\alpha+\beta<q}} q_x^{s-k}(0;p) p_{0x}^{i-s}(0,x) q_0^{l-i}(x;p) p_{0x}^{h-l}(x,0) \times q^{j-h}(0,x;\alpha) q_0^{m-j}(x;q,\beta) q^{r-m}(x,0;0,q-\alpha-\beta) r_{2n-r} r_k.$$

This summation is bounded by

$$q^2 \sum_{\substack{1\le k+s+i+l+h\\ +j+m+r\le 2n\\ x\neq 0}} f_s^{(p)} p_{0x}^i(0,x) f_l^{(p)} p_{0x}^h(x,0) p^j(0,x) \times f_m^{(q)} p_x^r(x,0) r_{2n-r-m-j-h-l-i-s-k} r_k,$$

the estimate of which is a constant multiple of $T^{1,1}_{1,1,0,1,1,0,2}(2n)$. Here we have used (2.1). Therefore II(7) is of order $n^2 r_n^7$ by Lemma 4.5. In the case (8), we need to give a bound of

$$\sum_{\substack{n<h<j<m<r\le 2n\\ 1\le k<i<s<l\le n\\ x\neq 0}} \sum_{\substack{0\le\alpha,\,\beta<p\\ 0\le\gamma+\delta<q}} q^{i-k}(0,x;\alpha) q^{s-i}(x,0;\beta,\,p-\alpha) \times q^{l-s}(0,x;0,\,p-\beta) p_{0x}^{h-l}(x,0) q^{j-h}(0,x;\gamma) \times q_0^{m-j}(x;q,\delta) q^{r-m}(x,0;0,q-\gamma-\delta) r_{2n-r} r_k,$$

which is not larger than

$$pq^2 \sum_{\substack{1\le k+i+s+l+h\\ +j+m+r\le 2n\\ x\neq 0\\ 0\le\alpha<p}} q^i(0,x;\alpha) p^s(x,0) p_0^l(0,x) p_{0x}^h(x,0) p^j(0,x) \times f_m^{(q)} p_x^r(x,0) r_{2n-r-m-j-h-l-s-i-k} r_k.$$

The method of calculation of this summation is quite similar to that of (5.5). Moreover, we have that II(9) is not larger than

$$\sum_{\substack{n<h<j<m<r\le 2n\\ 1\le k<i<l<s\le n\\ x\neq 0}} \sum_{\substack{0\le\alpha+\beta<p\\ 0\le\gamma+\delta<q}} q^{i-k}(0,x;\alpha)q_0^{l-i}(x;p,\beta)$$
$$\times q^{s-l}(x,0;0,p-\alpha-\beta)p_{0x}^{h-s}(0,0)q^{j-h}(0,x;\gamma)$$
$$\times q_0^{m-j}(x;q,\delta)q^{r-m}(x,0;0,q-\gamma-\delta)r_{2n-r}r_k,$$

which is bounded by

$$pq \sum_{\substack{1\le k+i+l+s+h\\ +j+m+r\le 2n\\ x\neq 0\\ 0\le\alpha<p\\ 0\le\gamma<q}} q^i(0,x;\alpha)f_l^{(p)}p_x^s(x,0)f_h q^j(0,x;\gamma)$$
$$\times f_m^{(q)}p_x^r(x,0)r_{2n-r-m-j-h-s-l-i-k}r_k.$$

This can be estimated in the same manner as (5.7). Then each term II(8) and II(9) is $O(n^2r_n^7)$. Their calculations are left to the reader. Thus we can conclude that II $\le C_{17}n^2r_n^7$.

Now we will estimate the term III and have that

$$\text{III} = \sum_{\substack{n<h<j\le 2n\\ 0<i<k\le n}} \sum_{\substack{h<r\le 2n\\ j<m\le 2n\\ i<l\le n\\ k<s\le n}} E\left[\Theta^n_{i,l,j,m}(p,q)\Theta^n_{k,s,h,r}(p,q)\right].$$

To obtain its order, we also need to divide the summation into the following nine parts:

(1) $1\le i<l<k<s\le n<h<r<j<m\le 2n$;
(2) $1\le i<k<l<s\le n<h<r<j<m\le 2n$;
(3) $1\le i<k<s<l\le n<h<r<j<m\le 2n$;
(4) $1\le i<l<k<s\le n<h<j<r<m\le 2n$;
(5) $1\le i<k<l<s\le n<h<j<r<m\le 2n$;
(6) $1\le i<k<s<l\le n<h<j<r<m\le 2n$;
(7) $1\le i<l<k<s\le n<h<j<m<r\le 2n$;
(8) $1\le i<k<l<s\le n<h<j<m<r\le 2n$;
(9) $1\le i<k<s<l\le n<h<j<m<r\le 2n$.

For $u=1,2,\dots,9$, the notations III(u) are defined in the same way as II(u). The method of calculating each term except III(1) is analogous with that used in estimating II, and we can conclude that they are of order $n^2r_n^7$. We can estimate the terms III(2), III(4) and III(7) by the same method that we have applied to estimate (5.5). Indeed, the term III(7), for example, is bounded by

$$\begin{aligned}
&\sum_{\substack{n<h<j<m<r\le 2n\\ 1\le i<l<k<s\le n\\ x\neq 0}} \sum_{0\le\alpha+\beta<q} q_x^{l-i}(0;p)p_{0x}^{k-l}(0,x)q_0^{s-k}(x;p)\\
&\qquad\qquad \times p_{0x}^{h-s}(x,x)q^{j-h}(x,0;\alpha)q_0^{m-j}(x;q,\beta)\\
&\qquad\qquad \times q^{r-m}(x,0;0,q-\alpha-\beta)r_{2n-r}r_k\\
&\le q \sum_{\substack{1\le i+l+k+s+h\\ +j+m+r\le 2n\\ x\neq 0\\ 0\le\alpha<q}} f_l^{(p)}p_{0x}^k(0,x)f_s^{(p)}f_h q^j(x,0;\alpha)\\
&\qquad\qquad \times f_m^{(q)}p_x^r(x,0)r_{2n-r-m-j-h-s-k-l-i}r_i.
\end{aligned}$$

Then we have that $\text{III}(7)=O(n^2r_n^7)$ by applying (5.2). The term III(3) can be estimated similarly to II(2). Moreover, we can estimate the terms III(5) and III(8) by the method which has been used in estimating (5.7) and can calculate III(6) and III(9) similarly to II(5). Their calculations are also left to the reader.

To bring the proof of Lemma 3.1 to an end, we must estimate the term III(1). In calculating the other terms, we need not consider that h is larger than n and s or l is less than or equal to n since this fact hardly has any effect on their estimates. However, the fact that h cannot be close to s when s is away from n affects the estimate of III(1) essentially, and the inequality

$$\sum_{j=1}^{m} f_{n+j} \le r_n \tag{5.8}$$

plays an important role to give an upper bound of III(1). The remainder of this section is devoted to the calculation of III(1). For $1\le i<l<k<s\le n<h<r<j<m\le 2n$, we have that

$$\begin{aligned}
&E\left[\Theta^n_{i,l,j,m}(p,q)\Theta^n_{k,s,h,r}(p,q)\right]\\
&\quad=\sum_{x\neq 0} q_x^{l-i}(0;p)p_{0x}^{k-l}(0,x)q_0^{s-k}(x;p)\\
&\qquad\quad \times p_{0x}^{h-s}(x,x)q_0^{r-h}(x;q)p_{0x}^{j-r}(x,0)q_x^{m-j}(0;q)\\
&\qquad\quad \times P_0(\tau_x>2n-m,\ \tau_0>2n-m)P_x(\tau_0>i,\ \tau_x>i).
\end{aligned}$$

Then it is sufficient to estimate

$$\sum_{\substack{n<h<r<j<m\le 2n\\ 1\le i<l<k<s\le n\\ x\neq 0}} f_{l-i}^{(p)}p_{0x}^{k-l}(0,x)f_{s-k}^{(p)}f_{h-s}f_{r-h}^{(q)}p_{0x}^{j-r}(x,0)f_{m-j}^{(q)}r_{2n-m}r_i. \tag{5.9}$$

Here the events $\{\tau_x>2m-n\}$ and $\{\tau_0>i\}$ have been neglected and the estimate (5.1) has been applied. We now observe the contribution for $2n-m\le$

$2nr_n^8$ in (5.9). Note that $\sum_{x\neq 0} p_0^u(0,x)p_0^w(x,0) = f_{u+w}$ and dominate r_{2n-m} and r_i by 1. Then we may consider the summation

$$\sum_{m=2n-2nr_n^8}^{2n} \sum_{\substack{1\le i<l<k<s\\ <h<r<j<m}} f_{l-i}^{(p)} f_{k-l+j-r} f_{s-k}^{(p)} f_{h-s} f_{r-h}^{(q)} f_{m-j}^{(q)}. \tag{5.10}$$

We first sum $f_{l-i}^{(p)}$ over i and dominate its summation by 1, and next sum $f_{k-l+j-r}$ over l and dominate its summation by 1. In a similar way, we sum over k, s, h and j in this order and dominate the summation by 1 each time. Then we have that (5.10) is not larger than $4n^2r_n^8$. Hence we can concentrate upon the calculation of (5.9) in the case that $2n-m > 2nr_n^8$, and can replace r_{2n-m} with a constant multiple of r_{2n} and also r_n in this case since $r_n \sim c(\log n)^{-1}$. Then we have that (5.9) is bounded by

$$\begin{aligned} C_{18}r_n \sum_{\substack{n<h<r<j<m\le 2n\\ 1\le i<l<k<s\le n\\ x\neq 0}} & f_{l-i}^{(p)} p_{0x}^{k-l}(0,x) f_{s-k}^{(p)} f_{h-s} f_{r-h}^{(q)} p_{0x}^{j-r}(x,0) f_{m-j}^{(q)} r_i \\ & + C_{19} n^2 r_n^8. \end{aligned} \tag{5.11}$$

In the first step, we calculate

$$\sum_{n<h<r<j<m\le 2n} f_{h-s} f_{r-h}^{(q)} p_{0x}^{j-r}(x,0) f_{m-j}^{(q)},$$

which is equal to

$$\sum_{1\le h+r+j+m\le n} f_{n+h-s} f_r^{(q)} p_{0x}^{j}(x,0) f_m^{(q)}. \tag{5.12}$$

The summations on m and r are dominated by 1. Using the inequality (5.8), a bound of (5.12) is

$$\sum_{1\le h+j\le n} f_{n+h-s} p_{0x}^j(x,0) \le \sum_{j=1}^{n} p_{0x}^j(x,0) r_{n-s},$$

and this is not larger than

$$\begin{aligned} \sum_{j=1}^{n} p_0^j(x,0) r_{n-j} r_{n-s} &+ \sum_{j=1}^{n} p_0^j(x,0) P_x(\tau_0 < \tau_x \le n-j) r_{n-s} \\ \le \sum_{j=1}^{n} & p^j(x,0) r_{n-j}^2 r_{n-s} \\ & + \sum_{1\le j+u+w\le n} p^j(x,0) p^u(0,x) p^w(x,0) r_{n-j-u-w}^3 r_{n-s}. \end{aligned}$$

Here we have used Lemmas 4.1 and 4.2 after applying Lemma 4.3. Hence the first term of (5.11) is bounded by

$$C_{18} r_n \sum_{\substack{1 \le i+l+k+s \le n \\ 1 \le j \le n \\ x \ne 0}} f_l^{(p)} p_{0x}^k(0,x) f_s^{(p)} p^j(x,0) r_{n-j}^2 r_{n-s-k-l-i} r_i$$
$$+ C_{18} r_n \sum_{\substack{1 \le i+l+k+s \le n \\ 1 \le j+u+w \le n \\ x \ne 0}} f_l^{(p)} p_{0x}^k(0,x) f_s^{(p)} p^j(x,0) \times p^u(0,x) p^w(x,0) r_{n-j-u-w}^3 r_{n-s-k-l-i} r_i.$$

In the next step, we estimate it and so can finish obtaining an upper bound of III. First summing over i and applying (4.1) and Lemma 2.2 and then summing over l and s, we have that it is not larger than

$$C_{20} n r_n^3 \sum_{\substack{1 \le k \le n \\ 1 \le j \le n \\ x \ne 0}} p_{0x}^k(0,x) p^j(x,0) r_{n-j}^2 \tag{5.13}$$

$$+ C_{20} n r_n^3 \sum_{\substack{1 \le k \le n \\ 1 \le j+u+w \le n \\ x \ne 0}} p_{0x}^k(0,x) p^j(x,0) p^u(0,x) p^w(x,0) r_{n-j-u-w}^3. \tag{5.14}$$

By Lemma 4.3, (5.13) is bounded by

$$C_{20} n r_n^3 \sum_{\substack{1 \le k,\ j \le n \\ x \ne 0}} p_x^k(0,x) p^j(x,0) r_{n-j}^2 r_{n-k}$$
$$+ C_{20} n r_n^3 \sum_{\substack{1 \le k,\ j \le n \\ x \ne 0}} p_x^k(0,x) P_0(\tau_x < \tau_0 \le n-k) p^j(x,0) r_{n-j}^2.$$

Now we will estimate these two terms by using Lemmas 4.1, 4.2, 4.4 and the method of changing r_{n-a} into a constant multiple of r_n which was used in (5.9). The first term is dominated by

$$C_{20} n r_n^3 \sum_{\substack{1 \le k,\ j \le n \\ x \ne 0}} p^k(0,x) p^j(x,0) r_{n-j}^2 r_{n-k}^2. \tag{5.15}$$

Considering the contributions for $n-k \le n r_n^7$ and $n-j \le n r_n^7$, it turns out that r_{n-j} and r_{n-k} can be replaced with r_n. Then (5.15) is of order $n r_n^7 \times T_{1,1}(n)$, which is $O(n^2 r_n^7)$ by Lemma 4.4. The second term is not larger than

$$C_{20} n r_n^3 \sum_{\substack{1\le k+v+t\le n\\ 1\le j\le n\\ x\neq 0}} p_x^k(0,x) p_0^v(x,0) p_x^t(0,x) p^j(x,0) r_{n-j}^2$$

$$\le C_{20} n r_n^3 \sum_{\substack{1\le k+v+t\le n\\ 1\le j\le n\\ x\neq 0}} p^k(0,x) p^v(x,0) p^t(0,x) p^j(x,0) r_{n-j}^2 r_{n-k-v-t}^3.$$

By the same argument as we used in estimating the first term, we can replace r_{n-j} and $r_{n-k-v-t}$ with r_n and it is of order $n r_n^8 \times T_{2,2}(n) = O(n^2 r_n^8)$. The remaining term (5.14) can be estimated by the same argument and so we omit its calculation.

Hence we have that III $\le C_{21} n^2 r_n^7$ and we immediately conclude that $E\left|N_n(p,q)\right|^2$ is $O(n^2 r_n^7)$ for $p, q \ge 2$.

We will next estimate $E\left|N_n(1,p)\right|^2$ for $p \ge 1$; however, the method is the same as that used in obtaining a bound of $E\left|N_n(p,q)\right|^2$ for $p, q \ge 2$. Moreover, we have that $E\left|N_n(1,p)\right|^2 = E\left|N_n(p,1)\right|^2$ for $p \ge 1$ by considering the reversed random walks. Then the remainder of the proof is left to the reader.

REMARK. To calculate more carefully, we can sharpen the result of Lemma 3.1, that is,

$$E\left|N_n^{0,1}(p,q)\right|^2 = O(n^2 r_n^8)$$

for $p, q \ge 1$.

6. The proof of Lemma 3.2. The method used to prove Lemma 3.2 is the same as we used in the proof of Lemma 3.1. However the calculation is somewhat more complicated. We can obtain (3.4) by considering the reversed random walks in (3.3), and so it is sufficient to prove (3.3).

For simplicity, we put $L_n(p) = L_n^{0,1}(p) = \left|S(0,n) \cap S^p(n,2n)\right|$. Recall the indicators Z_i^j, Y_j^i and $W_i^j(p)$ defined in Section 5. The random variable $L_n(p)$ can be expressed by summations of these indicator random variables, and then we have that for $p \ge 2$,

$$L_n(1) = \sum_{j=n+1}^{2n} \sum_{i=1}^{n} Z_i^n Y_j^n Z_j^{2n} \chi(S_i = S_j),$$

$$L_n(p) = \sum_{j=n+1}^{2n} \sum_{i=1}^{n} \sum_{l=j+1}^{2n} Z_i^n Y_j^n W_j^l(p-1) Z_l^{2n} \chi(S_i = S_j).$$

The idea of the calculation of (3.3) for $p \ge 2$ is quite similar to that for $p = 1$. So we will prove that only

$$E\left|L_n(1) - L_n(2)\right|^2 = O(n^2 r_n^7).$$

For $1 \leq i \leq n < j \leq 2n$, let

$$\Sigma_{i,j}^n(1) = Z_i^n Y_j^n Z_j^{2n} \chi(S_i = S_j),$$

$$\Sigma_{i,j}^n(2) = \sum_{l=j+1}^{2n} Z_i^n Y_j^n W_j^l(1) Z_l^{2n} \chi(S_i = S_j),$$

and then we have that

$$L_n(1) = \sum_{j=n+1}^{2n} \sum_{i=1}^{n} \Sigma_{i,j}^n(1), \qquad L_n(2) = \sum_{j=n+1}^{2n} \sum_{i=1}^{n} \Sigma_{i,j}^n(2).$$

Note that $\Sigma_{i,j}^n(1)$ and $\Sigma_{i,j}^n(2)$ are also indicator random variables. Then we have that

$$\begin{aligned} E\left|L_n(1) - L_n(2)\right|^2 &\\ &= \sum_{j=n+1}^{2n} \sum_{i=1}^{n} E\left\{\Sigma_{i,j}^n(1) - \Sigma_{i,j}^n(2)\right\}^2 \\ &\quad + 2 \sum_{\substack{n<h<j\leq 2n \\ 1\leq k<i\leq n}} E\left\{\Sigma_{i,j}^n(1) - \Sigma_{i,j}^n(2)\right\}\left\{\Sigma_{k,h}^n(1) - \Sigma_{k,h}^n(2)\right\} \\ &\quad + 2 \sum_{\substack{n<h<j\leq 2n \\ 1\leq i<k\leq n}} E\left\{\Sigma_{i,j}^n(1) - \Sigma_{i,j}^n(2)\right\}\left\{\Sigma_{k,h}^n(1) - \Sigma_{k,h}^n(2)\right\} \\ &=: \mathrm{I} + 2\mathrm{II} + 2\mathrm{III}. \end{aligned}$$

The term I can be estimated easily. Indeed, for $1 \leq i \leq n < j \leq 2n$,

$$E\left\{\Sigma_{i,j}^n(1)\right\}^2 = E\Sigma_{i,j}^n(1) = f_{j-i} r_{2n-j} \leq f_{j-i},$$

$$E\left\{\Sigma_{i,j}^n(2)\right\}^2 = E\Sigma_{i,j}^n(2) = \sum_{l=j+1}^{2n} f_{j-i} f_{l-j} r_{2n-l} \leq f_{j-i}.$$

Thus we obtain that $\mathrm{I} \leq 2n$.

We next calculate II by estimating

$$\mathrm{II}(1) := \sum_{\substack{n<h<j\leq 2n \\ 1\leq k<i\leq n}} E\left[\Sigma_{i,j}^n(1)\Sigma_{k,h}^n(1) - \Sigma_{i,j}^n(1)\Sigma_{k,h}^n(2)\right]$$

$$\mathrm{II}(2) := \sum_{\substack{n<h<j\leq 2n \\ 1\leq k<i\leq n}} E\left[\Sigma_{i,j}^n(2)\Sigma_{i,j}^n(2) - \Sigma_{i,j}^n(2)\Sigma_{k,h}^n(1)\right]$$

separately. Note that we have no need for lower bounds of II(1) and II(2) since $E\left|L_n(1) - L_n(2)\right|^2$ is nonnegative.

From now on, we give an upper bound of II(1). For $1 \le k < i < h < j \le 2n$,

$$E\left[\Sigma_{i,j}^n(1)\Sigma_{k,h}^n(1)\right] = \sum_{x\neq 0} p_0^{i-k}(0,x)p_{0x}^{h-i}(x,0)p_{0x}^{j-h}(0,x) \times P_x(\tau_0 > 2n-j,\ \tau_x > 2n-j), \tag{6.1}$$

$$\begin{aligned} E\left[\Sigma_{i,j}^n(1)\Sigma_{k,h}^n(2)\right] &= \sum_{\substack{h<l<j\\ x\neq 0}} p_0^{i-k}(0,x)p_{0x}^{h-i}(x,0)p_{0x}^{l-h}(0,0)p_{0x}^{j-l}(0,x) \\ &\qquad \times P_x(\tau_0 > 2n-j,\ \tau_x > 2n-j) \\ &\quad + \sum_{\substack{j<l\le 2n\\ x\neq 0}} p_0^{i-k}(0,x)p_{0x}^{h-i}(x,0)p_{0x}^{j-h}(0,x)p_{0x}^{l-j}(x,0) \\ &\qquad \times P_0(\tau_x > 2n-l,\ \tau_0 > 2n-l). \end{aligned} \tag{6.2}$$

Summing (6.1) over all indices, we have that

$$\begin{aligned} \sum_{\substack{n<h<j\le 2n\\ 1\le k<i\le n}} E\left[\Sigma_{i,j}^n(1)\Sigma_{k,h}^n(1)\right] &= \sum_{\substack{n<h\le 2n\\ 1\le k<i\le n\\ x\neq 0}} \sum_{j=1}^{2n-h} p_0^{i-k}(0,x)p_{0x}^{h-i}(x,0)p_{0x}^{j}(0,x) \\ &\qquad \times P_x(\tau_0 > 2n-j-h,\ \tau_x > 2n-j-h). \end{aligned} \tag{6.3}$$

By neglecting the second part of (6.2), we obtain that

$$\begin{aligned} \sum_{\substack{n<h<j\le 2n\\ 1\le k<i\le n}} E\left[\Sigma_{i,j}^n(1)\Sigma_{k,h}^n(2)\right] &\ge \sum_{\substack{n<h<l<j\le 2n\\ 1\le k<i\le n\\ x\neq 0}} p_0^{i-k}(0,x)p_{0x}^{h-i}(x,0)p_{0x}^{l-h}(0,0) \\ &\qquad \times p_{0x}^{j-l}(0,x)P_x(\tau_0 > 2n-j,\ \tau_x > 2n-j). \end{aligned} \tag{6.4}$$

The right-hand side of (6.4) is equal to

$$\begin{aligned} \sum_{\substack{n<h\le 2n\\ 1\le k<i\le n\\ x\neq 0}} \sum_{1\le j+l\le 2n-h} & p_0^{i-k}(0,x)p_{0x}^{h-i}(x,0)p_{0x}^{l}(0,0)p_{0x}^{j}(0,x) \\ &\times P_x(\tau_0 > 2n-j-l-h,\ \tau_x > 2n-j-l-h). \end{aligned}$$

Using the inequality that $P_z(\tau_w > a-b,\ \tau_z > a-b) \geq P_z(\tau_w > a,\ \tau_z > a)$ for any $a > b \geq 1$ and $z, w \in \mathbb{Z}^d$, we have that

$$\sum_{l=1}^{N} p_{xy}^{l}(x,x) P_z(\tau_w > N-l,\ \tau_z > N-l) \geq P_x(\tau_x \leq N,\ \tau_x < \tau_y) P_z(\tau_w > N,\ \tau_z > N) \tag{6.5}$$

for any $N \geq 1$ and $x, y, z, w \in \mathbb{Z}^d$ with $x \neq y$. Then the left-hand side of (6.4) is not less than

$$\begin{aligned}\sum_{\substack{n<h\leq 2n\\ 1\leq k<i\leq n\\ x\neq 0}} \sum_{1\leq j\leq 2n-h} & p_0^{i-k}(0,x) p_{0x}^{h-i}(x,0) p_{0x}^{j}(0,x)\\ & \times P_0(\tau_0 \leq 2n-h-j,\ \tau_0 < \tau_x)\\ & \times P_x(\tau_0 > 2n-j-h,\ \tau_x > 2n-j-h).\end{aligned} \tag{6.6}$$

Note that, for $a \geq 1$ and $x \neq y$,

$$1 - P_x(\tau_x \leq a,\ \tau_x < \tau_y) = r_a + P_x(\tau_y < \tau_x \leq a). \tag{6.7}$$

Combining (6.3) and (6.6), we have that

$$\begin{aligned}\mathrm{II}(1) \leq \sum_{\substack{n<h\leq 2n\\ 1\leq k<i\leq n\\ x\neq 0}} \sum_{j=1}^{2n-h} & p_0^{i-k}(0,x) p_{0x}^{h-i}(x,0) p_{0x}^{j}(0,x)\\ & \times \left\{ r_{2n-h-j} + P_0(\tau_x < \tau_0 \leq 2n-h-j)\right\} r_{2n-h-j}.\end{aligned}$$

In calculating this summation, we need not consider the contribution that h is larger than n and that i is not larger than n. In this case, the fact that h cannot become close to i if i is apart from n has a small effect in this summation, and so we can extend the range of the summation over k, i, and h to $\{1 \leq k < i < h \leq 2n\}$. Then we have that

$$\mathrm{II}(2) \leq T^{2,0}_{1,1,1,0,0,0,0}(2n) + T^{1,0}_{1,1,2,1,0,0,0}(2n) = O(n^2 r_n^7).$$

Here Lemma 4.1 and (2.1) have been applied and next Lemma 4.5 has been used.

We now show that II(2) is dominated by a constant multiple of $n^2 r_n^7$, which leads us to a bound for the term II of the form $C_{22} n^2 r_n^7$. For $1 \leq k < i < h < j \leq 2n$,

$$\begin{aligned} & E\left[\Sigma^n_{i,j}(2)\Sigma^n_{k,h}(1)\right]\\ & \quad = \sum_{\substack{j<l\leq 2n\\ x\neq 0}} p_0^{i-k}(0,x) p_{0x}^{h-i}(x,0) p_{0x}^{j-h}(0,x) p_{0x}^{l-j}(x,x)\\ & \qquad \times P_x(\tau_0 > 2n-l,\ \tau_x > 2n-l),\end{aligned} \tag{6.8}$$

$$
\begin{aligned}
&E\left[\Sigma^n_{i,j}(2)\Sigma^n_{k,h}(2)\right]\\
&\quad= \sum_{\substack{j<l\le 2n\\ h<m<j\\ x\neq 0}} p_0^{i-k}(0,x)p_{0x}^{h-i}(x,0)p_{0x}^{m-j}(0,0)p_{0x}^{j-m}(0,x)\\
&\qquad\qquad \times p_{0x}^{l-j}(x,x)P_x(\tau_0>2n-l,\ \tau_x>2n-l)\\
&\qquad+\sum_{\substack{j<l\le 2n\\ j<m<l\\ x\neq 0}} p_0^{i-k}(0,x)p_{0x}^{h-i}(x,0)p_{0x}^{j-h}(0,x)p_{0x}^{m-j}(x,0)\\
&\qquad\qquad \times p_{0x}^{l-m}(0,x)P_x(\tau_0>2n-l,\ \tau_x>2n-l)\\
&\qquad+\sum_{\substack{j<l\le 2n\\ l<m\le 2n\\ x\neq 0}} p_0^{i-k}(0,x)p_{0x}^{h-i}(x,0)p_{0x}^{j-h}(0,x)p_{0x}^{l-j}(x,x)\\
&\qquad\qquad \times p_{0x}^{m-l}(x,0)P_0(\tau_x>2n-m,\ \tau_0>2n-m).
\end{aligned}
\tag{6.9}
$$

Note that for $a>b\ge 1$ and $x\neq y$,

$$
\begin{aligned}
&P_x(\tau_y>a-b,\ \tau_x>a-b)\\
&\qquad\le P_x(\tau_y>a,\ \tau_x>a)+(r_{a-b}-r_a)+P_x(\tau_y<a<\tau_x).
\end{aligned}
\tag{6.10}
$$

Hence, employing (6.10) and $p_{0x}^{m-j}(0,0)\le f_{m-j}$, the sum of the first part of (6.9) over k, i, h and j is not larger than

$$
\begin{aligned}
&\sum_{\substack{n<h\le 2n\\ 1\le k<i\le n\\ x\neq 0}}\ \sum_{1\le m+j+l\le 2n-h} p_0^{i-k}(0,x)p_{0x}^{h-i}(x,0)f_m p_{0x}^{j}(0,x)p_{0x}^{l}(x,x)\\
&\qquad\qquad \times P_x(\tau_0>2n-l-j-h,\ \tau_x>2n-l-j-h)
\end{aligned}
\tag{6.11}
$$

$$
\begin{aligned}
&+\sum_{\substack{n<h\le 2n\\ 1\le k<i\le n\\ x\neq 0}}\ \sum_{1\le m+j+l\le 2n-h} p_0^{i-k}(0,x)p_{0x}^{h-i}(x,0)f_m p_{0x}^{j}(0,x)\\
&\qquad\qquad \times f_l(r_{2n-l-j-m-h}-r_{2n-l-j-h})
\end{aligned}
\tag{6.12}
$$

$$
\begin{aligned}
&+\sum_{\substack{n<h\le 2n\\ 1\le k<i\le n\\ x\neq 0}}\ \sum_{1\le m+j+l\le 2n-h} p_0^{i-k}(0,x)p_{0x}^{h-i}(x,0)f_m p_{0x}^{j}(0,x)\\
&\qquad\qquad \times f_l P_x(\tau_0<2n-l-j-h<\tau_x).
\end{aligned}
\tag{6.13}
$$

Taking the summation over m in (6.12), we have that, by Lemma 2.4, a bound of (6.12) is

$$
C_{23}\sum_{\substack{1\le k<i<h\le 2n\\ x\neq 0}}\ \sum_{1\le l+j\le 2n-h} p_0^{i-k}(0,x)p_{0x}^{h-i}(x,0)p_{0x}^{j}(0,x)f_l r^4_{2n-j-l-h},
$$

which turns out of order $T^{4,0}_{1,1,1,0,0,0,1}(2n)$ by (2.1). Then we have that (6.12) is $O(n^2r_n^9)$ by Lemma 4.5. We next estimate (6.13). We first sum the summands in (6.13) over m, dominate its summation by 1 and next apply (2.1) and Lemma 4.1. Then we obtain that (6.13) is bounded by

$$\begin{aligned}&\sum_{\substack{1\le k+i+h+l\le 2n\\ x\neq 0}} p_x^i(0,x)p_{0x}^h(x,0)p_{0x}^j(0,x)\\ &\qquad\qquad \times f_j P_x(\tau_0<2n-l-j-h-i-k<\tau_x)\\ &\qquad \le \sum_{\substack{1\le k+i+h+j\\ +l+s\le 2n\\ x\neq 0}} p_x^i(0,x)p_{0x}^h(x,0)p_{0x}^j(0,x)p_x^s(x,0)f_l r_{2n-l-j-h-i-k-s}\\ &\qquad = T^{1,0}_{1,1,1,1,0,0,1}(2n).\end{aligned}$$

Hence a bound of (6.13) is $O(n^2r_n^7)$. On the other hand, summing (6.8) over all indices, we have that

$$\begin{aligned}&\sum_{\substack{n<h<j\le 2n\\ 1\le k<i\le n}} E\left[\Sigma^n_{i,j}(2)\Sigma^n_{k,h}(1)\right]\\ &\qquad = \sum_{\substack{n<h\le 2n\\ 1\le k<i\le n\\ x\neq 0}}\ \sum_{1\le l+j\le 2n-h} p_0^{i-k}(0,x)p_{0x}^{h-i}(x,0)p_{0x}^j(0,x)p_{0x}^l(x,x)\\ &\qquad\qquad\qquad \times P_x(\tau_0>2n-l-j-h,\ \tau_x>2n-l-j-h).\end{aligned}\tag{6.14}$$

By taking the summation on m in (6.11), it is easy to obtain that (6.11) is dominated by (6.14). In other words, (6.11) minus (6.14) is not larger than zero. Our purpose is only to give an upper bound of II(2), and so we can neglect the contribution for the nonpositive part. Then we obtain that the sum of the first part of (6.9) minus (6.8) cannot exceed a constant multiple of $n^2r_n^7$. It remains to estimate the summations of the second and the third parts of (6.9). If we succeed in obtaining both summations are of order $n^2r_n^7$, we can conclude that an upper bound of II(2) is a constant multiple of $n^2r_n^7$. Dominating $P_x(\tau_0>2n-l,\ \tau_x>2n-l)$ by r_{2n-l}, we have that a bound of the sum of the second part of (6.9) over k, i, h and j is

$$\begin{aligned}&\sum_{\substack{1\le k+i+h+j\\ +m+l\le 2n\\ x\neq 0}} p_0^i(0,x)p_{0x}^h(x,0)p_{0x}^j(0,x)p_{0x}^m(x,0)p_{0x}^l(0,x)r_{2n-l-m-j-h-i-k}\\ &\qquad = T^{1,0}_{2,2,1,0,0,0,0}(2n)=O(n^2r_n^{10}).\end{aligned}$$

Here (2.1) and Lemma 4.5 have been applied. The summation of the third part of (6.9) is not larger than

$$\sum_{\substack{1\le k+i+h+j\\ +l+m\le 2n\\ x\neq 0}} p_0^i(0,x)p_{0x}^h(x,0)p_{0x}^j(0,x)f_l\,p_{0x}^m(x,0)r_{2n-m-l-j-h-i-k}$$
$$= T^{1,0}_{1,2,1,0,0,0,1}(2n),$$

which is of order $n^2r_n^8$. Therefore we can conclude that II $\le C_{24}n^2r_n^7$.

The remainder of this section is devoted to the calculation of an upper bound of term III. We aim to estimate the two summations

$$\sum_{\substack{n<h<j\le 2n\\ 1\le i<k\le n}} E\left[\Sigma^n_{i,j}(1)\Sigma^n_{k,h}(1)-\Sigma^n_{i,j}(1)\Sigma^n_{k,h}(2)\right], \tag{6.15}$$

$$\sum_{\substack{n<h<j\le 2n\\ 1\le i<k\le n}} E\left[\Sigma^n_{i,j}(2)\Sigma^n_{i,j}(2)-\Sigma^n_{i,j}(2)\Sigma^n_{k,h}(1)\right]. \tag{6.16}$$

Applying the same observation as II(1), it can be easily obtained that (6.15) is bounded by

$$\sum_{\substack{n<h<j\le 2n\\ 1\le i<k\le n\\ x\neq 0}} p_0^{k-i}(0,x)p_{0x}^{h-k}(x,x)p_{0x}^{j-h}(x,0)P_0(\tau_x>2n-j,\ \tau_0>2n-j)$$
$$-\sum_{\substack{n<h<j\le 2n\\ 1\le i<k\le n\\ x\neq 0}}\ \sum_{l=h+1}^{j-1} p_0^{k-i}(0,x)p_{0x}^{h-k}(x,x)p_{0x}^{l-h}(x,x)p_{0x}^{j-l}(x,0)$$
$$\times P_0(\tau_x>2n-j,\ \tau_0>2n-j)$$
$$=: \mathrm{III}(1)-\mathrm{III}(2).$$

We employ the same method used in estimating the left-hand side of (6.4), and can obtain that

$$\mathrm{III}(2)=\sum_{\substack{n<h\le 2n\\ 1\le k<i\le n\\ x\neq 0}}\ \sum_{1\le l+j\le 2n-h} p_0^{k-i}(0,x)p_{0x}^{h-k}(x,x)p_{0x}^{l}(x,x)p_{0x}^{j}(x,0)$$
$$\times P_0(\tau_x>2n-j-l-h,\ \tau_0>2n-j-l-h)$$
$$\ge \sum_{\substack{n<h\le 2n\\ 1\le i<k\le n\\ x\neq 0}}\ \sum_{j=1}^{2n-h} p_0^{k-i}(0,x)p_{0x}^{h-k}(x,x)p_{0x}^{j}(x,0)$$
$$\times P_x(\tau_x\le 2n-j-h,\ \tau_x<\tau_0)$$
$$\times P_0(\tau_x>2n-j-h,\ \tau_0>2n-j-h),$$

where (6.5) has been applied. By (6.7), we have that III(1) – III(2) is not larger than

$$\sum_{\substack{n<h\le 2n\\ 1\le i<k\le n\\ x\neq 0}} \sum_{j=1}^{2n-h} p_0^{k-i}(0,x) f_{h-k} p_{0x}^{j}(x,0) r_{2n-j-h}^2 \tag{6.17}$$

$$+ \sum_{\substack{n<h\le 2n\\ 1\le i<k\le n\\ x\neq 0}} \sum_{j=1}^{2n-h} p_0^{k-i}(0,x) f_{h-k} p_{0x}^{j}(x,0) r_{2n-j-h} \times P_x(\tau_0 < \tau_x \le 2n-j-h). \tag{6.18}$$

We make the substitutions that $h-n=m$ and $k-i=u$ in the summations on h and i in (6.17), respectively, and then have that (6.17) is equal to

$$\sum_{\substack{1\le m\le n\\ 1\le u<k\le n\\ x\neq 0}} \sum_{j=1}^{n-m} p_0^{u}(0,x) f_{n+m-k} p_{0x}^{j}(x,0) r_{n-j-m}^2.$$

By Lemmas 4.1 and 4.3, this summation is bounded by

$$\sum_{\substack{1\le m+j\le n\\ 1\le u<k\le n\\ x\neq 0}} p_0^{u}(0,x) f_{n+m-k} p_0^{j}(x,0) r_{n-j-m}^3 \tag{6.19}$$

$$+ \sum_{\substack{1\le m+j+v+w\le n\\ 1\le u<k\le n\\ x\neq 0}} p_0^{u}(0,x) f_{n+m-k} p_0^{j}(x,0) p_x^{v}(0,x) p_0^{w}(x,0) r_{n-j-m}^2. \tag{6.20}$$

The term (6.19) is equal to

$$\sum_{\substack{1\le m+j\le n\\ 1\le u<k\le n}} f_{u+j} f_{n+m-k} r_{n-j-m}^3. \tag{6.21}$$

Note that, by (2.3), it can be obtained that for $1\le a<b$,

$$\sum_{j=a+1}^{b} f_j \le C_{25} \frac{\log(a/b)}{(\log a)(\log b)}.$$

Take the summation on u and k over $[1,n]$ in (6.21). Then we have that (6.21) and also (6.19) are not larger than a constant multiple of

$$\sum_{1\le m+j\le n} \frac{\log((n+j)/j)\log((n+m)/m)}{\{\log(n-j-m)\}^3 \log j \log(n+j) \log m \log(n+m)}. \tag{6.22}$$

The contribution for $n-j-m\le nr_n^8$ in (6.22) is $O(n^2 r_n^8)$. Thus we can consider only the summation on $1\le m+j\le n-nr_n^8$ and can replace $\log(n-j-m)$ with $\log n$ in this case. Apply the same method to $\log j$ and $\log m$ in (6.22) by observing the contributions for $j \le nr_n^5$ and $m \le nr_n^5$, respectively. Hence a bound of (6.22) is

$$\begin{aligned} & C_{26}\frac{1}{(\log n)^7}\sum_{m=1}^{n}\sum_{j=1}^{n-m}\log\frac{n+j}{j}\log\frac{n+m}{m} \\ & \quad \sim C_{26}\frac{n^2}{(\log n)^7}\int_0^1 dx\int_0^{1-x}\log\frac{1+y}{y}\log\frac{1+x}{x}\,dy \\ & \quad \le C_{27}\frac{n^2}{(\log n)^7}, \end{aligned}$$

which implies that a bound of (6.19) is $O(n^2 r_n^7)$. We shall show that the term (6.20) is of order $n^2 r_n^7$. Since r_n is nonincreasing, we can bound r_{n-j-m} by $r_{n-j-m-v-w}$. We first sum on k and use an analogy of (5.8) and next apply Lemma 4.2 four times. Then the term (6.20) is dominated by

$$\sum_{\substack{1\le m+j+v+w\le n \\ 1\le u\le n \\ x\ne 0}} p^u(0,x)p^j(x,0)p^v(0,x)p^w(x,0)r^5_{n-j-m-v-w}r_{n-u}r_m. \tag{6.23}$$

Since the contribution for $n-u\le nr_n^5$ is $O(n^2r_n^8)$, we can regard r_{n-u} as a constant multiple of r_n. Applying (4.1) and Lemma 2.2 to the summation on m, a bound of (6.23) is

$$C_{28}nr_n^7\sum_{\substack{1\le j+v+w\le n \\ 1\le u\le n \\ x\ne 0}} p^u(0,x)p^j(x,0)p^v(0,x)p^w(x,0)\le C_{28}nr_n^7\times T_{2,2}(n).$$

Then (6.20), and also (6.17) are of order $n^2r_n^7$ by Lemma 4.4. The calculation of (6.18) is easier than that of (6.17). We must estimate (6.17) by noting the fact that h cannot be close to k when s is away from n. However, it has no effect to estimate (6.18). Indeed, making the substitution $k-i=u$ in the summation on i and dominating the summation on k by 1, we have that (6.18) is bounded by

$$\sum_{\substack{1\le u<h\le 2n \\ x\ne 0}}\sum_{j=1}^{2n-h} p_0^u(0,x)p_{0x}^j(x,0)r_{2n-j-h}P_x(\tau_0<\tau_x\le 2n-j-h),$$

which is equal to

$$\begin{aligned} & \sum_{\substack{1\le u+h+j \\ +v+w\le 2n \\ x\ne 0}} p_0^u(0,x)p_{0x}^j(x,0)r_{2n-j-h-u}p_x^v(0,x)p_0^w(x,0)r_{2n-j-h-u-v-w} \\ & \quad + \sum_{\substack{1\le u+h+j \\ +v+w\le 2n \\ x\ne 0}} p_0^u(0,x)p_{0x}^j(x,0)r_{2n-j-h-u} \\ & \qquad\qquad \times p_x^v(0,x)p_0^w(x,0)P_x(\tau_0<\tau_x\le 2n-j-h-u-v-w), \end{aligned}$$

where Lemma 4.1 has been applied. Noting the monotonicity of r_n, the first summation is not larger than $T^{0,2}_{0,1,2,1,0,0,0}(2n)$ and the second one is bounded by $T^{0,1}_{0,1,3,2,0,0,0}(2n)$ by applying Lemma 4.1 again. Then (6.18) is of order $n^2 r_n^7$ by Lemma 4.5. Therefore we conclude that III(1) − III(2) is not larger than $C_{29} n^2 r_n^7$.

We next calculate (6.16) and obtain that

$$\begin{aligned}
&\sum_{\substack{n<h<j\le 2n\\ 1\le i<k\le n}} E\left[\Sigma^n_{i,j}(2)\Sigma^n_{k,h}(1)\right]\\
&\quad= \sum_{\substack{n<h<j\le 2n\\ 1\le i<k\le n\\ x\neq 0}} \sum_{l=j+1}^{2n} p_0^{k-i}(0,x)p_{0x}^{h-k}(x,x)p_{0x}^{j-h}(x,0)p_{0x}^{l-j}(0,0)\\
&\qquad\times P_0(\tau_x>2n-l,\ \tau_0>2n-l).
\end{aligned}$$

It is clear that this summation is equal to

$$\begin{aligned}
&\sum_{\substack{n<h\le 2n\\ 1\le i<k\le n\\ x\neq 0}} \sum_{1\le l+j\le 2n-h} p_0^{k-i}(0,x)p_{0x}^{h-k}(x,x)p_{0x}^{j}(x,0)p_{0x}^{l}(0,0)\\
&\qquad\times P_0(\tau_x>2n-l-j-h,\ \tau_0>2n-l-j-h),
\end{aligned}$$

which is denoted by III(3). Moreover, we have that

$$\begin{aligned}
&\sum_{\substack{n<h<j\le 2n\\ 1\le i<k\le n}} E\left[\Sigma^n_{i,j}(2)\Sigma^n_{i,j}(2)\right]\\
&\quad= \sum_{\substack{n<h<j<l\le 2n\\ 1\le i<k\le n\\ x\neq 0}} \sum_{m=h+1}^{j-1} p_0^{k-i}(0,x)p_{0x}^{h-k}(x,x)p_{0x}^{m-h}(x,x)p_{0x}^{j-m}(x,0)\\
&\qquad\times p_{0x}^{l-j}(0,0)P_0(\tau_x>2n-l,\ \tau_0>2n-l)\\
&\quad+ \sum_{\substack{n<h<j<l\le 2n\\ 1\le i<k\le n\\ x\neq 0}} \sum_{m=j+1}^{l-1} p_0^{k-i}(0,x)p_{0x}^{h-k}(x,x)p_{0x}^{j-h}(x,0)p_{0x}^{m-j}(0,x)\\
&\qquad\times p_{0x}^{l-m}(x,0)P_0(\tau_x>2n-l,\ \tau_0>2n-l)\\
&\quad+ \sum_{\substack{n<h<j<l\le 2n\\ 1\le i<k\le n\\ x\neq 0}} \sum_{m=l+1}^{2n} p_0^{k-i}(0,x)p_{0x}^{h-k}(x,x)p_{0x}^{j-h}(x,0)p_{0x}^{l-j}(0,0)\\
&\qquad\times p_{0x}^{m-l}(0,x)P_x(\tau_0>2n-m,\ \tau_x>2n-m)\\
&\quad=: \mathrm{III}(4)+\mathrm{III}(5)+\mathrm{III}(6).
\end{aligned}$$

Hence (6.16) is equal to $-\mathrm{III}(3)+\mathrm{III}(4)+\mathrm{III}(5)+\mathrm{III}(6)$. Neglecting the event that $\{\tau_0 > 2n-l\}$ and using the inequality $p_{0x}^{h-k}(x,x) \le f_{h-k}$, we have that

$$\begin{aligned}\mathrm{III}(5) &\le \sum_{\substack{1\le i<k<h<j\\ <m<l\le 2n\\ x\neq 0}} p_0^{k-i}(0,x) f_{h-k} p_{0x}^{j-h}(x,0) p_{0x}^{m-j}(0,x) p_{0x}^{l-m}(x,0) r_{2n-l}\\ &= T^{1,0}_{1,2,1,0,0,0,1}(2n) = O(n^2 r_n^8).\end{aligned}$$

Then the term III(5) has little effect. Moreover, the term III(6) is dominated by

$$\sum_{\substack{n<h<j<l<m\le 2n\\ 1\le i<k\le n\\ x\neq 0}} p_0^{k-i}(0,x) f_{h-k} p_{0x}^{j-h}(x,0) f_{l-j} p_{0x}^{m-l}(0,x) r_{2n-m}. \tag{6.24}$$

To estimate (6.24), we need to adopt the same observation as we have used in estimating (5.9). Namely, we must derive the effect of the fact that h is not able to be close to i if i is away from n. We first observe the contribution for $2n-m \le 2nr_n^8$ in (6.24), and then need to calculate a bound of

$$\sum_{m=2n-2nr_n^8}^{2n} \sum_{\substack{1\le i<k<h<j<l<m\\ x\neq 0}} p_0^{k-i}(0,x) f_{h-k} p_{0x}^{j-h}(x,0) f_{l-j} p_{0x}^{m-l}(0,x) r_{2n-m}. \tag{6.25}$$

We dominate r_{2n-m} by 1, apply (2.1) to $p_0^{k-i}(0,x)$, take the summation over i and then have that (6.25) is bounded by

$$\sum_{m=2n-2nr_n^8}^{2n} \sum_{\substack{1\le k<h<j<l<m\\ x\neq 0}} P_0(\tau_x \le k) f_{h-k} p_0^{j-h}(x,0) f_{l-j} p_0^{m-l}(0,x).$$

Moreover, dominating $P_0(\tau_x \le k)$ by 1 and next summing over k and x, this summation is not larger than

$$\sum_{m=2n-2nr_n^8}^{2n} \sum_{1\le h<j<l<m} f_{j-h+m-l} f_{l-j}.$$

It is easy to obtain that this summation and also (6.25) are not larger than $4n^2 r_n^8$. Therefore we can concentrate on the case $2n-m > 2nr_n^8$ and then can dominate r_{2n-m} by a constant multiple of r_n under this situation. Consequently, we may estimate only

$$r_n \sum_{\substack{n<h<j<l<m\le 2n\\ 1\le i<k\le n\\ x\neq 0}} p_0^{k-i}(0,x) f_{h-k} p_{0x}^{j-h}(x,0) f_{l-j} p_{0x}^{m-l}(0,x) \tag{6.26}$$

instead of (6.24). We first calculate

$$\sum_{n<h<j<l<m\le 2n} f_{h-k} p_{0x}^{j-h}(x,0) f_{l-j} p_{0x}^{m-l}(0,x), \tag{6.27}$$

which is equal to

$$\sum_{1\le h+j+l+m\le n} f_{n+h-k}\, p_{0x}^{j}(x,0) f_l\, p_{0x}^{m}(0,x).$$

Dominating the summation on l by 1 and next summing over h, then (6.27) is bounded by

$$\sum_{1\le j+m\le n} p_{0x}^{j}(x,0) p_{0x}^{m}(0,x) r_{n-k}.$$

Therefore (6.26) is not larger than

$$\begin{aligned} &r_n \sum_{\substack{1\le j+m\le n\\ 1\le i+k\le n\\ x\neq 0}} p_0^k(0,x) p_{0x}^{j}(x,0) p_{0x}^{m}(0,x) r_{n-k-i}\\ &\quad\le r_n \sum_{\substack{1\le j+m\le n\\ 1\le i+k\le n\\ x\neq 0}} p^k(0,x) p_{0x}^{j}(x,0) p_{0x}^{m}(0,x) r_{n-k-i}^2. \end{aligned} \tag{6.28}$$

The fundamental calculations show that

$$\begin{aligned} &\sum_{1\le j+m\le n} p_{0x}^{j}(x,0) p_{0x}^{m}(0,x)\\ &\quad\le \sum_{1\le j+m\le n} p^j(x,0) p^m(0,x) r_{n-m-j}^4\\ &\qquad + 2 \sum_{1\le j+m+u+v\le n} p^j(x,0) p^m(0,x) p^u(0,x) p^v(x,0) r_{n-m-j-u-v}^5\\ &\qquad + \sum_{\substack{1\le j+m+u\\ +v+w+s\le n}} p^j(x,0) p^m(0,x) p^u(0,x) p^v(x,0)\\ &\qquad\qquad \times p^w(0,x) p^s(x,0) r_{n-m-j-u-v-w-s}^6, \end{aligned}$$

where Lemmas 4.1, 4.2 and 4.3 have been applied. Consequently, we can conclude that the right-hand side of (6.28) is of order $n^2 r_n^7$. Indeed, for example, we have that, by Lemma 2.2,

$$\begin{aligned} &r_n \sum_{\substack{1\le j+m\le n\\ 1\le i+k\le n\\ x\neq 0}} p^k(0,x) p^j(x,0) p^m(0,x) r_{n-k-i}^2 r_{n-m-j}^4\\ &\quad\le C_{30} n r_n^3 \sum_{\substack{1\le j+m\le n\\ 1\le k\le n\\ x\neq 0}} p^k(0,x) p^j(x,0) p^m(0,x) r_{n-m-j}^4, \end{aligned}$$

which is dominated by a constant multiple of

$$nr_n^7 \sum_{\substack{1\le j+m\le n\\ 1\le k\le n\\ x\ne 0}} p^k(0,x)p^j(x,0)p^m(0,x) \le nr_n^7 T_{2,1}(n) = O(n^2 r_n^7).$$

Here we have applied the fact that r_{n-m-j} can be replaced by r_n, obtained by observing the contribution for $n-m-j \le nr_n^8$. The remainder of the calculation of (6.26) is left to the reader. Then we have that III(6) is of order $n^2 r_n^7$.

It remains to calculate an upper bound of III(4) − III(3). If we succeed in proving that III(4)−III(3) $\le C_{31} n^2 r_n^7$, we have that III is not larger than $C_{32} n^2 r_n^7$ and can finish the proof of Lemma 3.2. By (6.10), the term III(4) is bounded by

$$\sum_{\substack{n<h\le 2n\\ 1\le i<k\le n\\ x\ne 0}} \sum_{1\le m+j+l\le 2n-h} p_0^{k-i}(0,x)p_{0x}^{h-k}(x,x)p_{0x}^m(x,x)p_{0x}^j(x,0)p_{0x}^l(0,0) \tag{6.29}$$
$$\times P_0(\tau_x > 2n-l-j-h,\ \tau_0 > 2n-l-j-h)$$

$$+ \sum_{\substack{n<h\le 2n\\ 1\le i<k\le n\\ x\ne 0}} \sum_{1\le m+j+l\le 2n-h} p_0^{k-i}(0,x) f_{h-k} f_m p_{0x}^j(x,0) f_l \tag{6.30}$$
$$\times (r_{2n-l-j-h-m} - r_{2n-l-j-h})$$

$$+ \sum_{\substack{n<h\le 2n\\ 1\le i<k\le n\\ x\ne 0}} \sum_{1\le m+j+l\le 2n-h} p_0^{k-i}(0,x) f_{h-k} f_m p_{0x}^j(x,0) f_l \tag{6.31}$$
$$\times P_0(\tau_x < 2n-l-j-h < \tau_0).$$

It is clear that (6.29) is not larger than III(3) by summing over m. In other words, the term (6.29) minus III(3) is nonpositive, and this difference can be neglected since we aim to obtain an upper bound of III. The method of estimating (6.30) is the same that we have used in (6.12). Indeed, applying Lemma 2.4 to the summation on m, a bound of (6.30) is

$$\sum_{\substack{n<h\le 2n\\ 1\le i<k\le n\\ x\ne 0}} \sum_{1\le l+j\le 2n-h} p_0^{k-i}(0,x) f_{h-k} p_{0x}^j(x,0) f_l r_{2n-l-j-h}^4,$$

which is of order $T^{4,0}_{0,1,1,0,0,0,2}(2n)$ by (2.1). Using Lemma 4.5, we obtain that (6.30) is $O(n^2 r_n^7)$. To end the proof, we may show that (6.31) is of order $n^2 r_n^7$. Dominating the summation on m by 1 and applying Lemma 4.1, a bound of

(6.31) is

$$\sum_{\substack{n<h\le 2n \\ 1\le i<k\le n \\ x\neq 0}} \sum_{1\le j+l\le 2n-h} p_0^{k-i}(0,x) f_{h-k} p_{0x}^{j}(x,0) f_l \sum_{u=1}^{2n-h-l-j} p_x^u(x,0) r_{2n-h-j-l-u}. \tag{6.32}$$

Make the substitution $v = k - i$ in the summation on i and $w = h - n$ in the summation on h. Then (6.32) is equal to

$$\sum_{\substack{1\le w\le n \\ 1\le v<k\le n \\ x\neq 0}} \sum_{1\le j+l+u\le n-w} p_0^{v}(0,x) f_{n+w-k} p_{0x}^{j}(x,0) f_l p_x^u(x,0) r_{n-w-j-l-u},$$

which is not larger than

$$\sum_{\substack{1\le w+j+l+u\le n \\ 1\le v<k\le n \\ x\neq 0}} p^{v}(0,x) f_{n+w-k} p_{0x}^{j}(x,0) f_l p^u(x,0) r_{n-w-j-l-u}^2 r_{k-v}$$

$$\le \sum_{\substack{1\le w+j+l+u\le n \\ 1\le v<k\le n \\ x\neq 0}} p^{v}(0,x) f_{n+w-k} p^{j}(x,0) \times f_l p^u(x,0) r_{n-w-j-l-u}^4 r_{k-v} \tag{6.33}$$

$$+ \sum_{\substack{1\le w+j+l+u\le n \\ 1\le v<k\le n \\ x\neq 0}} p^{v}(0,x) f_{n+w-k} p_{x}^{j}(x,0) f_l p^u(x,0) \times P_0(\tau_x<\tau_0\le n-w-j-l-u) \times r_{n-w-j-l-u}^2 r_{k-v}. \tag{6.34}$$

Here Lemmas 4.2 and 4.3 have been applied by noting (2.1). We first estimate (6.33). The contribution for $n-w-j-l-u \le nr_n^{10}$ is $O(n^2 r_n^8)$, and therefore we can replace $r_{n-w-j-l-u}$ with a constant multiple of r_n by Lemma 2.3. Then it is sufficient to calculate

$$r_n^4 \sum_{\substack{1\le w+j+u\le n \\ 1\le v<k\le n \\ x\neq 0}} p^{v}(0,x) p^{j}(0,x) p^u(x,0) f_{n+w-k} r_{k-v}. \tag{6.35}$$

Moreover, we investigate the contribution for $k-v\le nr_n^4$ in (6.35) and then we need to estimate

$$r_n^4 \sum_{v=1}^{n} \sum_{k=v}^{nr_n^4+v} \sum_{\substack{1\le w+j+u\le n \\ x\neq 0}} p^{v}(0,x) p^{j}(0,x) p^u(x,0) f_{n+w-k} r_{k-v}.$$

Dominating r_{k-v} and the summation on w by 1, we have that this summation is not larger than

$$nr_n^8 \sum_{\substack{1\le v,\, j,\, u\le n\\ x\ne 0}} p^v(0,x)p^j(0,x)p^u(x,0),$$

which is of order $n^2r_n^8$ by Lemma 4.4. Thus, in order to estimate (6.35) and also (6.33), we may give a bound of only the summation

$$r_n^5 \sum_{\substack{1\le w+j+u\le n\\ 1\le v<k\le n\\ x\ne 0}} p^v(0,x)p^j(0,x)p^u(x,0)f_{n+w-k},$$

which is dominated by

$$\begin{aligned} r_n^5 &\sum_{\substack{1\le v,\, j,\, u\le n\\ x\ne 0}} p^v(0,x)p^j(0,x)p^u(x,0) \sum_{1\le w,k\le n} f_{n+w-k} \\ &\le r_n^5 T_{2,1}(n) \sum_{1\le w,\, k\le n} f_{n+w-k}. \end{aligned}$$

Using (2.3),

$$\sum_{1\le w,\, k\le n} f_{n+w-k} \le C_{33} \sum_{2\le w\le n} \frac{\log((n+w)/w)}{\log w \log(n+w)} = O\left\{\frac{n}{(\log n)^2}\right\},$$

where we have investigated the contribution for $w \le n(\log n)^{-3}$ and it is possible to replace $\log w$ with a constant multiple of $\log n$. Hence, by Lemma 4.4, (6.33) is of order $n^2r_n^7$ since $r_n \sim c(\log n)^{-1}$. By Lemma 4.1 and the monotonicity of r_n, the summation (6.34) is bounded by

$$\begin{aligned} &\sum_{\substack{1\le w+j+l+u+s+t\le n\\ 1\le v<k\le n\\ x\ne 0}} p^v(0,x)f_{n+w-k}p_x^j(x,0)f_l \\ &\qquad \times p^u(x,0)p_0^s(x,0)p_x^t(0,x)r^2_{n-w-j-l-u-s-t}r_{k-v} \\ &\le \sum_{\substack{1\le w+j+l+u+s+t\le n\\ 1\le v<k\le n\\ x\ne 0}} p^v(0,x)f_{n+w-k}p^j(x,0)f_l \\ &\qquad \times p^u(x,0)p^s(x,0)p^t(0,x)r^5_{n-w-j-l-u-s-t}r_{k-v}. \end{aligned}$$

The method of estimating this summation is the same as that of (6.33). Noting that the influence of $n-w-j-l-u-s-t \le nr_n^{12}$ is $O(n^2r_n^8)$, it turns out that a constant multiple of r_n^5 can bound $r^5_{n-w-j-l-u-s-t}$. Moreover, the fact that the contribution for $k-v\le nr_n^3$ is $O(n^2r_n^8)$ assures the replacement of

r_{k-v} with a constant multiple of r_n. Then we obtain that (6.34) is not larger than

$$C_{34} r_n^6 \sum_{\substack{1 \le w+j+u+s+t \le n \\ 1 \le v < k \le n \\ x \ne 0}} p^v(0,x) f_{n+w-k} p^j(x,0) p^u(x,0) p^s(x,0) p^t(0,x)$$

$$\le C_{34} r_n^6 \sum_{\substack{1 \le j,\, u,\, s,\, t,\, v \le n \\ x \ne 0}} p^v(0,x) p^j(x,0) p^u(x,0) \times p^s(x,0) p^t(0,x) \sum_{1 \le w,\, k \le n} f_{n+w-k}$$

$$= O\{n r_n^8 \times T_{3,2}(n)\},$$

which is of order $n^2 r_n^8$ by Lemma 4.4. Hence $\mathrm{III}(4) - \mathrm{III}(3) \le C_{35} n^2 r_n^7$. Then we can conclude that $\mathrm{III} \le C_{36} n^2 r_n^7$. This completes the proof of Lemma 3.2.

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