The Annals of Probability
1997, Vol. 25, No. 1, 443–456

CENTRAL LIMIT THEOREM FOR LINEAR PROCESSES

BY MAGDA PELIGRAD[1] AND SERGEY UTEV

University of Cincinnati and Institute of Mathematics–Novosibirsk

In this paper we study the CLT for partial sums of a generalized linear process $X_n = \sum_{i=1}^n a_{ni}\xi_i$, where $\sup_n \sum_{i=1}^n a_{ni}^2 < \infty$, $\max_{1\le i\le n}|a_{ni}| \to 0$ as $n \to \infty$ and ξ_i's are in turn, pairwise mixing martingale differences, mixing sequences or associated sequences. The results are important in analyzing the asymptotical properties of some estimators as well as of linear processes.

1. Introduction. Let $\{\xi_k\}$ be a centered sequence of random variables and let $\{a_{ni},\ 1 \le i \le n\}$ be a triangular array of numbers. Many statistical procedures produce estimators of the type

$$S_n = \sum_{i=1}^n a_{ni}\xi_i. \tag{1.1}$$

To give an example let us consider the nonlinear regression model

$$y(x) = g(x) + \xi(x),$$

where $g(x)$ is an unknown function and $\xi(x)$ is the noise. Now we fix the design points x_{ni} and we get

$$y_{ni} = g(x_{ni}) + \xi(x_{ni}) = g(x_{ni}) + \xi_i,$$

where $\{\xi_i\}$ is a centered sequence of random variables. The nonparametric estimator of $g(x)$ is defined to be $\hat{g}_n(x) = \sum_{i=1}^n w_{ni}(x)y_{ni}$ where

$$w_{ni}(x) = K\left(\frac{x_{ni} - x}{h_n}\right)\bigg/\sum_{i=1}^n K\left(\frac{x_{ni} - x}{h_n}\right),$$

where K is a kernel function. It is obvious that $\hat{g}_n(x) - E\hat{g}_n(x)$ is of the type (1.1).

We shall see later on that the asymptotic behavior of the sums of variables of the form

$$X_k = \sum_{j=-\infty}^{\infty} a_{k+j}\xi_j, \quad \text{where } \sum_{j=-\infty}^{\infty} a_j^2 < \infty \tag{1.2}$$

can be obtained by the study of the sums of the form (1.1), and our results improve on some known results about CLT for sums of the form (1.2). Our paper is organized in the following way: Section 2 contains the definitions

Received April 1994; revised September 1996.
[1]Research supported in part by an NSF Grant and Taft Travel Grant.
AMS 1991 *subject classifications*. 60G09, 60F05.
Key words and phrases. Linear process, dependent random variables, central limit theorem.

and the results, Section 3 contains the proofs and the Appendix contains the statements of some known results used in the proofs.

2. Results. Our first theorem solves the problem described in the introduction for some sequences of martingale differences.

DEFINITION 2.1. We call $\{X_k\}$ a pairwise mixing sequence if for every x real,

$$\sup_k \left|\operatorname{cov}(I(X_k < x), I(X_{k+n} < x))\right| \to 0 \quad \text{as } n \to \infty.$$

THEOREM 2.1. *Let $\{\xi_k\}$ be a pairwise mixing martingale differences sequence of random variables, and let $\{a_{nk};\ 1 \le k \le n\}$ be a triangular array of real numbers such that*

$$\sup_n \sum_{k=1}^{n} a_{nk}^2 < \infty \quad \text{and} \quad \max_{1 \le k \le n} |a_{nk}| \to 0 \quad \text{as } n \to \infty. \tag{2.1}$$

Assume

$$\{\xi_k^2\} \text{ is an uniformly integrable family and } \operatorname{var}\left(\sum_{k=1}^{n} a_{nk}\xi_k\right) = 1. \tag{2.2}$$

Then

$$\sum_{k=1}^{n} a_{nk}\xi_k \to_D N(0,1) \quad \text{as } n \to \infty. \tag{2.3}$$

As a corollary of the above theorem we prove the following.

COROLLARY 2.1. *Let $\{\xi_j;\ j \in Z\}$ be a pairwise mixing martingale difference sequence of random variables which is uniformly integrable in L_2. Let $\{a_j;\ j \in Z\}$ be a sequence of real numbers such that $\sum_j a_j^2 < \infty$. Let $X_k = \sum_{j=-\infty}^{\infty} a_{k+j}\xi_j$ and $S_n = \sum_{i=1}^{n} X_i$. Assume $\sigma_n^2 = \operatorname{var}(S_n) \to \infty$ as $n \to \infty$ and $\inf_j \operatorname{var}(\xi_j) \equiv b > 0$. Then*

$$\frac{S_n}{\sigma_n} \to_D N(0,1) \quad \text{as } n \to \infty.$$

This result is an extension of Theorem 18.6.5 in Ibragimov and Linnik (1971) from i.i.d. to be the pairwise mixing martingale case.

It should be noted that the ergodicity cannot replace the condition of pairwise mixing in Theorem 2.1. We have the following example.

EXAMPLE 2.1. There is a sequence, $\{\xi_k\}$ of martingale differences which is strictly stationary and ergodic, having finite second moments and there are numbers $\{a_{nk},\ 1 \le k \le n\}$ satisfying (2.1) and (2.2) and such that $\sum_{k=1}^{n} a_{nk}\xi_k$ does not converge to a normal distribution.

We shall introduce now some measures of dependence between two σ-algebras.

DEFINITION 2.2. Let $\mathbf{A}$ and $\mathbf{B}$ be two σ-algebras of events and define

$$\varphi(\mathbf{A}, \mathbf{B}) = \sup_{A \in \mathbf{A},\, B \in \mathbf{B},\, P(A) \neq 0} |P(B|A) - P(B)|,$$

$$\rho(\mathbf{A}, \mathbf{B}) = \sup_{f \in L_2(\mathbf{A}),\, g \in L_2(\mathbf{B})} |\mathrm{corr}(f, g)|$$

and

$$\alpha(\mathbf{A}, \mathbf{B}) = \sup_{A \in \mathbf{A},\, B \in \mathbf{B}} |P(AB) - P(A)P(B)|$$

DEFINITION 2.3. Let $\{\xi_i\}$ be a stochastic sequence and let $F_n^m = \sigma(\xi_i, n \le i \le m)$.

(a) We call the sequence φ-mixing if $\varphi(n) \to 0$ where

$$\varphi(n) = \sup_k \varphi\left(F_1^k, F_{k+n}^\infty\right).$$

(b) We call the sequence ρ-mixing if $\rho(n) \to 0$ where

$$\rho(n) = \sup_k \rho\left(F_1^k, F_{k+n}^\infty\right).$$

(c) We call the sequence strongly mixing if $\alpha(n) \to 0$ where

$$\alpha(n) = \sup_k \alpha\left(F_1^k, F_{k+n}^\infty\right).$$

It is well known that the φ-mixing condition is the most restrictive and the strong mixing is the weakest among all. [See Bradley (1986) for a survey.] The next theorem solves the same problem as Theorem 2.1 for these three classes of dependent random variables. The conditions imposed to the moments and mixing rates are the same sufficient conditions, in some sense minimal, required for the validity of CLT for strictly stationary sequences. [See Peligrad (1986) for a survey, and Doukhan, Massart and Rio (1994) for a recent result on strong mixing sequences.] Therefore, the next theorem extends the known results for strictly stationary mixing sequences from equal weights to general weights, weakening at the same time the assumption of stationarity.

THEOREM 2.2. *Let $\{a_{ni}\}$ be a triangular array of real numbers satisfying* (2.1) *and let $\{\xi_k\}$ be a centered stochastic sequence satisfying* (2.2). *Assume that one of the following three conditions is satisfied*:

(a) $\{\xi_k\}$ *is φ-mixing.*

(b) $\{\xi_k\}$ *is ρ-mixing and $\sum_j \rho(2^j) < \infty$.*

(c) *For a certain $\delta > 0$, $\{\xi_k\}$ is strongly mixing, $\{|\xi_k|^{2+\delta}\}$ is uniformly integrable,* $\inf_k \operatorname{var} \xi_k > 0$ *and $\sum_n n^{2/\delta}\alpha(n) < \infty$.*

Then (2.3) *holds.*

REMARK 2.1. In Theorem 2.2(c) the condition $\inf_k \operatorname{var} \xi_k > 0$ can be removed but this requires further additional work and it will be considered elsewhere.

Our last theorem refers to associated sequences of random variables.

DEFINITION 2.4. We call the family $(X_1, \dots, X_n)$ associated if for any coordinatewise nonincreasing functions $f(x_1, \dots, x_n)$ and $g(x_1, \dots, x_n)$ we have

$$\operatorname{cov}\big(f(X_1, \dots, X_n), \qquad g(X_1, \dots, X_n)\big) \geq 0$$

whenever this is defined.

A sequence $\{X_i\}$ is called associated if every finite family of variables is associated.

The next theorem extends the well known CLT of Newman and Wright (1981) to general weights while keeping the same conditions as in the classical case.

THEOREM 2.3. *Assume* $\{a_{ni}\}$ *are nonnegative numbers satisfying* (2.1). *Let* $\{\xi_k\}$ *be an associated sequence of random variables which is centered, satisfies* (2.2) *and*

$$\sum_{j:\,|k-j| \geq u} \operatorname{cov}\big(\xi_k, \xi_j\big) \to 0 \quad \textit{as } u \to \infty \textit{ uniformly in } k \geq 1. \tag{2.4}$$

Then (2.3) *holds.*

3. Proofs.

3.1. *Proof of Theorem* 2.1. In order to prove Theorem 2.1 we shall apply Theorem 3.2 from Hall and Heyde (1980) which is stated in the Appendix for convenience (Theorem A). It is easy to see that under (2.1) and (2.2) for every $\varepsilon > 0$,

$$\sum_{i=1}^{n} a_{ni}^2 E\xi_i^2 I\big(|a_{ni}\xi_i| > \varepsilon\big) \to 0 \quad \text{as } n \to \infty. \tag{3.1}$$

and therefore, for every $\varepsilon > 0$,

$$E \max_{1 \leq i \leq n} a_{ni}^2 \xi_i^2 I\big(|a_{ni}\xi_i| > \varepsilon\big) \to 0 \quad \text{as } n \to \infty,$$

which proves (a) and (c) of Theorem A. In order to verify (b) we shall prove the following lemma which will conclude the proof of Theorem 2.1.

LEMMA 3.1. *Suppose that f is a continuous function such that* $\{X_k \equiv f(\xi_k)\}$ *is a uniformly integrable family in* L_1. *Let* $\{t_{nk};\ 1 \leq k \leq n\}$ *be a triangular array of real numbers such that*

$$\sup_n \sum_{k=1}^{n} |t_{nk}| \equiv T < \infty \tag{3.2}$$

and

$$\lim_{n\to\infty}\max_{1\le k\le n}|t_{nk}|=0. \tag{3.3}$$

Then $\sum_{k=1}^{n}(X_k - EX_k)t_{nk}\to_P 0$ *as* $n\to\infty$.

PROOF. Let $M>1$ and $0<\varepsilon<1$ be two real numbers and define

$$M_\varepsilon=[M/\varepsilon]+1,\qquad \delta(\varepsilon)=\sup_{|t|,|s|\le 2M;\,|t-s|\le\varepsilon}|f(t)-f(s)|.$$

For $1\le k\le n$, denote by $Y_k^\varepsilon=\sum_{j=-M_\varepsilon}^{M_\varepsilon}f(j\varepsilon)I(j\varepsilon\le\xi_k<(j+1)\varepsilon)$. From the definition of Y_k^ε we deduce that for every $k\ge 1$,

$$\big|\big[X_k I(-M_\varepsilon\varepsilon\le\xi_k<M_\varepsilon\varepsilon+\varepsilon)\big]-Y_k^\varepsilon\big|\le\delta(\varepsilon)\quad\text{a.s.},$$

whence, by Jensen inequality and (3.2),

$$\begin{aligned}
&E\left|\left(\sum_{k=1}^{n}(X_k-Y_k^\varepsilon)t_{nk}\right)-\left(\sum_{k=1}^{n}(EX_k-EY_k^\varepsilon)t_{nk}\right)\right|\\
&\quad\le 2E\left|\sum_{k=1}^{n}(X_k-Y_k^\varepsilon)t_{nk}\right|\le 2T\sup_k E|X_k-Y_k^\varepsilon|\\
&\quad\le 2T\sup_k E\big[|X_k|I(|\xi_k|>M)\big]+2T\delta(\varepsilon),
\end{aligned}$$

whence the facts that $\{X_k\}$ is a uniformly integrable family in L_1, $\{\xi_k\}$ is bounded in probability and f is continuous show that it is enough to prove the validity of the lemma for Y_k^ε with M and ε fixed.

Denote by $X_k^{\varepsilon,j}=I(\xi_k<j\varepsilon)$. We have

$$\begin{aligned}
\sum_{k=1}^{n}(Y_k^\varepsilon-EY_k^\varepsilon)t_{nk}=\sum_{j=-M_\varepsilon}^{M_\varepsilon}f(j\varepsilon)\Bigg[&\sum_{k=1}^{n}\big(X_k^{\varepsilon,j+1}-EX_k^{\varepsilon,j+1}\big)t_{nk}\\
&-\sum_{k=1}^{n}\big(X_k^{\varepsilon,j}-EX_k^{\varepsilon,j}\big)t_{nk}\Bigg]
\end{aligned}$$

and therefore it remains to establish the lemma for $X_k^{\varepsilon,j}$ with ε and j fixed.

Denote by $\Gamma_i=\sup_k\big|\operatorname{cov}\big(X_k^{\varepsilon,j},X_{k+i}^{\varepsilon,j}\big)\big|$ and by the condition of pairwise mixing

$$\Gamma_i\to 0\quad\text{as } i\to\infty. \tag{3.4}$$

With the above notation we have the following estimate:

$$\operatorname{var}\left(\sum_{k=1}^{n}t_{nk}X_k^{\varepsilon,j}\right)\le\sum_{k=1}^{n}t_{nk}^2E\big(X_k^{\varepsilon,j}\big)^2+2\sum_{i=1}^{n-1}\Gamma_i\sum_{k=1}^{n-i}|t_{nk}t_{n,k+i}| \tag{3.5}$$

Obviously we have

$$\sum_{k=1}^{n}t_{nk}^2E\big(X_k^{\varepsilon,j}\big)^2\le T\max_{1\le k\le n}|t_{nk}|. \tag{3.6}$$

To estimate the second term in (3.5) we split the sum in two, one up to h and another after h, where h is an integer:

$$
\begin{aligned}
\sum_{i=1}^{n-1} \Gamma_i \sum_{k=1}^{n-i} |t_{nk} t_{n,k+i}| &\le h \max_{1\le i\le h} \sum_{k=1}^{n-i} |t_{nk}|\,|t_{n,k+i}| \\
&\quad + \max_{h\le i\le n} |\Gamma_i| \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} |t_{nk}|\,|t_{n,k+i}| \\
&\le hT \max_{1\le k\le n} |t_{nk}| + T^2 \max_{h\le i\le n} |\Gamma_i|.
\end{aligned} \tag{3.7}
$$

By (3.5), (3.6) and (3.7),

$$
\operatorname{var}\left(\sum_{k=1}^{n} t_{nk} X_k^{\varepsilon, j} \right) \le T(2h+1) \max_{1\le k\le n} |t_{nk}| + 2T^2 \max_{h\le i\le n} |\Gamma_i|
$$

and the result follows by (3.3) and (3.4) by letting first, $n \to \infty$, and after, $h \to \infty$. $\square$

3.2. *Proof of Corollary* 2.1. Without restricting the generality, we can assume $\sup_k E\xi_k^2 = 1$. We have

$$
S_n = \sum_{k=1}^{n} X_k = \sum_{j=-\infty}^{\infty} \left(\sum_{k=1}^{n} a_{k+j} \right) \xi_j .
$$

In order to apply Theorem 2.1, we fix W_n such that $\sum_{|j|>W_n} a_j^2 < n^{-3}$ and take $k_n = W_n + n$. Then

$$
S_n/\sigma_n = \sum_{|j|\le k_n} \left(\sum_{k=1}^{n} a_{k+j} \right) \xi_j/\sigma_n + \sum_{|j|>k_n} \left(\sum_{k=1}^{n} a_{k+j} \right) \xi_j/\sigma_n = T_n + U_n
$$

and we have the following estimate

$$
\begin{aligned}
\operatorname{var}(U_n) &\le \sum_{|j|>k_n} \left(\sum_{k=1}^{n} a_{k+j}/\sigma_n \right)^2 \\
&\le n\sigma_n^{-2} \sum_{|j|>k_n} \sum_{k=1}^{n} a_{k+j}^2 \le n^2\sigma_n^{-2} \sum_{|j|>k_n - n} a_j^2 \\
&\le n^2\sigma_n^{-2} \sum_{|j|>W_n} a_j^2 \le n^{-1}\sigma_n^{-2} \to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{3.8}
$$

Therefore we have only to prove that $T_n \to_D N(0,1)$ as $n \to \infty$. According to Theorem 2.1 it is sufficient to show that

$$
\frac{\sup_{-\infty<k<\infty} \left|\sum_{j=1}^{n} a_{k+j}\right|}{\sigma_n} \to 0 \quad \text{as } n \to \infty.
$$

Let us suppose on the contrary that for some $\varepsilon > 0$ there exist a subsequence (j', n'), $n' \to \infty$ such that

$$
\left| \sum_{k=1}^{n'} a_{k+j'} \right| > \varepsilon \sigma_{n'} .
$$

Denote by $A = \sup_{-\infty < k < \infty} |a_k|$ and notice that for $r > j'$,

$$\left| \sum_{k=1}^{n'} a_{k+r} \right| \geq \varepsilon \sigma_{n'} - 2A(r - j').$$

Hence

$$\frac{\sigma_{n'}^2}{b} \geq \sum_{i=j'}^{j'+W} \left(\sum_{k=1}^{n'} a_{k+i} \right)^2 \geq W\varepsilon^2 \sigma_{n'}^2 - 4A\sigma_{n'}\varepsilon \left(\sum_{i=j'}^{j'+W} (i - j') \right)$$
$$\geq W\varepsilon^2 \sigma_{n'}^2 - 4A\sigma_{n'}\varepsilon W^2.$$

Taking W to be the least integer greater than or equal to $3/b\varepsilon^2$ and because $\sigma_{n'} \to \infty$, we obtain for n' sufficiently large,

$$\frac{\sigma_{n'}^2}{b} \geq \frac{3\sigma_{n'}^2}{b} - \sigma_{n'} \frac{36A}{b^2 \varepsilon^3} \geq \frac{2\sigma_{n'}^2}{b},$$

which is a contradiction. □

3.3. *Construction of Example* 2.1. Let $\{Z_i\}$ and $\{Y_i\}$ be two independent sequences of random variables such that $P(Z_1 = 0) = P(Z_1 = 1) = 1/2$, $Z_{i+1} := 1 - Z_i$ for all $i \geq 1$ and $\{Y_i\}$ is an i.i.d. sequence of standard normal variables. Define $\xi_i := Z_i Y_i$ for $i \geq 1$. It is easy to verify that $\{\xi_i\}$ is a strictly stationary sequence of martingale differences with $\operatorname{var}(\xi_i) = 1/2$. We shall prove that $\{\xi_i\}$ is ergodic by verifying [cf. Shiryayev (1984), chapter V] that for every measurable bounded function f and every $k \geq 1$ positive integer we have

$$\frac{1}{n} \sum_{i=1}^{n} f(\xi_{i+1}, \ldots, \xi_{i+k}) \to Ef(\xi_1, \ldots, \xi_k) \quad \text{a.s.}$$

as $n \to \infty$.

Denote by $X_i = f(\xi_{i+1}, \ldots, \xi_{i+k})$. If we fix Z_1, say $Z_1 = 1$, then the sequence $\{\xi_i\}$ becomes $Y_1, 0, Y_3, 0, \ldots$. If $Z_1 = 0$, the sequence $\{\xi_i\}$ consists of $0, Y_2, 0, Y_4, \ldots$.

Therefore for Z_1 fixed, the sequence $\{X_i\}$ becomes stationary and k-dependent, therefore ergodic, and we have

$$I(Z_1 = 1) \frac{1}{n} \sum_{i=1,\, i:\,\text{even}}^{n} X_i \to I(Z_1 = 1) \frac{1}{2} Ef(Y_1, 0, Y_3, \ldots) \quad \text{a.s.},$$

$$I(Z_1 = 1) \frac{1}{n} \sum_{i=1,\, i:\,\text{odd}}^{n} X_i \to I(Z_1 = 1) \frac{1}{2} Ef(0, Y_3, 0, \ldots) \quad \text{a.s.},$$

$$I(Z_1 = 0) \frac{1}{n} \sum_{i=1,\, i:\,\text{even}}^{n} X_i \to I(Z_1 = 0) \frac{1}{2} Ef(0, Y_2, 0, \ldots) \quad \text{a.s.},$$

$$I(Z_1 = 0) \frac{1}{n} \sum_{i=1,\, i:\,\text{odd}}^{n} X_i \to I(Z_1 = 0) \frac{1}{2} Ef(Y_2, 0, Y_4, \ldots) \quad \text{a.s.}$$

Now we add all these relations, and by stationarity and construction we have:

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to \frac{1}{2}\left[Ef(Y_1, 0, Y_3, \dots) + Ef(0, Y_2, 0, Y_4, \dots)\right]$$
$$= Ef(X_1) \quad \text{a.s.}$$

Let us now consider the numbers $\{a_{ni}\}$ satisfying (2.1), (2.2) and in addition $\sum_{i=1,\, i:\text{odd}}^{n} a_{ni}^2 = 1/3$ for all $n \geq 2$. Set $S_n := \sum_{i=1}^{n} a_{ni}\xi_i$ and let Y be a standard normal random variable independent of $\{Z_i\}$. By the construction S_n has the same distribution as $Y\sqrt{\sum_{i=1}^{n} a_{ni}^2 Z_i}$. Because $Z_{i+2} = Z_i$ for every $i \geq 1$ we have

$$\sum_{i=1}^{n} a_{ni}^2 Z_i = Z_1 \sum_{i=1,\, i:\text{odd}}^{n} a_{ni}^2 + Z_2 \sum_{i=1,\, i:\text{even}}^{n} a_{ni}^2 = \tfrac{1}{3}Z_1 + \tfrac{5}{3}Z_2 = \tfrac{5}{3} - \tfrac{4}{3}Z_1.$$

Therefore S_n has a fixed nongaussian distribution for all $n \geq 2$.

3.4. *Proof of Theorem* 2.2(a), (b). In order to prove Theorem 2.3 under the assumptions (a) and (b), according to Theorem 2.1 and Theorem 4.1, respectively, in Utev (1990) we have only to verify the Lindeberg's condition

$$\sum_{i=1}^{n} Ea_{ni}^2 \xi_i^2 I(|a_{ni}\xi_i| > \varepsilon) \to 0 \quad \text{as } n \to \infty \text{ for every } \varepsilon > 0$$

and this follows exactly as (3.1) by (2.1) and (2.2). □

3.5. *Proof of Theorem* 2.2(c). In order to prove this part of Theorem 2.2, we shall first use a truncation argument and after that we shall apply Theorem B from the Appendix. The proof requires the following auxiliary lemma.

LEMMA 3.2. *Assume* $\{X_i\}$ *satisfies the conditions of Theorem* 2.2(c) *then*

$$\operatorname{var}\left(\sum_{i=a}^{b} a_{ni} X_i\right) \leq C \sum_{i=a}^{b} a_{ni}^2$$

where $C = \sup_k EX_k^2 + c(\delta)(\sum_n n^{2/\delta}\alpha(n))^{\delta/(2+\delta)}\sup_k \|X_k\|_{2+\delta}^2$, *and* $c(\delta)$ *is a numerical constant depending on* δ.

PROOF. We have

$$\operatorname{var}\left(\sum_{i=a}^{b} a_{ni} X_i\right) \leq \sum_{i=a}^{b} a_{ni}^2 \operatorname{var} X_i + 2\sum_{j=a}^{b-1}\sum_{i=j+1}^{b} a_{ni}a_{nj}, \operatorname{cov}(X_j, X_i)$$
$$\leq \sum_{i=a}^{b} a_{ni}^2 \operatorname{var} X_i + 2\sum_{i=a}^{b} a_{ni}^2 \sum_{j=a,\, j\neq i}^{b} \left|\operatorname{cov}(X_i, X_j)\right|$$

whence the result follows by Lemma B from the Appendix.

In order to prove Theorem 2.2(c), we shall truncate first the variables at the level $A > 0$ and denote

$$\xi_i' = \xi_i I(|\xi_i| \le A) - E\xi_i I(|\xi_i| \le A)$$

and

$$\xi_i'' = \xi_i I(|\xi_i| > A) - E\xi_i I(|\xi_i| > A).$$

By Lemma 3.2 there is a positive constant C such that

$$\operatorname{var}\left(\sum_{i=1}^{n} a_{ni}\xi_i''\right) \le C\left[\sup_k \left(E\xi_k^2 I(|\xi_k| > A)\right) + \sup_k \left(E|\xi_k|^{2+\delta} I(|\xi_k| > A)\right)^{2/(2+\delta)}\right]$$

and notice that by the uniform integrability of $\{|\xi_k|^{2+\delta}\}$

$$\lim_{A\to\infty} \sup_n \operatorname{var}\left(\sum_{i=1}^{n} a_{ni}\xi_i''\right) = 0. \tag{3.9}$$

By (2.2) and (3.9) we obtain

$$\lim_{A\to\infty} \operatorname{var}\left(\sum_{i=1}^{n} a_{ni}\xi_i'\right) = 1 \quad \text{uniformly in } n. \tag{3.10}$$

By (3.9), (3.10) and Theorem 4.2 in Billingsley (1968), in order to prove the theorem it is enough to show that for every fixed positive A we have

$$\frac{\sum_{i=1}^{n} a_{ni}\xi_i'}{\operatorname{st\,dev}\left(\sum_{i=1}^{n} a_{ni}\xi_i'\right)} \to_D N(0,1) \quad \text{as } n \to \infty.$$

To prove this we shall verify the conditions of Theorem B given in the Appendix. The condition (a) requires a uniform bound on the variance of partial sums which follows by Lemma 3.2. The Lindeberg condition (d) is satisfied by (2.1), (2.2) and (3.10). Condition (b) is a consequence of (2.1), (2.2) and (3.10). We have only to verify condition (c). Let $\varepsilon = 1/(2+2\delta)$.

Notice that by Lemma A in the Appendix, we can find a constant K_1 such that

$$\begin{aligned}
&\left|\operatorname{cov}\left(\left[\exp\left\{it\sum_{j=a}^{b} a_{nj}\xi_j'\right\}\right], \left[\exp\left\{it\sum_{k=b+u}^{c} a_{nk}\cdot\xi_k'\right\}\right]\right)\right| \\
&\quad = \left|\operatorname{cov}\left(\left[\exp\left\{it\sum_{j=a}^{b} a_{nj}\xi_j'\right\} - 1\right], \left[\exp\left\{it\sum_{k=b+u}^{c} a_{nk}\xi_k'\right\} - 1\right]\right)\right| \\
&\quad \le K_1 t^2 \alpha^{\delta/(1+\delta)}(u) \left\|\sum_{j=a}^{b} a_{nj}\xi_j'\right\|_{2+2\delta} \left\|\sum_{j=b+u}^{c} a_{nj}\xi_j'\right\|_{2+2\delta}.
\end{aligned} \tag{3.11}$$

Notice now that by Lemma 3.2 and the Hölder inequality, we find a constant K_2 such that

$$\begin{aligned}
E\left|\sum_{i=a}^{b} a_{ni}\xi_i'\right|^{2+2\delta} &\le (2A)^{2\delta} E\left(\sum_{i=a}^{b} a_{ni}\xi_i'\right)^2 \left(\sum_{i=a}^{b} |a_{ni}|\right)^{2\delta} \\
&\le K_2 (2A)^{2\delta} \left(\sum_{i=a}^{b} a_{ni}^2\right) \left(\sum_{i=a}^{b} a_{ni}^2\right)^{\delta} (b-a+1)^{\delta} \\
&= K_2 (2A)^{2\delta} \left(\sum_{i=a}^{b} a_{ni}^2\right)^{(1+\delta)} (b-a+1)^{\delta}.
\end{aligned}$$

Therefore the right-hand side of (3.11) is bounded above by

$$K_3 t^2 [\alpha(u)(c-a)]^{\delta/(1+\delta)} \sum_{i=a}^{c} a_{ni}^2,$$

where $u = (c-a)^{1-\varepsilon}$ and K_3 is a constant.

Due to our condition on $\alpha(n)$ and the selection of ε, we have

$$\begin{aligned}
\sum_i \left(\alpha(2^i) 2^{i/(1-\varepsilon)}\right)^{\delta/(1+\delta)} &\le K_4 \sum_i \left([2^i]^{-(1+\delta)/\delta} 2^{i/(1-\varepsilon)}\right)^{\delta/(1+\delta)} \\
&\le K_5 \sum_i \left(2^{1/(1+2\delta)}\right)^{-i} < \infty,
\end{aligned}$$

where K_4 and K_5 are constants, which completes the proof of Theorem 2.2 under (c). □

3.6. *Proof of Theorem* 2.3. Without loss of generality we assume that $a_{ni} = 0$ for all $i > n$. For every $1 \le a < b \le n$ and $1 \le u \le b - a$ we have, after simple manipulations,

$$\begin{aligned}
0 &\le \left(\sum_{i=a}^{b-u} a_{ni} \sum_{j=i+u}^{b} a_{nj} \operatorname{cov}(\xi_i, \xi_j)\right) \\
&\le \sup_k \left(\sum_{j:\,|k-j|\ge u}^{\infty} \operatorname{cov}(\xi_k, \xi_j)\right)\left(\sum_{i=a}^{b} a_{ni}^2\right).
\end{aligned} \tag{3.12}$$

In particular by (2.4), there exists a constant C such that for every $1 \le a \le b \le n$,

$$\operatorname{var}\left(\sum_{i=a}^{b} a_{ni}\xi_i\right) \le C \sum_{i=a}^{b} a_{ni}^2.$$

We shall construct now a triangular array of random variables $\{Z_{ni}, 1 \le i \le n\}$ for which we shall make use of Theorem C in the Appendix. Fix a small positive ε and find a positive integer $u = u_\varepsilon$ such that

$$0 \le \left(\sum_{i=1}^{n-u} a_{ni} \sum_{j=i+u}^{b} a_{nj} \operatorname{cov}(\xi_i, \xi_j)\right) \le \varepsilon \quad \text{for every } n \ge u+1.$$

This is possible because of (3.12) and (2.4). Denote by $[x]$ the integer part of x and define

$$K = \left[\frac{1}{\varepsilon}\right],$$

$$Y_{nj} = \sum_{i=uj+1}^{u(j+1)} a_{ni}\xi_i, \qquad j = 0, 1, \dots,$$

$$A_j = \left\{ i \colon 2Kj \le i < 2Kj + K, \operatorname{cov}(Y_{ni}, Y_{n,i+1}) \le \frac{2}{K} \sum_{i=2Kj}^{2Kj+K} \operatorname{var}(Y_{ni}) \right\}.$$

Since $2\operatorname{cov}(Y_{ni}, Y_{n,i+1}) \le \operatorname{var}(Y_{ni}) + \operatorname{var}(Y_{n,i+1})$, we get that for every j the set A_j is not empty. Now we define the integers $m_1, m_2, \dots, m_n$ recurrently by $m_0 = 0$:

$$m_{j+1} = \min\{m; m > m_j, m \in A_j\}$$

and define

$$Z_{nj} = \sum_{i=m_j+1}^{m_{j+1}} Y_{ni}, \qquad j = 0, 1, \dots,$$

$$\Delta_j = \{u(m_j + 1) + 1, \dots, u(m_{j+1} + 1)\}.$$

We observe that

$$Z_{nj} = \sum_{k \in \Delta_j} a_{nk} X_k, \qquad j = 0, 1, \dots .$$

It is easy to see that every set Δ_j contains no more than $3Ku$ elements. Hence, for every fixed positive ε by (2.1) and (2.2) the array $\{Z_{ni} \colon i = 1, \dots, n; n \ge 1\}$ satisfies the Lindeberg's condition. It remains to observe that by Theorem C and construction

$$\begin{aligned}
&\left| E\exp\left(it\sum_{j=1}^{n} Z_{nj}\right) - \prod_{j=1}^{n} E\exp(itZ_{nj}) \right| \\
&\quad \le ct^2\left(\operatorname{var}\left(\sum_{j=1}^{n} Z_{nj}\right) - \sum_{j=1}^{n} \operatorname{var}(Z_{nj}) \right) \\
&\quad \le ct^2\left(2\left(\sum_{i=1}^{n} \operatorname{cov}(Z_{ni}, Z_{n,i+1})\right) + 2\left(\sum_{i=1}^{n-2}\sum_{j=i+2}^{n} \operatorname{cov}(Z_{ni}, Z_{nj})\right)\right) \\
&\quad \le ct^2\left(4\sum_{i=1}^{n-u} a_{ni} \sum_{j=i+u}^{n} a_{nj} \operatorname{cov}(\xi_i, \xi_j) + 2\sum_{j=1}^{n} \operatorname{cov}(Y_{nm_j}, Y_{n,m_j+1}) \right)
\end{aligned}$$

$$\leq ct^2\left(4\varepsilon + \frac{8}{K}\sum_{i=1}^{n}\operatorname{var}(Y_{ni})\right)$$
$$\leq c_1(t)\,\varepsilon\left(1 + \operatorname{var}\left(\sum_{i=1}^{n} a_{ni}\xi_i\right)\right)$$
$$\leq c_2(t)\,\varepsilon.$$

Now the proof is complete by Theorem 4.2 in Billingsley (1968). □

APPENDIX

This section contains some of the theorems which were used in the proofs of the results of Section 3.

The following lemma is a variant of Theorem 17.2.2 in Ibragimov and Linnik (1971). The constant comes from Theorema 1.1 in Bradley and Bryc (1985).

LEMMA A. *Let ξ, η be two complex-valued random variables measurable with respect to $A = \sigma(\xi)$ and $B = \sigma(\eta)$. Let $\delta > 0$ and assume $E|\xi|^{2+\delta} < \infty$ and $E|\eta|^{2+\delta} < \infty$; then*

$$|E(\xi\eta) - E(\xi)E(\eta)| \leq 2\pi\alpha^{\delta/(2+\delta)}(A, B)\|\xi\|_{2+\delta}\|\eta\|_{2+\delta}.$$

By the application of Theorem 1.2 in Rio (1993), followed by the Cauchy–Schwarz inequality, we formulate the following result.

LEMMA B. *Let $\{X_n\}$ be a strongly mixing sequence of random variables such that $E|X_k|^{2+\delta} < \infty$ for a certain $\delta > 0$ and every $k \geq 1$. Then there is a numerical constant $c(\delta)$ depending only on δ such that for every $k \geq 1$ we have*

$$\sum_{n>k}^{k+m} |\operatorname{cov}(X_k, X_n)| \leq c(\delta)\left(\sum_{n=1}^{m} n^{2/\delta}\alpha(n)\right)^{\delta/(2+\delta)} \sup_j \|X_j\|_{2+\delta}^2.$$

The following theorem is a simplified version of Theorem 3.2 from Hall and Heyde (1980).

THEOREM A. *Let $\{X_{ni},\ 1 \leq i \leq k_n\}$ be a square integrable martingale difference sequence under the natural filtrations.*

Suppose that:

(a) $\max_i |X_{ni}| \to_P 0$;
(b) $\sum_{i=1}^{k_n} X_{ni}^2 \to_P 1$;
(c) $E \max_i X_{ni}^2$ *is bounded in* n.

Then $S_n \to_D N(0,1)$ as $n \to \infty$ where $S_n = \sum_{i=1}^{k_n} X_{ni}$.

A careful analysis of the proof of Theorem 4.1 in Utev (1990) gives the following statement.

THEOREM B. *Let* $\{X_{ni},\ 1 \le i \le k_n\}$ *be a triangular array of random variables such that the following hold.*

(a) $\operatorname{var}(\sum_{j=a}^{b} X_{nj}) \le C\sum_{j=a}^{b} \operatorname{var}(X_{nj})$ *for every* $0 \le a \le b \le k_n$ *where* C *is a universal constant*;

(b)
$$\liminf_{n\to\infty} \frac{\operatorname{var}\left(\sum_{j=1}^{k_n} X_{nj}\right)}{\sum_{j=1}^{k_n} \operatorname{var}\left(X_{nj}\right)} > 0;$$

(c)
$$\left|\operatorname{cov}\left(\exp\left(it\sum_{j=a}^{b} X_{nj}\right), \exp\left(it\sum_{j=b+u}^{c} X_{nj}\right)\right)\right| \le h_t(u)\sum_{j=a}^{c} \operatorname{var} X_{nj}$$

for every $1 \le a \le b \le c \le k_n$ *where* $h_t(u) \ge 0$, $\sum h_t(2^i) < \infty$ *and* u *is of the form* $u = [(c-a)^{1-\varepsilon}]$ *for a certain* $0 < \varepsilon < 1$;

(d) $\sigma_n^{-2}\sum_{i=1}^{k_n} EX_{ni}^2 I(|X_{ni}| > \varepsilon\sigma_n) \to 0$ *as* $n \to \infty$ *for every* $\varepsilon > 0$, *where* σ_n^2 *denotes* $\operatorname{var}(\sum_{i=1}^{k_n} X_{ni})$. *Then* $S_n/\sigma_n \to_D N(0,1)$ *as* $n \to \infty$.

The next theorem is taken from Newman and Wright (1981).

THEOREM C. *Let* $\{Z_i,\ 1 \le i \le k\}$ *be an associated family of random variables. Then*

$$\left|E\exp\left(it\sum_{j=1}^{n} Z_j\right) - \prod_{j=1}^{n} E\exp\left(itZ_j\right)\right| \le ct^2\left(\operatorname{var}\left(\sum_{j=1}^{n} Z_j\right) - \sum_{j=1}^{n} \operatorname{var}\left(Z_j\right)\right).$$

Acknowledgments. We would like to thank Emmanuel Rio for pointing out to us that point (c) of Theorem 2.2 holds in this given form, which is more general than in an earlier version of the paper. Many thanks go to Richard Bradley who commented about the pairwise mixing condition. He provided us with two examples proving that neither does pairwise mixing imply ergodicity nor does ergodicity imply pairwise mixing. His examples motivated and inspired Example 2.1. We also want to thank Qi-Man Shao who noticed that uniform integrability could replace the strictly stationarity assumption which was used in a previous version of this paper. We thank also the referee who pointed out a gap in the proof of Theorem 2.1 and for his many useful suggestions which improved the presentation of this paper.

REFERENCES

BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York.

BRADLEY, R. C. (1986). Basic properties of strong mixing conditions. *Progr. Probab. Statist.* **11** 165–192.

BRADLEY, R. C. and BRYC, W. (1985). Multilinear forms and measures of dependence between random variables. *J. Multivariate Anal.* **16** 335–367.

COX, J. T. and GRIMMET, G. (1984). Central limit theorems for associated random variables and the percolation model. *Ann. Probab.* **12** 514–528.

DOUKHAN, P., MASSART, P. and RIO, E. (1994). The functional central limit theorem for strongly mixing processes. *Ann. Inst. H. Poincaré* **30** 63–82.

HALL, P. and HEYDE, C. C. (1980). *Martingale Limit Theory and Its Applications*. Academic Press, New York.

IBRAGIMOV, I. A. and LINNIK, YU. V. (1971). *Independent and Stationary Sequences of Random Variables*. Wolters, Groningen.

NEWMAN, C. M. and WRIGHT, A. L. (1981). An invariance principle for certain dependent sequences. *Ann. Probab.* **9** 671–675.

PELIGRAD, M. (1986). Recent advances in the central limit theorem and its weak invariance principle for mixing sequences of random variables. *Prog. Probab. Statist.* **11** 193–224.

RIO, E. (1993). Covariance inequalities for strongly mixing processes. *Ann. Inst. H. Poincaré* **29** 589–597.

RIO, E. (1995). About the Lindeberg method for strongly mixing sequences. *ESIAM: Probability and Statistics* **1** 35–61.

SHIRYAYEV, A. N. (1984). *Probability*. Springer, New York.

UTEV, S. A. (1990). Central limit theorem for dependent random variables. *Probab. Theory Math. Statist.* **2** 519–528.

DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF CINCINNATI
P.O. BOX 210025
CINCINNATI, OHIO 45221-0025

INSTITUTE OF MATHEMATICS
630090 NOVOSIBIRSK
RUSSIA
AND
DEPARTMENT OF MATHEMATICS AND STATISTICS
LATROBE UNIVERSITY
BUNDOORA, VIC 3084
AUSTRALIA