The Annals of Probability
1997, Vol. 25, No. 1, 259–284

MODERATE DEVIATIONS FOR EMPIRICAL MEASURES OF MARKOV CHAINS: LOWER BOUNDS

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We obtain lower bounds for moderate deviations of empirical measures of a Markov chain with general state space under the assumption of ergodicity of degree 2. We derive an explicit expression for the rate function.

1. Introduction. Let $M(S)$ be the space of finite signed measures on a measurable space $(S, \mathcal{S})$ and let $\{X_j; j \geq 0\}$ be an S-valued ergodic Markov chain with invariant probability measure π. Let $\{b_n\}$ be a positive sequence such that $b_n \to \infty$, and let M_n be the random element of $M(S)$ defined by

$$M_n = \frac{1}{b_n} \sum_{j=0}^{n-1} \left(\delta_{X_j} - \pi \right).$$

Let $U = \{\lambda \in M(S): |\int f_i \, d\lambda| < \varepsilon, \ i = 1, \dots, k\}$, where the f_i's are bounded measurable functions on S; that is, U is a neighborhood of the zero measure in the τ-topology (see Section 3). If $b_n = n$ for all n, then by ergodicity as $n \to \infty$,

$$\mathbf{P}\{ M_n \in U^c \} \leq \sum_{i=1}^{k} \mathbf{P}\left\{ \left| \frac{1}{n} \sum_{j=0}^{n-1} f_i(X_j) - \int f_i \, d\pi \right| \geq \varepsilon \right\} \to 0. \tag{1.1}$$

The estimation of the order of magnitude of the small probabilities in (1.1) falls under what is usually called large deviation theory. Since in this case $M_n = L_n - \pi$, where $L_n = (1/n)\sum_{j=0}^{n-1} \delta_{X_j}$ is the empirical measure, the question is usually framed in terms of L_n as a random element of the space $P(S)$ of probability measures on $(S, \mathcal{S})$ and an extensive literature exists (see, e.g., [1], [6], [7], [9]).

Suppose now that the Markov chain $\{X_j, j \geq 0\}$ is ergodic of degree 2; this is a strengthening of the ergodicity assumption (see Section 2). It is known that under this hypothesis, the following central limit property holds: for every bounded measurable function f on S,

$$L\left(n^{-1/2} \sum_{j=0}^{n-1} \left[f(X_j) - \int f \, d\pi \right] \right) \to N\left(0, \sigma_f^2\right)$$

Received August 1995; revised May 1996.
AMS 1991 *subject classification*. 60F10.
Key words and phrases. Moderate deviations, Markov chains, ergodicity of degree 2.

for a certain variance σ_f^2 (see Section 2). Now assume that $\{b_n\}$ satisfies

$$b_n/n^{1/2} \to \infty, \qquad b_n/n \to 0. \tag{1.2}$$

By the first condition in (1.2) we have again as $n \to \infty$,

$$\mathbf{P}\{M_n \in U^c\} \le \sum_{i=1}^{k} \mathbf{P}\left\{\left|\frac{1}{b_n}\sum_{j=0}^{n-1}\left[f_i(X_j) - \int f_i\,d\pi\right]\right| \ge \varepsilon\right\} \to 0, \tag{1.3}$$

but because of the second condition in (1.2) the order of magnitude of the probabilities in (1.3) is larger than that in (1.1). The estimation of the probabilities in (1.3) is usually called a moderate deviation problem.

This is the problem addressed in the present paper. More specifically, we are concerned with lower bounds. Roughly speaking, we show in our main result (Theorem 3.2) that, under the assumption of ergodicity of degree 2, for $A \subset M(S)$ an asymptotic lower bound for $\mathbf{P}\{M_n \in A\}$ is

$$\exp\left\{-\frac{b_n^2}{n}\inf_{\lambda\in A} I_0(\lambda)\right\}$$

for a certain functional I_0: $M(S) - \mathbf{R}^+$.

We have also obtained the corresponding upper bounds. However, the upper bound result requires stronger assumptions and a different technique of proof and will be reported elsewhere.

Moderate deviations for empirical measures of Markov chains have been previously studied in [8] and [11] under assumptions which are considerably more restrictive than ours.

In Section 2 we develop some useful consequences of the assumption of ergodicity of degree 2; the assumption appears to be minimal for our problem. Section 3 contains the main result of the paper, Theorem 3.2. As a preliminary step, we prove a moderate deviation lower bound for functionals of the chain taking values in a finite-dimensional space. In spite of the difference between the two situations, the technique of proof is partly inspired by [6], in which it is elegantly shown how to use directly the minorization property of irreducible Markov kernels to obtain large deviation lower bounds. Paper [5] also provided some useful ideas.

In Section 4 we derive an explicit formula for the rate function I_0. As it turns out, this requires an analytically intricate argument.

2. Preliminary results: some consequences of the assumption of ergodicity of degree 2. Let $(S, \mathcal{S})$ be a measurable space: we assume that $\mathcal{S}$ is countably generated. $\mathbf{N}$ will denote the set of nonnegative integers. Given a Markov kernel P on $(S, \mathcal{S})$ and a probability measure μ on $(S, \mathcal{S})$, $\mathbf{P}_\mu$ will denote the Markovian probability measure on $(S^{\mathbf{N}}, \mathcal{S}^{\mathbf{N}})$ determined by P and μ. Let $\{X_j, j \ge 0\}$ be the coordinate projections on $S^{\mathbf{N}}$; then $(S^{\mathbf{N}}, \mathcal{S}^{\mathbf{N}}, \mathbf{P}_\mu, \{X_j, j \ge 0\})$ is the canonical Markov chain with state space S, transition kernel P and initial distribution μ. Let us recall (see pages 114 and 118 in [12]) that $\{X_j, j \geq 0\}$ (or P) is as defined in the following.

1. It is *ergodic* if there exists a probability measure π on $(S, \mathcal{S})$ such that
$$\lim_n \|P^n(x,\cdot) - \pi\|_v = 0$$
for all $x \in S$, where $\|\cdot\|_v$ is the total variation norm.
2. It is *ergodic of degree 2* if it is ergodic and for all $B \in \mathcal{S}$ such that $\pi(B) > 0$,
$$\int_B (\mathbf{E}_x \tau_B^2)\pi(dx) < \infty, \tag{2.1}$$
where $\mathbf{E}_x$ is the $\mathbf{P}_x$-expectation and $\tau_B = \inf\{j \geq 1\colon X_j \in B\}$ [assumption (2.1) is a strengthening of the condition
$$\int_B (\mathbf{E}_x \tau_B)\pi(dx) < \infty,$$
which holds for every B such that $\pi(B) > 0$ if P is ergodic].

For the purposes of this paper, there are two important consequences of the assumption of ergodicity of degree 2, namely Propositions 2.1 and 2.2. We denote by $B_0(S)$ [resp., $L_0^\infty(\pi)$; resp., $L_0^1(\pi)$] the subspace of the space $B(S)$ of bounded measurable functions on S [resp., of the space $L^\infty(\pi)$ of π-essentially bounded functions; resp., of the space $L^1(\pi)$ of π-integrable functions] consisting of functions f such that $\int f\,d\pi = 0$.

PROPOSITION 2.1. *If P is ergodic of degree* 2, *then we have the following*:

(i) $\sum_{k=0}^\infty \int \|P^k(x,\cdot) - \pi\|_v \pi(dx) < \infty$.

(ii) *For each $f \in L_0^\infty(\pi)$, the series $\{\sum_{k=0}^n P^k f\}$ converges in $L^1(\pi)$ and if $Gf = \sum_{k=0}^\infty P^k f$, then G is a bounded operator from $L_0^\infty(\pi)$ into $L_0^1(\pi)$.*

PROOF. (i) In what follows, $g \in B(S)$. By the invariance of π, we have for $k \geq 1$,
$$\begin{aligned}
&\int \pi(dx)\|P^k(x,\cdot) - \pi\|_v \\
&\quad = \int \pi(dx) \sup_{|g|\leq 1} \left| \int P^k(x,dy)g(y) - \int g\,d\pi \right| \\
&\quad = \int \pi(dx) \sup_{|g|\leq 1} \left| \int P^k(x,dy)g(y) - \int \pi(dz)\int P^k(z,dy)g(y) \right| \\
&\quad = \int \pi(dx) \sup_{|g|\leq 1} \left| \int \pi(dz)\int \left[P^k(x,dy) - P^k(z,dy)\right] g(y) \right| \\
&\quad \leq \int\int \pi(dx)\pi(dz)\|P^k(x,\cdot) - P^k(z,\cdot)\|_v
\end{aligned}$$
and now (1) follows from Corollary 6.9, page 118 of [12].

(ii) Let $f \in L_0^\infty(\pi)$ and let $\tilde{f} = f I_{\{|f| \le \|f\|_\infty\}}$; then $\tilde{f} \in B_0(S)$. By the invariance of π, it is easily seen that $P^n f$ is defined a.s. $[\pi]$ and in fact $P^n f = P^n \tilde{f}$ a.s. $[\pi]$. Therefore

$$\begin{aligned}\sum_{n=0}^{\infty} \int |P^n f|\, d\pi &= \sum_{n=0}^{\infty} \int |P^n \tilde{f}|\, d\pi \\ &= \sum_{n=0}^{\infty} \int \Big|P^n \tilde{f} - \int \tilde{f}\, d\pi\Big|\, d\pi \\ &\le \sum_{n=0}^{\infty} \int \|P^n(x, \cdot) - \pi\|_v \|f\|_\infty \pi(dx)\end{aligned}$$

and therefore (ii) follows from (i). □

The next preliminary result is a central limit theorem.

PROPOSITION 2.2. *If P is ergodic of degree 2, then for every $f \in B_0(S)$, every probability measure μ on $(S, \mathcal{S})$,*

$$L_{\mathbf{P}_\mu}\left(n^{-1/2} \sum_{j=0}^{n-1} f(X_j)\right) \to_w N\left(0, \sigma_f^2\right),$$

where

$$\sigma_f^2 = \int f^2\, d\pi + 2 \int f P G f\, d\pi. \tag{2.2}$$

PROOF. This result is Corollary 7.3(ii), page 140 of [12], except for the expression for σ_f^2. We will show that the form of σ_f^2 in [12] (which prima facie depends on certain noncanonical objects) in fact coincides with (2.2).

As in [12], our assumption implies that it is possible to choose a number $m_0 \in \mathbf{N}$, a function $s \in B(S)$ such that $0 \le s \le 1$ and $\int s\, d\pi > 0$, and a probability measure ν on $(S, \mathcal{S})$ satisfying $\int s\, d\nu > 0$, such that the minorization condition

$$P^{m_0} \ge s \otimes \nu \tag{2.3}$$

holds, where $s \otimes \nu$ is the kernel defined by $(s \otimes \nu)(x, A) = s(x)\nu(A)$. The expression for σ_f^2 in [12] is

$$\int f^2\, d\pi + 2 m_0^{-1} \left[\sum_{j=1}^{m_0} (m_0 - j) \int f P^j f\, d\pi + \sum_{j=1}^{m_0} \int f P^j \overline{G}_{m_0, s, \nu} f\, d\pi \right], \tag{2.4}$$

where $\overline{G}_{m_0, s, \nu} = \sum_{n=0}^{\infty} (P^{m_0} - s \otimes \nu)^n (\sum_{i=1}^{m_0 - 1} P^i)$. To establish the equality of (2.2) and (2.4), we first recall the following algebraic formula (used, e.g., in

[12]); if a and b are elements of a ring, then for $n \geq 1$,

$$(a+b)^n = a^n + \sum_{i=1}^{n} (a+b)^{i-1} b a^{n-i}. \tag{2.5}$$

The proof is

$$\begin{aligned}(a+b)^n - a^n &= \sum_{i=1}^{n} \left[(a+b)^{n-(i-1)} a^{i-1} - (a+b)^{n-1} a^i \right] \\ &= \sum_{i=1}^{n} (a+b)^{n-i} [a+b-a] a^{i-1} \\ &= \sum_{i=1}^{n} (a+b)^{n-i} b a^{i-1},\end{aligned}$$

which is the same as (2.5). Taking $a = P^{m_0} - s \otimes \nu$, $b = s \otimes \nu$ in (2.5), we have for $n \geq 1$,

$$P^{m_0 n} = (P^{m_0} - s \otimes \nu)^n + \sum_{i=1}^{n} P^{m_0(i-1)} (s \otimes \nu)(P^{m_0} - s \otimes \nu)^{n-i}$$

and therefore for any $N \geq 1$, $j \geq 1$,

$$\begin{aligned}P^j \sum_{n=0}^{N} P^{m_0 n} &= P^j \sum_{n=0}^{N} (P^{m_0} - s \otimes \nu)^n \\ &\quad + P^j \sum_{n=1}^{N} \sum_{i=1}^{n} P^{m_0(i-1)} (s \otimes \nu)(P^{m_0} - s \otimes \nu)^{n-i} \\ &= P^j \sum_{n=0}^{N} (P^{m_0} - s \otimes \nu)^n \\ &\quad + P^j \sum_{i=1}^{N} P^{m_0(i-1)} (s \otimes \nu) \sum_{k=0}^{N-i} (P^{m_0} - s \otimes \nu)^k.\end{aligned}$$

For $f, g \in B_0(S)$,

$$\begin{aligned}&\int f P^j \sum_{n=0}^{N} P^{m_0 n} g \, d\pi \\ &\quad = \int f P^j \sum_{n=0}^{N} (P^{m_0} - s \otimes \nu)^n g \, d\pi \\ &\qquad + \int f P^j \sum_{i=1}^{N} P^{m_0(i-1)} (s \otimes \nu) \sum_{k=0}^{N-i} (P^{m_0} - s \otimes \nu)^k g \, d\pi.\end{aligned} \tag{2.6}$$

The second term on the right-hand side of (2.6) may be written (taking into account that $\int f\,d\pi = 0$ and setting $\bar{s} = s - \int s\,d\pi$),

$$\sum_{i=1}^{N}\left[\int \pi(dx) f(x) \int P^{j+m_0(i-1)}(x, dy)\bar{s}(y)\right] \times\left[\int \nu(dz) \sum_{k=0}^{N-i} (P^{m_0} - s\otimes\nu)^k g(z)\right]. \tag{2.7}$$

According to [12], Corollary 5.2, page 74, the measure

$$\sum_{k=0}^{\infty} \nu(P^{m_0} - s\otimes\nu)^k$$

is proportional to π (and, in particular, finite); therefore for each i,

$$\begin{aligned}&\lim_{N\to\infty} \int \nu(dz) \sum_{k=0}^{N-i} (P^{m_0} - s\otimes\nu)^k g(z)\\ &\quad = \int g d\left(\sum_{k=0}^{\infty} \nu(P^{m_0} - s\otimes\nu)^k\right) = 0.\end{aligned} \tag{2.8}$$

The ith term in the series in (2.7) is dominated by

$$d_i = \|f\|_\infty \int |g|\, d\left(\sum_{k=0}^{\infty} \nu(P^{m_0} - s\otimes\nu)^k\right) \int \|P^{j+m_0(i-1)}(x,\cdot) - \pi\|_v \pi(dx),$$

and $\{d_i\}$ is summable by Proposition 2.1; therefore (2.8) implies that the second term on the right-hand side of (2.6) converges to 0 as $N\to\infty$. Letting $N\to\infty$ in (2.6) we obtain: for all $j \geq 1$,

$$\int fP^j \sum_{n=0}^{\infty} P^{m_0 n} g\,d\pi = \int fP^j \sum_{n=0}^{\infty} (P^{m_0} - s\otimes\nu)^n g\,d\pi. \tag{2.9}$$

and the common value is finite; the meaningfulness and finiteness of the left-hand side, as well as the passage to the limit leading to it, are justified by Proposition 2.1, and the corresponding points for the right-hand side are justified by the fact that by [12], Proposition 5.16, page 85, the assumption of ergodicity of degree 2 implies that

$$\pi \overline{G}_{m_0, s, \nu}(S) < \infty.$$

Next, (2.9) implies that for $f, g \in B_0(s)$, $j \geq 1$,

$$\begin{aligned}\int fP^j \sum_{n=0}^{\infty} P^n g\,d\pi &= \int fP^j \sum_{n=0}^{\infty} P^{m_0 n} \sum_{i=0}^{m_0-1} P^i g\,d\pi\\ &= \int fP^j \sum_{n=0}^{\infty} (P^{m_0} - s\otimes\nu)^n \left(\sum_{i=0}^{m_0-1} P^i\right) g\,d\pi.\end{aligned} \tag{2.10}$$

We are now ready to transform (2.4). By (2.10),

$$\begin{aligned}\sum_{j=1}^{m_0}\int fP^j\overline{G}_{m_0,s,\nu}f\,d\pi &= \sum_{j=1}^{m_0}\int fP^j\sum_{n=0}^{\infty}P^nf\,d\pi\\ &= \sum_{j=1}^{m_0}\int f\sum_{n=j}^{\infty}P^nf\,d\pi\\ &= \int f\sum_{n=1}^{\infty}\sum_{j=1}^{\min\{n,m_0\}}P^nf\,d\pi\\ &= \int f\sum_{n=1}^{m_0}P^nfn\,d\pi+\int f\sum_{n=m_0+1}^{\infty}P^nfm_0\,d\pi,\end{aligned}$$

so finally the term in brackets in (2.4) equals

$$\begin{aligned}&\sum_{n=1}^{m_0}(m_0-n)\int fP^nf\,d\pi+\sum_{n=1}^{m_0}n\int fP^nf\,d\pi+m_0\int f\sum_{n=m_0+1}^{\infty}P^nf\,d\pi\\ &\quad=\left(\int f\sum_{n=1}^{\infty}P^nf\,d\pi\right)m_0\\ &\quad=\left(\int fPGf\,d\pi\right)m_0,\end{aligned}$$

proving the equality of (2.2) and (2.4). □

3. Moderate deviation lower bounds. Let V be a finite-dimensional real vector space, and let $\{b_n\}$ be a positive sequence such that

$$b_n/n^{1/2}\to\infty,\qquad b_n/n\to 0.$$

We first consider moderate deviation lower bounds for $\{L_{\mathbf{P}_\mu}(S_n/b_n)\}$, where $S_n=\sum_{j=0}^{n-1}f(X_j)$ and $f\colon S\to V$ is a bounded measurable function such that $\int f\,d\pi=0$. We denote by V^* the dual space of V.

THEOREM 3.1. *If P is ergodic of degree 2, then for any probability measure μ on $(S,\mathcal{S})$ and any open set U in V,*

$$\liminf_n\frac{n}{b_n^2}\log\mathbf{P}_\mu\left\{\frac{S_n}{b_n}\in U\right\}\geq-\inf_{z\in U}J_f(z),$$

where

$$J_f(z)=\sup_{\xi\in V^*}\left[\langle z,\xi\rangle-\tfrac{1}{2}\left(\int\langle f,\xi\rangle^2\,d\pi+2\int\langle f,\xi\rangle PG\langle f,\xi\rangle\,d\pi\right)\right].$$

PROOF. In the minorization condition (2.3), s may be taken to be of the form $s=\beta I_C$, where $0<\beta\leq 1$, $C\in\mathcal{S}$ and $\pi(C)>0$; thus for any $x\in S$, $A\in\mathcal{S}$,

$$P^{m_0}(x,A)\geq\beta I_C(x)\nu(A).\tag{3.1}$$

It follows from (3.1) that for every nonnegative measurable function Φ defined on $S^{\mathbf{N}}$, any $x \in S$,

$$\mathbf{E}_x(\Phi \circ \theta^{m_0}) \geq \beta I_C(x) \mathbf{E}_\nu \Phi, \tag{3.2}$$

where θ is the shift operator on $S^{\mathbf{N}}$.

As is easily seen, in order to prove the theorem it is enough to prove that for any open convex set U in V, any $z \in U$,

$$\liminf_n \frac{n}{b_n^2} \log \mathbf{P}_\mu \left\{ \frac{S_n}{b_n} \in U \right\} \geq -J_f(z). \tag{3.3}$$

We will assume henceforth that U is an open convex set. For fixed $t > 0$, let

$$p_n = \left[\frac{n^2 t^2}{b_n^2} \right], \qquad q_n = \left[\frac{n}{p_n} \right],$$

where $[\cdot]$ is the integer part function.

Claim I. For $\varepsilon > 0$, let $U_\varepsilon = \{x \in V: d(x, U^c) > \varepsilon\}$, where d is the metric associated to some norm $\|\cdot\|$ on V. Then for any probability measure μ on $(S, \mathscr{S})$,

$$\liminf_n \frac{n}{b_n^2} \log \mathbf{P}_\mu \left\{ \frac{S_n}{b_n} \in U \right\}$$
$$\geq \liminf_n \frac{n}{b_n^2} \log \mathbf{P}_\mu \left\{ b_n^{-1} S_{p_n q_n - m_0 + 1} \in U_\varepsilon, X_{p_n q_n - m_0} \in C \right\}.$$

The claim will obviously follow if we can prove that

$$\mathbf{P}_\mu\{ S_n / b_n \in U \} \geq \mathbf{P}_\mu \left\{ b_n^{-1} S_{p_n q_n - m_0 + 1} \in U_\varepsilon, X_{p_n q_n - m_0} \in C \right\} a_n, \tag{3.4}$$

where $\{a_n\}$ is a positive sequence bounded away from zero. To prove (3.4), we notice first that

$$0 \leq n - p_n q_n \leq p_n,$$

$$\frac{n - p_n q_n}{b_n^2} \leq \frac{p_n}{b_n^2} \leq t^2 \left(\frac{n}{b_n^2} \right)^2 \to 0. \tag{3.5}$$

It follows from Proposition 2.2 that $\{ L_{\mathbf{P}_\mu}(S_n / n^{1/2}) \}$ converges to a Gaussian measure on V and therefore (3.5) implies

$$b_n^{-1} S_{n - p_n q_n} \to_{\mathbf{P}_\mu} 0. \tag{3.6}$$

If $n = p_n q_n - m_0 + 1$, then (3.4) is obvious. Assume $n > p_n q_n - m_0 + 1$. Let $B_\varepsilon = \{z \in V: \|z\| < \varepsilon\}$. Writing $S_n = S_{p_n q_n - m_0 + 1} + (S_n - S_{p_n q_n - m_0 + 1})$, we have by the Markov property

$$\begin{aligned} \mathbf{P}_\mu\{ S_n / b_n \in U \} &\geq \mathbf{P}_\mu \left\{ b_n^{-1} S_{p_n q_n - m_0 + 1} \in U_\varepsilon, X_{p_n q_n - m_0} \in C, b_n^{-1} T_n \in B_\varepsilon \right\} \\ &= \mathbf{E}_\mu \left\{ I_{U_\varepsilon}\left(b_n^{-1} S_{p_n q_n - m_0 + 1} \right) I_C\left(X_{p_n q_n - m_0} \right) \mathbf{P}_{X_{p_n q_n - m_0}}(A_n) \right\}, \end{aligned} \tag{3.7}$$

where $T_n = S_n - S_{p_n q_n - m_0 + 1}$ and

$$A_n = \left\{ (x_j)_{j \geq 0} \in S^{\mathbf{N}} \colon b_n^{-1} \sum_{j=1}^{n - p_n q_n + m_0 - 1} f(x_j) \in B_\varepsilon \right\}.$$

Choose now n_0 such that $n \geq n_0$ implies $b_n^{-1}(m_0 - 1) \sup_{x \in S} \| f(x) \| < \varepsilon/2$. Suppose $n \geq n_0$. If $1 \leq n - p_n q_n$, then by (3.2) with

$$\Phi\big((x_j)_{j \geq 0}\big) = I_{B_{\varepsilon/2}}\left(b_n^{-1} \sum_{j=0}^{n - p_n q_n - 1} f(x_j) \right),$$

we have for $x \in C$

$$\begin{aligned} \mathbf{P}_x(A_n) &\geq \mathbf{P}_x\left\{ b_n^{-1} \sum_{j=m_0}^{n - p_n q_n + m_0 - 1} f(X_j) \in B_{\varepsilon/2} \right\} \\ &= \mathbf{E}_x(\Phi \circ \theta^{m_0}) \\ &\geq \beta \mathbf{E}_\nu \Phi \\ &= \beta \mathbf{P}_\nu\big\{ b_n^{-1} S_{n - p_n q_n} \in B_{\varepsilon/2} \big\}, \end{aligned} \tag{3.8}$$

which converges to 1 by (3.6). If $n = p_n q_n$, then

$$A_n = S^{\mathbf{N}}. \tag{3.9}$$

It is clear now from (3.7) to (3.9) that (3.4) holds, proving Claim I.

CLAIM II. Let $r_n = b_n / t q_n$. Then for every open convex set U in V, $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that for $n \geq n_0$,

$$\begin{aligned} &\mathbf{P}_\nu\big\{ b_n^{-1} S_{p_n q_n - m_0 + 1} \in U,\ X_{p_n q_n - m_0} \in C \big\} \\ &\qquad \geq \Big(\beta \mathbf{P}_\nu\big\{ r_n^{-1} S_{p_n - m_0 + 1} \in tU_\varepsilon,\ X_{p_n - m_0} \in C \big\} \Big)^{q_n}. \end{aligned}$$

We proceed to prove this claim. Letting $F_k = \sigma\{X_j, j \leq k\}$ we have by the convexity of U and the Markov property

$$\begin{aligned} &\mathbf{P}_\nu\big\{ b_n^{-1} S_{p_n q_n - m_0 + 1} \in U,\ X_{p_n q_n - m_0} \in C \big\} \\ &\quad = \mathbf{E}_\nu \mathbf{E}_\nu\left[I_{t r_n q_n U}\left(S_{p_n(q_n - 1) - m_0 + 1} + \sum_{j = p_n(q_n - 1) - m_0 + 1}^{p_n q_n - m_0} f(X_j) \right) \right. \\ &\qquad\qquad \left. \times I_C\big(X_{p_n q_n - m_0}\big) \Big| F_{p_n(q_n - 1) - m_0} \right] \\ &\quad \geq \mathbf{E}_\nu \mathbf{E}_\nu\left[I_{t r_n (q_n - 1) U}\big(S_{p_n(q_n - 1) - m_0 + 1} \big) I_{t r_n U}\left(\sum_{j = p_n(q_n - 1) - m_0 + 1}^{p_n q_n - m_0} f(X_j) \right) \right. \\ &\qquad\qquad \left. \times I_C\big(X_{p_n q_n - m_0}\big) I_C\big(X_{p_n(q_n - 1) - m_0}\big) \Big| F_{p_n(q_n - 1) - m_0} \right] \\ &\quad = \mathbf{E}_\nu\Big\{ I_{t r_n (q_n - 1) U}\big(S_{p_n(q_n - 1) - m_0 + 1} \big) I_C\big(X_{p_n(q_n - 1) - m_0} \big) \mathbf{P}_{X_{p_n(q_n - 1) - m_0}}(E_n) \Big\}, \end{aligned} \tag{3.10}$$

where

$$E_n = \left\{ (x_j)_{j \geq 0} \in S^{\mathbf{N}} \colon \sum_{j=1}^{p_n} f(x_j) \in tr_n U,\, x_{p_n} \in C \right\}.$$

We notice that, as $n \to \infty$,

$$r_n = \frac{b_n}{t[n/p_n]} = b_n t^{-1} \left[\frac{n}{\left[\frac{n^2 t^2}{b_n^2} \right]} \right]^{-1} \sim t\left(\frac{n}{b_n} \right).$$

Choose now n_0 such that $n \geq n_0$ implies $(tr_n)^{-1}(m_0 - 1)\sup_{x \in S} \| f(x) \| < \varepsilon$. Then by (3.2) with

$$\Phi\big((x_j)_{j \geq 0}\big) = I_{U_\varepsilon}\left((tr_n)^{-1} \sum_{j=0}^{p_n - m_0} f(x_j) \right) I_C(x_{p_n - m_0}),$$

we have for $x \in C$, $n \geq n_0$,

$$\begin{aligned} \mathbf{P}_x(E_n) &\geq \mathbf{P}_x\left\{ (tr_n)^{-1} \sum_{j=m_0}^{p_n} f(X_j) \in U_\varepsilon,\, X_{p_n} \in C \right\} \\ &= \mathbf{E}_x(\Phi \circ \theta^{m_0}) \\ &\geq \beta \mathbf{E}_\nu \Phi \\ &= \beta \mathbf{P}_\nu\left\{ (tr_n)^{-1} \sum_{j=0}^{p_n - m_0} f(X_j) \in U_\varepsilon,\, X_{p_n - m_0} \in C \right\}. \end{aligned} \tag{3.11}$$

By (3.10) and (3.11), we have for $n \geq n_0$:

$$\begin{aligned} &\mathbf{P}_\nu\{ S_{p_n q_n - m_0 + 1} \in tr_n q_n U,\, X_{p_n q_n - m_0} \in C \} \\ &\quad \geq \mathbf{P}_\nu\{ S_{p_n(q_n - 1) - m_0 + 1} \in tr_n(q_n - 1)U,\, X_{p_n(q_n - 1) - m_0} \in C \} \\ &\quad\quad \times \beta \mathbf{P}_\nu\{ S_{p_n - m_0 + 1} \in tr_n U_\varepsilon,\, X_{p_n - m_0} \in C \}. \end{aligned} \tag{3.12}$$

Iterating (3.12), we obtain

$$\begin{aligned} &\mathbf{P}_\nu\{ S_{p_n q_n - m_0 + 1} \in tr_n q_n U,\, X_{p_n q_n - m_0} \in C \} \\ &\quad \geq \mathbf{P}_\nu\{ S_{p_n - m_0 + 1} \in tr_n U,\, X_{p_n - m_0} \in C \} \\ &\quad\quad \times \left(\beta \mathbf{P}_\nu\{ S_{p_n - m_0 + 1} \in tr_n U_\varepsilon,\, X_{p_n - m_0} \in C \} \right)^{q_n - 1}, \end{aligned}$$

and Claim II is proved.

CLAIM III. Let $\gamma_f = \lim_n L_{\mathbf{P}_\mu}(n^{-1/2} S_n)$ (See Proposition 2.2). Then for every open set $W \subset V$,

$$\liminf_n \mathbf{P}_\mu\{ n^{-1/2} S_n \in W,\, X_{n-1} \in C \} \geq \gamma_f(W)\pi(C).$$

To prove this claim, given $\varepsilon > 0$, choose k_0 such that

$$\int \pi(dx)\|P^{k_0}(x,\cdot) - \pi\|_v < \varepsilon. \tag{3.13}$$

Then given $\delta > 0$, choose n_0 such that $n_0^{-1/2} k_0 \sup_{x\in S}\|f(x)\| < \delta$. Then for $n \geq n_0$,

$$\begin{aligned}
&\mathbf{P}_\mu\Big\{n^{-1/2}S_n \in W,\, X_{n-1} \in C\Big\}\\
&\quad\geq \mathbf{P}_\mu\Big\{n^{-1/2}S_{n-k_0} \in W_\delta,\, X_{n-1} \in C\Big\}\\
&\quad= \mathbf{E}_\mu \mathbf{E}_\mu\Big[I_{W_\delta}\Big(n^{-1/2}S_{n-k_0}\Big)I_C(X_{n-1})|F_{n-k_0-1}\Big]\\
&\quad= \mathbf{E}_\mu\Big\{I_{W_\delta}\Big(n^{-1/2}S_{n-k_0}\Big)P^{k_0}\Big(X_{n-k_0-1}, C\Big)\Big\}.
\end{aligned} \tag{3.14}$$

Next,

$$\begin{aligned}
&\Big|\mathbf{E}_\mu\Big\{I_{W_\delta}\Big(n^{-1/2}S_{n-k_0}\Big)P^{k_0}\Big(X_{n-k_0-1}, C\Big)\Big\} - \mathbf{E}_\mu\Big\{I_{W_\delta}\Big(n^{-1/2}S_{n-k_0}\Big)\pi(C)\Big\}\Big|\\
&\quad\leq \int \big(\mu P^{n-k_0-1}\big)(dx)\|P^{k_0}(x,\cdot) - \pi\|_v.
\end{aligned} \tag{3.15}$$

By (3.13) to (3.15) and since $\lim_n \|\mu P^n - \pi\|_v = 0$ by ergodicity, we have

$$\begin{aligned}
\liminf_n \mathbf{P}_\mu\Big\{n^{-1/2}S_n \in W,\, X_{n-1} \in C\Big\} &\geq \liminf_n \mathbf{P}_\mu\Big\{n^{-1/2}S_{n-k_0} \in W_\delta\Big\}\pi(C) - \varepsilon\\
&\geq \gamma_f(W_\delta)\pi(C) - \varepsilon.
\end{aligned}$$

Since ε and δ are arbitrary, Claim III follows.

We shall now finish the proof. First we note that for an arbitrary probability measure μ on $(S, \mathcal{S})$, it follows from (3.2) that for any nonnegative measurable function Φ on $S^{\mathbf{N}}$, $m_1 \in \mathbf{N}$,

$$\mathbf{E}_\mu(\Phi \circ \theta^{m_0+m_1}) \geq \beta\mu P^{m_1}(C)\mathbf{E}_\nu\Phi. \tag{3.16}$$

Since $\pi(C) > 0$ and $\|\mu P^n - \pi\|_v \to 0$, one may choose m_1 such that $\mu P^{m_1}(C) > 0$. Given $\varepsilon > 0$, let n_0 be such that $n \geq n_0$ implies $b_n^{-1}(m_0 + m_1)\sup_{x\in S}\|f(x)\| < \varepsilon$. Then by (3.16),

$$\begin{aligned}
\mathbf{P}_\mu\{S_n/b_n \in U\} &\geq \beta\mu P^{m_1}(C)\mathbf{P}_\nu\left\{b_n^{-1}\sum_{j=m_0+m_1}^{n-1} f(X_j) \in U_\varepsilon\right\}\\
&\geq \beta\mu P^{m_1}(C)\mathbf{P}_\nu\{S_n/b_n \in U_{2\varepsilon}\}.
\end{aligned} \tag{3.17}$$

By (3.17) and Claims (I) to (III), for any open convex set U and any $t > 0$, $\varepsilon > 0$, we have

$$\begin{aligned}\liminf_n \frac{n}{b_n^2} \log \mathbf{P}_\mu\left\{\frac{S_n}{b_n} \in U\right\} &\geq \liminf_n \frac{n}{b_n^2} \log \mathbf{P}_\nu\left\{\frac{S_n}{b_n} \in U_{2\varepsilon}\right\} \\ &\geq \liminf_n \frac{n}{b_n^2} \log \mathbf{P}_\nu\left\{b_n^{-1} S_{p_n q_n - m_0 + 1} \in U_{3\varepsilon}, X_{p_n q_n - m_0} \in C\right\} \\ &\geq \liminf_n \frac{n}{b_n^2} q_n \log \beta \mathbf{P}_\nu\left\{r_n^{-1} S_{p_n - m_0 + 1} \in tU_{3\varepsilon}, X_{p_n - m_0} \in C\right\} \\ &\geq t^{-2} \log\left\{\beta\gamma_f(tU_{3\varepsilon})\pi(C)\right\},\end{aligned}$$

since

$$\frac{n}{b_n^2} q_n = \frac{n}{b_n^2} \frac{n}{\left[\dfrac{n^2 t^2}{b_n^2}\right]} \to t^{-2} \quad \text{and} \quad r_n \sim p_n^{-1/2}.$$

Letting $t \to \infty$ it follows from Lemma 3.2, page 107 of [5], and the fact that ε is arbitrary, that for every $z \in U$,

$$\liminf_n \frac{n}{b_n^2} \log \mathbf{P}_\mu\left\{\frac{S_n}{b_n} \in U\right\} \geq -\frac{1}{2}\|z\|_{\gamma_f}^2,$$

where $\|\cdot\|_{\gamma_f}$ is the reproducing kernel Hilbert space norm associated to γ_f. But, as is well known,

$$\tfrac{1}{2}\|z\|_{\gamma_f}^2 = \sup_{\xi \in V^*}\left[\langle z, \xi\rangle - \tfrac{1}{2}\int \xi^2 \, d\gamma_f\right],$$

which equals $J_f(z)$ since, by Proposition 2.2,

$$\int \xi^2 \, d\gamma_f = \int \langle f, \xi\rangle^2 \, d\pi + 2\int \langle f, \xi\rangle PG\langle f, \xi\rangle \, d\pi. \qquad \square$$

REMARK 1. It is clear from the proof of Theorem 3.1 that the result is valid under the weaker assumption that P is ergodic if further restrictions, as in the central limit theorem in [12], are imposed on $f \in B(S)$ [or $L_0^\infty(\pi)$].

REMARK 2. It is easily seen from the proof of Theorem 3.1 that if D is a set of probability measures on $(S, \mathcal{S})$ such that, as $n \to \infty$,

$$\sup_{\mu \in D} \|\mu P^n - \pi\|_v \to 0,$$

then we have the following uniformity result:

$$\liminf_n \frac{n}{b_n^2} \log \inf_{\mu \in D} \mathbf{P}_\mu \left\{ \frac{S_n}{b_n} \in U \right\} \geq - \inf_{z \in U} J_f(z).$$

If follows, for example, that the result will hold for $D = P(S)$ if P is uniformly ergodic (see [8]).

The next proposition is one of the main results of this paper. The proof is based on the projective system method and is in fact an implementation of Remark (1) following Theorem 3.3 of [4]. Similar arguments may be found in [2] and in [3], where we studied moderate deviations of empirical measures in the i.i.d. case.

Let us recall that the τ-topology on the space $M(S)$ of finite signed measures on $(S, \mathcal{S})$ is the smallest topology such that for each $f \in B(S)$, the map $\nu \to \int f\,d\nu$ $[\nu \in M(S)]$ is continuous. For $B \subset M(S)$, we denote by $\operatorname{int}_\tau(B)$ the interior of B in the τ-topology. The σ-algebra $\mathcal{B}$ on $M(S)$ is defined to be the smallest σ-algebra such that for each $f \in B(S)$, the map $\nu \to \int f\,d\nu$ is measurable. Recall that

$$M_n = b_n^{-1} \sum_{j=0}^{n-1} \left(\delta_{X_j} - \pi \right).$$

THEOREM 3.2. *If P is ergodic of degree* 2, *then for every probability measure μ on $(S, \mathcal{S})$, every $B \in \mathcal{B}$,*

$$\liminf_n \frac{n}{b_n^2} \log \mathbf{P}_\mu\{ M_n \in B\} \geq - \inf_{\lambda \in \operatorname{int}_\tau(B)} I_0(\lambda), \tag{3.18}$$

where for $\lambda \in M(S)$, setting $\bar{f} = f - \int f\,d\pi$,

$$I_0(\lambda) = \sup_{f \in B(S)} \left[\int f\,d\lambda - \tfrac{1}{2}\left(\int \bar{f}^2\,d\pi + 2\int \bar{f} PG\bar{f}\,d\pi \right)\right].$$

Moreover, for each $a \geq 0$ the level set $L_a = \{\lambda \in M(S)\colon I_0(\lambda) \leq a\}$ is τ-compact.

Note. In the next section we compute the variational expression in the definition of I_0 and obtain an explicit formula for I_0.

PROOF. Let $\mathcal{F}$ be the family of finite subsets of $B(S)$, directed upward by inclusion. For each $F \in \mathcal{F}$, let $\Pi_F\colon M(S) \to \mathbf{R}^F$ be the map $\Pi_F(\lambda)(f) = \int f\,d\lambda$ $(f \in F)$. Then

$$L_{\mathbf{P}_\mu}(\Pi_F(M_n)) = L_{\mathbf{P}_\mu}(S_n/b_n),$$

where

$$S_n = \left\{ \sum_{j=0}^{n-1} \left(f(X_j) - \int f\,d\pi \right) \right\}_{f \in F} = \sum_{j=0}^{n-1} \left(h(X_j) - \int h\,d\pi \right),$$

where $h: S \to \mathbf{R}^F$ is defined by $h(x) = \{f(x)\}_{f \in F}$. Now let $B \in B$, and suppose $\lambda \in \operatorname{int}_\tau(B)$. By the definition of the τ-topology, there exist $F \in F$, U open in $\mathbf{R}^F$ such that $\lambda \in \Pi_F^{-1}(U) \subset B$. Then $z = \Pi_F(\lambda) \in U$ and by Theorem 3.1,

$$\begin{aligned}\liminf_n \frac{n}{b_n^2} \log \mathbf{P}_\mu\{M_n \in B\} &\geq \liminf_n \frac{n}{b_n^2} \log \mathbf{P}_\mu\{\Pi_F(M_n) \in U\} \\ &= \liminf_n \frac{n}{b_n^2} \log \mathbf{P}_\mu\left\{\frac{S_n}{b_n} \in U\right\} \\ &\geq -J_h(z).\end{aligned} \tag{3.19}$$

Writing $\bar{h} = h - \int h\, d\pi$, $\bar{g} = g - \int g\, d\pi$, we have, since $z = \int h\, d\lambda$,

$$\begin{aligned}J_h(z) = \sup_{\alpha \in \mathbf{R}_F}&\Big[\int \langle h, \alpha\rangle\, d\lambda \\ &- \tfrac{1}{2}\Big(\int \langle \bar{h}, \alpha\rangle^2\, d\pi + 2\int \langle \bar{h}, \alpha\rangle PG \langle \bar{h}, \alpha\rangle\, d\pi\Big)\Big] \\ \leq \sup_{g \in B(S)}&\Big[\int g\, d\lambda - \tfrac{1}{2}\Big(\int \bar{g}^2\, d\pi + 2\int \bar{g} PG \bar{g}\, d\pi\Big)\Big] \\ = I_0(\lambda)&.\end{aligned} \tag{3.20}$$

Now (3.18) follows from (3.19) and (3.20).

Next we show that if $I_0(\lambda) < \infty$, then

$$\lambda(S) = 0 \quad \text{and} \quad \lambda \ll \pi. \tag{3.21}$$

In fact, setting $g = tI_A$ for $A \in S$, $t \in \mathbf{R}$, we have by the definition of I_0,

$$t\lambda(A) = \int (tI_A)\, d\lambda \leq \tfrac{1}{2}t^2\left\{\int \varphi_A^2\, d\pi + 2\int \varphi_A PG \varphi_A\, d\pi\right\} + I_0(\lambda),$$

where $\varphi_A = I_A - \pi(A)$. Now if $A = S$, then $t\lambda(S) \leq I_0(\lambda)$ for all $t \in \mathbf{R}$ and therefore $\lambda(S) = 0$. If $\pi(A) = 0$, then again $t\lambda(A) \leq I_0(\lambda)$ for all $t \in R$, implying $\lambda(A) = 0$. This proves (3.21).

Let $H_a = \{g \in L_0^1(\pi): \sup_{f \in B(S)}[\int fg\, d\pi - \frac{1}{2}\Phi(f, f)] \leq a\}$, where $\Phi(f, f) = \int \bar{f}^2\, d\pi + 2\int \bar{f} PG \bar{f}\, d\pi$ and $\bar{f} = f - \int f\, d\pi$. Define $T: L^1(\pi) \to M(S)$ by $T(g) = g\, d\pi$. Then T is continuous from the weak topology $\sigma(L^1(\pi), L^\infty(\pi))$ to the τ-topology and by (3.21), $T(H_a) = L_a$. Therefore it suffices to prove that H_a is $\sigma(L^1(\pi), L^\infty(\pi))$ compact.

(i) H_a is bounded in $L^1(\pi)$. To prove this, we observe first that

$$\sup_{f \in B(S)}\Big[\int fg\, d\pi - \tfrac{1}{2}\Phi(f, f)\Big] = \sup_{f \in L^\infty(\pi)}\Big[\int fg\, d\pi - \tfrac{1}{2}\Phi(f, f)\Big]$$

(see Proposition 2.1). Therefore for any $f \in L^\infty(\pi)$,

$$\sup_{g \in H_a}\left|\int fg\, d\pi\right| \leq \tfrac{1}{2}\Phi(f, f) + a.$$

The result follows now from the Banach–Steinhaus theorem.

(ii) H_a is uniformly integrable. To prove this, let $g \in H_a$ and set $f = b(\operatorname{sgn} g) I_{\{|g| > t\}}$ where $b, t > 0$. Then

$$b \int |g| I_{\{|g| > t\}} \, d\pi = \int fg \, d\pi \le \tfrac{1}{2} \Phi(f, f) + a. \tag{3.22}$$

Now

$$\begin{aligned} \Phi(f, f) &\le b^2 \int I_{\{|g| > t\}} \, d\pi \\ &\quad + 2b^2 \int I_{\{|g|>t\}}(x) \sum_{n=1}^{\infty} \|P^n(x, \cdot) - \pi\|_v 2\pi(dx) \\ &\le \frac{b^2}{t} \int |g| \, d\pi + \frac{8b^2 m}{t} \int |g| \, d\pi \\ &\quad + 4b^2 \sum_{n=m+1}^{\infty} \int \|P^n(x, \cdot) - \pi\|_v \pi(dx). \end{aligned} \tag{3.23}$$

Dividing by b in (3.22) and using (3.23) and (i), we have for each $b > 0$, $m \in \mathbf{N}$:

$$\limsup_{t \to \infty} \sup_{g \in H_a} \int |g| I_{\{|g| > t\}} \, d\pi \le 4b \sum_{n=m+1}^{\infty} \int \|P^n(x, \cdot) - \pi\|_v \pi(dx) + \frac{a}{b}.$$

Since b, m are arbitrary we conclude that (ii) holds. Therefore by the Dunford Pettis theorem (see, e.g., [10], page 20), H_a is $\sigma(L^1(\pi), L^\infty(\pi))$ relatively compact. Since clearly H_a is $\sigma(L^1(\pi), L^\infty(\pi))$ closed, it follows that it is $\sigma(L^1(\pi), L^\infty(\pi))$ compact. □

4. Identification of the rate function. The purpose of this section is to obtain an explicit formula for I_0. It is necessary to discuss first the concept of adjoint of P (see [13], Chapter 4). We recall first that by the formula

$$Pf(x) = \int P(x, dy) f(y),$$

P acts as an operator of norm 1 on each of the function spaces $B(S)$, $L^1(\pi)$, $L^2(\pi)$ and $L^\infty(\pi)$ (in the three latter cases, this is of course due to the invariance of π). Now for $f \in L^1(\pi)$, define for $A \in \mathcal{S}$,

$$(fP)(A) = \int P(x, A) f(x) \pi(dx).$$

Then fP is a finite signed measure and $fP \ll \pi$; for, if $\pi(A) = 0$, then $\int P(x, A)\pi(dx) = 0$, so $P(\cdot, A) = 0$ a.s. $[\pi]$ and therefore $(fP)(A) = 0$. We define

$$P^* f = \frac{d(fP)}{d\pi} \tag{4.1}$$

in the Radon–Nikodym sense. We call the operator P^* the *adjoint* of P. It has the following easily proved properties.

1. P^* is an operator of norm 1 on both $L^1(\pi)$ and $L^\infty(\pi)$.
2. For all $f \in L^\infty(\pi)$, $g \in L^1(\pi)$ (or $f \in L^1(\pi)$, $g \in L^\infty(\pi)$,

$$\int Pfg\,d\pi = \int fP^*g\,d\pi. \tag{4.2}$$

3. P^* is an operator of norm 1 on $L^2(\pi)$ and if $f, g \in L^2(\pi)$ then (4.2) holds; that is, P^* operating on $L^2(\pi)$ is the standard $L^2(\pi)$-adjoint of P. Moreover, for $f \in L_0^2(\pi)$, $\|(P^*)^n f\|_2 \to 0$ [this is proved by first doing it for $f \in L_0^\infty(\pi)$ and then approximating].
4. Let $G_n = \sum_{j=0}^n P^j$. If $(P^n)^*$ and G_n^* are defined as P^* was, then $(P^n)^* = (P^*)^n$ and $G_n^* = \sum_{j=0}^n (P^*)^j$.

We are now ready to state the main result of this section.

THEOREM 4.1. *If P is ergodic of degree 2, then for every $\lambda \in M(S)$,*

$$I_0(\lambda) = \begin{cases} \frac{1}{2}\sum_{j=0}^\infty \int \left\{ |(P^*P)^j(I-P^*)g|^2 + |(PP^*)^j P(I-P^*)g|^2 \right\} d\pi, \\ \qquad\qquad\qquad\qquad \text{if } \lambda(S) = 0 \text{ and } \lambda \ll \pi, \\ \infty, \qquad \text{otherwise}, \end{cases}$$

where $g = d\lambda/d\pi$.

The key part of the proof is the following inequality.

LEMMA 4.2. *For all $f \in L_0^\infty(\pi)$, $g \in L_0^1(\pi)$,*

$$\int fg\,d\pi \le \tfrac{1}{2}\Phi(f,f) + \tfrac{1}{2}\psi(g,g), \tag{4.3}$$

where

$$\Phi(f,f) = \int f^2\,d\pi + 2\int fPGf\,d\pi$$

$$\psi(g,g) = \sum_{j=0}^\infty \int \left\{ |(P^*P)^j(I-P^*)g|^2 + |(PP^*)^j P(I-P^*)g|^2 \right\} d\pi.$$

PROOF.

Step I. It is enough to prove the statement for $g \in L_0^1(\pi)$, $f = (I-P)h$ with $h \in L_0^\infty(\pi)$.

PROOF OF STEP I. Assume that (4.3) is valid for functions of the form $(I-P)h$. Given $f \in L_0^\infty(\pi)$, let $f_n = (I-P)G_n f$, where $G_n = \sum_{j=0}^n P^j$. Then we have the following:

(i) $f - f_n = P^{n+1} f \to 0$ pointwise by the ergodicity of P;
(ii) $|f - f_n| = |P^{n+1} f| \le \|f\|_\infty$ a.s. $[\pi]$, $|f_n| \le 2\|f\|_\infty$ a.s. $[\pi]$.

Therefore, by the dominated convergence theorem

$$\left|\int f_n g\, d\pi - \int fg\, d\pi\right| \le \int |f_n - f||g|\, d\pi \to 0, \tag{4.4}$$

$$\int f_n^2\, d\pi \to \int f^2\, d\pi. \tag{4.5}$$

Also,

$$\int f_n PGf_n\, d\pi = \int f_n PG(I-P)G_n f\, d\pi = \int f_n PG_n f\, d\pi$$

and therefore,

$$\begin{aligned}\left|\int f_n PGf_n\, d\pi - \int fPGf\, d\pi\right| &\le \left|\int f_n PG_n f\, d\pi - \int fPG_n f\, d\pi\right| \\ &\quad + \left|\int fPG_n f\, d\pi - \int fPGf\, d\pi\right| \\ &\le \int |f_n - f|(x) \sum_{j=1}^\infty \|P^j(x,\cdot) - \pi\|_v \|f\|_\infty\, d\pi \\ &\quad + \int |f(x)| \sum_{j=n+1}^\infty \|P^j(x,\cdot) - \pi\|_v \|f\|_\infty\, d\pi\end{aligned}$$

and by dominated convergence,

$$\int f_n PGf_n\, d\pi \to \int fPGf\, d\pi. \tag{4.6}$$

Since by assumption (4.3) holds for f_n, it follows from (4.4) to (4.6) that (4.3) holds for $f \in L_0^\infty(\pi)$. □

Step II. For $f, g \in L^2(\pi)$, $\alpha, \beta \in (0,1)$, define

$$\Phi_{\alpha,\beta}(f,g) = \int fG_\alpha^*(I - \beta\alpha^2 P^* P)G_\alpha g\, d\pi,$$

where $G_\alpha = \sum_{n=0}^\infty \alpha^n P^n$ [which operates in $L^2(\pi)$], and G_α^* is its $L^2(\pi)$-adjoint. For $f = (I-P)h$ with $h \in L_0^\infty(\pi)$, define

$$\Phi_\beta(f,f) = \int f^2\, d\pi + 2\int fPGf\, d\pi + (1-\beta)\int |PGf|^2\, d\pi.$$

[Note that $\int |PGf|^2 \, d\pi = \int |PG(I-P)h|^2 \, d\pi = \int |Ph|^2 \, d\pi < \infty$.] Then for all $f = (I-P)h$ with $h \in L_0^\infty(\pi)$,

$$\lim_{\alpha \to 1} \Phi_{\alpha,\beta}(f, f) = \Phi_\beta(f, f).$$

PROOF OF STEP II. For $f = (I-P)h$,

$$\begin{aligned}
\Phi_{\alpha,\beta}(f,f) &= \int f G_\alpha^* (I - \beta\alpha^2 P^*P) G_\alpha f \, d\pi \\
&= \int G_\alpha f (I - \beta\alpha^2 P^*P) G_\alpha f \, d\pi \\
&= \int |G_\alpha f|^2 \, d\pi - \beta \int G_\alpha f \alpha^2 P^* P G_\alpha f \, d\pi \qquad (4.7)\\
&= \int |f + \alpha P G_\alpha f|^2 \, d\pi - \beta \int |\alpha P G_\alpha f|^2 \, d\pi \\
&= \int f^2 \, d\pi + 2 \int f \alpha P G_\alpha f \, d\pi + (1-\beta) \int |\alpha P G_\alpha f|^2 \, d\pi.
\end{aligned}$$

Now

$$\left| \int f \alpha P G_\alpha f \, d\pi - \int f P G f \, d\pi \right| \le \|f\|_\infty^2 \sum_{j=1}^\infty (1 - \alpha^j) \int \|P^j(x, \cdot) - \pi\|_v \, d\pi \to 0 \qquad (4.8)$$

by Proposition 2.1 and dominated convergence. Also

$$\begin{aligned}
\alpha P G_\alpha f = \alpha P G_\alpha (I-P) h &\to PG(I-P)h = Ph \text{ a.s. } [\pi], \\
|\alpha P G_\alpha (I-P) h| &\le |\alpha P G_\alpha (I - \alpha P) h| + |\alpha P G_\alpha (\alpha - 1) P h| \\
&\le \|Ph\|_\infty + \|h\|_\infty \\
&\le 2\|h\|_\infty
\end{aligned}$$

since $\|G_\alpha\|_v = (1-\alpha)^{-1}$. By dominated convergence it follows that

$$\int |\alpha P G_\alpha f|^2 \, d\pi \to \int |Ph|^2 \, d\pi = \int |PGf|^2 \, d\pi. \qquad (4.9)$$

By (4.7) to (4.9), the proof of Step II is complete. □

Step III. For $f, g \in L^2(\pi)$,

$$\int fg \, d\pi \le \tfrac{1}{2} \Phi_{\alpha,\beta}(f, f) + \tfrac{1}{2} \sum_{j=0}^\infty \beta^j \int (I - \alpha P)(\alpha^2 P^* P)^j (I - \alpha P^*) g g \, d\pi. \qquad (4.10)$$

PROOF OF STEP III. First we express $\langle f, g\rangle = \int fg\, d\pi$ in terms of $\Phi_{\alpha,\beta}$. Let $L_{\alpha,\beta} = G_\alpha^*(I - \beta\alpha^2 P^*P)G_\alpha$; then $L_{\alpha,\beta}$ is an invertible operator on $L^2(\pi)$ and its inverse is

$$L_{\alpha,\beta}^{-1} = (I - \alpha P)\left(\sum_{j=0}^{\infty} \beta^j \alpha^{2j} (P^*P)^j\right)(I - \alpha P^*).$$

Then

$$\langle f, g\rangle = \langle f, L_{\alpha,\beta} L_{\alpha,\beta}^{-1} g\rangle = \Phi_{\alpha,\beta}\big(f, L_{\alpha,\beta}^{-1} g\big).$$

Next we observe that $\Phi_{\alpha,\beta}$ is positive semidefinite; for,

$$\begin{aligned}\Phi_{\alpha,\beta}(f, f) &= \langle f, L_{\alpha,\beta} f\rangle = \big\langle f, G_\alpha^*\big(I - \beta\alpha^2 P^*P\big)G_\alpha f\big\rangle\\ &= \big\langle G_\alpha f, \big(I - \beta\alpha^2 P^*P\big)G_\alpha f\big\rangle\\ &= \|G_\alpha f\|_2^2 - \beta\alpha^2 \|PG_\alpha f\|_2^2\\ &\geq 0\end{aligned}$$

by the invariance of π and Jensen's inequality. Since $\Phi_{\alpha,\beta}$ is also symmetric, it follows that

$$\begin{aligned}\langle f, g\rangle &= \Phi_{\alpha,\beta}\big(f, L_{\alpha,\beta}^{-1} g\big)\\ &\leq \tfrac{1}{2}\Phi_{\alpha,\beta}(f, f) + \tfrac{1}{2}\Phi_{\alpha,\beta}\big(L_{\alpha,\beta}^{-1} g, L_{\alpha,\beta}^{-1} g\big)\\ &= \tfrac{1}{2}\Phi_{\alpha,\beta}(f, f) + \tfrac{1}{2}\langle g, L_{\alpha,\beta}^{-1} g\rangle\\ &= \tfrac{1}{2}\Phi_{\alpha,\beta}(f, f) + \tfrac{1}{2}\sum_{j=0}^{\infty} \beta^j \int (I - \alpha P)(\alpha^2 P^*P)^j (I - \alpha P^*) g\, g\, d\pi,\end{aligned}$$

proving (4.10). □

Step IV. For $f = (I - P)h$ with $h \in L_0^\infty(\pi)$, $g \in L^2(\pi)$,

$$\begin{aligned}\int fg\, d\pi \leq \tfrac{1}{2}\Phi_\beta(f, f) + \tfrac{1}{2}\sum_{j=0}^{\infty}\Big\{&\beta^{2j}\int |(P^*P)^j (I - P^*) g|^2\, d\pi\\ &+ \beta^{2j+1}\int |(PP^*)^j P(I - P^*) g|^2\, d\pi\Big\}.\end{aligned} \tag{4.11}$$

PROOF OF STEP IV. First let $\alpha \to 1$ in (4.10). Then:

(i) $(I - \alpha P)(\alpha^2 P^*P)^j (I - \alpha P^*) g \to_{L^2(\pi)} (I - P)(P^*P)^j (I - P^*) g$,

(ii)
$$\begin{aligned}&\left|\int (I - \alpha P)(\alpha^2 P^*P)^j (I - \alpha P^*) g\, g\, d\pi\right|\\ &\quad\leq \|(I - \alpha P)(P^*P)^j (I - \alpha P^*) g\|_2 \|g\|_2\\ &\quad\leq 4\|g\|_2^2\end{aligned}$$

and therefore by dominated convergence,

$$\sum_{j=0}^{\infty} \beta^j \int (I - \alpha P)(\alpha^2 P^* P)^j (I - \alpha P^*) g g \, d\pi$$
$$\to \sum_{j=0}^{\infty} \beta^j \int (I - P)(P^* P)^j (I - P^*) g g \, d\pi.$$

Now by (4.10) and Step II, for $f = (I - P)h$ with $h \in L_0^\infty(\pi)$, $g \in L^2(\pi)$, we have

$$\int fg \, d\pi \le \tfrac{1}{2}\Phi_\beta(f, f) + \tfrac{1}{2} \sum_{j=0}^{\infty} \beta^j \int (I - P)(P^* P)^j (I - P^*) g g \, d\pi. \tag{4.12}$$

Next, for $j = 2k$ we have

$$\begin{aligned}
&\int (I - P)(P^* P)^{2k} (I - P^*) g g \, d\pi \\
&\quad = \int (I - P^*) g (P^* P)(P^* P)^{2k-1} (I - P^*) g \, d\pi \\
&\quad = \int P^* P (I - P^*) g (P^* P)^{2k-1} (I - P^*) g \, d\pi \\
&\qquad \vdots \\
&\quad = \int (P^* P)^k (I - P^*) g (P^* P)^k (I - P^*) g \, d\pi \\
&\quad = \int |(P^* P)^k (I - P^*) g|^2 \, d\pi.
\end{aligned} \tag{4.13}$$

and for $j = 2k + 1$,

$$\begin{aligned}
&\int g (I - P)(P^* P)^{2k+1} (I - P^*) g \, d\pi \\
&\quad = \int (I - P^*) g (P^* P)(P^* P)^{2k} (I - P^*) g \, d\pi \\
&\quad = \int P (I - P^*) g (P P^*)^{2k} P (I - P^*) g \, d\pi \\
&\qquad \vdots \\
&\quad = \int |(P P^*)^k P (I - P^*) g|^2 \, d\pi.
\end{aligned} \tag{4.14}$$

Now (4.11) follows from (4.12) to (4.14). □

Step V. For $f = (I - P)h$ with $h \in L_0^\infty(\pi)$, $g \in L_0^1(\pi)$, (4.11) holds.

PROOF OF STEP V. If $\int |(I - P^*)g|^2 \, d\pi = \infty$ there is nothing to prove, so assume $\int |(I - P^*)g|^2 \, d\pi < \infty$. Let $\varphi = (I - P^*)g$, and let $\{\varphi_n\}$ be a sequence of bounded measurable functions such that

$$|\varphi_n| \le |\varphi|, \qquad \varphi_n \to \varphi \text{ pointwise.}$$

Define $g_n = G_n^* \varphi_n$. Then (i) $g_n \in L^2(\pi)$, (ii) $(I - P^*)g_n = \varphi_n - (P^*)^{n+1}\varphi_n$. Now $\varphi_n \to_{L^2(\pi)} \varphi$ by dominated convergence and since

$$\begin{aligned} \|(P^*)^{n+1}\varphi_n\|_2 &\le \|(P^*)^{n+1}(\varphi_n - \varphi)\|_2 + \|(P^*)^{n+1}\varphi\|_2 \\ &\le \|\varphi_n - \varphi\|_2 + \|(P^*)^{n+1}\varphi\|_2 \end{aligned}$$

and $\|(P^*)^{n+1}\varphi\|_2 \to 0$ by property (3) of the adjoint operator, we have

$$(I - P^*)g_n \to_{L^2(\pi)} (I - P^*)g.$$

Therefore, for all $j \ge 0$,

$$(P^*P)^j (I - P^*)g_n \to_{L^2(\pi)} (P^*P)^j (I - P^*)g$$

and in particular

$$\int |(P^*P)^j (I - P^*)g_n|^2 \, d\pi \to \int |(P^*P)^j (I - P^*)g|^2 \, d\pi. \tag{4.15}$$

Similarly,

$$\int |(PP^*)^j P(I - P^*)g_n|^2 \, d\pi \to \int |(PP^*)^j P(I - P^*)g|^2 \, d\pi. \tag{4.16}$$

Also

$$\begin{aligned} \|(P^*P)^j (I - P^*)g_n\|_2^2 &\le \|(I - P^*)g_n\|_2^2 \le 4\|\varphi\|_2^2, \\ \|(PP^*)^j P(I - P^*)g_n\|_2^2 &\le \|(I - P^*)g_n\|_2^2 \le 4\|\varphi\|_2^2. \end{aligned}$$

By (4.15) to (4.17) and dominated convergence,

$$\begin{aligned} &\sum_{j=0}^{\infty} \Big\{ \beta^{2j} \int |(P^*P)^j (I - P^*)g_n|^2 \, d\pi \\ &\qquad + \beta^{2j+1} \int |(PP^*)^j P(I - P^*)g_n|^2 \, d\pi \Big\} \\ &\quad \to \sum_{j=0}^{\infty} \Big\{ \beta^{2j} \int |(P^*P)^j (I - P^*)g|^2 \, d\pi \\ &\qquad + \beta^{2j+1} \int |(PP^*)^j P(I - P^*)g|^2 \, d\pi \Big\}. \end{aligned} \tag{4.18}$$

Also

$$\int fg_n \, d\pi = \int (I-P)hG_n^*\varphi_n \, d\pi$$

$$(4.19) \qquad = \int G_n(I-P)h\,\varphi_n \, d\pi$$

$$= \int h\varphi_n \, d\pi - \int P^{n+1}h\varphi_n \, d\pi.$$

But $\varphi_n \to_{L^1(\pi)} \varphi$, and therefore

$$\int h\varphi_n \, d\pi \to \int h\varphi \, d\pi = \int h(I-P^*)g \, d\pi$$

$$(4.20) \qquad = \int (I-P)hg \, d\pi$$

$$= \int fg \, d\pi,$$

while

$$(4.21) \qquad \left|\int P^{n+1}h\,\varphi_n \, d\pi\right| \le \int |P^{n+1}h||\varphi| \, d\pi \to 0$$

by dominated convergence, since $P^{n+1}h \to 0$ pointwise by ergodicity and $|P^{n+1}h||\varphi| \le \|h\|_\infty|\varphi|$.

From (4.19) to (4.21) we conclude that

$$(4.22) \qquad \int fg_n \, d\pi \to \int fg \, d\pi.$$

Finally, since (4.11) holds for g_n, passing to the limit as $n \to \infty$ and using (4.18) and (4.22), we indeed obtain (4.11) for $f = (I-P)h$ with $h \in L_0^\infty(\pi)$, $g \in L_0^1(\pi)$. Now letting $\beta \to 1$ in the general form of (4.11) and taking into account Step I, the proof of (4.3) is complete. □

PROOF OF THEOREM 4.1. We have already shown in the proof of Theorem 3.2 that $I_0(\lambda) < \infty$ implies $\lambda(S) = 0$ and $\lambda \ll \pi$. In view of Lemma 4.2, it only remains to prove that for every $g \in L_0^1(\pi)$,

$$\sup_{f \in L_0^\infty(\pi)} \left[\int fg \, d\pi - \tfrac{1}{2}\Phi(f,f)\right] \ge \tfrac{1}{2}\psi(g,g).$$

We consider two cases.

Case I. $\int |(I-P^*)g|^2 \, d\pi < \infty$.

By Abel's theorem,

$$(4.23) \qquad \psi(g,g) = \lim_{\beta \uparrow 1} \sum_{j=0}^{\infty} \left\{ \beta^{2j} \int |(P^*P)^j(I-P^*)g|^2 \, d\pi + \beta^{2j+1} \int |(PP^*)^j P(I-P^*)g|^2 \, d\pi \right\}.$$

Let $\{g_n\}$ be as in the proof of Lemma 4.2. Then, arguing as in that proof,

$$\int\Big\{(P^*P)^j(I-P^*)g\,(P^*P)^j(I-P^*)g_n$$
$$+(PP^*)^jP(I-P^*)g\,(PP^*)^jP(I-P^*)g_n\Big\}\,d\pi$$
$$-\tfrac{1}{2}\int\Big\{|(P^*P)^j(I-P^*)g_n|^2+|(PP^*)^jP(I-P^*)g_n|^2\Big\}\,d\pi$$
$$\to\tfrac{1}{2}\int\Big\{|(P^*P)^j(I-P^*)g|^2+|(PP^*)^jP(I-P^*)g|^2\Big\}\,d\pi.$$

Let $u^{(j)}=(P^*P)^j(I-P^*)g$, $u_n^{(j)}=(P^*P)^j(I-P^*)g_n$. Then by (4.17),

$$\begin{aligned}|\langle u^{(j)},u_n^{(j)}\rangle-\tfrac{1}{2}\langle u_n^{(j)},u_n^{(j)}\rangle| &\le \|u^{(j)}\|_2\|u_n^{(j)}\|_2+\tfrac{1}{2}\|u_n^{(j)}\|_2^2\\ &\le 6\|(I-P^*)g\|_2^2,\end{aligned}$$

and a similar bound holds for

$$v^{(j)}=(PP^*)^jP(I-P^*)g,\qquad v_n^{(j)}=(PP^*)^jP(I-P^*)g_n.$$

Therefore, by dominated convergence,

$$\begin{aligned}
&\tfrac{1}{2}\psi(g,g)\\
&\quad=\lim_{\beta\to1}\lim_{n\to\infty}\sum_{j=0}^{\infty}\bigg\{\beta^{2j}\int\Big[(P^*P)^j(I-P^*)g(P^*P)^j(I-P^*)g_n\\
&\qquad\qquad-\tfrac{1}{2}|(P^*P)^j(I-P^*)g_n|^2\Big]\,d\pi\\
&\qquad+\beta^{2j+1}\int\Big[(PP^*)^jP(I-P^*)g\,(PP^*)^jP(I-P^*)g_n\\
&\qquad\qquad-\tfrac{1}{2}|(PP^*)^jP(I-P^*)g_n|^2\Big]\,d\pi\bigg\}\\
&\quad=\lim_{\beta\to1}\lim_{n\to\infty}\sum_{j=0}^{\infty}\bigg\{\beta^{2j}\int\Big[g\,(I-P)(P^*P)^{2j}(I-P^*)g_n\\
&\qquad\qquad-\tfrac{1}{2}g_n(I-P)(P^*P)^{2j}(I-P^*)g_n\Big]\,d\pi\\
&\qquad+\beta^{2j+1}\int\Big[g(I-P)(P^*P)^{2j+1}(I-P^*)g_n\\
&\qquad\qquad-\tfrac{1}{2}g_n(I-P)(P^*P)^{2j+1}(I-P^*)g_n\Big]\,d\pi\bigg\}
\end{aligned}\tag{4.24}$$

$$= \lim_{\beta\to 1}\lim_{n\to\infty}\int\Bigg[g(I-P)\Bigg(\sum_{j=0}^{\infty}\beta^j(P^*P)^j\Bigg)(I-P^*)g_n$$

$$-\tfrac{1}{2}g_n(I-P)\Bigg(\sum_{j=0}^{\infty}\beta^j(P^*P)^j\Bigg)(I-P^*)g_n\Bigg]\,d\pi$$

$$= \lim_{\beta\to 1}\lim_{n\to\infty}\lim_{\alpha\to 1}\left[\int g h_\beta^{(n)}\,d\pi - \tfrac{1}{2}\int h_{\alpha,\beta}^{(n)} G_\alpha^*(I-\beta\alpha^2P^*P)G_\alpha h_{\alpha,\beta}^{(n)}\,d\pi\right],$$

where

$$h_\beta^{(n)} = (I-P)\Bigg(\sum_{j=0}^{\infty}\beta^j(P^*P)^j\Bigg)(I-P^*)g_n,$$

$$h_{\alpha,\beta}^{(n)} = (I-\alpha P)\Bigg(\sum_{j=0}^{\infty}\beta^j\alpha^{2j}(P^*P)^j\Bigg)(I-\alpha P^*)g_n,$$

so

$$g_n = G_\alpha^*(I-\beta\alpha^2P^*P)G_\alpha h_{\alpha,\beta}^{(n)}.$$

Arguing as in Step II of the proof of Lemma 4.2 and using the easily proved fact that for each fixed n, β, as $\alpha\to 1$,

$$\|h_{\alpha,\beta}^{(n)} - h_\beta^{(n)}\|_\infty \to 0,$$

we have, as $\alpha \to 1$,

$$\int h_{\alpha,\beta}^{(n)} G_\alpha^*(I-\beta\alpha^2P^*P)G_\alpha h_{\alpha,\beta}^{(n)}\,d\pi$$

$$= \int\left(h_{\alpha,\beta}^{(n)}\right)^2 d\pi + 2\int h_{\alpha,\beta}^{(n)}\alpha P G_\alpha h_{\alpha,\beta}^{(n)}\,d\pi$$

$$+(1-\beta)\int|\alpha P G_\alpha h_{\alpha,\beta}^{(n)}|^2\,d\pi \tag{4.25}$$

$$\to \int\left(h_\beta^{(n)}\right)^2 d\pi + 2\int h_\beta^{(n)} P G h_\beta^{(n)}\,d\pi$$

$$+(1-\beta)\int\left|P\Bigg(\sum_{j=0}^{\infty}\beta^j(P^*P)^j\Bigg)(I-P^*)g_n\right|^2 d\pi.$$

Now by (4.24) and (4.25),

$$\tfrac{1}{2}\psi(g,g) \le \lim_{\beta\to 1}\liminf_n\left[\int g h_\beta^{(n)}\,d\pi - \tfrac{1}{2}\left(\int\left(h_\beta^{(n)}\right)^2 d\pi + 2\int h_\beta^{(n)} P G h_\beta^{(n)}\,d\pi\right)\right]$$

$$\le \sup_{f\in L_0^\infty(\pi)}\left[\int fg\,d\pi - \tfrac{1}{2}\Phi(f,f)\right].$$

Case II. $\int |(I - P^*)g|^2 \, d\pi = \infty$.

In this case $\psi(g, g) = \infty$, so we must show

$$\sup_{f \in L_0^\infty(\pi)} \left[\int fg \, d\pi - \tfrac{1}{2}\Phi(f, f) \right] = \infty. \tag{4.26}$$

Let $f = (I - P)h$, with $h \in L^\infty(\pi)$. Then $f \in L_0^\infty(\pi)$ and

$$\begin{aligned} &\int fg \, d\pi - \tfrac{1}{2}\Phi(f, f) \\ &\quad = \int g(I - P)h \, d\pi - \tfrac{1}{2}\int \Big([(I - P)h]^2 + 2(I - P)h\, Ph \Big) \, d\pi \\ &\quad = \int h(I - P^*)g \, d\pi - \tfrac{1}{2}\int \Big(h^2 - (Ph)^2 \Big) \, d\pi \\ &\quad \geq \int h\varphi \, d\pi - \tfrac{1}{2}\int h^2 \, d\pi, \end{aligned} \tag{4.27}$$

where $\varphi = (I - P^*)g$. Now let $h_n = \varphi I_{\{|\varphi| \leq n\}}$. Then

$$\int h_n \varphi \, d\pi - \tfrac{1}{2}\int h_n^2 \, d\pi = \tfrac{1}{2}\int \varphi^2 I_{\{|\varphi| \leq n\}} \, d\pi \to \infty. \tag{4.28}$$

But (4.27) and (4.28) imply (4.26). This completes the proof. □

ADDENDUM. After the present paper had been submitted for publication, paper [14] appeared. This work contains some results related to our Theorems 3.1 and 3.2, namely Theorem 2.1 and 2.3 of [14]. The assumptions in [14] are, however, substantially stronger than those in our results. For, let P be ergodic. It is assumed in [14] that for the operator P: $L^2(\pi) \to L^2(\pi)$, 1 is an isolated simple eigenvalue and there is no other point of the spectrum on $\{z: |z| = 1\}$. It is not difficult to show that under this condition, there exists $\lambda > 1$ such that

$$\sum_{n=0}^{\infty} \lambda^n \| P^n - 1 \otimes \pi \|_2 < \infty, \tag{4.29}$$

where $\|\cdot\|_2$ is the operator norm in $L(L^2(\pi))$. But it is easy to verify that condition (4.29) is stronger than geometric ergodicity (see [12]), which is in turn stronger than ergodicity of degree 2.

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