QUANTUM OPERATORS IN CLASSICAL PROBABILITY THEORY: IV. QUASI-DUALITY AND THINNINGS OF INTERACTING PARTICLE SYSTEMS

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Duality has proved to be a powerful technique in the study of interacting particle systems (IPS). This concept can be enlarged and a "quasi-duality" defined between various pairs of IPS previously thought unrelated. Consequently, theorems of a similar style to those involving duality can be deduced.

The concept of quasi-duality follows naturally from our previous studies into the use of "single-site operators" (an idea borrowed from quantum physics) in paper II of this series. It is shown that a necessary condition for quasi-duality is that the eigenvalues of the corresponding two-site infinitesimal generators be the same, and, using this observation, a number of quasi-dual pairs have been found and studied.

It is further shown that if two different IPS share a common dual, then one can be considered as a "thinning" of the other.

1. **Introduction.** This is paper IV in a series of papers showing how methods drawn from theoretical physics may be applied to problems in interacting particle systems (IPS). In paper II, Lloyd and Sudbury (1995), it was shown how the concept of duality in IPS could be treated by algebraic methods rather than in the more usual way using Harris diagrams. This paper extends the work in paper II and cannot be readily understood unless most of Sections 2, 3 and 4 of that paper have been absorbed.

In this paper we shall at first confine ourselves to IPS on a graph G with n sites, although, as was shown at the end of Section 4 in paper II, as long as one of the initial configurations is finite, the results do extend to infinite graphs. Section 2 of paper II explains the manner in which probability distributions on the 2^n possible configurations can be expressed as nonnegative linear sums of the 2^n basis vectors $|i_1, \ldots, i_n\rangle$, $i_j = 0, 1$, each of which represents a configuration.

There is a 1:1 correspondence between subsets of G and configurations. If A is a subset, then we write $|A\rangle$ for the basis vector with $i_j = 1$ iff $j \in A$. We write $A(j) = 1$ if $j \in A$, $A(j) = 0$ otherwise. Notice that $|A|$ means the number of sites in A.

Section 2 of paper II further explains the special role of the single-site operators, O_i, which only change the configuration at site i, and demonstrates that all such operators can be expressed as linear sums of four fundamental single-site operators, $n_i, \bar{n}_i, S_i^+, S_i^-$. Section 2 ends by expressing the infinitesimal generators of several well-known IPS in terms of these four operators.

Duality is closely related to time reversal with occupied sites changed into empty sites and vice versa, that is, the "particle-hole" transformation. If this were the definition, then we would have $P(\xi_t^A \cap B^c) = P(\zeta_t^B \cap A^c)$ for all A, B, t, where the notation ξ_t^A means the set of occupied sites at time t when the initial set of occupied sites is A. However, the only solutions to these conditions involve "doubly-stochastic" infinitesimal generators, such as those for the exclusion process and the annihilating branching process.

For this reason the standard definition for two processes to be dual is the modification $P(\zeta_t^A \cap B = \varnothing) = P(\xi_t^B \cap A = \varnothing)$. This definition of duality is usually analysed geometrically by tracing paths through Harris diagrams.

However, Section 4 of paper II demonstrates that there is a more general type of duality than coalescing duality, $P(\zeta_t^A \cap B = \varnothing) = P(\xi_t^B \cap A = \varnothing)$, and annihilating duality, $P(|\zeta_t^A \cap B| \text{ is odd}) = P(|\xi_t^B \cap A| \text{ is odd})$, which are both special cases of the equation

$$(1) \qquad\qquad E(a^{|\zeta_t^A \cap B|}) = E(a^{|\xi_t^B \cap A|}),$$

with $a = 0$ and $a = -1$, respectively. The technique was to express (1) as an algebraic equation. Adopting the convention that Q_ζ is the infinitesimal generator of ζ, (1) was shown to be equivalent to

$$(2) \qquad \langle A| \exp(Q_\zeta^T t) UC|B\rangle = \langle B| \exp(Q_\xi^T t) UC|A\rangle,$$

where U and C were the special products of single-site operators

$$(3) \qquad U = \prod_{i \in G}(1 + S_i^- + aS_i^+), \qquad C = \prod_{i \in G}(S_i^- + S_i^+).$$

A sufficient condition for (2) to be true for all A, B is that $Q_\zeta^T UC = C^T U^T Q_\xi$, and paper II demonstrated how this condition could be satisfied.

The "ket" notation $e^{Qt}|B\rangle$, drawn from quantum mechanics, means the distribution at time t of the process with infinitesimal generator Q starting from B. If A and B are configurations, then $|A\rangle$ and $|B\rangle$ are basis vectors and the inner product $\langle A|B\rangle = \delta_{AB}$. Thus $\langle A|e^{Qt}|B\rangle$ is the probability that at time t the configuration is B, when the initial configuration was A. This paper considers choices of products of single-site operators other than U and C and thus derives a number of new relationships between IPS.

In this paper we shall be looking at equations that are similar to (2) but of the form

$$(4) \qquad\qquad \langle A|U \exp(Q_\zeta t)|B\rangle = \langle B|V \exp(Q_\xi t)|A\rangle.$$

If this identity is to be true for all $|A\rangle, |B\rangle$, we require $\exp(Q_\zeta^T t)U^T = V \exp(Q_\xi t)$ for all t. Thus $U^T = V$. We are therefore going to seek pairs of infinitesimal generators Q_ζ, Q_ξ such that

$$(5) \qquad\qquad Q_\zeta^T V = VQ_\xi.$$

We shall restrict ourselves to V being a product of single-site operators, so that $V = \prod V_i$.

In this paper we restrict ourselves to Q_ζ and Q_ξ which can be expressed as sums of the infinitesimal generators governing neighboring pairs of sites, so that $Q = \sum Q(ij)$. Equation (5) then becomes

$$(6) \qquad \sum Q_\zeta^T(ij)\prod V_i = \prod V_i \sum Q_\xi(ij).$$

Since single-site operators commute with operators at other sites, a sufficient condition for (6) is

$$(7) \qquad Q_\zeta^T(ij)V_iV_j = V_iV_jQ_\xi(ij).$$

In Section 2 we shall show that (7) is a necessary and sufficient condition for quasi-duality, a relationship which is a generalization of the usual dual relationship (Theorem 1). We shall further show that IPS can only be quasi-dual if the two-site infinitesimal generators (4×4 matrices) have the same eigenvalues.

In Sections 3–6 we give examples of quasi-duality. In Sections 7–9 it is shown that if two IPS have the same dual, then one can be considered a thinning of the other. Sections 10–12 collect some further results about the biased voter model, the double-flipping model and the biased annihilating branching process.

To give a flavor of the sorts of results that are in the remainder of this paper, we mention a nice result for the biased voter model (BVM) noted in paper II. If ξ is a BVM with transitions $10 \to_1 00$, $01 \to_1 00$, $10 \to_x 11$, $01 \to_x 11$, then

$$(8) \qquad E\big(x^{-|\xi_t^A \cap B|}\big) = E\big(x^{-|\xi_t^B \cap A|}\big).$$

This may be directly derived from (20) of paper II putting $a = x^{-1}$ and representing the two-site infinitesimal generators of ξ as

$$(9) \qquad \begin{pmatrix} \cdot & x & x & \cdot \\ \cdot & -(1+x) & \cdot & \cdot \\ \cdot & \cdot & -(1+x) & \cdot \\ \cdot & 1 & 1 & \cdot \end{pmatrix}.$$

It should be noted that, as in paper II, the positive elements of the matrix are the transition rates from states j to i, λ_{ij}, and that states 1, 2, 3 and 4 are $|11\rangle$, $|10\rangle$, $|01\rangle$ and $|00\rangle$, respectively. We do not always choose to include the negative diagonal terms which ensure that each column of the matrix has zero sum.

2. A necessary condition for quasi-duality. In this section we shall drop the ij in $Q(ij)$ and all the Q-matrices considered will be 4×4 matrices. Notice that $V_{(2)} = V_iV_j$ is also a 4×4 matrix, but is special in that it is the direct product of two single-site operators. In fact, if the 2×2 matrix corresponding to V_i and V_j is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = an + bS^+ + cS^- + d\bar{n},$$

which we shall assume to be nonsingular ($bc \neq ad$), then

$$
(10) \qquad V_{(2)} = \begin{pmatrix} a^2 & ab & ab & b^2 \\ ac & ad & bc & bd \\ ac & bc & ad & bd \\ c^2 & cd & cd & d^2 \end{pmatrix}.
$$

If $|A\rangle$, $|B\rangle$ are basis vectors (i.e., configurations) with $A(j) = B(j)$, $j \neq i$, then

$$
(11) \qquad \langle B|V_i|A\rangle = \begin{cases} a, & \text{if } A(i) = B(i) = 1 \\ b, & \text{if } A(i) = 0, B(i) = 1 \\ c, & \text{if } A(i) = 1, B(i) = 0 \\ d, & \text{if } A(i) = B(i) = 0. \end{cases}
$$

From this we see that

$$
(12) \qquad \langle B|V|A\rangle = a^{|A \cap B|} b^{|B \cap A^c|} c^{|A \cap B^c|} d^{|A^c \cap B^c|}.
$$

Thus, remembering ξ_t^B has distribution $\exp(Q_\zeta t)|B\rangle$ and so on, we see (4) is equivalent to

$$
\begin{aligned}
(13) \qquad & E\big(a^{|A \cap \zeta_t^B|} b^{|\zeta_t^B \cap A^c|} c^{|A \cap (\zeta_t^B)^c|} d^{|A^c \cap (\zeta_t^B)^c|}\big) \\
& = E\big(a^{|\xi_t^A \cap B|} b^{|B \cap (\xi_t^A)^c|} c^{|\xi_t^A \cap B^c|} d^{|(\xi_t^A)^c \cap B^c|}\big).
\end{aligned}
$$

DEFINITION. Two IPS ζ and ξ are said to be *quasi-dual* when they satisfy (13) for all pairs of sets A and B on all finite graphs.

THEOREM 1. *Two IPS ζ and ξ are quasi-dual iff $Q_\zeta^T V_{(2)} = V_{(2)} Q_\xi$, where Q_ζ, Q_ξ are the 4×4 matrices representing the infinitesimal generators on a two-site graph and $V_{(2)}$ is given by (10).*

Further, if ζ and ξ are quasi-dual, then (13) holds for any infinite graph, where A, B are such that the expectations exist.

PROOF. Equation (13) is equivalent to (6), which, when true for all $|A\rangle$, $|B\rangle$, is equivalent to (5). Here $Q_\zeta^T V_{(2)} = V_{(2)} Q_\xi$ is the special case of (5) for a two-site graph. It is equivalent to (7) which we have seen is sufficient for (5) on all finite graphs.

The argument for going from finite to infinite graphs is almost identical to that given in Section 4 of paper II and is not repeated here. Informally, we note that if B is a finite subset of an infinite graph, G, then Theorem 1 has been shown to be true on every finite subset of G, however large. Fix t. Then, given ε, there is a finite $B(t) \subset G$ containing B sufficiently large that the probability any sites outside $B(t)$ can influence B within time t is less than ε. Now apply the theorem for finite graphs to $B(t)$. By taking ever larger $B(t)$ equation (13) can be approximated as close as is wished when the expectations

are bounded. Usually this will require that for each of the eight terms in (13) of the form $u^{|D|}, u \neq 1, |D|$ is a.s finite.

For special choices of a, b, c, d considerable simplifications occur. The coalescing dual is the case $a = 0, b = c = d = 1$ and the annihilating dual $a = -1, b = c = d = 1$.

Thus we are looking for solutions of

$$(14) \qquad\qquad Q_\zeta^T V_{(2)} = V_{(2)} Q_\xi.$$

If x is an eigenvector of Q_ξ, then (14) implies that $V_{(2)}$x is an eigenvector of Q_ζ^T with the same eigenvalue. Suppose Q is "right–left symmetric" so that the rates $\lambda_{2j} = \lambda_{3j}$ and $\lambda_{j2} = \lambda_{j3}, j = 1, 4,$ and $\lambda_{23} = \lambda_{32}$. It is simple to check that $(0 \quad 1 \quad -1 \quad 0)^T$ is an eigenvector of Q and Q^T with eigenvalue $-(\lambda_{12} + \lambda_{42} + 2\lambda_{23})$, and an eigenvector of $V_{(2)}$ with eigenvalue $(ad - bc)$. So, to satisfy (14), the eigenvalue corresponding to $(0 \quad 1 \quad -1 \quad 0)^T$ must be the same for Q_ζ and Q_ξ.

THEOREM 2. *If ζ and ξ are "right–left symmetric," then a necessary condition for Q_ζ and Q_ξ to be quasi-dual is that they have the same eigenvalues. Further, their values of $\lambda_{12} + \lambda_{42} + 2\lambda_{23}$ must be equal.*

Theorem 2 enables us to see which IPS may be quasi-dual to each other and shows that certain pairs of IPS cannot be so related. We therefore give a list of well-known IPS with their corresponding 4×4 matrices and their eigenvectors and eigenvalues where these are simple.

(a) Biased voter model (BVM):

$$\begin{pmatrix} \cdot & x & x & \cdot \\ \cdot & -(1+x) & \cdot & \cdot \\ \cdot & \cdot & -(1+x) & \cdot \\ \cdot & 1 & 1 & \cdot \end{pmatrix}.$$

The eigenvectors are $(1 \quad 0 \quad 0 \quad 0)^T, (0 \quad 0 \quad 0 \quad 1)^T, (0 \quad 1 \quad -1 \quad 0)^T$ and $(2x \quad -(1+x) \quad -(1+x) \quad 2)^T$, with eigenvalues $0, 0, -(1+x), -(1+x),$ respectively.

(b) Annihilating/coalescing random walk (A/CRW):

$$\begin{pmatrix} -2 & \cdot & \cdot & \cdot \\ J_C & -1 & 1 & \cdot \\ J_C & 1 & -1 & \cdot \\ 2J_A & \cdot & \cdot & \cdot \end{pmatrix}.$$

Above $J_A + J_C = 1$. The eigenvectors are $(0 \quad 0 \quad 0 \quad 1)^T, (0 \quad 1 \quad 1 \quad 0)^T, (0 \quad 1 \quad -1 \quad 0)^T$ and $(1 \quad -J_C/2 \quad -J_C/2 \quad -J_A)^T$, with eigenvalues $0, 0, -2, -2,$ respectively.

(c) Double-flipping process (DFP):

$$\begin{pmatrix} -z & \cdot & \cdot & x \\ \cdot & -y & y & \cdot \\ \cdot & y & -y & \cdot \\ z & \cdot & \cdot & -x \end{pmatrix}$$

The eigenvectors are $(x \quad 0 \quad 0 \quad z)^T$, $(0 \quad 1 \quad 1 \quad 0)^T$, $(0 \quad 1 \quad -1 \quad 0)^T$ and $(1 \quad 0 \quad 0 \quad -1)^T$, with eigenvalues 0, 0, $-2y$, $-(z+x)$, respectively.

(d) Contact process (CP):

$$\begin{pmatrix} -2 & x & x & \cdot \\ 1 & -(1+x) & \cdot & \cdot \\ 1 & \cdot & -(1+x) & \cdot \\ \cdot & 1 & 1 & \cdot \end{pmatrix}.$$

The eigenvalues are 0, $-(1+x)$ and $(-(3+x) \pm \sqrt{x^2 - 10x + 1})/2$.

(e) Branching annihilating random walk (BARW):

$$\begin{pmatrix} -2 & x & x & \cdot \\ 0 & -(1+x) & 1 & \cdot \\ 0 & 1 & -(1+x) & \cdot \\ 2 & \cdot & \cdot & \cdot \end{pmatrix}.$$

The eigenvectors are $(1 \quad 0 \quad 0 \quad -1)^T$, $(0 \quad 0 \quad 0 \quad 1)^T$, $(0 \quad 1 \quad -1 \quad 0)^T$ and $(-2x \quad x-2 \quad x-2 \quad 4)^T$, with eigenvalues -2, 0, $-(2+x)$ and $-x$, respectively.

(f) Branching coalescing random walk (BCRW):

$$\begin{pmatrix} -2 & x & x & \cdot \\ 1 & -(1+x) & 1 & \cdot \\ 1 & 1 & -(1+x) & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

The eigenvectors are $(x \quad 1 \quad 1 \quad 0)^T$, $(0 \quad 0 \quad 0 \quad 1)^T$, $(0 \quad 1 \quad -1 \quad 0)^T$ and $(1 \quad 0 \quad -1 \quad 0)^T$, with eigenvalues 0, 0, $-(2+x)$ and $-(2+x)$, respectively.

(g) Biased annihilating branching process (BABP):

$$\begin{pmatrix} -2 & x & x & \cdot \\ 1 & -x & \cdot & \cdot \\ 1 & \cdot & -x & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

The eigenvectors are $(x \quad 1 \quad 1 \quad 0)^T$, $(0 \quad 0 \quad 0 \quad 1)^T$, $(0 \quad 1 \quad -1 \quad 0)^T$ and $(2 \quad -1 \quad -1 \quad 0)^T$, with eigenvalues 0, 0, $-x$ and $-(2+x)$, respectively.

The most obvious division of IPS is into those with one zero eigenvalue and those with two. The DFP is flexible enough to have the same eigenvalues as any IPS with two zero eigenvalues, but it does not follow that there is a DFP quasi-dual to every such IPS. [There will be a matrix V formed from the respective eigenvectors that satisfies (14), but it may not be a product of single-site matrices.] We can see that the BVM can only be quasi-dual to the A/CRW when $x = 1$ (i.e., the VM). The coalescing duality between the VM and the CRW and the annihilating duality between the VM and the ARW are well known.

The CP cannot be dual to any of the other processes listed when x is not large enough for the eigenvalues to be real. The BVM can only be quasi-dual to the BCRW when its x is one more than the x of the BCRW. This duality was given in paper II and has appeared before [Durrett (1988)].

We shall now investigate the special role of the exclusion process (EP) with diffusion rate y. This only allows the interchange $01 \leftrightarrow_y 10$, that is, has $\lambda_{23} = \lambda_{32} = y$. The matrix for this IPS is yE, where

$$(15) \qquad E = \begin{pmatrix} \cdot & \cdot & & \cdot & \cdot \\ \cdot & -1 & & 1 & \cdot \\ \cdot & 1 & & -1 & \cdot \\ \cdot & \cdot & & \cdot & \cdot \end{pmatrix}.$$

It is simple to check that $V_{(2)}$ given in (10) commutes with E for all values of a, b, c, d. If $Q_{\zeta'} = Q_\zeta + yE$, we say that the IPS ζ' is ζ with added diffusion at rate y.

THEOREM 3. *If the IPS ξ, ζ are quasi-dual, and ξ', ζ' have the same rates as ξ, ζ except that both have added diffusion at rate y, then ξ', ζ' have an identical quasi-duality relationship to ξ, ζ.*

PROOF. There exists a product of single-site operators $V_{(2)}$ s.t. $Q_\xi^T V_{(2)} = V_{(2)} Q_\zeta$. Since E and $V_{(2)}$ commute and $E = E^T$, $(Q_\xi^T + yE^T) V_{(2)} = V_{(2)} (Q_\zeta + yE)$, and the result follows. □

Further, by adding diffusion, it is possible to make one IPS quasi-dual to a given IPS when some of the eigenvalues are already the same.

THEOREM 4. *Suppose ζ has "right–left symmetry" and $Q_\xi = Q_\zeta + yE$. Let $(0 \quad 1 \quad -1 \quad 0)^T$ have eigenvalue a. Then the following hold:*

(i) *Every eigenvector of Q_ζ which has eigenvalue $b \neq a$ is an eigenvector of Q_ξ with eigenvalue b.*
(ii) *Every eigenvector of Q_ζ which has eigenvalue a of the form $(u \quad v \quad v \quad w)^T$ is an eigenvector of Q_ξ with eigenvalue a.*
(iii) *$(0 \quad 1 \quad -1 \quad 0)$ is an eigenvalue of Q_ξ with eigenvalue $a - 2y$.*

PROOF. Since $(0 \quad 1 \quad -1 \quad 0)$ is both a left and right eigenvector for each Q-matrix which has "right–left symmetry," it is normal to every eigenvector of Q which does not have the same eigenvalue. They are thus of the form $(u \quad v \quad v \quad w)$. $E(u \quad v \quad v \quad w)^T = 0$. Parts (i) and (ii) of the theorem follow. It is clear that eigenvectors sharing an eigenvalue with $(0 \quad 1 \quad -1 \quad 0)^T$ can be spanned by $(0 \quad 1 \quad -1 \quad 0)^T$ and vectors of the form $(u \quad v \quad v \quad w)$. Part (iii) follows because $(0 \quad 1 \quad -1 \quad 0)^T$ is an eigenvector of E with eigenvalue $-2y$. $\square$

3. The case $V_i = n_i + d\bar{n}_i$ $(a = 1, b = c = 0)$. The above 2×2 representation of V_i is $\mathrm{diag}(1, d)$ so that $V_{(2)} = \mathrm{diag}(1, d, d, d^2)$. If (leaving out the negative diagonal terms)

$$
Q_\zeta = \begin{pmatrix} \cdot & d(x + y/2) & d(x + y/2) & d^2 z \\ x & \cdot & t & d^2 y/2 \\ x & t & \cdot & d^2 y/2 \\ y + z & \cdot & \cdot & \cdot \end{pmatrix},
$$

(16)

$$
Q_\xi = \begin{pmatrix} \cdot & dx & dx & d^2(y + z) \\ x + y/2 & \cdot & t & \cdot \\ x + y/2 & t & \cdot & \cdot \\ z & dy/2 & dy/2 & \cdot \end{pmatrix},
$$

then $Q_\zeta^T V_{(2)} = V_{(2)} Q_\xi$. The effect of having $b = c = 0$ is to require that the exponents of b, c are 0 in (13), that is, that $\zeta_t^B = A$, $\xi_t^A = B$. Equation (13) then becomes

(17) $$d^{n-|A|} P(\zeta_t^B = A) = d^{n-|B|} P(\xi_t^A = B).$$

THEOREM 5. *If an IPS has a Q-matrix with either of the forms given in (16), then the measure $\mu(A) = d^{|A|}$ is an equilibrium measure.*

PROOF. Inserting the fact that $\sum_A P(\zeta_t^B = A) = 1$ into (17), we obtain

$$\sum_A d^{|A|} P(\xi_t^A = B) = d^{|B|}. \qquad \square$$

The special case $y = z = t = 0$, $x = 1$ is the biased annihilating branching process (BABP). If η is the BABP, then (17) implies the following result.

THEOREM 6. *If η is the BABP, then*

$$d^{-|A|} P(\eta_t^B = A) = d^{-|B|} P(\eta_t^A = B).$$

If ξ is a branching coalescing random walk (BCRW), then

$$d^{-|A|} P(\xi_t^B = A) = d^{-|B|} P(\xi_t^A = B).$$

The latter equation is (17) when $y = z = 0$, $x = t = 1$. The IPS with $x = y = 0$ is a DFP and will turn up in Section 6.

4. The case $V_i = an_i + b(S_i^+ + \bar{n}_i) + S_i^-$ ($d = b$, $c = 1$). This choice of V_i gives

$$(18) \qquad V_{(2)} = \begin{pmatrix} a^2 & ab & ab & b^2 \\ a & ab & b & b^2 \\ a & b & ab & b^2 \\ 1 & b & b & b^2 \end{pmatrix}.$$

With $b = d$, $c = 1$, Theorem 1 gives

$$(19) \qquad b^{n-|A|} E(a^{|\zeta_t^B \cap A|}) = E(a^{|\xi_t^A \cap B|} b^{|n - \xi_t^A|}),$$

n being the number of sites on the lattice. Putting $a = b = -1$ in $V_{(2)}$, it is elementary to show that

$$(20) \qquad Q_\zeta = \begin{pmatrix} \cdot & 1-x & 1-x & 2x \\ \cdot & \cdot & x & \cdot \\ \cdot & x & \cdot & \cdot \\ \cdot & 1+x & 1+x & \cdot \end{pmatrix}$$

and

$$(21) \qquad Q_\xi = \begin{pmatrix} \cdot & 2x & 2x & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 2 & \cdot & \cdot & \cdot \end{pmatrix}$$

satisfy (14). (Note that the diagonal terms in the Q-matrices have been omitted.) Here ξ is the branching annihilating random walk (BARW) of Bramson and Gray (1983). However, the case $a = b$ is essentially the same as the dual given in (1) except that the ζ above is the ζ in (1) with "holes" and "particles" interchanged. Thus we have the following result.

THEOREM 7. *If*

$$Q_\zeta = \begin{pmatrix} \cdot & 1+x & 1+x & \cdot \\ \cdot & \cdot & x & \cdot \\ \cdot & x & \cdot & \cdot \\ 2x & 1-x & 1-x & \cdot \end{pmatrix}, \qquad Q_\xi = \begin{pmatrix} \cdot & 2x & 2x & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 2 & \cdot & \cdot & \cdot \end{pmatrix},$$

then

$$P(|\zeta_t^B \cap A| \text{ is odd}) = P(|\xi_t^A \cap B| \text{ is odd}).$$

This is (21) of paper II with $a = 1$, $b = 2x$, although this dual was not considered there. Theorem 4 may be considered a generalization of the well-known dual between the ARW and the voter model which arises when $x = 0$.

A rather more surprising case is with $b = \sqrt{x}$, $a = -1$ and

(22)
$$Q_\zeta = \begin{pmatrix} \cdot & \cdot & \cdot & \dfrac{(\sqrt{x}-1)^2}{2} \\ \cdot & \cdot & \dfrac{1+x}{2} & \cdot \\ \cdot & \dfrac{1+x}{2} & \cdot & \cdot \\ \dfrac{(\sqrt{x}+1)^2}{2} & \cdot & \cdot & \cdot \end{pmatrix},$$

$$Q_\xi = \begin{pmatrix} \cdot & x & x & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot \end{pmatrix}.$$

Equation (19) then gives the following result.

THEOREM 8.   *With ζ, ξ defined by (22),*

$$(\sqrt{x})^{-|A|}[1 - 2P(|\zeta_t^B \cap A| \text{ is odd})] = E\big((-1)^{|\xi_t^A \cap B|}(\sqrt{x})^{-|\xi_t^A|}\big).$$

It is quite simple to deduce the limiting distribution of ζ_t on an infinite lattice from Theorem 5. For $x > 1$, $|\xi_t^A| \to 0$ with probability $x^{-|A|}$ and $|\xi_t^A| \to \infty$ with probability $1 - x^{-|A|}$ since the number of particles performs an embedded random walk with probability $x/(1+x)$ of going up and absorption at 0. Thus

$$1 - 2P\big(|\zeta_t^B \cap A| \text{ is odd}\big) \to (\sqrt{x})^{-|A|}.$$

Since this is true for all A, we obtain the following result.

COROLLARY.   *For all B,*

$$\zeta_t^B \to_d \nu_{(1 - 1/\sqrt{x})/2}.$$

It should be noted that Q_ζ in (22) is a special case of Q_ζ in (16) with $d = (\sqrt{x}-1)/(\sqrt{x}+1)$. Theorem 5 tells us that $d^{|A|}$ is an equilibrium measure for the process, and this corresponds to the product measure given in the corollary to Theorem 5. Equation (17) can then be written as follows.

THEOREM 9.   *When ζ is the DFP with rates given by Q_ζ in (22),*

$$\frac{P(\zeta_t^A = B)}{((\sqrt{x}-1)/(\sqrt{x}+1))^{|B|}} = \frac{P(\zeta_t^B = A)}{((\sqrt{x}-1)/(\sqrt{x}+1))^{|A|}}.$$

5. The case $V_i = an_i + S_i^+$ ($b = 1$, $c = d = 0$).   The effect of having $c = d = 0$ is that their exponents in (13) must be 0. This requires that both ζ_t^B, B are the configurations "all occupied" which we shall designate by $|\Omega\rangle$. Equation (13) then becomes

(23) $$a^{|A|} P(\zeta_t^B = |\Omega\rangle) = P(B = |\Omega\rangle) E(a^{|\xi_t^A|}).$$

The two sides will not be 0 when ζ is an IPS which does not change $|\Omega\rangle$, and the simplest to choose is the IPS which makes no changes, that is, $Q_\zeta = 0$. The 4×4 matrix $V_{(2)}$ is 0 except for its top line which is $(a^2 \quad a \quad a \quad 1)$. Multiplying this by the general 4×4 right–left symmetric stochastic matrix so that $V_{(2)} Q_\xi = 0$, we obtain the following result.

THEOREM 10.   *If ξ is an IPS whose transition rates satisfy*

$$2\lambda_{24} + (1 + a)\lambda_{14} = \lambda_{42} - a\lambda_{12} = 2a\lambda_{21} + (1 + a)\lambda_{41} = 0,$$

then

$$E(a^{|\xi_t^A|}) = a^{|A|},$$

so that $a^{|\xi_t^A|}$ is a martingale for the process ξ.

EXAMPLE 1 (BVM).

$$\lambda_{42} = 1, \qquad a = x^{-1}, \qquad \lambda_{12} = x,$$

$$E\left(\left(\frac{1}{x}\right)^{|\xi_t^A|}\right) = \left(\frac{1}{x}\right)^{|A|}.$$

This is the well-known martingale that leads to a solution of the gambler's ruin.

EXAMPLE 2 (A/CRW).

$$\lambda_{21} = J_C, \qquad \lambda_{41} = 2J_A, \qquad a = -J_A/(J_A + J_C).$$

It is well known that in one-dimension a.s. the A/CRW ends with one or zero particles, and this martingale enables us to calculate

$$p_0 = \lim_{t \to \infty} P(|\xi_t^A| = 0).$$

Theorem 10 implies that

$$\left(\frac{-J_A}{(J_A + J_C)}\right)^{|A|} = p_0 - \frac{(1 - p_0)J_A}{(J_A + J_C)},$$

giving

$$\lim_{t \to \infty} P(|\xi_t^A| = 0) = \frac{J_A}{2J_A + J_C}\left(1 - \left(\frac{-J_A}{J_A + J_C}\right)^{|A|-1}\right).$$

We note that when $J_C = 0$, the right-hand side is 0 or 1 according as $|A|$ is even or odd as would be expected for the ARW.

EXAMPLE 3 (DFP).

$$a = -1, \qquad \lambda_{14}, \lambda_{41}, \lambda_{23} \neq 0.$$

This simply expresses the fact that a DFP preserves parity.

6. The case $V_i = an_i + bS_i^+ + \bar{n}_i + S_i^-$ ($c = d = 1$). It was shown in paper II that the BABP (β) was self-dual, and this enabled some new results about the process to be derived. To obtain a DFP (δ) which is quasi-dual to the BABP, it is necessary to match eigenvalues. This means we have

$$Q_\delta = \begin{pmatrix} & \cdot & & \cdot & \ddots & \frac{1}{2}(1 + \sqrt{x+1})^2 \\ & & \cdot & & \frac{x}{2} & \\ & & \cdot & & \frac{x}{2} & \cdot & \cdot \\ & & \cdot & \frac{x}{2} & \cdot & & \cdot \\ \frac{1}{2}(\sqrt{x+1} - 1)^2 & \cdot & \cdot & & \cdot & \end{pmatrix}, \qquad Q_\beta = \begin{pmatrix} \cdot & x & x & \cdot \\ 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

and, putting $y = \sqrt{x+1}$, we need $a = (y+1)^{-1}$, $b = -(y-1)^{-1}$, $c = d = 1$ in order to satisfy $Q_\beta^T V_{(2)} = V_{(2)} Q_\delta$. Theorem 1 then gives

$$(24) \quad E\left(\left(\frac{1}{y+1} \right)^{|A \cap \beta_t^B|} \left(\frac{-1}{y-1} \right)^{|A^c \cap \beta_t^B|} \right) = E\left(\left(\frac{1}{y+1} \right)^{|B \cap \delta_t^A|} \left(\frac{-1}{y-1} \right)^{|B \cap (\delta_t^A)^c|} \right).$$

We shall first note some special features of the relationship. When $B = \varnothing$, the BABP cannot grow, so that $\beta_t^B = \varnothing$. Both sides of (24) are then always 1. Here $B = \nu_{x/(1+x)}$ is an equilibrium measure for the BABP, thus $A \cap \beta_t^B$ is the random set obtained by independently retaining each site of A with probability $x/(1+x)$, that is, an $x/(1+x)$-thinning of A. It has p.g.f. $((1 + xs)/(1+x))^{|A|}$. The left-hand side of (24) then equals $(-1)^{n-|A|} y^{-n}$, remembering $1 + x = y^2$. Similarly, the right-hand side is $(-1)^{n-|\delta_t^A|} y^{-n}$. However, since parity is preserved by any DFP, these are the same.

It is simple to check that $\nu_{(1+y^{-1})/2}$ is an equilibrium measure for the DFP, and with this as the initial measure (A), both sides are always 0.

7. Thinnings. In paper II it was shown that duals of the form

$$(25) \qquad\qquad E(a^{|\zeta_t^A \cap B|}) = E(a^{|\xi_t^B \cap A|})$$

existed for many pairs of processes. Given the transition rates of the process ζ, a formula was given [(21)] for the transition rates of a dual process ξ. Usually some of these transition rates were negative and no dual existed, but several examples were given in which there was a dual. We now show that when two processes possess the same dual (albeit with a different value of a), then one can be considered a thinning of the other.

Suppose α_t, β_t and ϕ_t are IPS on G and that

$$E(a^{|\alpha_t^A \cap B|}) = E(a^{|\phi_t^B \cap A|}), \tag{26}$$

$$E(b^{|\beta_t^A \cap B|}) = E(b^{|\phi_t^B \cap A|}). \tag{27}$$

Let the infinitesimal generators of α_t, β_t, ϕ_t be Q_α, Q_β, Q_ϕ, respectively. Then it was shown in (15) of paper II that (26) and (27) were equivalent to

$$Q_\alpha^T U_a C = U_a C Q_\phi, \qquad Q_\beta^T U_b C = U_b C Q_\phi, \tag{28}$$

where $U_a = \prod_{i \in G}(1 + S_i^- + aS_i^+)$, similarly for U_b and $C = \prod_{i \in G}(S_i^+ + S_i^-)$. Equation (28) implies

$$CQ_\phi C^{-1} = U_a^{-1} Q_\alpha^T U_a = U_b^{-1} Q_\beta^T U_b,$$

giving

$$Q_\beta = T^{-1} Q_\alpha T, \tag{29}$$

where $T = (U_b U_a^{-1})^T$. Now

$$(1 + S_i^- + bS_i^+)^{-1} = (1-b)^{-1}(1 - S_i^- - bS_i^+),$$

so, putting $T = \prod_{i \in G} T_i$, where

$$(T_i^T)^{-1} = (1 + S_i^- + aS_i^+)(1-b)^{-1}(1 - S_i^- - bS_i^+),$$

and remembering $S^+ S^- = n$, $S^- S^+ = \bar{n}$, we obtain

$$T_i^{-1} = \frac{1-a}{1-b}n_i + \frac{a-b}{1-b}S_i^- + \bar{n}_i. \tag{30}$$

The distribution of α at time t is designated by a ket vector $|\alpha_t\rangle$. Thus, if $|\beta_0\rangle = T^{-1}|\alpha_0\rangle$, it follows from (29) that

$$|\beta_t\rangle = \exp(Q_\beta t)|\beta_0\rangle = T^{-1}|\alpha_t\rangle. \tag{31}$$

Now (35) implies that, at a site i,

$$T_i^{-1}|1\rangle = \frac{1-a}{1-b}|1\rangle + \frac{a-b}{1-b}|0\rangle, \qquad T_i^{-1}|0\rangle = |0\rangle. \tag{32}$$

So, if site i is occupied and $b < a$, the effect of T_i^{-1} is to select the particle at i with probability $(1-a)/(1-b)$, and $T^{-1} = \prod T_i^{-1}$ represents a $(1-a)/(1-b)$-thinning. Equation (36) may then be interpreted as showing that if the initial distribution for the β-process is a $(1-a)/(1-b)$-thinning of the initial distribution of the α-process, then the distributions at time t have the same relationship.

There is a similar relationship based on $T|\beta_t\rangle$, but this has no obvious probabilistic interpretation. In paper III the new objects formed for this relationship are called quasi-particles. We have shown the following result.

THEOREM 11. *Suppose α_t, β_t and ϕ_t are IPS on G and that*

$$E(a^{|\alpha_t^A \cap B|}) = E(a^{|\phi_t^B \cap A|}),$$

$$E(b^{|\beta_t^A \cap B|}) = E(b^{|\phi_t^B \cap A|}).$$

Then, if $b < a$,

$$(\alpha_t^\mu)_p = \beta_t^{\mu_p},$$

where $p = (1 - a)/(1 - b)$.

8. Examples of thinnings. The following dual is of particular interest. If $\xi(J_A)$ is an A/CRW and ζ the voter model (VM), which is the BVM with $\lambda = 1$, then

(33) $$E((-J_A)^{|\zeta_t^C \cap B|}) = E((-J_A)^{|\xi_t^B(J_A) \cap C|}).$$

This may be checked from (21) of paper II.

Equation (33) shows that the A/CRW is dual to the VM for all $0 \le J_A \le 1$. Applying Theorem 11, we obtain $\xi(J_2)$ as a thinning of $\xi(J_1)$ with $p = (1 + J_1)/(1 + J_2)$.

THEOREM 12.

$$(\xi_t^\mu(J_1))_p = \xi_t^{\mu_p}(J_2), \qquad J_1 < J_2,$$

where $p = (1 + J_1)/(1 + J_2)$.

Thus, if we start the A/CRW (J_1) with distribution μ and the A/CRW (J_2) with a p-thinning of μ, then the distribution of the A/CRW (J_2) remains a p-thinning of the distribution of the A/CRW (J_1) at time t. In particular, putting $J_1 = 0, J_2 = 1$, we see that the ARW will remain a 1/2-thinning of the CRW. Arratia (1981) showed that asymptotically the ARW is a 1/2-thinning of the CRW.

Further, if ζ stands for the BVM with $01 \to_{\lambda+1} 11$ and γ for the BCRW, we have

(34) $$E((1/(\lambda + 1))^{|\zeta_t^A \cap B|}) = E((1/(\lambda + 1))^{|\zeta_t^B \cap A|})$$

and

(35) $$E((-1/\lambda)^{|\gamma_t^A \cap B|}) = E((-1/\lambda)^{|\gamma_t^B \cap A|})$$

and, putting $a = 0$,

(36) $$P(\zeta_t^B \cap A = \varnothing) = P(\gamma_t^A \cap B = \varnothing).$$

We may use either (35) and (36) or (34) and (36) to show that the BCRW is a $\lambda/(1 + \lambda)$-thinning of the BVM. For (35) and (36), we have $a = 0$, $b = -1/\lambda$, and, for (34) and (36), $a = 1/(1+\lambda)$, $b = 0$. Both give $(1-a)/(1-b) = \lambda/(1+\lambda)$ and we have the following result.

THEOREM 13.

$$(\zeta_t^\mu)_p = \gamma_t^{\mu_p},$$

where $p = \lambda/(1 + \lambda)$.

In particular, we note that the two equilibrium distributions for the BVM are "all occupied" or δ_G and "all unoccupied," $\delta_\varnothing$. If the initial configuration is A, then the probability the BVM reaches δ_G is $((1+\lambda)^{-|A|} - 1)/((1+\lambda)^{-n} - 1)$ by the standard theory of the gambler's ruin. This is because whatever the configuration (as long as it is not G or $\varnothing$) the probability of increasing the number of particles by 1 is $(1 + \lambda)/(2 + \lambda)$. Thus the distribution of $\gamma_t^{\mu_p}$ tends to a mixture of $\nu_{\lambda/(1+\lambda)}$ and $\delta_\varnothing$, the latter only occurring when the initial configuration is $\varnothing$.

The contact process (CP) has transitions $01 \to_\lambda 11$, $1 \to_1 0$. It is known that if ρ_t represents a CP, then

(37) $$P(\rho_t^B \cap A = \varnothing) = P(\rho_t^A \cap B = \varnothing).$$

A less interesting relationship is with the process σ_t defined by the transitions

$$01 \to_{\lambda-1} 11, \qquad 11 \to_{(2\lambda-3)/(\lambda-1)} 10, \qquad 11 \to_{2/(\lambda-1)} 00, \qquad 01 \leftrightarrow_1 01.$$

Equation (20) of paper II implies

(38) $$E\left(\left(\frac{-1}{\lambda - 1}\right)^{|\sigma_t^A \cap B|}\right) = E\left(\left(\frac{-1}{\lambda - 1}\right)^{|\rho_t^B \cap A|}\right).$$

It follows with $a = 0$, $b = -1/(\lambda - 1)$ that σ_t is a $(\lambda - 1)/\lambda$-thinning of the CP, ρ_t.

Unfortunately, σ_t is not a well-known process although when $\lambda = 2$ or $3/2$ the transition rates simplify. It is not clear how this will aid in the analysis of the CP or of the branching annihilating random walk of Bramson and Gray (1985) which σ_t most resembles.

9. A general formula for thinnings. In this section we exhibit the Q-matrix Q_β, which is a p-thinning of Q_α in the sense of (30) and (31). Here T_i^{-1} has the 2×2 matrix representation

$$T_i^{-1} = \begin{pmatrix} p & 0 \\ 1 - p & 1 \end{pmatrix}.$$

The procedure is now very similar to that of paper II, Section 4. Putting $Q_\alpha = \sum Q_\alpha(ij)$, where i, j are neighbors, $Q_\beta = T^{-1}Q_\alpha T$ is satisfied if each of the corresponding two-site equations is satisfied, that is, if $Q_\beta(ij) = T_{(2)}^{-1}Q_\alpha(ij)T_{(2)}$, where the direct product

$$T_{(2)}^{-1} = T_j^{-1}T_i^{-1} = \begin{pmatrix} p^2 & \cdot & \cdot & \cdot \\ p(1-p) & p & \cdot & \cdot \\ p(1-p) & \cdot & p & \cdot \\ (1-p)^2 & 1-p & 1-p & 1 \end{pmatrix}.$$

If we put

$$Q_\alpha(ij) = \begin{pmatrix} \cdot & b & b & \cdot \\ c & \cdot & e & \cdot \\ c & e & \cdot & \cdot \\ a & d & d & \cdot \end{pmatrix}, \qquad Q_\beta(ij) = \begin{pmatrix} \cdot & b' & b' & \cdot \\ c' & \cdot & e' & \cdot \\ c' & e' & \cdot & \cdot \\ a' & d' & d' & \cdot \end{pmatrix},$$

where a, b, c, d, e stand for the annihilation, birth, coalescence, death and exclusion rates, respectively, then $Q_\beta(ij) = T_{(2)}^{-1} Q_\alpha(ij) T_{(2)}$ gives

(39)
$$a' = (2p^{-1} - 1)a + (2p + 2p^{-1} - 4)b + (2p^{-1} - 2)c + (2 - 2p^{-1})d,$$
$$b' = pb,$$
$$c' = (1 - p^{-1})a + (3 - p^{-1} - 2p)b + (2 - p^{-1})c + (p^{-1} - 1)d,$$
$$d' = (p - 1)b + d,$$

(40) $$e' = (1 - p)b + e.$$

THEOREM 14. *Given a, b, c, d, e, β is a p-thinning of α if $a', b', c', d', e' \geq 0$.*

10. A thinning-type relationship between the BABP and the DFP. It was shown in paper II that the BABP is self-dual. In fact, $Q_\beta^T(UC) = (UC)Q_\beta$, where $UC = -x^{-1}n + S^+ + S^- + \bar{n}$ (see paper II, Section 4). In Section 6 we showed $Q_\beta^T V_{(2)} = V_{(2)} Q_\delta$. Solving these equations gives a relationship somewhat similar to thinning:

(41) $$Q_\delta = TQ_\beta T^{-1},$$

where $T = V^{-1}UC$. From UC given above and the values of a, b, c, d given earlier in this section, we obtain

(42) $$T_i = \begin{pmatrix} \frac{1}{2} & \frac{1}{2}(1 + y) \\ \frac{1}{2} & -\frac{1}{2}(y - 1) \end{pmatrix},$$

where $T = \prod T_i$.

As always occurs, the columns of T add to 1. If the elements are nonnegative, then, as we have seen, T has a probabilistic interpretation as a "thinning." This is not so in this case; nevertheless, T is always a single-site transform.

Suppose $|p_t\rangle$ is the distribution of a BABP at time t, then $\exp(Q_\beta t)|p_0\rangle = |p_t\rangle$, giving

$$T^{-1} T \exp(Q_\beta t) T^{-1} T |p_0\rangle = |p_t\rangle.$$

Using (41) we have

(43) $$\exp(Q_\delta t) T |p_0\rangle = T |p_t\rangle.$$

Thus, if the initial distribution of the DFP is $T|p_0\rangle$, then the distribution at time t is $T|p_t\rangle$, so that the relationship between the time-dependent distributions of the BABP and the DFP then stays the same. We illustrate this with equilibrium distributions.

If we take $|p_0\rangle = \nu_{x/(1+x)}$, then we know $|p_t\rangle = |p_0\rangle$. Independently at each site,

$$T_i|p_0\rangle = \begin{pmatrix} \frac{1}{2} & \frac{1}{2}(1+y) \\ \frac{1}{2} & -\frac{1}{2}(y-1) \end{pmatrix} \begin{pmatrix} x/(1+x) \\ 1/(1+x) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(1+y^{-1}) \\ \frac{1}{2}(1-y^{-1}) \end{pmatrix},$$

which is $\nu_{(1+y^{-1})/2}$ at a single site and an equilibrium distribution for the DFP.

However, $\delta_\varnothing$ is also an equilibrium for the BABP. In that case at a single site,

$$T_i|p_0\rangle = \begin{pmatrix} \frac{1}{2} & \frac{1}{2}(1+y) \\ \frac{1}{2} & -\frac{1}{2}(y-1) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(1+y) \\ \frac{1}{2}(1-y) \end{pmatrix}.$$

This is clearly not a positive measure, but still must have the property of being unchanged when acted upon by the infinitesimal generator of the DFP. In fact, in this case on the whole graph,

$$T|p_0\rangle = y^n \prod \otimes \left(\tfrac{1}{2}(1+y^{-1})|1\rangle - \tfrac{1}{2}(1-y^{-1})|0\rangle\right).$$

The direct product gives positive values when the number of holes is even and negative when it is odd. Because the DFP preserves parity, both of these distributions are equilibrium measures.

11. The BVM and BABP with diffusion. The BVM is essentially a struggle between two types, usually labeled 0 and 1. It is well known that the equilibrium states are either all sites 0 or all sites 1. If we allowed the types to diffuse, that is, swap states, could the rates be so high that some other equilibrium would be possible? One could imagine the clones breaking up so fast that no clone eventually filled space. We shall now use a simple argument employed by Mountford and Sudbury (1993) to show that this does not happen.

We are considering models with the following transitions: $10 \to_{1+x} 11$, $10 \to_1 00$, $10 \leftrightarrow_y 01$. If ζ_t is the configuration at time t, then define

$$f(\zeta_t) = \sum_{u \in Z^d} \zeta_t(u) r^{|u|}, \qquad r < 1,$$

where $|u| = u_1 + \cdots + u_d$ if $u = (u_1, u_2, \ldots, u_d)$. We show that $f(\zeta_t)$ is a submartingale for a suitable choice of r.

Note that $f(\zeta_t)$ can only change where there are neighboring 0's and 1's. There are two situations:

1. 0 at u, 1 at v, $|v| = |u| + 1$. If there is a change at this site, the expected change in $f(\zeta_t)$ is then, putting $s = |u|$,

$$y(r^s - r^{s+1}) + (1+x)r^s - r^{s+1} = r^s((1+x+y) - r(1+y)) > 0.$$

2. 1 at distance s, 0 at distance $s + 1$. The expected change is then

$$y(r^{s+1} - r^s) + (1 + x)r^{s+1} - r^s = r^s(r(1 + x + y) - (1 + y)) > 0$$

for $r > (1 + y)/(1 + x + y)$. Thus $f(\zeta_t)$ is a submartingale, and since f is bounded, $f(\zeta_t)$ tends to a limit. This implies ζ_∞ is a.s. one of the two states "all 0" or "all 1," because, for any configuration with a neighboring pair 01, $f(\zeta_t)$ can change by an amount bounded away from 0.

THEOREM 15. *On Z^d the BVM with diffusion tends either to the state "all 0" or the state "all 1."*

Sudbury (1993) shows that the duality between the BVM and the BCRW implies the convergence of the BCRW. The BCRW is equivalent to the BABP with added diffusion at rate 1. Theorem 3 shows that adding diffusion to two processes preserves duality. Since the limiting behavior of the BVM with diffusion is the same as for the BVM (Theorem 15), the argument of Sudbury (1993) may be repeated to show the following result.

THEOREM 16. *The BABP with diffusion rate greater than 1 ($10 \rightarrow_x 11$, $11 \rightarrow_1 10$, $01 \leftrightarrow_{1+y} 10$) has limiting measure $\nu_{x/(1+x)}$.*

12. **Some limiting results for the BABP and the DFP.** The first theorem is easy to check.

THEOREM 17. *$\nu_{\sqrt{y}/(\sqrt{x}+\sqrt{y})}$ is an equilibrium measure for a DFP with rates $11 \rightarrow_x 00$, $00 \rightarrow_y 11$, $01 \leftrightarrow_z 10$.*

When $z \geq (x + y)/2$ the DFP is ergodic.

THEOREM 18. *$\nu_{\sqrt{y}/(\sqrt{x}+\sqrt{y})}$ is the limit measure when $z \geq (x + y)/2$.*

We have seen that when $z = (x + y)/2$ such a DFP is quasi-dual to a BVM. Theorem 3 shows that when $z \geq (x + y)/2$ the DFP is quasi-dual to a BVM with added diffusion $z - (x + y)/2$. It was shown in Section 13 that diffusion does not alter the probabilities with which the BVM ends either by dying out or by filling space, and thus the argument used to prove the corollary to Theorem 8 may be used to prove Theorem 18.

When $x = y$ we may show the following result.

THEOREM 19. *On Z^d, $\nu_{1/2}$ is the limiting measure of a DFP with $11 \leftrightarrow_x 00$, $10 \leftrightarrow_z 01$.*

PROOF. For $x \in Z^d$, define $|x| = x_1 + x_2 + \cdots + x_d$, the sum of the components of x. Then the DFP ζ with rates $11 \leftrightarrow_x 00$, $10 \leftrightarrow_z 01$ defines another DFP ζ^*

s.t.

$$\zeta_t^*(x) = \zeta_t(x), \qquad |x| \text{ is even,}$$

$$\zeta_t^*(x) = 1 - \zeta_t(x), \qquad |x| \text{ is odd.}$$

Notice that ζ^* has rates $11 \leftrightarrow_z 00$, $10 \leftrightarrow_x 01$. If $x \geq z$, Theorem 18 implies ζ is ergodic, if $z \geq x$, then the theorem implies ζ^* is ergodic. $\square$

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