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THE SECOND LOWEST EXTREMAL INVARIANT MEASURE OF THE CONTACT PROCESS

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We study the ergodic behavior of the contact process on infinite connected graphs of bounded degree. We show that the fundamental notion of complete convergence is not as well behaved as it was thought to be. In particular there are graphs for which complete convergence holds in any number of separated intervals of values of the infection parameter and fails for the other values of this parameter. We then introduce a basic invariant probability measure related to the recurrence properties of the process, and an associated notion of convergence that we call "partial convergence." This notion is shown to be better behaved than complete convergence, and to hold in certain cases in which complete convergence fails. Relations between partial and complete convergence are presented, as well as tools to verify when these properties hold. For homogeneous graphs we show that whenever recurrence takes place (i.e., whenever local survival occurs) there are exactly two extremal invariant measures.

1. Introduction and main results.

Preliminaries. We consider the contact process on fairly arbitrary graphs. The only restrictions on the graphs are given by (G1)–(G3) in the next subsection. We present results addressing the characterization of the invariant probability measures and their domains of attraction. These are classical issues in the field of interacting particle systems, and are sometimes referred to as the study of the "ergodic behavior" of the process. Although our methods are fairly elementary, the results presented here seem to have remained hidden for the more than 20 years in which the contact process has been extensively studied. We believe this is because most of the efforts have been concentrated on cases in which the graph is homogeneous. In these cases most of the rich structure presented here collapses into other known results. Still, some of the new results refer to homogeneous graphs [these results appear as Theorem 2(i)] and include the fact that for such graphs under local survival there are exactly two extremal invariant measures.

Since its introduction by Harris (1974) and until about 1990, the contact process was mostly studied on the d-dimensional cubic lattice $\mathbb{Z}^d$. (In a

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harmless abuse of notation, we also denote by $\mathbb{Z}^d$ the graph with this set as its vertex set and edges connecting each pair of vertices separated by Euclidean distance 1.) The ergodic behavior of the contact process on such graphs was completely characterized in the fundamental paper by Bezuidenhout and Grimmett (1990), who built on the extensive work of many others during the previous 15 years (see references in that paper). A short time after that paper appeared, interest in the behavior of the contact process on other graphs, especially trees, was raised by Pemantle (1992). In that latter paper most of the analysis concerned the homogeneous trees of degree $d + 1$, which we denote by $\mathbb{T}_d$. It was shown that (when $d \geq 3$) the contact process on such trees has a subtler behavior than the one on $\mathbb{Z}^d$, in that there are at least two different critical points and between them the system can survive in a global sense but not in a local sense. From that paper and subsequent work by Madras and Schinazi (1992), Morrow, Schinazi and Zhang (1994), Durrett and Schinazi (1995), Liggett (1996a), Stacey (1996), and Liggett (1996b) a great deal of information became available about the ergodic behavior of these systems. In particular, the results have now been extended to all values of $d \geq 2$. Between the two critical points just mentioned, it is known that there are infinitely many extremal invariant probability measures, while above the second one of these points there are only two such measures and the complete convergence theorem, to be reviewed later, holds.

Our results basically show that on less regular graphs the ergodic behavior of the contact process can be richer than on $\mathbb{Z}^d$ or $\mathbb{T}^d$. An arbitrarily large number of critical points, separating intervals where the ergodic behavior is qualitatively distinct, can occur. A fundamental invariant probability measure will be identified and studied. It will be shown that this measure is always an extremal invariant measure, and while on $\mathbb{Z}^d$ and $\mathbb{T}_d$ it always coincides either with the lower or the upper invariant measure, this is not the case for other graphs. Finally a notion, which we call "partial convergence," will be introduced. While partial convergence is not as sharp a property as complete convergence, it will be shown to be nevertheless a better-behaved notion. Moreover, we show that in some situations while complete convergence fails, partial convergence holds.

Notation and background. We need to introduce a certain amount of notation. We will also review in greater detail some basic facts about the contact process. The graphs considered in this paper will be supposed to have the following characteristics.

(G1) Be infinite, since otherwise the issues discussed in this paper trivialize.

(G2) Be connected, since otherwise the features of interest can be studied on each connected component.

(G3) Be of bounded degree, that is, each vertex belongs to at most κ edges, for some $\kappa < \infty$. (This restriction is actually more than what we need, and could be replaced by the assumption that the process started from a finite set does not explode for all values of the infection parameter λ introduced below.)

We will denote by $\mathcal{G}$ the class of graphs which satisfy (G1), (G2) and (G3). For a graph $G \in \mathcal{G}$ we denote by V_G its set of vertices, also called sites in this paper, to stick to the usual interacting particle system terminology. Pairs of sites which belong to a common edge of G will be said to be neighbors in G. One of the sites of G will be distinguished from the others and called its root, denoted simply by 0; in this paper the choice of the root will usually be arbitrary, in that the statements made will depend on the graph G but not on the choice of its root. We measure the distance between sites in V_G by the length of the minimal path along neighboring sites which joins them. The ball of center $x \in V_G$ and radius N is denoted by $B(x, N)$. Clearly (G2) and (G3) imply that V_G is a countable set for all $G \in \mathcal{G}$.

A subgraph of a graph G is another graph which has its set of vertices contained in the set of vertices of G and its set of edges contained in the set of edges of G. An isomorphism between two graphs, G_1 and G_2, is a one-to-one mapping from V_{G_1} onto V_{G_2} which preserves the graph structure, that is, such that the set of edges of G_2 can be obtained as the set of pairs of images of vertices of G_1 which form edges. An isomorphism between a graph G and itself is called an automorphism of G. We will say that a graph G_1 can be embedded as a subgraph of another graph G_2 in case there is an isomorphism between G_1 and a subgraph of G_2. A graph is said to be homogeneous if for each pair x and y of its vertices there is an automorphism of the graph which maps x into y. The class of homogeneous graphs in $\mathcal{G}$ will be denoted by $\mathcal{H}$. Typical examples of graphs in $\mathcal{H}$ are $\mathbb{Z}^d$ and $\mathbb{T}_d$, but, of course, there are others. For instance if $G \in \mathcal{H}$ and we add edges to G, connecting all pairs of vertices which are at a given prescribed distance from each other, then the resulting graph is also in $\mathcal{H}$. Homogeneous graphs are also called transitive graphs. Everything that we say about homogeneous graphs in this paper applies also (with essentially the same proofs) to the larger class of almost transitive graphs, defined as those graphs in $\mathcal{G}$ for which there is a finite set of vertices, V_0, with the property that each vertex of the graph can be mapped into one of the vertices of V_0 by an automorphism.

The contact process on the graph $G \in \mathcal{G}$ with infection parameter $\lambda > 0$ is a continuous time Markov process with state space $\{0, 1\}^{V_G}$. Elements of this state space are called configurations. When the configuration at a given site is 1, one says that there is a particle there or that the site is occupied or that the site is infected. Otherwise one says that the site is vacant or healthy. The contact process evolves according to the following local prescription.

1. A particle at a site gives birth to new ones at each neighboring vacant site at rate λ.
2. Particles die at rate 1.

The assumption that G has a bounded degree assures us that there is a well-defined unique Markov process with these features; moreover, it will satisfy the Feller property. For constructions of such processes and also for the proofs of the basic facts reviewed below, the reader can consult, for example, Liggett (1985) or Durrett (1988).

We can think of an element η of $\{0,1\}^{V_G}$ either as a function from V_G to $\{0,1\}$, in which case the notation $\eta(x)$ will be used for the value of this function at $x \in V_G$, or as the subset of V_G where this function takes the value 1. As usual, we will take advantage of this flexibility in our notation, and no confusion should arise from this common practice.

The set $\{0,1\}$ is endowed with the discrete topology and $\{0,1\}^{V_G}$ with the corresponding product topology and corresponding Borel σ-field. Probability distributions on the configuration space are determined then by their finite-dimensional distributions, and the notion of weak convergence corresponds to the convergence of these finite-dimensional distributions. We use the double arrow, $\Rightarrow$, to denote weak convergence. The probability measure which puts all mass on the configuration η will be denoted by δ_η.

We denote by $(\xi_t^\mu: t \geq 0)$ the version of the contact process starting from a configuration which is randomly chosen according to the law μ. When μ is concentrated on the configuration η we write simply $(\xi_t^\eta: t \geq 0)$. Abusing notation one step further, we also write $(\xi_t^x: t \geq 0)$ for the contact process started from a single particle at $x \in V_G$. Similar conventions on the notation will be used systematically without further notice. When there is need to specify the graph G or the value of λ in the notation, this will be done in the following fashion: $(\xi_{G,\lambda:t}^\mu: t \geq 0)$.

For fixed $t \geq 0$, the law of ξ_t^μ will be denoted by $\mu_t = \mu S(t) = \mu S_{G,\lambda}(t)$. The set of invariant probability measures will be denoted by $I = \{\mu: \mu_t = \mu$ for all $t \geq 0\}$. This is a convex set, and the set of its extremal points will be denoted by I_e. It is obvious that $\delta_\varnothing \in I_e$, regardless of the value of λ.

A basic property of the contact process is attractiveness. Endow $\{0,1\}^{V_G}$ with the partial order given by writing $\eta \leq \zeta$ in case $\eta(x) \leq \zeta(x)$ for all $x \in V_G$. Next endow the set of probability measures on $\{0,1\}^{V_G}$ with the partial order given by writing $\mu_1 \leq \mu_2$ in case

$$\int f \, d\mu_1 \leq \int f \, d\mu_2,$$

for all continuous nondecreasing function $f: \{0,1\}^{V_G} \to \mathbb{R}$. This is called the stochastic order. Attractiveness means that the stochastic order is preserved by the time evolution, that is, if $\mu_1 \leq \mu_2$ then $\mu_1 S(t) \leq \mu_2 S(t)$ for all $t \geq 0$.

Attractiveness easily implies the following results. As $t \to \infty$, $\delta_{V_G} S(t) \Rightarrow \bar{\nu}$. Here $\bar{\nu} \in I_e$ is called the upper invariant measure, while $\delta_\varnothing$ is called the lower invariant measure. Having $\delta_\varnothing = \bar{\nu}$ is equivalent to having $\mu S(t) \Rightarrow \delta_\varnothing$ for all laws μ; the process is in this case said to be ergodic. If this happens, in particular, $I = \{\delta_\varnothing\}$.

The contact process enjoys also a property which is stronger than attractiveness. This property is called additivity, and it states that the collection of processes $(\xi_t^A: t \geq 0)$, $A \subset V_G$, can be constructed on a common probability space in such a way that the following relation holds:

$$\xi_t^{A \cup B} = \xi_t^A \cup \xi_t^B,$$

for all pairs of sets A and B, and $t \geq 0$.

Two types of monotonicity are closely related to attractiveness and additivity. One is monotonicity in λ and the other is monotonicity in the graph G. They can be combined in the following single statement. If $\lambda_1 \leq \lambda_2$, and G_1 is a subgraph of G_2, then for all $A \subset V_{G_1}$, the processes $(\xi^A_{G_1, \lambda_1; t}: t \geq 0)$ and $(\xi^A_{G_2, \lambda_2; t}: t \geq 0)$ can be constructed on the same probability space in such a way that $\xi^A_{G_1, \lambda_1; t} \subset \xi^A_{G_2, \lambda_2; t}$ for all $t \geq 0$.

Another basic tool in the study of contact processes is their self-duality. This property can be expressed by

$$\mathbb{P}\big(\xi_t^A \cap B \neq \varnothing\big) = \mathbb{P}\big(\xi_t^B \cap A \neq \varnothing\big),$$

for all pairs of sets A and B, and $t \geq 0$.

In order to introduce two basic critical points for the contact process on a graph, we define $\Omega_\infty^A = \{\xi\colon \xi_t^A \neq \varnothing, \text{ for all } t \geq 0\}$, as the event that the process $(\xi_t^A\colon t \geq 0)$ lives forever; and we set $\rho(A, \lambda) = \rho(A) = \mathbb{P}(\Omega_\infty^A)$. Also we define $\Omega_r^A = \{\xi\colon \xi_t^A(0) = 1, \text{ for an unbounded set of values of } t\}$, as the event that there is recurrence; and we set $\beta(A, \lambda) = \beta(A) = \mathbb{P}(\Omega_r^A)$. When the argument A is omitted in the functions β and ρ, it should be understood that we are taking the set $A = \{0\}$. The positivity of $\rho(A, \lambda)$ for one finite set A clearly implies its positivity for all other such sets, and a similar remark is valid for $\beta(A, \lambda)$. When $\rho(\lambda)$ is positive one says that the contact process survives at λ, or, more precisely, that it survives globally at λ. Otherwise one says that the contact process dies out at λ. When $\beta(\lambda)$ is positive, one says that the contact process is recurrent at λ, or that it survives locally at λ. Next we define the critical values

$$\lambda_s = \lambda_s(G) = \inf\{\lambda\colon \rho(\lambda) > 0\},$$

and

$$\lambda_r = \lambda_r(G) = \inf\{\lambda\colon \beta(\lambda) > 0\}.$$

Of course, the choice of the root for the graph G is irrelevant in the definition of these critical points. Obviously we always have $\lambda_s \leq \lambda_r$. A standard comparison with a branching process shows that for all graphs in G we have $0 < \lambda_s$, and the remark that all these graphs have $\mathbb{Z}_+$ embedded into them gives $\lambda_r \leq \lambda_r(\mathbb{Z}_+) < \infty$. [It is known that $\lambda_r(\mathbb{Z}_+) = \lambda_s(\mathbb{Z}_+) = \lambda_r(\mathbb{Z}) = \lambda_s(\mathbb{Z})$. For this see, e.g., Durrett and Griffeath (1983), or write down a proof based on the renormalization procedure of Bezuidenhout and Grimmett (1990).] We refer to $\lambda_s(G)$ as the survival point of the graph G and to $\lambda_r(G)$ as the recurrence point of this graph.

One should be careful with the distinction between finite and infinite sets above. Even when $\rho(\lambda) = 0$ we still trivially have $\rho(\eta, \lambda) = 1$ for all infinite sets η. Similarly, even when $\beta(\lambda) = 0$ we may have $\beta(\eta, \lambda) > 0$ for some infinite η. This happens, for example, whenever $\rho(\lambda) > 0$ and $\eta = V_G$, as can be easily checked using self-duality to see that $\mathbb{P}(\xi_t^{V_G}(0) = 1) \geq \rho(\lambda)$ for all $t \geq 0$.

A fundamental notion is the following.

Complete convergence (*cc*).

$$\text{For any finite } A \subset V_G,\ \xi_t^A \Rightarrow (1 - \rho(A))\delta_{\varnothing} + \rho(A)\bar{\nu} \quad \text{as } t \to \infty.$$

Or equivalently,

$$\text{for any finite } A, B \subset V_G,\ \mathbb{P}\big(\xi_t^A \cap B \neq \varnothing\big) \to \rho(A)\rho(B) \quad \text{as } t \to \infty.$$

For the equivalence between the two statements one should note that self-duality implies that

$$\bar{\nu}(\zeta : \zeta \cap A \neq \varnothing) = \mathbb{P}\big(\Omega_\infty^A\big) = \rho(A).$$

In the way that *cc* is being defined we are not requiring the system to survive, that is, $\rho(\lambda)$ to be positive. With this definition of *cc* it holds trivially in case $\rho(\lambda) = 0$, or, in other words, when the system is ergodic. We introduce the notation *s* & *cc* (for "survival with complete convergence") to denote the statement that not only *cc* holds, but also $\rho(\lambda) > 0$.

We are not sure about the origin of the term "complete convergence," but it may be because on $\mathbb{Z}^d$, if the statements in the definition of *cc* above hold in the way that they are presented, that is, for finite initial configurations A, then the same is also true for all initial configurations. We see in Theorem 2(h) that this has to be replaced by a more general statement for general graphs in G. Ironically, we see then that it is still true that when *cc* holds, weak convergence always takes place for all initial configurations, justifying, therefore, the name "complete convergence."

The ergodic behavior of the contact process on $\mathbb{Z}^d$ can be summarized by saying that for every dimension d, $0 < \lambda_s = \lambda_r =: \lambda_c < \infty$; for $\lambda \leq \lambda_c$ the process is ergodic; while for $\lambda > \lambda_c$ there are exactly two extremal invariant measures, $\delta_{\varnothing}$ and $\bar{\nu}$, and *cc* holds.

The ergodic behavior of the contact process on $\mathbb{T}_d$ is richer. It is now known that $0 < \lambda_s < \lambda_r < \infty$ for each $d \geq 2$. For $\lambda \leq \lambda_s$ the process is ergodic; for $\lambda_s < \lambda \leq \lambda_r$ there are infinitely many measures in I_e, but $\beta(\lambda) = 0$, so if the process is started from a finite set $A \subset V_G$ then $\xi_t^A \Rightarrow \delta_{\varnothing}$; finally for $\lambda > \lambda_r$ there are exactly two extremal invariant measures, $\delta_{\varnothing}$ and $\bar{\nu}$, and *cc* holds; moreover, $\beta(\lambda) = \rho(\lambda)$.

Results. We started the present investigation by asking ourselves some questions regarding *cc* and *s* & *cc*.

(Q1) Is it the case that, for all graphs in G, *cc* holds for all $\lambda > \lambda_r$?
(Q2) If *s* & *cc* holds for G at λ, does it also hold for G at every $\lambda' > \lambda$?
(Q3) If G_0 can be embedded as a subgraph of G and *s* & *cc* holds for G_0 at λ, is it the case that *s* & *cc* also holds for G at the same λ?

From what is known about the contact process on $\mathbb{Z}^d$ and on $\mathbb{T}_d$, it is clear that in these cases the answer to each one of these questions is "yes" [regarding (Q3), we mean here that we take both G_0 and G as cubic lattices or as homogeneous trees). Also, the need to talk about $s\,\&\,cc$, rather than cc, in (Q2) and (Q3) should be clear, since otherwise the answers are trivially "no," for a spurious reason.] Pemantle (1992) had conjectured that the answer to the first question would be "yes" for generic trees. Nevertheless the answer to the three questions above is in general "no," even if we restrict ourselves to trees.

The example that we present to answer the three questions is actually surprisingly simple; we refer to it as the "basic example" later on. We use the following notation: if G_1 and G_2 are two graphs which have disjoint sets of vertices, then $G_1 \vee G_2$ will denote the graph obtained by connecting their roots, or more precisely, the graph in which the set of vertices is the union of the sets of vertices of G_1 and G_2 and the set of edges is the union of the set of edges of these two graphs plus an edge connecting their roots. Our example is $\mathbb{T}_j \vee \mathbb{T}_k$ with $j > 2$ and k sufficiently larger than j, so that we have

$$\lambda_s(\mathbb{T}_k) < \lambda_r(\mathbb{T}_k) < \lambda_s(\mathbb{T}_j) < \lambda_r(\mathbb{T}_j). \tag{1.1}$$

That these inequalities can all be satisfied by such a choice is an immediate consequence of what we have reviewed about the contact process on these graphs and the fact that as $k \to \infty$, $\lambda_r(\mathbb{T}_k) \to 0$, as proved by Pemantle (1992). Under these conditions, the contact process on $\mathbb{T}_j \vee \mathbb{T}_k$ has the following features:

(BE1) $\lambda_r(\mathbb{T}_j \vee \mathbb{T}_k) \leq \lambda_r(\mathbb{T}_k)$.
(BE2) In the interval $(\lambda_s(\mathbb{T}_j), \lambda_r(\mathbb{T}_j)]$ cc fails.
(BE3) In the intervals $(\lambda_r(\mathbb{T}_k), \lambda_s(\mathbb{T}_j)]$ and $(\lambda_r(\mathbb{T}_j), \infty)$ cc holds.

Together with the fact that $s\,\&\,cc$ holds for $\mathbb{T}_k$ in the interval to which (BE2) refers, these features of $\mathbb{T}_j \vee \mathbb{T}_k$ answer the three questions above in the negative.

The truth of (BE1) is clear from monotonicity in λ. The conditions (BE2) and (BE3) can be proved using Griffeath's equivalence, reviewed in Section 3, but they are also particular cases of more general results stated in Theorem 6 and proved in Section 2, using the machinery developed in this paper. We will nevertheless present now the intuitive reasons for (BE2) and also a heuristic which makes (BE3) at least plausible. We first observe that, intuitively, cc means that if the system survives, then we eventually see $\bar{\nu}$. But in the situation of (BE2), the process can survive in $\mathbb{T}_j$ without ever reaching its root, since we are in the regime where on this graph there is a positive probability of survival without recurrence. If this event happens, we would have survival, but certainly not convergence to the nontrivial measure $\bar{\nu}$. Regarding (BE3), we start with the lower of the two intervals included there (the bounded one). On this interval the process dies out on $\mathbb{T}_j$, so if there is going to be survival on $\mathbb{T}_j \vee \mathbb{T}_k$, then $\mathbb{T}_k$ must contain occupied sites at arbitrarily large times. But because the probability of survival without recurrence on a homogeneous tree is null above its recurrence point, the process will return to the root at arbitrarily large times. This means that for $\mathbb{T}_j \vee \mathbb{T}_k$ survival will also a.s. imply recurrence, that is, $\rho(A) = \beta(A)$. This is not yet the same as saying that *cc* holds, but is a strong indication that if survival occurs then there should be convergence to some nontrivial invariant measure. Regarding the unbounded interval included in (BE3), it seems reasonable to expect *cc* to hold there since it holds then for both homogeneous trees, $\mathbb{T}_j$ and $\mathbb{T}_k$. Under survival there will be recurrence a.s., and so the occurrence of *cc* in this interval is at least as believable as the corresponding statement on the other interval discussed above. To wrap up this heuristic discussion, it may be worth saying that what happens in the region covered by (BE2) is that the process, started from a finite set, can survive but hide in $\mathbb{T}_j$, where survival without recurrence is a possibility. In the regions covered by (BE3) there is no place to hide.

One way to rephrase the negative answer to questions (Q2) and (Q3) is by saying that $s\,\&\,cc$ is not a monotone increasing property of either λ or of the graph. For future reference we define the following notion.

MONOTONE INCREASING PROPERTY. A property of the contact process is said to be monotone increasing when both of the following hold.

(a) If the property holds for the contact process on a graph $G \in \mathbf{G}$ at some λ, then it also holds for the same graph for all $\lambda' > \lambda$.

(b) If the property holds for the contact process on some subgraph $G_0 \in \mathbf{G}$ of some graph $G \in \mathbf{G}$ at some value of λ, then it also holds for G at the same λ.

We also say that a property is $\mathbf{F}$-monotone increasing, for some family of graphs $\mathbf{F} \subset \mathbf{G}$, in case the statements in the definition above are true when $\mathbf{G}$ is replaced by $\mathbf{F}$ in each place where it appears in the definition.

It is clear that the basic example above can be generalized to produce a tree with any number of critical points, separating alternating intervals on which *cc* holds or fails. To this end, it is enough to glue by one site the trees $\mathbb{T}_{d_i}$, $i = 1, \dots, n$, with an appropriate choice of $d_1, \dots, d_n$, so that

$$\begin{aligned} \lambda_s(\mathbb{T}_{d_n}) &< \lambda_r(\mathbb{T}_{d_n}) < \lambda_s(\mathbb{T}_{d_{n-1}}) \\ &< \lambda_r(\mathbb{T}_{d_{n-1}}) < \cdots < \lambda_s(\mathbb{T}_{d_1}) < \lambda_r(\mathbb{T}_{d_1}). \end{aligned} \tag{1.2}$$

It is natural to ask what the ergodic behavior of, for example, the basic example is in the region covered by (BE2), where there is recurrence, but *cc* fails. The theory developed in this paper will answer this question to a great extent.

The main contributions of this paper are the introduction of two objects. The first one is an extremal invariant probability measure for the contact process which is distinct from $\delta_{\emptyset}$ and $\bar{\nu}$ at values of λ where there is recurrence but $\beta(\lambda) < \rho(\lambda)$. This measure will be denoted by ν_r and is defined by setting, for each finite A,

$$\nu_r(\zeta : \zeta \cap A \neq \varnothing) = \mathbb{P}(\Omega_r^A) = \beta(A).$$

The fact that ν_r is a probability measure is not immediately obvious; one needs an argument which shows that the probability of any cylinder set of configurations (i.e., any set of configurations in which the values at a finite set of sites are specified) is positive. The best argument we have found for this is as follows: consider the additive coupling of the processes $(\xi_t^A\colon t \geq 0)$, $A \subset V_G$, and note that ν_r is the law of the random field indexed by V_G which takes the value 1 or 0 at $x \in V_G$ according to whether Ω_r^x happens or not, respectively.

The invariance of ν_r derives from the following computation. For any finite $A \subset V_G$, by self-duality and the Markov property,

$$\begin{aligned}
\mathbb{P}(\xi_t^{\nu_r} \cap A \neq \varnothing) &= \sum_{\substack{B \subset V_G \\ \text{finite}}} \mathbb{P}(\xi_t^A = B)\nu_r(\zeta\colon \zeta \cap B \neq \varnothing) \\
&= \sum_{\substack{B \subset V_G \\ \text{finite}}} \mathbb{P}(\xi_t^A = B)\mathbb{P}(\Omega_r^B) \\
&= \mathbb{P}(\Omega_r^A) = \nu_r(\zeta\colon \zeta \cap A \neq \varnothing).
\end{aligned}$$

The basic properties of ν_r are collected in Theorem 1. For future reference we introduce the following terminology, where $r = s$ stands for "recurrence equals survival."

CRITERION $r = s$.

$$\nu_r = \bar{\nu}.$$

Or equivalently,

$$\text{for any finite } A \subset V_G,\ \beta(A) = \rho(A).$$

Or still equivalently,

$$\text{for some finite nonempty } A \subset V_G,\ \beta(A) = \rho(A).$$

The equivalence between the last two statements is a very simple and standard matter. In any case, we derive it next. If the third statement above holds, then, using the Markov property at time 1, for any finite $B \subset V_G$,

$$0 = \rho(A) - \beta(A) = \mathbb{P}(\Omega_\infty^A \setminus \Omega_r^A) \geq \mathbb{P}(\Omega_\infty^B \setminus \Omega_r^B)\mathbb{P}(\xi_1^A = B).$$

This leads to

$$\rho(B) - \beta(B) = \mathbb{P}(\Omega_\infty^B \setminus \Omega_r^B) = 0.$$

It is relevant that in the definition above we restrict A to be a finite set. When A is infinite, clearly $\rho(A) = 1$, but even if $r = s$ holds, we may have $\beta(A) < 1$. One example with this feature is the tree $\mathbb{T}_k \vee \mathbb{Z}^+$, with $\lambda_r(\mathbb{T}_k) < \lambda_c(\mathbb{Z}^+) = \lambda_c(\mathbb{Z})$. To simplify an argument below, we suppose that k is large enough for $\lambda_r(\mathbb{T}_k) < 1$. Take λ between $\lambda_r(\mathbb{T}_k)$ and 1 and as initial configuration take $\mathbb{Z}^+$. Because $\lambda < 1$, a simple comparison with a biased random walk (obtained by not letting particles die unless they have at least one vacant neighboring site) shows that there is positive probability that $\mathbb{T}_k$ will never become infected, and while infection will always be present somewhere, it will disappear from every finite set eventually. Therefore $\beta(\mathbb{Z}^+) < 1 = \rho(\mathbb{Z}^+)$. That $r = s$ holds, nevertheless, will be a consequence of Theorem 6(b) and the fact that $r = s$ holds for $\mathbb{T}_k$ above its recurrence point. An informal argument for the validity of $r = s$ is contained in our discussion of why (BE3) should hold. The present example has the feature that, in spite of *cc* holding, the process started from the infinite set $\mathbb{Z}^+$ does not converge to $(1 - \rho(\mathbb{Z}^+))\delta_\varnothing + \rho(\mathbb{Z}^+)\bar{\nu}$, as one could naively expect [see Theorem 2(h)].

THEOREM 1. *For each graph $G \in G$ and each value of $\lambda > 0$, the following statements are true.*

(a) (0 − 1 law) *If* $\mathbb{P}(\Omega_r^0) > 0$ (*resp.* $= 0$) *then* $\lim_{N\to\infty} \mathbb{P}(\Omega_r^{B(0,N)}) = 1$ (*resp.* $= 0$).

In particular, if $\mathbb{P}(\Omega_r^0) > 0$, *then* $\lim_{N\to\infty}\nu_r(\zeta\colon \zeta \cap B(0, N) = \varnothing) = 0$, *and* $\nu_r \perp \delta_\varnothing$.

(b) *For every $\mu \in I$ such that $\mu \perp \delta_\varnothing$, the following order relation holds*:

$$\text{for every finite } A \subset V_G,\ \nu_r(\zeta\colon \zeta \cap A \neq \varnothing) \leq \mu(\zeta\colon \zeta \cap A \neq \varnothing).$$

In particular this is the case for all $\mu \in I_e \setminus \{\delta_\varnothing\}$.

(c) $\nu_r \in I_e$.

(d) *If the criterion $r = s$ is satisfied, then* $I_e = \{\delta_\varnothing, \bar{\nu}\}$.

The order relation stated in Theorem 1(b) is responsible for the title of this paper. The notion of partial order there is known not to be equivalent to the more commonly considered stochastic order, reviewed in the introduction of this paper. The order in Theorem 1(b) is weaker than the stochastic order, and the following question is therefore raised.

(Q4) Can Theorem 1(b) be strengthened by replacing the order which appears there with the stochastic order?

A partial result in this direction is provided by Theorem 2(e). (See *Note added in revision* at the end of the introduction.)

Regarding Theorem 1(d), we see in Theorem 2(f) below that under *cc*, the condition $r = s$ holds. On the other hand, this is not an interesting use of Theorem 1(d), since in Theorem 2(h) we show that *cc* is actually stronger than the conclusion in Theorem 1(d). One can ask the following.

(Q5) Are there examples in which $r = s$ is satisfied, but *cc* is not?

The possible existence of such examples was one of the main motivations for singling Theorem 1(d) out as an item of Theorem 1. (See *Note added in revision at the end of this section*.) The main application that we have at the moment for Theorem 1(d) is contained in Theorem 2(i), where statements about homogeneous graphs are made.

A curious application of Theorem 1(d) is the following. If G_1 and G_2 are graphs on which the contact process dies out at a certain value of λ, we tend to believe that also the contact process on $G_1 \vee G_2$ would die out at this value of λ. We are not able to prove it, but it is clear that survival without recurrence is impossible in this situation. Hence the criterion $r = s$ is satisfied and we can conclude that there are at most two extremal invariant probability measures. [See Theorem 6(b) for details.]

The second main object introduced in this paper is the following statement of weak convergence.

PARTIAL CONVERGENCE (pc).

$$\text{For any finite } A \subset V_G,\ \xi_t^A \Rightarrow (1 - \beta(A))\delta_{\emptyset} + \beta(A)\nu_r \quad \text{as } t \to \infty.$$

Or equivalently,

$$\text{for any finite } A, B \subset V_G,\ \mathbb{P}\big(\xi_t^A \cap B \neq \emptyset\big) \to \beta(A)\beta(B) \quad \text{as } t \to \infty.$$

The term "Partial convergence" indicates that even when pc holds so that convergence takes place for the process started from any finite set $A \subset V_G$, the same can fail for initial sets which are infinite [see Theorem 2(g)].

In analogy with $s\,\&\,cc$, we define $r\,\&\,pc$ (for "recurrence with partial convergence") as the property that pc holds and recurrence takes place. The following theorem gives the basic properties of pc and its relations with cc. Recall that $B(0, N)$ is the ball of center 0 and radius N.

THEOREM 2. *For each graph $G \in \mathcal{G}$ and each value of $\lambda > 0$, the following statements are true.*

(a) *For any finite $A, B \subset V_G$,*

$$\limsup_{t\to\infty} \mathbb{P}\big(\xi_t^A \cap B \neq \emptyset\big) \le \beta(A)\beta(B). \tag{1.3}$$

For any $\eta \subset V_G$ and for any finite $B \subset V_G$,

$$\limsup_{t\to\infty} \mathbb{P}(\xi_t^\eta \cap B \neq \emptyset) \le \beta(\eta)\rho(B). \tag{1.4}$$

(b) *The property $r\,\&\,pc$ is equivalent to*

$$\lim_{N\to\infty} \liminf_{t\to\infty} \mathbb{P}\big(\xi_t^{B(0,N)} \cap B(0, N) \neq \emptyset\big) = 1. \tag{1.5}$$

(c) *The property $r\,\&\,pc$ is monotone increasing.*

(d) *If pc holds, then for any law μ and any continuous nonnegative nondecreasing function $f\colon \{0,1\}^{V_G} \to \mathbb{R}$,*

$$\liminf_{t\to\infty} \int f(\zeta)\, d\mu_t(\zeta) \ge \int \beta(\eta)\, d\mu(\eta) \int f(\zeta)\, d\nu_r(\zeta).$$

In particular, for any $\eta \subset V_G$ and for any finite $B \subset V_G$,

$$\liminf_{t\to\infty} \mathbb{P}(\xi_t^\eta \cap B \neq \emptyset) \ge \beta(\eta)\beta(B).$$

(e) *If pc holds, then for every* $\mu \in I$ *such that* $\mu \perp \delta_\emptyset$, *the following order relation holds*: $\nu_r \leq \mu$, *in the stochastic sense. In particular this is the case for all* $\mu \in I_e \setminus \{\delta_\emptyset\}$.

(f) *cc is equivalent to having simultaneously pc and* $r = s$. *In particular, if* s & cc *holds, then* r & pc *also holds.*

(g) *If* $r = s$ *fails, then there is a configuration* η *with infinitely many particles for which* ξ_t^η *does not converge weakly as* $t \to \infty$. *By* (f), *this happens in particular if pc holds but cc fails.*

(h) *If cc holds, then*

$$\text{for any } \eta \subset V_G,\ \xi_t^\eta \Rightarrow (1 - \beta(\eta))\delta_\emptyset + \beta(\eta)\bar{\nu} \quad \text{as } t \to \infty.$$

Or equivalently,

$$\text{for any } \eta \subset V_G \text{ and any finite } B \subset V_G,$$
$$\mathbb{P}(\xi_t^\eta \cap B \neq \emptyset) \to \beta(\eta)\rho(B) \quad \text{as } t \to \infty.$$

(i) *If* $G \in H$, *then whenever* $\beta(\lambda) > 0$, *the criterion* $r = s$ *is satisfied. In particular we have the following for homogeneous graphs*: (a) r & pc *is equivalent to* s & cc; (b) s & cc *is an H-monotone increasing property*; (c) ν_r *coincides with* $\delta_\emptyset$ *when* $\beta(\lambda) = 0$ *and with* $\bar{\nu}$ *when* $\beta(\lambda) > 0$; (d) *if* $\beta(\lambda) > 0$, *then* $I_e = \{\delta_\emptyset, \bar{\nu}\}$.

The monotonicity of r & pc, as stated in Theorem 2(c), is very useful, since it provides a way for proving that pc holds, as we will see later on, when we prove some of the properties of the basic example and some other more general results.

Because of this monotonicity, it is natural to define

$$\lambda_{r\&pc} = \inf\{\lambda\colon r \,\&\, pc \text{ holds}\}.$$

The following question can then be raised.

(Q6) Is it always the case that $\lambda_r = \lambda_{r\&pc}$?

[See *Note added in revision* at the end of this section.]

Next we use Theorem 2 to show that for our basic example the following is true.

(BE4) For all $\lambda > \lambda_r(\mathbb{T}_k)$, pc holds, that is, throughout the regions covered by the statements (BE2) and (BE3).

Thanks to Theorem 2(c), to prove this claim, it is enough to see that pc holds for the subgraph $\mathbb{T}_k$ of our graph $\mathbb{T}_k \vee \mathbb{T}_j$. But this is true since for homogeneous trees above the recurrence point we have cc and therefore, by Theorem 2(f), also pc.

The parallels between the definitions of $\bar{\nu}$ and cc on one hand and of ν_r and pc on the other are evident and aesthetically appealing (at least to the authors). When $r = s$ holds, the parallel notions collapse into each other. The parallel is nevertheless broken by the fact that s & cc is not a monotone increasing property while $r \,\&\, pc$ is; we wonder if there is any intuitive reason behind this difference.

The features of the contact process on homogeneous graphs, summarized in Theorem 2(i), and the emphasis of the study of the contact process being put on the homogeneous graphs $\mathbb{Z}^d$ and more recently $\mathbb{T}_d$, may have been the reason why ν_r and pc have never been identified before as separate entities.

Because of the H-monotonicity of $s \,\&\, cc$, as stated in Theorem 2(i), it is natural to define for the homogeneous graphs,

$$\lambda_{s\&cc} = \inf\{\lambda\colon\ s \,\&\, cc \text{ holds}\}$$

and to ask the following question.

(Q7) Is it always the case that for homogeneous graphs $\lambda_r = \lambda_{s\&cc}$?

The answer is known to be positive in basic cases of cubic lattices [Bezuidenhout and Grimmett (1990)], and of homogeneous trees [Zhang (1996)]. Nevertheless, the corresponding proofs are substantially different, and while each of these proofs generalizes to some other homogeneous graphs, the complexity of the proofs, and the use of the special structure of the graphs in them, make (Q7) seem a very difficult question to settle rigorously.

Another related natural question is the following.

(Q8) Is it always the case that, for homogeneous graphs, if $\lambda_s < \lambda_r$ then for $\lambda_s < \lambda < \lambda_r$ there are infinitely many extremal invariant measures?

The answer is known to be positive in the case of the homogeneous trees [Durrett and Schinazi (1995)]. Liggett (1996b) has further results in this direction, which in particular indicate that the set of all extremal invariant measures may be difficult to characterize even for these relatively simple graphs. If the answers to (Q7) and (Q8) turn out to be positive, as we tend to expect, just for simplicity, then the qualitative ergodic behavior of the contact process on homogeneous graphs would basically always be the one found for homogeneous trees, but allowing also for the possibility that the survival and the recurrence points may coincide, with the intermediate phase then absent, as is the case for the cubic lattices.

It is worth stressing that for homogeneous trees the fact that $I_e = \{\delta_\varnothing, \bar{\nu}\}$ above the recurrence point has its proof now greatly simplified. [See Theorem 2(i).] The only other proof that we are aware of goes by proving first that cc holds [Zhang (1996)]; this approach gives an important extra result, but is much more complicated.

The next result provides another sufficient condition for the $r = s$ criterion.

THEOREM 3. *Suppose that for a graph G there exists $\delta > 0$ so that for every $x \in V_G$,*

$$\mathbb{P}\big(0 \in \xi_t^x \text{ for some } t > 0\big) \geq \delta > 0.$$

Then $r = s$ holds.

This is a very intuitive result, since in the event of survival, the hypothesis of the theorem assures us that there is "a constant push towards the root." The proof will be a direct rigorization of this intuition.

Theorem 3 will be used to prove the part of Theorem 4 that refers to recurrence, when $d \geq 2$; the part which refers to survival and the case $d = 1$ are already in the literature. Theorem 4 refers to the tree $\mathbb{T}_d^+$, obtained from the tree $\mathbb{T}_d$ by removing one of the neighbors of the root and defining the new tree as the remaining connected component of $\mathbb{T}_d$ which contains its root.

THEOREM 4. *For each $d \geq 1$, $\mathbb{T}_d^+$ has the same survival point and the same recurrence point as $\mathbb{T}_d$. Moreover, above the recurrence point cc is satisfied by the contact process on $\mathbb{T}_d^+$.*

Theorem 3 will be used also in the proof of Theorem 5. As before, $\lambda_c(\mathbb{Z})$ denotes the common value of $\lambda_s(\mathbb{Z})$ and $\lambda_r(\mathbb{Z})$.

THEOREM 5. *For every graph $G \in G$, s & cc holds for $\lambda > \lambda_c(\mathbb{Z})$.*

This theorem may at first sight seem totally intuitive, but for the wrong reason. It is true that all graphs in G have $\mathbb{Z}^+$ embedded in them, and that, for $\mathbb{Z}^+$, *cc* holds above its critical point, which coincides with $\lambda_c(\mathbb{Z})$. But as we know, the answer to question (Q3) is negative, and therefore we cannot immediately conclude the statement in Theorem 5. In other words, Theorem 5 states that the answer to (Q3) becomes "yes" if $G_0 = \mathbb{Z}_+$; the nature of $\mathbb{Z}_+$ is crucial in this theorem.

The features (BE2) and (BE3) of the basic example are particular cases of the results stated in Theorem 6. The claim made about the graph obtained by gluing n homogeneous trees which satisfy (1.2) can also be obtained from this theorem, by induction on n.

THEOREM 6. *Suppose that $G = G_1 \vee G_2$. Everything below refers to a common fixed value of λ.*

(a) *If the contact process on G_1 survives but does not satisfy $r = s$, then $r = s$, and hence also cc, fail for the contact process on G.*

(b) *If the contact processes on both G_1 and G_2 satisfy $r = s$, then the contact process on G also satisfies $r = s$.* (*This includes the cases in which the contact process on G_1 or on G_2 dies out.*)

(c) *If the contact process on G_1 satisfies $r = s$ and the contact process on G_2 satisfies s & cc, then the contact process on G also satisfies s & cc.* (*This includes the case in which the contact process on G_1 dies out.*)

It is natural to ask what the survival and the recurrence points of the basic example are. We conjecture that they are equal to, respectively, $\lambda_s(\mathbb{T}_k)$ and $\lambda_r(\mathbb{T}_k)$. Unfortunately we do not have a complete proof of these conjectures. Nevertheless consider the graph $G = \mathbb{T}_k^+ \vee \mathbb{T}_j^+$ with $j < k$ being chosen so that (1.1) is satisfied. This is a variant of the basic example, for which

Theorem 4 allows us to obtain the same properties, (BE1)–(BE4), as that one. Moreover, we have also $\lambda_s(G) = \lambda_s(\mathbb{T}_k)$ and $\lambda_r(G) = \lambda_r(\mathbb{T}_k)$, since $\mathbb{T}_k^+$ is a subgraph of G which is a subgraph of $\mathbb{T}_k$.

In Section 2 we prove the claims made in this section. In Section 3 we discuss further the relations between the results in this paper and related results.

Note added in revision. This is an update on the status of questions (Q1)–(Q8) raised in this paper. (Q1), (Q2) and (Q3) are answered here; (Q4) has been answered affirmatively by Andjel. After this paper was completed and submitted for publication, we found an example which answers (Q5) positively and (Q6) negatively; this example will be included in a later publication, since its presentation and the proofs of its properties are relatively long. Finally, we are not aware of any progress on (Q7) and (Q8).

2. Proofs. For every configuration η and positive integer N, define the stopping time $S_N^\eta = \inf\{t \geq 0: \xi_t^\eta \supset B(0, N)\}$ and let $F_{S_N^\eta}$ be the associated σ-field. Define also the event $\Omega^\eta(s, N) = \{S_N^\eta < s\} = \{\xi_t^\eta \supset B(0, N)$ for some $t < s\}$. To prove the 0–1 law in Theorem 1(a), the following lemma will be used.

LEMMA 1. *For any configuration η the following holds*:

$$\lim_{N \to \infty} \lim_{s \to \infty} \mathbb{P}\big(\Omega_r^\eta \,\triangle\, \Omega^\eta(s, N)\big) = 0.$$

PROOF. The event $\Omega^\eta(s, N)$ increases to $\Omega^\eta(N) = \{\xi_t^\eta \supset B(0, N)$, for some $t\}$ as $s \to \infty$. The event $\Omega^\eta(N)$ decreases to $\Omega^\eta(\infty) = \{\forall N,\ \xi_t^\eta \supset B(0, N)$, for some $t\}$ as $N \to \infty$. But it is easy to see that $\mathbb{P}(\Omega^\eta(\infty) \triangle \Omega_r^\eta) = 0$, so the result follows. □

PROOF OF THEOREM 1(a). The statements about ν_r follow from the 0–1 law since $\nu_r(\zeta\colon \zeta \cap B(0, N) = \varnothing) = 1 - \mathbb{P}(\Omega_r^{B(0,N)})$, and $\nu_r(\varnothing) = \lim_{N \to \infty} \nu_r(\zeta\colon \zeta \cap B(0, N) = \varnothing) = 0$. To prove the nontrivial part of the 0–1 law, set $\lim_{N \to \infty} \mathbb{P}(\Omega_r^{B(0,N)}) = \alpha$. Note that by attractiveness this is a monotone limit and for all finite $A \in V_G$, $\mathbb{P}(\Omega_r^A) \leq \alpha$. By the previous lemma, the Markov property and the last inequality,

$$\begin{aligned}
\mathbb{P}\big(\Omega_r^0\big) &= \lim_{N \to \infty} \lim_{s \to \infty} \mathbb{P}\big(\Omega^0(s, N) \cap \Omega_r^0\big) \\
&= \lim_{N \to \infty} \lim_{s \to \infty} \mathbb{P}\big(\Omega^0(s, N)\big)\mathbb{P}\big(\Omega_r^0 | \Omega^0(s, N)\big) \\
&= \lim_{N \to \infty} \lim_{s \to \infty} \mathbb{P}\big(\Omega^0(s, N)\big) \sum_{\substack{A \subset V_G \\ \text{finite}}} \mathbb{P}\big(\Omega_r^A\big)\mathbb{P}\big(\xi_s^0 = A | \Omega^0(s, N)\big) \\
&\leq \lim_{N \to \infty} \lim_{s \to \infty} \mathbb{P}\big(\Omega^0(s, N)\big)\alpha = \mathbb{P}\big(\Omega_r^0\big)\alpha.
\end{aligned}$$

If $\mathbb{P}(\Omega_r^0) > 0$, then the inequality above implies $\alpha \geq 1$. Hence $\alpha = 1$. □

The following lemma will play a key role.

LEMMA 2. *For every law* μ, $A \subset V_G$, $N \in \mathbb{N}$, $0 \le \alpha \le 1$, *and* $t \ge 0$,

$$\mu_t(\zeta\colon \zeta \cap A \neq \emptyset) \ge \mathbb{P}\big(S_N^A < \alpha t\big) \inf_{u > (1-\alpha)t} \mu_u(\zeta\colon \zeta \cap B(0, N) \neq \emptyset).$$

PROOF. By self-duality,

$$\begin{aligned}\mu_t(\zeta\colon \zeta \cap A \neq \emptyset) &= \int d\mu(\eta)\mathbb{P}\big(\xi_t^A \cap \eta \neq \emptyset\big)\\ &\ge \int d\mu(\eta)\mathbb{P}\big(\xi_t^A \cap \eta \neq \emptyset,\ S_N^A < \alpha t\big)\\ &= \int d\mu(\eta)\mathbb{E}\Big(\mathbb{E}\big(\mathbf{1}_{\{\xi_t^A \cap \eta \neq \emptyset\}}\mathbf{1}_{\{S_N^A < \alpha t\}}|F_{S_N^A}\big)\Big)\\ &= \mathbb{E}\bigg(\mathbf{1}_{\{S_N^A < \alpha t\}}\int d\mu(\eta)\mathbb{E}\big(\mathbf{1}_{\{\xi_t^A \cap \eta \neq \emptyset\}}|F_{S_N^A}\big)\bigg).\end{aligned} \tag{2.1}$$

On $\{S_N^A < \alpha t\}$, the strong Markov property gives us

$$\mathbb{E}\big(\mathbf{1}_{\{\xi_t^A \cap \eta \neq \emptyset\}}|F_{S_N^A}\big) = g\big(\xi_{S_N^A}^A, S_N^A, t, \eta\big), \tag{2.2}$$

where $g(\zeta, s, t, \eta) = \mathbb{E}(\mathbf{1}_{\{\xi_{t-s}^\zeta \cap \eta \neq \emptyset\}})$. But on $\{S_N^A < \alpha t\}$ one has $\xi_{S_N^A}^A \supset B(0, N)$, so that by attractiveness

$$\begin{aligned}\int d\mu(\eta)\, g\big(\xi_{S_N^A}^A, S_N^A, t, \eta\big) &\ge \int d\mu(\eta)\mathbb{P}\big(\xi_{t-S_N^A}^{B(0,N)} \cap \eta \neq \emptyset\big)\\ &\ge \inf_{u > (1-\alpha)t} \int d\mu(\eta)\mathbb{P}\big(\xi_u^{B(0,N)} \cap \eta \neq \emptyset\big).\end{aligned} \tag{2.3}$$

From (2.1), (2.2), (2.3) and self-duality,

$$\begin{aligned}\mu_t(\zeta\colon \zeta \cap A \neq \emptyset) &\ge \mathbb{P}\big(S_N^A < \alpha t\big) \inf_{u > (1-\alpha)t} \int d\mu(\eta)\mathbb{P}\big(\xi_u^{B(0,N)} \cap \eta \neq \emptyset\big)\\ &= \mathbb{P}\big(S_N^A < \alpha t\big) \inf_{u > (1-\alpha)t} \mu_u(\zeta\colon \zeta \cap B(0, N) \neq \emptyset). \qquad \square\end{aligned}$$

PROOF OF THEOREM 1(b). By Lemma 2 and the invariance of μ it follows that

$$\mu(\zeta\colon \zeta \cap A \neq \emptyset) \ge \mathbb{P}\big(S_N^A < \alpha t\big)\mu(\zeta\colon \zeta \cap B(0, N) \neq \emptyset).$$

Letting $t \to \infty$, then $N \to \infty$ and using Lemma 1 and the assumption that $\mu(\emptyset) = 0$, we obtain

$$\mu(\zeta\colon \zeta \cap A \neq \emptyset) \ge \mathbb{P}\big(\Omega_r^A\big) = \nu_r(\zeta\colon \zeta \cap A \neq \emptyset). \qquad \square$$

PROOF OF THEOREM 1(c). We already know that $\nu_r \in I$. To see now that $\nu_r \in I_e$, note that Theorem 1(a) implies that either $\nu_r = \delta_\emptyset$, or else $\nu_r \perp \delta_\emptyset$.

In the former case the proof is complete. In the latter case suppose that $\nu_r = \alpha\mu_1 + (1-\alpha)\mu_2$, $\mu_i \perp \delta_\emptyset$, $\mu_i \in I$, for some $\alpha \in (0,1)$. By Theorem 1(b) for all finite $A \subset V_G$

$$\mu_i(\zeta\colon \zeta \cap A \neq \emptyset) \geq \nu_r(\zeta\colon \zeta \cap A \neq \emptyset), \qquad i = 1, 2.$$

But this is impossible unless $\mu_1 = \mu_2 = \nu_r$. $\square$

The proof of Theorem 1(d) is an immediate consequence of Theorem 1(b) and the fact that $\bar{\nu}$ is the largest element of I in the stochastic sense, and hence also in the sense of Theorem 1(b).

LEMMA 3. *For any finite $A \subset V_G$ and any $\eta \subset V_G$*, $\lim_{t\to\infty} \mathbb{P}(\xi_t^\eta \cap A \neq \emptyset, (\Omega_r^\eta)^c) = 0$.

PROOF. On $(\Omega_r^\eta)^c$, $\xi_t^\eta \cap A = \emptyset$ a.s., for t large enough. So on this set, $\mathbf{1}_{\{\xi_t^\eta \cap A \neq \emptyset\}} \to 0$ a.s., as $t \to \infty$. The result follows then by the dominated convergence theorem. $\square$

PROOF OF THEOREM 2(a). From Lemmas 1 and 3,

$$\begin{aligned}
&\limsup_{t\to\infty} \mathbb{P}\big(\xi_t^A \cap B \neq \emptyset\big) \\
&\quad \leq \limsup_{N\to\infty} \limsup_{u\to\infty} \limsup_{t\to\infty} \mathbb{P}\big(\Omega^A(u,N), \xi_t^A \cap B \neq \emptyset\big).
\end{aligned} \tag{2.4}$$

Using the Markov property and attractiveness, for $u < t$ and arbitrary M,

$$\begin{aligned}
&\mathbb{P}\big(\Omega^A(u,N), \xi_t^A \cap B \neq \emptyset\big) \\
&\quad \leq \mathbb{P}\big(\Omega^A(u,N)\big)\Big\{\mathbb{P}\big(\xi_{t-u}^{B(0,M)} \cap B \neq \emptyset\big) \\
&\qquad\qquad + \mathbb{P}\big(\xi_u^A \cap (B(0,M))^c \neq \emptyset | \Omega^A(u,N)\big)\Big\}.
\end{aligned}$$

Using self-duality for the first term inside the braces and Lemma 3, we obtain

$$\begin{aligned}
&\limsup_{t\to\infty} \mathbb{P}\big(\Omega^A(u,N), \xi_t^A \cap B \neq \emptyset\big) \\
&\quad \leq \mathbb{P}\big(\Omega^A(u,N)\big)\Big\{\beta(B) + \mathbb{P}\big(\xi_u^A \cap (B(0,M))^c \neq \emptyset | \Omega^A(u,N)\big)\Big\}.
\end{aligned}$$

If in this inequality we let $M \to \infty$, then $u \to \infty$, then $N \to \infty$, the right-hand side converges to $\beta(A)\beta(B)$, thanks to Lemma 1. Therefore the first statement in Theorem 2(a) follows from (2.4).

To prove the second statement, we use again Lemmas 1 and 3, the Markov property, attractiveness and self-duality to write

$$\begin{aligned}
&\limsup_{t\to\infty} \mathbb{P}(\xi_t^\eta \cap B \neq \varnothing)\\
&\quad\le \limsup_{N\to\infty}\limsup_{u\to\infty}\limsup_{t\to\infty} \mathbb{P}\big(\Omega^\eta(u,N), \xi_t^\eta \cap B \neq \varnothing\big)\\
&\quad\le \limsup_{N\to\infty}\limsup_{u\to\infty}\limsup_{t\to\infty} \mathbb{P}(\Omega^\eta(u,N))\mathbb{P}\big(\xi_{t-u}^{V_G} \cap B \neq \varnothing\big)\\
&\quad= \beta(\eta)\rho(B). \qquad\square
\end{aligned}$$

PROOF OF THEOREM 2(b). Suppose first that (1.5) holds. Then for large N, using (1.3), we have

$$(\beta(B(0,N)))^2 \ge \limsup_{t\to\infty} \mathbb{P}\big(\xi_t^{B(0,N)} \cap B(0,N) \neq \varnothing\big) \ge \tfrac{1}{2},$$

which clearly implies recurrence.

To see that (1.5) implies pc, we use Lemma 2 twice. In the first application we replace A with B and take $\mu = \delta_A$ and $\alpha \in (0,1)$. Using also self-duality, we have

$$\mathbb{P}\big(\xi_t^A \cap B \neq \varnothing\big) \ge \mathbb{P}\big(S_N^B < \alpha t\big) \inf_{u>(1-\alpha)t} \mathbb{P}\big(\xi_u^{B(0,N)} \cap A \neq \varnothing\big). \tag{2.5}$$

Using Lemma 2 again with $\mu = \delta_{B(0,N)}$,

$$\begin{aligned}
&\mathbb{P}\big(\xi_u^{B(0,N)} \cap A \neq \varnothing\big)\\
&\qquad \ge \mathbb{P}\big(S_N^A < \alpha u\big) \inf_{v>(1-\alpha)u} \mathbb{P}\big(\xi_v^{B(0,N)} \cap B(0,N) \neq \varnothing\big).
\end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6), it follows that

$$\begin{aligned}
\mathbb{P}\big(\xi_t^A \cap B \neq \varnothing\big) &\ge \mathbb{P}\big(S_N^B < \alpha t\big)\mathbb{P}\big(S_N^A < \alpha(1-\alpha)t\big)\\
&\qquad\times \inf_{v>(1-\alpha)^2 t} \mathbb{P}\big(\xi_v^{B(0,N)} \cap B(0,N) \neq \varnothing\big).
\end{aligned} \tag{2.7}$$

Now pc follows from (2.7), (1.5), Lemma 1 and (1.3).

To prove the other direction of Theorem 2(b), note that pc implies

$$\lim_{t\to\infty} \mathbb{P}\big(\xi_t^{B(0,N)} \cap B(0,N) \neq \varnothing\big) = (\beta(B(0,N)))^2.$$

Under recurrence $\beta(B(0,N)) \to 1$ as $N \to \infty$, by Theorem 1(a), completing the proof. □

For the proof of Theorem 2(c), note that the condition (1.5), which in Theorem 2(b) is presented as equivalent to $r\,\&\,pc$, is manifestly monotone increasing.

PROOF OF THEOREM 2(d). Thanks to Fatou's lemma, there is no loss in generality in taking $\mu = \delta_\eta$, for some η. We can also suppose that the probability of recurrence is positive, since otherwise $\nu_r = \delta_\varnothing$ and there is nothing to prove. With these assumptions we have

$$
\begin{aligned}
\liminf_{t\to\infty}\int f(\zeta)\,d\mu_t(\zeta) &\geq \liminf_{N\to\infty}\liminf_{u\to\infty}\liminf_{t\to\infty}\mathbb{E}(f(\xi_t^\eta)|S_N^\eta < u)\mathbb{P}(S_N^\eta < u)\\
&\geq \liminf_{N\to\infty}\liminf_{u\to\infty}\Big[\mathbb{P}(S_N^\eta < u)\liminf_{s\to\infty}\mathbb{E}\big(f\big(\xi_s^{B(0,N)}\big)\big)\Big]\\
&\geq \liminf_{N\to\infty}\liminf_{u\to\infty}\mathbb{P}(S_N^\eta < u)\beta(B(0,N))\int f(\zeta)\,d\nu_r(\zeta)\\
&= \beta(\eta)\int f(\zeta)\,d\nu_r(\zeta),
\end{aligned}
$$

as desired. In the first inequality we used the fact that f is nonnegative. In the second inequality we used the strong Markov property at time S_N^η and attractiveness. In the third one we used the hypothesis that *pc* holds and that f is nonnegative. In the final step we used Lemma 1 and Theorem 1(a) with the assumption that the probability of recurrence is positive. □

LEMMA 4. *If $\mu \in I$ and $\mu \perp \delta_\emptyset$, then*

$$
\int \beta(\eta)\,d\mu(\eta) = 1. \tag{2.8}
$$

PROOF. Taking $B = B(0, N)$, (1.4) can be rewritten as

$$
1 - \beta(\eta)\rho(B(0,N)) \leq \liminf_{t\to\infty}\mathbb{P}\big(\xi_t^\eta \cap B(0,N) = \emptyset\big).
$$

Integrating η with respect to μ, using Fatou's lemma and the invariance of μ, we obtain

$$
1 - \left(\int \beta(\eta)\,d\mu(\eta)\right)\rho(B(0,N)) \leq \mu(\zeta\colon \zeta \cap B(0,N) = \emptyset).
$$

By letting $N \to \infty$ we obtain, from the hypothesis $\mu \perp \delta_\emptyset$,

$$
1 - \left(\int \beta(\eta)\,d\mu(\eta)\right)\lim_{N\to\infty}\rho(B(0,N)) \leq 0.
$$

But this can only happen if (2.8) holds. □

PROOF OF THEOREM 2(e). From Lemma 4 and Theorem 2(d) we obtain, since $\mu \in I$,

$$
\int f(\zeta)\,d\mu(\zeta) \geq \int f(\zeta)\,d\nu_r(\zeta),
$$

for all continuous nonnegative and nondecreasing functions $f\colon \{0,1\}^{V_G} \to \mathbb{R}$. The restriction to nonnegative f is irrelevant at this point, and we have proven the desired inequality between μ and ν_r. □

PROOF OF THEOREM 2(f). We will show that *cc* implies $r = s$; the rest is then immediate. Under *cc*, for all finite $A, B \subset V_G$,

$$
\beta(A)\beta(B) \leq \rho(A)\rho(B) = \lim_{t\to\infty}\mathbb{P}\big(\xi_t^A \cap B \neq \emptyset\big) \leq \beta(A)\beta(B),
$$

where the last inequality follows from (1.3). Taking $A = B$ we obtain $\beta(A) = \rho(A)$. $\square$

In the proof of Theorem 2(g) the following lemma will be used. The notation $|A|$ will be used for the cardinality of the set A.

LEMMA 5. *If $|\eta^c| < \infty$, then $\xi_t^\eta \Rightarrow \bar{\nu}$, as $t \to \infty$.*

PROOF. By self-duality, for any finite $A \subset V_G$,

$$\begin{aligned}\mathbb{P}(\xi_t^\eta \cap A \neq \varnothing) &= \mathbb{P}\big(\xi_t^A \cap \eta \neq \varnothing\big) \geq \mathbb{P}\big(|\xi_t^A| > |\eta^c|\big)\\ &\geq \mathbb{P}\big(|\xi_t^A| > |\eta^c|, \Omega_\infty^A\big) \to \rho(A),\end{aligned}$$

as $t \to \infty$, since on Ω_∞^A, $\lim_{t\to\infty}|\xi_t^A| = \infty$ a.s.

But also

$$\mathbb{P}(\xi_t^\eta \cap A \neq \varnothing) \leq \mathbb{P}\big(\xi_t^{V_G} \cap A \neq \varnothing\big) \to \rho(A),$$

as $t \to \infty$, completing the argument. $\square$

PROOF OF THEOREM 2(g). From the hypothesis there is a finite set $C \subset V_G$ such that

$$\nu_r(\zeta: \zeta \cap C \neq \varnothing) = \beta(C) < \rho(C) = \bar{\nu}(\zeta: \zeta \cap C \neq \varnothing).$$

For $0 < M < N$ we use the notation $A(M, N) = B(0, N) \setminus B(0, M)$ for an annulus centered at 0. The configuration η is taken as

$$\eta = \bigcup_{i=1}^{\infty} A(N_{2i-1}, N_{2i}),$$

where $0 = N_0 < N_1 < N_2 \cdots$ will be chosen properly. Also a sequence of times $t_j \nearrow \infty$ will be chosen, and the goal is to have for a small $\varepsilon > 0$,

$$\limsup_{i\to\infty} \mathbb{P}\big(\xi_{t_{2i}}^\eta \cap C \neq \varnothing\big) \leq \beta(C) + \varepsilon \tag{2.9}$$

and

$$\liminf_{i\to\infty} \mathbb{P}\big(\xi_{t_{2i+1}}^\eta \cap C \neq \varnothing\big) \geq \rho(C) - \varepsilon. \tag{2.10}$$

Intuitively we want to let N_j and t_j increase very fast with j, so that: (i) at time t_{2i} the set C is not affected by what happens beyond $B(0, N_{2i+1})$ at time 0, so that this region could as well be totally vacant initially and (1.3) leads to (2.9); (ii) at time t_{2i+1} the set C is not affected by what happens beyond $B(0, N_{2i+2})$ at time 0, so that this region could as well be totally occupied initially, and (2.10) follows from the previous lemma.

To make this intuition precise define the following two sequences of configurations which approximate η from opposite sides:

$$\eta^{(2k)} = \bigcup_{i=1}^{k} A(N_{2i-1}, N_{2i}) = \eta \cap B(0, N_{2k}), \qquad k = 1, 2, \ldots$$

and

$$\eta^{(2k+1)} = \eta^{(2k)} \cup \left(B(0, N_{2k+1})\right)^c, \qquad k = 1, 2, \dots .$$

Choose t_1, N_1 and N_2 arbitrarily and proceed recursively as follows. Given $t_1, \dots, t_{2i-1}$ and $N_1, \dots, N_{2i}$, take t_{2i} such that

$$\mathbb{P}\left(\xi_{t_{2i}}^{\eta^{(2i)}} \cap C \neq \varnothing\right) \leq \beta(C) + \varepsilon/2,$$

which is possible by (1.3). Next take N_{2i+1} such that

$$\mathbb{P}\left(\xi_{t_{2i}}^{\eta} \cap C \neq \varnothing\right) \leq \mathbb{P}\left(\xi_{t_{2i}}^{\eta^{(2i)}} \cap C \neq \varnothing\right) + \varepsilon/2,$$

which is possible by the Feller property. Therefore (2.9) is assured.

Next take t_{2i+1} such that

$$\mathbb{P}\left(\xi_{t_{2i+1}}^{\eta^{(2i+1)}} \cap C \neq \varnothing\right) \geq \rho(C) - \varepsilon/2,$$

which is possible by the previous lemma. Finally take next N_{2i+2} such that

$$\mathbb{P}\left(\xi_{t_{2i+1}}^{\eta} \cap C \neq \varnothing\right) \geq \mathbb{P}\left(\xi_{t_{2i+1}}^{\eta^{(2i+1)}} \cap C \neq \varnothing\right) - \varepsilon/2.$$

Thus (2.10) is also assured. □

Half of the claim for the proof of Theorem 2(h) is in (1.4). The other half is in the combination of Theorem 2(f), which assures that pc and $r = s$ hold, and Theorem 2(d).

PROOF OF THEOREM 2(i). Suppose that $\beta(\lambda) > 0$. A proof that $r = s$ holds in case G is a homogeneous tree can be found in the proof of Theorem 4 in Madras and Schinazi (1992). Their simple and intuitive argument applies as well to any homogeneous graph. It turns out that we can also present an alternative self-contained and equally simple proof, and for the reader's benefit we do it next.

Fix N and let F_N be the event that eventually a ball of radius N centered somewhere will become fully occupied. Clearly $\mathbb{P}(\Omega_\infty^0 \cap (F_N)^c) = 0$. Using the strong Markov property, attractiveness and the hypothesis that $G \in H$, we can write

$$\begin{aligned} \rho(\lambda) - \beta(\lambda) &= \mathbb{P}\left(\Omega_\infty^0 \cap \left(\Omega_r^0\right)^c\right) \leq \mathbb{P}\left(F_N \cap \left(\Omega_r^0\right)^c\right) \\ &\leq \sup_{x \in V_G} \mathbb{P}\left(\left(\Omega_r^{B(x, N)}\right)^c\right) = \mathbb{P}\left(\left(\Omega_r^{B(0, N)}\right)^c\right). \end{aligned}$$

Letting $N \to \infty$, $r = s$ follows now from Theorem 1(a), since $\beta(\lambda) > 0$.

With this part done, the rest of the claim follows easily.

Statement (a) is immediate now, using Theorem 2(f).

Statement (b) follows easily from (a) and Theorem 2(c).

Statement (c) is immediate from the definitions of ν_r and of $\bar{\nu}$.

Statement (d) is immediate from Theorem 1(d). □

For use in the next proof, for each configuration η, define the stopping time $\tau^\eta = \inf\{t\colon \xi_t^\eta = \varnothing\}$.

PROOF OF THEOREM 3. Let η be the initial configuration. For every $x \in V_G$ take $T(x)$ such that

$$\mathbb{P}\big(0 \in \xi_t^x \text{ for some } t \le T(x)\big) \ge \delta/2. \tag{2.11}$$

Order the sites in G and define a sequence of stopping times $(S_i)_{i=0,1,\dots}$ by first setting

$$S_0 = 0.$$

Then we proceed recursively in the following way. On $\{\tau^\eta > S_i + 1\}$ we define

$$S_{i+1} = S_i + 1 + T(x_i) \qquad \text{where } x_i \text{ is the first site in } \xi^\eta_{S_i+1}.$$

On $\{\tau^\eta \le S_i + 1\}$ we define, quite arbitrarily, $S_{i+1} = S_i + 1$. Define the events

$$F_i = \big\{0 \in \xi_t^\eta \text{ for some } t \in [S_i + 1, S_{i+1})\big\}.$$

With this notation, it follows that

$$\begin{aligned}\mathbb{P}\Big(\Omega_\infty^\eta, (\Omega_r^\eta)^c\Big) &\le \mathbb{P}\Big(\{\tau^\eta = \infty\}, \{F_i \text{ infinitely often}\}^c\Big)\\ &\le \sum_{j\ge 0} \mathbb{P}\bigg(\bigcap_{i \ge j}\big(\{\tau^\eta > S_i + 1\} \cap (F_i)^c\big)\bigg) = 0,\end{aligned}$$

where the last step uses the Markov property and (2.11). $\square$

PROOF OF THEOREM 4. As mentioned before the theorem was stated, the only parts that remain to be proven are the ones that concern $d \ge 2$ and the recurrence point and recurrence regime. By graph monotonicity, it is clear that $\lambda_r(\mathbb{T}_d) \le \lambda_r(\mathbb{T}_d^+)$, and we prove next the complementary inequality.

Think of $\mathbb{T}_d^+$ as a subgraph of $\mathbb{T}_d$, and take an arbitrary site $x \in V_{\mathbb{T}_d^+}$. It is clear that when $\lambda > \lambda_r(\mathbb{T}_d)$,

$$\mathbb{P}\big(0 \in \xi^x_{\mathbb{T}_d^+;t} \text{ for some } t > 0\big) \ge \beta_{\mathbb{T}_d}(\{x\}) = \beta_{\mathbb{T}_d}(\{0\}) > 0.$$

Therefore Theorem 3 implies that $r = s$ holds for $\mathbb{T}_d^+$, for such values of λ. The proof that $\lambda_r(\mathbb{T}_d^+) \le \lambda_r(\mathbb{T}_d)$ is now complete by recalling that it is known that $\rho_{\mathbb{T}_d^+}(\{0\}, \lambda) > 0$, for $\lambda > \lambda_r(\mathbb{T}_d)$, since it is known that $\lambda_s(\mathbb{T}_d^+) = \lambda_s(\mathbb{T}_d) < \lambda_r(\mathbb{T}_d)$.

The argument above gave us already the validity of $r = s$ when $\lambda > \lambda_r(\mathbb{T}_d^+)$, and therefore, by Theorem 2(f), the proof of *cc* will be complete once we show that *pc* holds also in this regime. The argument for this is a good illustration of the usefulness of Theorem 2(b). From Proposition 5 in Zhang (1996), we know that

$$\inf_{t\ge 0} \mathbb{P}\big(0 \in \xi^0_{\mathbb{T}_d^+;t}\big) > 0. \tag{2.12}$$

(In that paper, the set U is a version of $\mathbb{T}_d^+$ and the $\liminf_{t\to\infty}$ which appears there can clearly be replaced with $\inf_{t\ge 0}$.) From this it is fairly easy to see that the condition (1.5), which in Theorem 2(b) is presented as equivalent to

r & pc, holds. First observe that each one of the d^N sites in $\mathbb{T}_d^+$ which are exactly at distance N from 0 can be thought of as the root of a subgraph of $\mathbb{T}_d^+$ which is isomorphic to $\mathbb{T}_d^+$. These d^N subgraphs have disjoint sets of vertices. We are interested in the process $(\xi_{\mathbb{T}_d^+;t}^{B(0,N)})$, and at time zero each one of the subgraphs that we just mentioned has therefore its root occupied. If we let the process in each one of these subgraphs evolve without being allowed to infect or be infected by other sites, then we are looking simply at a large number of independent copies of the process $(\xi_{\mathbb{T}_d^+;t}^0)$. The validity of (1.5) follows now at once from (2.12). □

PROOF OF THEOREM 5. Each graph $G \in G$ has $\mathbb{Z}^+$ embedded into it as a subgraph. Since s & cc holds for $\mathbb{Z}^+$ above its recurrence point, which coincides with $\lambda_c(\mathbb{Z})$, we can conclude from Theorem 2(f) and Theorem 2(c) that r & pc holds for G. Thanks to Theorem 2(f) we also know that our task has been reduced to showing that $r = s$ holds for G.

Let G_0 be a graph which is isomorphic to $\mathbb{Z}^+$ and has no vertex in common with G. Set $G' = G \vee G_0$. Every site $x \in V_G$ can be connected to the root of G by a chain of neighboring sites in G. Therefore in G', x belongs to an infinite linear chain of neighboring sites in which only x itself has one single neighbor, that is, a subgraph G_x isomorphic to $\mathbb{Z}^+$, with x playing the role of its origin.

Because in $\mathbb{Z}^+$ survival without ever hitting a given fixed site is impossible (otherwise we would have survival without growth), we obtain for $\lambda > \lambda_s(\mathbb{Z}^+)$,

$$\begin{aligned}\mathbb{P}\big(0 \in \xi_{G;t}^x \text{ for some } t > 0\big) &\geq \mathbb{P}\big(0 \in \xi_{G';t}^x \text{ for some } t > 0\big)\\ &\geq \mathbb{P}\big(0 \in \xi_{G_x;t}^x \text{ for some } t > 0\big) \geq \rho_{\mathbb{Z}^+}(\{0\}, \lambda) > 0.\end{aligned}$$

Therefore Theorem 3 assures us that $r = s$ holds for G provided $\lambda > \lambda_s(\mathbb{Z}^+) = \lambda_c(\mathbb{Z})$, as we wanted to show. □

LEMMA 6. *The validity of $r = s$ is equivalent to the statement that for all finite $A \subset V_G$, $\mathbb{P}(\Omega_\infty^A,\ 0 \notin \xi_t^A$ for all $t > 0) = 0$.*

PROOF. Clearly $r = s$ implies the other statement.

For the converse, suppose that $r = s$ fails. Then $\mathbb{P}(\Omega_\infty^0, (\Omega_r^0)^c) > 0$ and therefore, for some large enough T, $\mathbb{P}(\Omega_\infty^0, 0 \notin \xi_t^0$ for all $t > T) > 0$. Hence, for some finite $A \subset V_G$, $\mathbb{P}(\Omega_\infty^0, 0 \notin \xi_t^0$ for all $t > T$, $\xi_T^0 = A) > 0$. Using the Markov property at time T, this inequality leads to $\mathbb{P}(\Omega_\infty^A,\ 0 \notin \xi_t^A$ for all $t > 0) > 0$. □

For the proof of Theorem 6(a): the previous lemma states that there is a finite $A \subset V_{G_1}$, such that for the contact process on G_1 starting from A, there is a positive probability of surviving without ever infecting the root. But then it is clear that the same property is true for the contact process on G starting from the same set A. The proof is now complete by using the same lemma in the opposite direction.

PROOF OF THEOREM 6(b). For arbitrary finite $A \subset V_G$,

$$0 \le \rho(A) - \beta(A) \le \mathbb{P}\left(\bigcup_{n=0}^{\infty} F_n\right),$$

where F_n is the event that $(\xi^A_{G;t}\colon t \ge 0)$ survives, but from time n on the roots of G_1 and G_2 are never occupied. But on the event F_n we must have after time n survival starting from a finite set without ever hitting the root either for the process restricted to G_1 or to G_2. Both are impossible events, since the contact processes on G_1 and G_2 satisfy $r = s$. □

PROOF OF THEOREM 6(c). By Theorem 2(f), s & cc for G_2 implies r & pc and $r = s$ for G_2. By Theorem 2(c), r & pc must then hold for G as well. From Theorem 6(b) we have that $r = s$ holds for G, and invoking Theorem 2(f) again we see that our task has been completed. □

3. Notes. In this section we try to clarify how this paper came into existence and in so doing we emphasize the connections with some work of several years ago and also state a further theorem which was left out of the main part of the paper because it seems that it is less useful than the ones that appear there.

In a first stage of this work, we are trying to understand better the notion of complete convergence for the contact process on graphs. We wanted to decide whether Conjecture 1 in Pemantle (1992), which states that, on trees, cc should always hold above λ_r was true, and we also wanted to prove what in our paper is now called Theorem 5. For this second task we checked that the technique introduced in the papers Schonmann (1987a) and Schonmann (1987b) could indeed be used. This technique relies on a fundamental little result in Griffeath (1978), which states that cc is equivalent to the following: for all pairs of finite subsets of V_G, A and B, if (ξ_t^A) and $(\bar{\xi}_t^B)$ are two independent versions of the contact process, starting from A and B, respectively, then

$$\lim_{t\to\infty} \mathbb{P}\left(\Omega^A_\infty, \overline{\Omega}^B_\infty, \xi_t^A \cap \bar{\xi}_t^B = \varnothing\right) = 0. \tag{3.1}$$

To prove Theorem 5 using this result, one can roughly say that on the intersection of the events Ω^A_∞ and $\overline{\Omega}^B_\infty$, each process will eventually contain lots of particles on some subgraph of G which is isomorphic to $\mathbb{Z}^+$, with these two subgraphs being such that they have an intersection also isomorphic to $\mathbb{Z}^+$. At the random time when this happens, one restricts each contact process to the corresponding subgraph and uses graph monotonicity for comparisons with the true processes. We are above $\lambda_c(\mathbb{Z}^+) = \lambda_c(\mathbb{Z})$, so that the contact process on $\mathbb{Z}^+$ satisfies cc, and hence also (3.1). Using then the fact that above $\lambda_c(\mathbb{Z}^+) = \lambda_c(\mathbb{Z})$, the contact process on $\mathbb{Z}^+$ starting with many particles is likely to survive, one can finish the proof that (3.1) also holds for the contact process on G.

At this point our attention was called to the fact that the special structure of $\mathbb{Z}^+$ was crucial in such an argument, and that the answer to the corresponding question (Q3), when G_0 is arbitrary could be negative. This led us to examples similar to what we now call our basic example and eventually to that one and to the understanding that the answers to (Q1), (Q2) and (Q3) are negative in general.

The question then arose as to what could be said about the ergodic behavior of the contact process for values of λ above λ_r for which *cc* fails. The introduction of ν_r and of the notion of *pc* came naturally at this point, by replacing Ω_∞^A with Ω_r^A in *cc*, and trying to prove an analogue to Griffeath's result quoted above, concerning *pc*. This could be done, and we proved that *pc* is equivalent to the following: for all pairs of finite subsets of V_G, A and B, if (ξ_t^A) and $(\bar{\xi}_t^B)$ are two independent versions of the contact process, starting from A and B, respectively, then

$$\lim_{t\to\infty} \mathbb{P}\Big(\Omega_r^A, \overline{\Omega}_r^B, \xi_t^A \cap \bar{\xi}_t^B = \varnothing\Big) = 0. \tag{3.2}$$

Our first proof of the monotonicity of $r\,\&\,pc$ relied on the equivalence above and arguments of the nature of those presented above to prove Theorem 5 based on Griffeath's equivalence. By successive simplifications of this proof, we eventually obtained Theorem 2(b). We omit the equivalence between *pc* and (3.2) from the main part of this paper, because it seems to us now that any other application that this may have should also be obtainable from Theorem 2(b), in an easier fashion.

In this connection we would like to stress that the machinery introduced in this paper can be used to simplify proofs of *cc*. To our knowledge, *cc* has always been proved for the contact process on particular graphs either using Griffeath's equivalence above or some "shape theorem" for the region of V_G where the contact process started from a finite set agrees with the contact process started from the fully occupied lattice. This second method is more delicate, but it also provides a much more refined result than *cc*. Regarding the first method, one can now sometimes simplify such proofs by checking that *pc* holds, via the equivalent condition in Theorem 2(b), and that $r = s$ also holds [recall Theorem 2(f)]. For instance, the proof by Bezuidenhout and Grimmett (1990) of *cc* for the contact process on $\mathbb{Z}^d$ above the unique critical point becomes easier to understand: by Theorem 2(i)(a) and Theorem 2(b) it is enough to verify that (1.5) holds. But this clearly follows from the dynamic renormalization scheme in Bezuidenhout and Grimmett (1990) (compare with the argument for *cc* in Section 5 of that paper). Also the proof by Zhang (1996) of *cc* for the contact process on homogeneous trees for $\lambda > \lambda_r$ is simplified. The use of Theorem 2(b) in this case was presented in our proof of Theorem 4, where we considered $\mathbb{T}_d^+$. The case of $\mathbb{T}_d$ is analogous (compare with Zhang's approach to verifying that Griffeath's equivalence holds).

REFERENCES

BEZUIDENHOUT, C. and GRIMMETT, G. (1990). The critical contact process dies out. *Ann. Probab.* **18** 1462–1482.

DURRETT, R. (1988). *Lecture Notes on Interacting Particle Systems and Percolation*. Wadsworth & Brooks/Cole, Belmont, CA.

DURRETT, R. and GRIFFEATH, D. (1983). Supercritical contact processes on $\mathbb{Z}$. *Ann. Probab.* **11** 1–15.

DURRETT, R. and SCHINAZI, R. B. (1995). Intermediate phase for the contact process on a tree. *Ann. Probab.* **23** 668–673.

GRIFFEATH, D. (1978). Limit theorems for non-ergodic set-valued Markov processes. *Ann. Probab.* **6** 379–387.

HARRIS, T. E. (1974). Contact interactions in a lattice. *Ann. Probab.* **2** 969–988.

LIGGETT, T. M. (1985). *Interacting Particle Systems*. Springer, New York.

LIGGETT, T. M. (1996a). Multiple transition points for the contact process on the binary tree. *Ann. Probab.* **24** 1675–1710.

LIGGETT, T. M. (1996b). Branching random walks and contact processes on homogeneous trees. *Probab. Theory Related Fields* **106** 495–519.

MADRAS, N. and SCHINAZI, R. B. (1992). Branching random walk on trees. *Stochastic Process. Appl.* **42** 255–267.

MORROW, G. J., SCHINAZI, R. B. and ZHANG, Y. (1994). The critical contact process on a homogeneous tree. *J. Appl. Probab.* **31** 250–255.

PEMANTLE, R. (1992). The contact process on trees. *Ann. Probab.* **20** 2089–2116.

SCHONMANN, R. H. (1987a). A new proof of the complete convergence theorem for contact processes in several dimensions with large infection parameter. *Ann. Probab.* **15** 382–387.

SCHONMANN, R. H. (1987b). A new look at contact processes in several dimensions. In *Percolation Theory and Ergodic Theory of Infinite Particle Systems* (H. Kesten, ed.) 245–250. Springer, New York.

STACEY, A. M. (1996). The existence of an intermediate phase for the contact process on trees. *Ann. Probab.* **24** 1711–1726.

ZHANG, Y. (1996). The complete convergence theorem of the contact process on trees. *Ann. Probab.* **24** 1408–1443.

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