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THE MEAN VELOCITY OF A BROWNIAN MOTION IN A RANDOM LÉVY POTENTIAL

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A Brownian motion in a random Lévy potential V is the informal solution of the stochastic differential equation

$$dX_t = dB_t - \tfrac{1}{2}V'(X_t)\,dt,$$

where B is a Brownian motion independent of V.

We generalize some results of Kawazu–Tanaka, who considered for V a Brownian motion with drift, by proving that X_t/t converges almost surely to a constant, the mean velocity, which we compute in terms of the Lévy exponent ϕ of V, defined by $\mathbb{E}[e^{mV(t)}] = e^{-t\phi(m)}$.

1. Introduction

EXAMPLE 1. Given a Poisson cloud $(\sigma_i,\ i \in \mathbb{Z})$ on the real line, we consider a process X such that we have the following (see Figure 1):

1. X behaves like a Brownian motion between adjacent barriers σ_i and σ_{i+1}.
2. When X hits a barrier, he flips a coin and goes to the right with probability p, to the left with probability $q = 1 - p$.

Observe that it is natural to require stationarity of the random media, that is, invariance in law under translations, so that the intervals between barriers have a distribution with a lack of memory property; hence, they are exponentially distributed, with parameter $\lambda > 0$.

It turns out that the process X has a mean velocity. That is, almost surely

$$\lim_{t\to\infty} \frac{X_t}{t} = \begin{cases} \lambda \dfrac{p-q}{2p}, & \text{if } p > q; \\ \lambda \dfrac{p-q}{2q}, & \text{if } p < q. \end{cases}$$

EXAMPLE 2. We are led to wonder what would happen if we consider a more sophisticated process; for example, if we ask X to behave like a Brownian motion with drift between barriers:

$$X_t = B_t - \mu t \quad \text{between barriers.}$$

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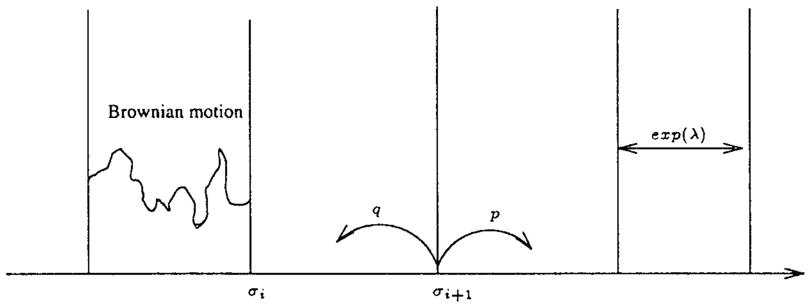


FIG. 1. *Brownian motion in a Poissonian potential.*

We show that almost surely,

$$\lim_{t\to\infty}\frac{X_t}{t}=\begin{cases}\lambda\dfrac{p-q}{2p}-\mu, & \text{if } \lambda\dfrac{p-q}{2p}-\mu>0;\\ \lambda\dfrac{p-q}{2q}-\mu, & \text{if } \lambda\dfrac{p-q}{2q}-\mu<0;\\ 0, & \text{if } \lambda\dfrac{p-q}{2p}\le\mu\le\lambda\dfrac{p-q}{2q}.\end{cases}$$

Therefore, we observe a phase transition phenomenon. The mean velocity is zero when the drift balances the impulses given by the elastic barriers.

EXAMPLE 3. We may increase the complexity of our model by substituting, for the fixed probability p, independent identically distributed random variables $(p_i,\ i\in\mathbb{Z})$ located at each barrier. We may even ask what happens in the following limiting scheme: λ increases to infinity and at the same time the p_i goes to $\frac{1}{2}$?

All these models fall into a unique framework: X is a Brownian motion in a random Lévy potential, that is, the informal solution of the stochastic differential equation

$$dX_t=dB_t-\tfrac{1}{2}V'(X_t)\,dt,$$

where B is a standard Brownian motion, $(V(x),\ x\in\mathbb{R})$ is a spatial Lévy process, that is, a process with independent homogeneous increments and V is independent of B.

1. In Example 1, $V(x)=\log(q/p)N(x)$, where N is a Poisson process on the line, with intensity λ.
2. In Example 2, V is a Poisson Process with drift: $V(x)=\log(q/p)N(x)+2\mu x$.
3. In Example 3, V is a compound Poisson process with drift.

We now see clearly what the limiting scheme of Example 3 consists of: we take for the random potential the limit in distribution of a compound Poisson process, that is, a general Lévy process.

This model for a one-dimensional Brownian motion in a random potential V was introduced by Brox [2], and thoroughly studied by numerous authors (e.g., [6, 7, 8, 11, 12, 13]. This paper generalizes, to some extent, the beautiful work of Kawazu and Tanaka [8], who considered for V a Brownian motion with drift, by giving the mean velocity in terms of the Lévy exponent ϕ of V, defined by $\mathbb{E}[e^{mV(t)}] = e^{-t\phi(m)}$,

$$\frac{X_t}{t} \text{ converges a.s. to } \begin{cases} \dfrac{\phi(1)}{2}, & \text{if } \phi(1) > 0; \\ -\dfrac{\phi(-1)}{2}, & \text{if } \phi(-1) > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Observe that when the mean velocity is zero, a remarkable phenomenon may happen: almost surely $X_t \to \infty$ but $(X_t/t) \to 0$, that is, sublinear growth. Kawazu and Tanaka [8] proved that for a Brownian motion with drift environment $V(x) = W(x) - (\kappa/2)x$, we have the following.

1. If $\kappa = 1$, then $(\log t/t)X_t$ converges in probability to a constant;
2. If $0 < \kappa < 1$, then X_t/t^κ converges in law to $1/S_\kappa^\kappa$, where S_κ is a one-sided stable random variable of index κ, that is, for a positive constant c, we have, for all $\lambda > 0$, $\mathbb{E}[e^{-\lambda S_\kappa}] = e^{-c\lambda}$.

We conjecture that this behavior is fairly general. More precisely, if we assume that the Laplace exponent is defined on an open interval containing $[-1, 1]$ and satisfies $\phi(1) \le 0$, $\phi(-1) \le 0$ and $\phi'(0) > 0$ (see Figure 2), then we think we shall observe the same asymptotics as in the Brownian motion with drift environment case, if κ denotes now the unique root in $(0, 1\,]$ of $\phi(m) = 0$.

Sections 2 and 3 of this paper aim at proving that Brownian motion in a Lévy potential is the true continuous analogue of the random walk in a random media considered by Kozlov [10], Solomon [16] and Kesten, Kozlov and Spitzer [9]. Also, the local time process is the appropriate object if we want to take into account, in our stochastic differential equation, the discontinuities of the potential V.

We may have to consider two simultaneous renormalizations (one on the random walk, one on the random media) to obtain (heuristically) in the limit our Brownian motion in a Lévy potential. We think that this justifies our work, since our results do not seem to be easily derived from those renormalizations and classical results concerning random walks in random environments.

Section 4 is devoted to the construction of Brownian motion in a Lévy potential and to the proof of our main result.

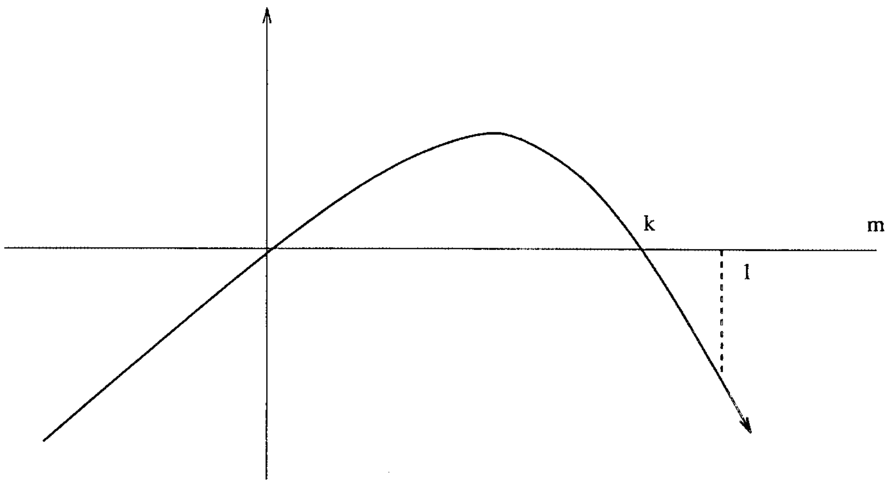


FIG. 2. *The Laplace exponent: sublinear growth.*

2. A random walk with one barrier at 0—Walsh's Brownian motion. Let $p = 1 - q \in (0, 1)$ and consider the Markov process $(X_n,\ n \in \mathbb{N})$ with transition kernel

$$K(x, x+1) = 1 - K(x, x-1) = \begin{cases} \frac{1}{2}, & \text{if } x \neq 0, \\ p, & \text{if } x = 0. \end{cases}$$

In other words, X is a simple random walk away from level zero, and each time it crosses level zero, it throws a biased coin: if the coin shows heads—this event happens with probability p—it chooses to go upwards; otherwise it goes downwards.

Let θ_n be the number of visits to 0 up to time n:

$$\theta_n = \#\{p: 0 \leq p \leq n,\ X_p = 0\} \tag{1}$$

It is easy to show that

$$S_n = X_n - (p - q)\theta_{n-1} \tag{2}$$

is a martingale that satisfies (see Appendix A)

$$\mathbb{E}[S_n^2] \sim n. \tag{3}$$

Hence, we have the convergence in distribution to a standard Brownian motion:

$$\left(n^{-1/2} S_{\lfloor nt \rfloor};\ t \geq 0\right) \underset{n \to \infty}{\overset{d}{\mapsto}} (B_t;\ t \geq 0) \tag{4}$$

(from now on B will denote a standard Brownian motion).

Considering the relations (2) and (4), we expect, and this is indeed the case (see, e.g., [4]), the convergence in distribution of $(n^{-1/2} X_{\lfloor nt \rfloor};\ t \geq 0)$ to a continuous semimartingale X which is a solution of the stochastic equation

$$X_t = B_t + (p - q)\tilde{L}_t,$$

where $(\tilde{L}_t;\ t \geq 0)$ denotes the symmetric local time of X at level 0:

$$\tilde{L}_t = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1_{(|X_s| \leq \varepsilon)}\, ds.$$

The limiting process is called Walsh's Brownian motion, or skew Brownian motion, with skewness parameter p, introduced by Walsh in [17]. We may characterize or define it in the following ways (see, e.g., [15], Exercises 2.22, page 390; 1.23, page 270; 1.16, page 82; 2.16, page 449 and also [1]).

DEFINITION 1. Given $x \in \mathbb{R}$ the unique solution of the stochastic equation

$$X_t = x + B_t + (p - q)\tilde{L}_t \tag{5}$$

is a Walsh Brownian motion starting from x.

DEFINITION 2. Walsh's Brownian motion is the unique Markov linear process X such that the processes $(|X_t|;\ t \geq 0)$ and $(|B_t|;\ t \geq 0)$ are identically distributed and when it starts from zero, we have

$$\mathbb{P}(X_t > 0) = p \quad \text{for all } t > 0.$$

DEFINITION 3. Let $n_+(de)$ (resp. $n_-(de)$) be Itô's measure of the positive (resp. negative) excursions of a Brownian motion away from zero (n_- is the image of n_+ under the transformation $e \to -e$). Then, Walsh's Brownian motion is the unique continuous Markov process for which 0 is a regular point, with the corresponding Itô excursion measure: $n(de) = pn_+(de) + qn_-(de)$. Intuitively, when Walsh's Brownian motion crosses level 0, it chooses with probability p a positive Brownian excursion, and with probability q a negative Brownian excursion.

DEFINITION 4. The semigroup of Walsh's Brownian motion is a contraction semigroup on the space $C_0(\mathbb{R})$ of continuous functions vanishing at infinity, with infinitesimal generator given by

$$Lf(x) = (1/2) f''(x)$$

on the domain

$$\begin{aligned} D(\mathrm{L}) = \{ f\colon\ & f \text{ and } f' \text{ have a continuous derivative on } (-\infty, 0) \\ & \text{and } (0, +\infty),\ pf'(0+) = qf'(0-) \text{ and } f''(0+) = f''(0-)\}. \end{aligned}$$

Definition 5. Walsh's Brownian is a continuous linear Markov process with scale function s_V and speed measure m_V given by

$$s_V(x) = \int_0^x e^{V(y)}\, dy, \qquad m_V(dy) = 2e^{-V(y)}\, dy, \tag{6}$$

with

$$V(x) = \begin{cases} \log(1/p), & \text{if } x \geq 0; \\ \log(1/q), & \text{if } x < 0. \end{cases}$$

Hence, its infinitesimal generator is given by

$$Lf(x) = \frac{d}{dm_V}\left(\frac{df}{ds_V}\right)(x) = \frac{1}{2} f''(x)$$

on the space $D(L)$ defined above.

REMARK 1. Definition 3 is a consequence of our construction and the limiting procedure. Likewise, Definition 2 should not upset us, since we have the following:

(i) it is clear that $(|X_n|,\ n \in \mathbb{N})$ has the same distribution as the absolute value of a simple random walk;

(ii) it is not very difficult to prove that for every n, $\mathbb{P}_0(X_{2n+1} > 0) = p$.

3. A random walk–diffusion with infinitely many barriers. The random environment consists of barriers $\{\tau_n,\ n \in \mathbb{Z}\}$ such that the distances between barriers $\{\tau_n - \tau_{n-1},\ n \in \mathbb{Z}\}$ are i.i.d. geometric random variables of parameter $\alpha \in (0,1)$. It can be realized from a Bernoulli trial $(\beta_\alpha(n),\ n \in \mathbb{Z})$ in the following way: let $\{Z_n,\ n \in \mathbb{Z}\}$ be i.i.d. Bernoulli r.v. with $\mathbb{P}(Z_1 = 1) = 1 - \mathbb{P}(Z_1 = 0) = \alpha$ and let

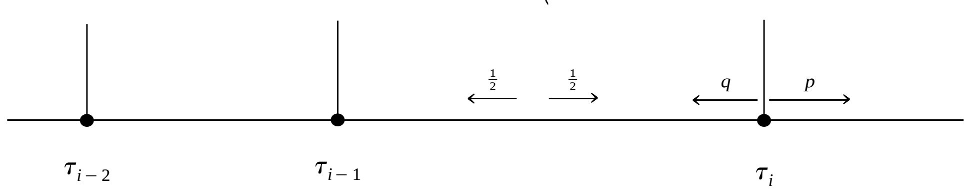
$$\beta_\alpha(n) = \begin{cases} Z_1 + \cdots + Z_n, & \text{if } n > 0, \\ 0, & \text{if } n = 0, \\ -(Z_{-1} + \cdots + Z_n), & \text{if } n < 0. \end{cases}$$

Then,

$$\tau_n = \begin{cases} \inf\{p:\ \beta_\alpha(p) = n\}, & \text{if } n \geq 0, \\ \inf\{p:\ \beta_\alpha(-p) = -n\}, & \text{if } n \leq 0. \end{cases}$$

Between the barriers, the process $(X_n,\ n \in \mathbb{N})$ is a simple random walk. When it crosses a barrier, it chooses with probability p to go to the right and $q = 1 - p$ to go to the left. In other words, the random transition function of X is given by

$$K(x, x+1) = 1 - K(x, x-1) = \begin{cases} \frac{1}{2}, & \text{if } x \notin \{\tau_n,\ n \in \mathbb{Z}\}, \\ p, & \text{otherwise}. \end{cases} \tag{7}$$



To obtain the continuous analogue we have to consider the normalized process

$$X^n = \left(\frac{1}{\sqrt{n}} X_{\lfloor nt \rfloor};\ t \geq 0\right),$$

and the corresponding normalized random environment, for a fixed $\lambda > 0$,

$$\beta^n = \Big(\beta_{\lambda/\sqrt{n}}(\lfloor \sqrt{n}\, t \rfloor);\ t \geq 0 \Big).$$

Since β^n converges in distribution to a Poisson process with parameter λ, $(N_x,\ x \in \mathbb{R})$ (see Appendix B), according to Section 2, we expect X^n to converge in distribution to a continuous semimartingale which is the solution of the stochastic equation

$$X_t = B_t + (p - q) \sum_{n \in \mathbb{Z}} \tilde{L}_t^{\sigma_n}, \tag{8}$$

where $(B_t;\ t \geq 0)$ is a standard Brownian motion, $(\sigma_n,\ n \in \mathbb{N})$ denotes the sequence of jump times of a Poisson process of parameter λ, and $(\tilde{L}_t^x,\ x \in \mathbb{R};\ t \geq 0)$ is the process of symmetric local times of X.

In the setting of Section 4, X is a diffusion process in the random Lévy potential

$$V(t) = \log(q/p)\, N_t.$$

The Laplace exponent of V, defined by $\mathbb{E}[e^{mV_t}] = e^{-t\phi(m)}$, is given by

$$\phi(m) = \lambda\left(1 - \left(\frac{q}{p}\right)^m\right).$$

Suppose $p > q$; then $\phi(1) > 0$, and by Theorem 6, almost surely,

$$\frac{X_t}{t} \underset{t\to\infty}{\mapsto} \frac{\phi(1)}{2} = \frac{\lambda(p - q)}{2p}. \tag{9}$$

REMARK 2. An open problem is to prove the convergence in distribution of the normalized random walk in the normalized random environment to the diffusion solution of the equation (8).

4. The diffusion model. Let $\mathcal{V}$ be the space of cadlag functions $V\colon \mathbb{R} \to \mathbb{R}$ vanishing at the origin: $V(0) = 0$. Let $\mathbb{P}(dV)$ be the law of a spatial real Lévy process vanishing at 0. In other words, under $\mathbb{P}(dV)$, the coordinate process v is such that the following hold:

1. $v(0) = 0$;
2. if $x_0, x_1, \ldots, x_n$ are real numbers, then the random variables $(v(x_{i+1}) - v(x_i),\ 0 \leq i \leq n - 1)$ are independent;
3. for all real numbers x, y, the law of $v(x + y) - v(x)$ depends only on y.

Let Ω be the space of continuous functions $\omega\colon \mathbb{R}_+ \to \mathbb{R}$, let $(X_t;\ t \geq 0)$ be the coordinate process on Ω: $X_t(\omega) = \omega(t)$, and let $(\mathcal{F}_t;\ t \geq 0)$ be the natural σ-field of X.

Given a sample function $V \in \mathcal{V}$, we consider probability measures P_V^x on $(\Omega, \mathcal{F}_\infty)$ such that $(X_t, \mathcal{F}_t;\ t \geq 0;\ P_V^x,\ x \in \mathbb{R})$ is a diffusion process with infinitesimal generator

$$L_V f(x) = \frac{d}{dm_V}\left(\frac{df}{ds_V}(x)\right).$$

In other words, it is a diffusion process with scale function s_V and speed measure m_V given by

$$s_V(x) = \int_0^x e^{V(y)}\,dy \quad \text{and} \quad m_V(dx) = 2e^{-V(x)}\,dx.$$

Recall that the law of the process is unchanged when we add a constant to the potential V. Hence, it is natural to force the condition $V(0) = 0$. When V is considered random, X is regarded as a process defined on the probability space $(V \times \Omega,\ P^x)$ with

$$P^x(dV\,d\omega) = \mathbb{P}(dV)\mathbb{P}_V^x(d\omega).$$

The process $(X_t;\ t \geq 0;\ P^x)$ is called a *diffusion process in a random Lévy environment*, starting from x. The environment is spatially homogeneous in the following sense: since, under P, V is a Lévy process on $\mathbb{R}$, the probability measure $\mathbb{P}(dV)$ is invariant and ergodic under the action of the group of transformations $\{\Gamma_x,\ x \in \mathbb{R}\}$ defined by

$$\Gamma_x V(y) = V(x+y) - V(x), \qquad x, y \in \mathbb{R}$$

(see Appendix C for a proof).

To state our main result we shall need the following notations. Let $\phi(m)$ be the Laplace exponent of V:

$$\mathbb{E}\left[e^{mV(t)}\right] = e^{-t\phi(m)}, \qquad m \in \mathbb{R},\ t \geq 0.$$

For $x \in \mathbb{R}$, let

$$T_x = \inf\{t > 0:\ X_t = x\}.$$

THEOREM 1. (i) *Assume that* $\phi(1) > 0$. *Then,* P^0 *almost surely, we have*

$$\lim_{x \to +\infty} \frac{T_x}{x} = \frac{2}{\phi(1)},$$

$$\lim_{t \to +\infty} \frac{X_t}{t} = \frac{\phi(1)}{2}.$$

(ii) *Assume that* $\phi(-1) > 0$. *Then,* P^0 *almost surely, we have*

$$\lim_{x \to +\infty} \frac{T_{-x}}{x} = \frac{2}{\phi(-1)},$$

$$\lim_{t \to +\infty} \frac{X_t}{t} = -\frac{\phi(-1)}{2}.$$

(iii) *Assume that* $\phi(1)$ *and* $\phi(-1)$ *are in* $[-\infty, 0]$. *Then,* P^0 *almost surely, we have*

$$\lim_{t \to +\infty} \frac{X_t}{t} = 0.$$

REMARK 3. (a) The three cases (i), (ii) and (iii) are mutually exclusive. Indeed, Jensen's inequality implies that

$$\frac{1}{\mathbb{E}[e^{V(1)}]} \leq \mathbb{E}\left[\frac{1}{e^{V(1)}}\right] \quad \text{i.e., } \phi(1) \leq -\phi(-1).$$

Thus, (i), (ii) and (iii) correspond respectively to $\mathbb{E}[e^{V(1)}] < 1$, $\mathbb{E}[e^{-V(1)}] < 1$ and $(1/\mathbb{E}[e^{V(1)}]) \leq 1 \leq \mathbb{E}[e^{-V(1)}]$.

(b) We may check our results against those of Theorem 1 of [8]. There, V is a Brownian motion with negative drift $-\kappa/2$, so that

$$\phi(m) = \tfrac{1}{2}(\kappa m - m^2) \quad \text{and} \quad \phi(1) = \tfrac{1}{2}(\kappa - 1).$$

Hence, if $\kappa > 1$, we have P^0 almost surely:

$$\lim_{t\to+\infty} \frac{X_t}{t} = \frac{\kappa - 1}{4}.$$

(c) We recall here some classical relationships between the asymptotic behavior of a linear continuous Markov process X and its scale function s (see, e.g., [14], Chapter VII, Section 3, Exercise 3.21).

1. If $s(-\infty) = -\infty$ and $s(+\infty) = +\infty$, then X is recurrent, and almost surely $\liminf X_t = -\infty$ and $\limsup X_t = +\infty$.
2. If $s(-\infty) = -\infty$ and $s(+\infty) < +\infty$, then $X_t \to +\infty$ almost surely and for all x, $\mathbb{P}^0(\inf_{t\geq 0} X_t < x) = (s(+\infty) - s(0)/s(+\infty) - s(x))$.
3. If $s(-\infty) > -\infty$ and $s(+\infty) = +\infty$, then $X_t \to -\infty$ almost surely and for all x, $\mathbb{P}^0(\sup_{t\geq 0} X_t > x) = (s(0) - s(-\infty)/(s(x) - s(-\infty))$.

4.1. *Construction for the probability measures* $\mathbb{P}_V^x$. They are constructed from a standard Brownian motion $(B_t;\ t \geq 0)$ in the following way (see, e.g., Chapter VII of [15]).

Let $(\gamma_t;\ t \geq 0)$ be the right continuous inverse of the continuous additive functional

$$\Gamma_t = \int_0^t \exp\bigl(-2V \circ s_V^{-1}(B_s)\bigr)\, ds.$$

That is,

$$\gamma_t = \inf\{u > 0\colon \Gamma_u > t\}.$$

Let $\mathbb{P}_V^0$ be the law of the process

$$\bigl(s_V^{-1}(B_{\gamma_t});\ t \geq 0\bigr).$$

Then $\mathbb{P}_V^x$ is defined by

$$\mathbb{E}_V^x\bigl[F(X_t;\ t \geq 0)\bigr] = \mathbb{E}^0_{\Gamma_x V}\bigl[F(x + X_t;\ t \geq 0)\bigr].$$

With these definitions, we have the strong Markov property, with obvious notation,

$$\mathbb{E}_V^x\bigl[Z \circ \theta_T | F_T\bigr] = \mathbb{E}_V^{X_T}\bigl[Z\bigr].$$

4.2. *Proof of Theorem* 1. (i) For $n = 1, 2, \dots,$ consider $\tau_n = T_n - T_{n-1}$.

CLAIM 1. *Under* P^0, *the process* $((\tau_n, \Gamma_n V),\ n \geq 1)$ *is a stationary ergodic Markov chain.*

We show that its transition kernel is given, on bounded measurable functions $\phi(t, V)$, by

$$R\phi(t, V) = Q\phi(V) = \mathbb{E}_V^0[\phi(\tau_1, \Gamma V)].$$

Indeed, let $(G_n,\ n \geq 1)$ be the natural filtration of this process and let Z be a bounded G_{n-1}-measurable random variable. Then we have

$$E^0(\phi(\tau_n, \Gamma_n V) Z) = \int \mathbb{P}(dV) \mathbb{E}_V^0[\phi(\tau_n, \Gamma_n V) Z]$$

$$= \int \mathbb{P}(dV) \mathbb{E}_V^0\Big[\mathbb{E}_V^0[\phi(\tau_n, \Gamma_n V) | F_{\tau_{n-1}}] Z\Big].$$

By the strong Markov property (for a deterministic V), we have

$$\begin{aligned}
\mathbb{E}_V^0[\phi(\tau_n, \Gamma_n V) | F_{\tau_{n-1}}] &= \mathbb{E}_V^0[\phi(\tau_1 \circ \theta_{\tau_{n-1}}, \Gamma_n V) | F_{\tau_{n-1}}] \\
&= \mathbb{E}_V^{X_{\tau_{n-1}}}[\phi(\tau_1, \Gamma_n V)] \\
&= \mathbb{E}_V^{n-1}[\phi(\tau_1, \Gamma_n V)] \\
&= \mathbb{E}_{\Gamma_{n-1} V}[\phi(\tau_1, \Gamma_n V)] \\
&= Q\phi(\Gamma_{n-1} V).
\end{aligned}$$

Hence,

$$E^0(\phi(\tau_n, \Gamma_n V) Z) = \int \mathbb{P}(dV) \mathbb{E}_V^0[Q\phi(\Gamma_{n-1} V) Z] = E^0(Q\phi(\Gamma_{n-1} V) Z).$$

The Markov chain is stationary, since for every bounded measurable ϕ,

$$\begin{aligned}
E^0(R\phi(\tau_1, \Gamma V)) &= \int \mathbb{P}(dV) \mathbb{E}_{\Gamma V}^0[\phi(\tau_1, \Gamma V)] \\
&= \int \mathbb{P}(dV) \mathbb{E}_V^0[\phi(\tau_1, \Gamma V)] \quad \text{(by invariance of } \mathbb{P} \text{ under } \Gamma) \\
&= E^0(\phi(\tau_1, \Gamma V)).
\end{aligned}$$

The chain is ergodic because if a bounded measurable $\phi(t, V)$ satisfies $R\phi = \phi$, then ϕ is P^0-a.s. constant. Indeed,

$$\phi(t, V) = R\phi(t, V) = \mathbb{E}_V^0[\phi(\tau_1, \Gamma V)]$$

is a function that does not depend on variable t; thus, $\phi(t, V) = h(V)$ with

$$h(V) = R\phi(t, V) = \mathbb{E}_V^0[\phi(\tau_1, \Gamma V)] = h(\Gamma V).$$

Since $\mathbb{P}(dV)$ is ergodic under the action of Γ (see Appendix C), the function $h(V)$ is $\mathbb{P}(dV)$ a.s. constant, and that implies that ϕ is P^0-a.s. constant.

CLAIM 2. $\lim_{x\to+\infty}(T_x/x) = (2/\phi(1))$.

Birkhoff's ergodic theorem yields that P^0-a.s.

$$\lim_{n\to+\infty}\frac{T_n}{n} = \lim_{n\to+\infty}\frac{1}{n}(\tau_1+\tau_2+\cdots+\tau_n) = E^0(\tau_1).$$

Since $x\to T_x$ is increasing, all that remains to show is that $E^0(\tau_1) = 2/\phi(1)$. Observe that P^0-a.s., $s_V(0)=0$ and $s_V(-\infty)=-\infty$. Hence, after taking limits and integrating by parts in Corollary 3.8 of Revuz–Yor [15], we obtain

$$\mathbb{E}_V^0[\tau_1] = \int_0^1 ds_V(x)\int_{-\infty}^x m_V(dy).$$

Therefore,

$$\begin{aligned} E^0(\tau_1) &= 2\int_0^1 dx\int_{-\infty}^x dy\,\mathbb{E}[e^{V(x)-V(y)}] \\ &= 2\int_0^1 dx\int_{-\infty}^x dy\, e^{-(x-y)\phi(1)} \\ &= \frac{2}{\phi(1)}. \end{aligned}$$

CLAIM 3. *Let* $\overline{X}_t = \sup_{0\le s\le t} X_s$. *Then*, P^0-*a.s. we have*

$$\lim_{t\to+\infty}\frac{\overline{X}_t}{t} = \frac{\phi(1)}{2}. \tag{10}$$

PROOF. Since $\lim_{x\to+\infty}(T_x/x) = (2/\phi(1))$, almost surely, for all n big enough, $T_n < +\infty$. Therefore,

$$\lim_{t\to+\infty}\overline{X}_t = +\infty \quad P^0\text{-a.s.}$$

For every $\varepsilon > 0$ we have

$$\frac{T_{\overline{X}_t}}{\overline{X}_t} \le \frac{t}{\overline{X}_t} \le \frac{T_{\overline{X}_t+\varepsilon}}{\overline{X}_t}.$$

Letting $t\uparrow+\infty$, and then $\varepsilon\downarrow 0$ gives the limit (10). □

To resume our proof of Theorem 1, all that remains to show is the following.

CLAIM 4. *We have*, P^0-*a.s.*,

$$\liminf_{t\to+\infty}\frac{1}{t}\inf_{s\ge t} X_s \ge \frac{\phi(1)}{2}.$$

PROOF. For $\varepsilon\in(0,1)$ and $n\ge 1$, let $\theta(n) = n(1-\varepsilon)\frac{1}{2}\phi(1)$. From the strong Markov property of X and the invariance of $\mathbb{P}(dV)$ under the action of

$\{\Gamma_x,\ x \in \mathbb{R}\}$, we derive

$$\begin{aligned} P^0\Bigl(\inf_{s \geq T_{\theta(n)}} X_s - \theta(n) < -\sqrt{n}\Bigr) &= \int \mathbb{P}(dV)\mathbb{P}_V^{\theta(n)}\Bigl(\inf_{s \geq 0} X_s - \theta(n) < -\sqrt{n}\Bigr) \\ &= \int \mathbb{P}(dV)\mathbb{P}^0_{\Gamma_{\theta(n)}V}\Bigl(\inf_{s \geq 0} X_s < -\sqrt{n}\Bigr) \\ &= \int \mathbb{P}(dV)\mathbb{P}^0_V\Bigl(\inf_{s \geq 0} X_s < -\sqrt{n}\Bigr) \\ &= P^0\Bigl(\inf_{s \geq 0} X_s < -\sqrt{n}\Bigr). \end{aligned}$$

Proposition 4.9 of [3] states that

$$P^0\Bigl(\inf_{s \geq 0} X_s < -\sqrt{n}\Bigr) = o\bigl(\exp\bigl(-\sqrt{n}\,\phi(1)\bigr)\bigr).$$

Hence, this is the general term of a convergent series, and an application of the Borel–Cantelli lemma yields that P^0-a.s., for $n \geq n_0(V, \omega)$ we have

$$\inf_{s \geq T_{\theta(n)}} X_s - \theta(n) \geq -\sqrt{n}\,.$$

Note that, P^0-a.s.,

$$\frac{T_{\theta(n)}}{n} \underset{n \to \infty}{\mapsto} 1 - \varepsilon.$$

For $t > 0$, we let $n = \lfloor t \rfloor$. Therefore, for $t \geq n_1(V, \omega)$, we have

$$\inf_{s \geq t} X_s \geq \inf_{s \geq n} X_s \geq \inf_{s \geq T_{\theta(n)}} X_s \geq \theta(n) - \sqrt{n}\,.$$

Hence,

$$\liminf_{t \to +\infty} \frac{1}{t} \inf_{s \geq t} X_s \geq \lim_{n \to +\infty} \frac{\theta(n) - \sqrt{n}}{n} \geq \frac{\phi(1)}{2}(1 - \varepsilon).$$

Letting $\varepsilon \downarrow 0$ concludes our proof. $\square$

(ii) This is a direct consequence of (i), since $\tilde{X} = -X$ is a diffusion process in the random potential $\tilde{V}(x) = V(-x)$.

(iii) As in Claim 2, we prove that $E^0(\tau_1) = +\infty$, and use Birkhoff's theorem to infer that, P^0-a.s.,

$$\lim_{x \to +\infty} \frac{T_x}{x} = +\infty.$$

Considering the diffusion $-X$, we obtain similarly that P^0-a.s.,

$$\lim_{x \to +\infty} \frac{T_{-x}}{x} = +\infty.$$

For every $t \geq 0$, we note $\overline{X}_t = \sup_{s \leq t} X_s$ and $\underline{X}_t = \inf_{s \leq t} X_s$. Let $\varepsilon > 0$. From the inequality, $t \leq T_{\overline{X}_t + \varepsilon}$ we deduce that P^0-a.s.,

$$\lim_{t \to +\infty} \overline{X}_t = +\infty.$$

We may then take the limit as $t \to \infty$ in

$$\frac{T_{\overline{X}_t}}{\overline{X}_t} \leq \frac{t}{\overline{X}_t} \leq \frac{T_{\overline{X}_t + \varepsilon}}{\overline{X}_t}$$

to obtain that P^0-a.s.,

$$\lim_{t \to +\infty} \frac{\overline{X}_t}{t} = 0.$$

In the same way, we prove that P^0-a.s.,

$$\lim_{t \to +\infty} \frac{\underline{X}_t}{t} = 0.$$

APPENDIX A

Proof of the equivalence $\mathbb{E}[\mathbf{S}_\mathbf{n}^2] \sim \mathbf{n}$. Recall that $Lf(x) = Kf(x) - f(x)$ is the infinitesimal generator of the process X; thus, for every well-behaved function f, the process

$$M_n^f = f(X_n) - \sum_{p=0}^{n-1} Lf(X_p)$$

is a martingale with previsible quadratic variation

$$\langle M^f, M^f \rangle_n = \sum_{p=0}^{n-1} \Gamma(f, f)(X_p),$$

where Γ denotes the *opérateur carré du champ*

$$\Gamma(f, g) = L(fg) - fLg - gLf - (Lf)(Lg).$$

We recall that $\langle M^f, M^f \rangle$ is the unique increasing, previsible process A such that $((M_n^f)^2 - A_n,\ n \in \mathbb{N})$ is a martingale. The relations (2) and (3) are obtained by applying these results to the function $f(x) = x$. Indeed, we have $Lf(x) = (p-q)1_{(x=0)}$, and $\Gamma(f, f) = 1 - (p-q)^2 1_{(x=0)}$. Therefore,

$$\mathbb{E}\left[S_n^2\right] = n - (p-q)^2 \mathbb{E}[\theta_{n-1}].$$

Now relation (2) implies that

$$0 \leq \mathbb{E}[\theta_{n-1}] = \frac{\mathbb{E}[X_n]}{p-q} \leq \frac{\mathbb{E}[|X_n|]}{p-q}$$

Observe that $(|X_n|,\ n \in \mathbb{N})$ is simple reflected random walk; its law does not depend on p. Hence, $(1/\sqrt{n})\mathbb{E}[|X_n|] \to \sqrt{\pi/2}$ and $(1/n)\mathbb{E}[S_n^2] \to 1$.

APPENDIX B

Convergence of renormalized Bernoulli trials to a Poisson process. Since β^n is a process with independent increments (and fixed discontinuities), that is, a PII in the terminology used by Jacod and Shiryaev in Section 4, Chapter II of [5], and since the limiting process is a Poisson process N, a stationary PII (thus without fixed discontinuities), to obtain the convergence in distribution of β^n to N in the Skorokhod topology, we only have to check the uniform convergence of characteristic functions on finite intervals (see Theorem 4.3, Chapter VII of [5]):

$$\sup_{0 \le s \le t,\ |u| \le \theta} \left| \mathbb{E}\left[\exp^{(iu\beta_s^n)}\right] - \mathbb{E}\left[\exp^{(iuN_s)}\right] \right| \underset{n \to \infty}{\mapsto} 0.$$

That is,

$$\sup_{0 \le s \le t,\ |u| \le \theta} \left| \left(1 + \frac{\lambda}{\sqrt{\lfloor n \rfloor}}(\exp(iu) - 1)\right)^{\lfloor \sqrt{n}\, s \rfloor} - \exp(\lambda s(\exp(iu) - 1)) \right| \underset{n \to \infty}{\mapsto} 0.$$

APPENDIX C

The law of a Lévy process is ergodic under the action of the group $\{\Gamma_x, x \in \mathbb{R}\}$. Let A be an event invariant under the action of $\{\Gamma_x,\ x \in \mathbb{R}\}$. We are going to show that $\mathbb{P}(A) \in \{0, 1\}$. Let $\varepsilon > 0$; there exists an event A_ε, depending on a finite number of coordinates, such that $\mathbb{P}(A \Delta A_\varepsilon) < \varepsilon$, where Δ denotes the symmetric difference operator. To fix ideas, let

$$A_\varepsilon = \{v\colon v(x_i) \in B_i,\ 1 \le i \le n\},$$

where $x_1 \le x_2 \le \cdots \le x_n$, B_i, $1 \le i \le n$ are Borel sets, and v is the coordinate process. Remember that $v(0) = 0$ so that for all x,

$$\Gamma_x A_\varepsilon = \{v\colon v(x + x_i) - v(x) \in B_i,\ 1 \le i \le n\}.$$

Since, under $\mathbb{P}(dV)$, V is a Lévy process, it has independent increments; for x big enough, the events A_ε and $\Gamma_x A_\varepsilon$ are independent. We now infer, from this independence and from the invariance of V under Γ_x, the equality

$$\mathbb{P}(A_\varepsilon \cap \Gamma_x A_\varepsilon) = \mathbb{P}(A_\varepsilon)\mathbb{P}(\Gamma_x A_\varepsilon) = \mathbb{P}(A_\varepsilon)^2.$$

Taking limits, as ε goes to 0, gives, since $\Gamma_x A = A$,

$$\mathbb{P}(A) = \mathbb{P}(A \cap \Gamma_x A) = \mathbb{P}(A)^2.$$

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