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ON RANDOM WALKS ON WREATH PRODUCTS

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Wreath products are a type of semidirect product. They play an important role in group theory. This paper studies the basic behavior of simple random walks on such groups and shows that these walks have interesting, somewhat exotic behaviors. The crucial fact is that the probability of return to the starting point of certain walks on wreath products is closely related to some functionals of the local times of a walk taking place on a simpler factor group.

1. Introduction. The aim of this paper is to study simple random walks on certain groups that are all obtained through the same algebraic construction known as a wreath product. In the random walk literature, examples of such groups were considered in a paper of Kaimanovich and Vershik [27]. They also appear in two papers of Varopoulos [48, 49], in [32] and, more recently, in [31]. In particular, Varopoulos's articles are the roots of the present paper. We show that these groups exhibit, as far as random walk is concerned, some interesting—in some sense exotic—behaviors. Some of our results were announced in [38] where applications of these results to isoperimetric inequalities on groups are described. Somewhat weaker but similar results were obtained independently by Hebisch [24]. Wreath products also appear in recent works by Erschler (Dyubina) [13, 14], Grigorchuck and Żuk [18] and Revelle [41, 42].

1.1. *Background.* To put the results of this paper in perspective, let us recall what is known regarding the basic behavior of simple random walks on finitely generated groups and let G be such a group. Let S be a symmetric generating set. The simple random walk on G associated to S is the G-valued process that evolves as follows: if $X_n = x$ is the position at time n, then $X_{n+1} = xs$, where s is chosen uniformly at random in S. More generally, the random walk associated with a probability measure q on G is the similar G-valued process where s is chosen according to q. One of the most basic quantities of interest in this context is the probability of return to the starting point after a given number of steps. If the walk is driven by q as above, this probability is equal to $q^{(n)}(e)$ where e is the neutral element of G and $q^{(n)}$ denotes the n-fold convolution power of q. The question to be addressed here is that of the behavior of $q^{(n)}(e)$ as n tends to ∞. For the most

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part, we will be interested in characterizing this behavior up to the equivalence relation $\approx$ defined as follows. For two functions $f, g\colon (0,\infty) \to (0,\infty)$ we write $f \approx g$ if there are constants c_i such that $c_1 f(c_2 t) \le g(t) \le c_3 f(c_4 t)$ for all $t > 0$ (or all integers t, replacing $c_i t$ by its integer part $[c_i t]$). This notation will be used mostly for monotone functions. The question of to what extent the behavior of the sequence $q^{(n)}(e)$ depends on q when q is finitely supported will be discussed briefly later on.

The subject started with an investigation of Pólya who showed that, for simple random walks on $\mathbb{Z}^d$, the probability of return is $\approx n^{-d/2}$. The book of Spitzer [45] describes in detail many aspects of the theory of random walks on $\mathbb{Z}^d$. The first study of random walks on general discrete groups appeared in Kesten's Ph.D. thesis [29]. For a review of the subject, we refer the reader to Woess's survey [55] which contains a long list of references, to the recent book [56] by the same author, and to [44].

One of the interesting aspects of the subject (and one of its difficulties) comes from the interplay between random walk behavior and algebraic structure. In what follows we will need to use the notions of nilpotent, polycyclic, solvable and metabelian groups. We refer to [43], Chapter 5, for a detailed introduction to these notions, but for the convenience of the reader we recall the following definitions. See also Figure 1.

For $h, k \in G$, set $[h,k] = h^{-1}k^{-1}hk$. For two subsets A, B of G, let $[A, B]$ be the subgroup of G generated by all the elements of the form $[a,b]$, $a \in A$, $b \in B$. The lower central series of G is the nonincreasing sequence of subgroups defined by $G_1 = G$, $G_2 = [G,G]$, $G_{i+1} = [G_i, G]$. By construction, G_i/G_{i+1} lies in the center of G/G_{i+1}. A group is *nilpotent* if there is an integer k such that $G_k = \{e\}$.

The derived series is defined by $G^1 = G$, $G^2 = [G,G]$, $G^{i+1} = [G^i, G^i]$. By construction, G^i/G^{i+1} is abelian. A group is *solvable* if there is an integer k such that $G^k = \{e\}$. A basic result is that any nilpotent group is solvable.

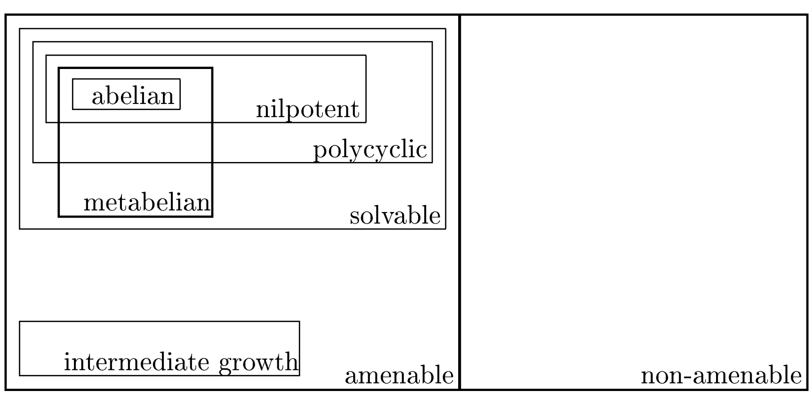

FIG. 1. *Inclusion relations between various classes of finitely generated groups.*

A group is *polycyclic* if it admits a finite decreasing sequence of subgroups $H_1 = G \supset H_2 \supset \cdots \supset H_{k-1} \supset H_k = \{e\}$ such that H_{i+1} is normal in H_i and H_i/H_{i+1} is cyclic. Polycyclic groups are always solvable. Nilpotent groups are always polycyclic.

A group G is *metabelian* if its commutator group $[G, G]$ is abelian. Observe that metabelian groups are obviously solvable. They can be polycyclic or not, nilpotent or not. The group of 3×3 upper-triangular unipotent matrices with integer entries is nilpotent and metabelian. The group $\mathbb{Z} \ltimes \mathbb{Z}^2$ whose product is given by

$$(x, u) \cdot (y, v) = (x + y, u + A^x v),$$

where

$$u = (u_1, u_2), \qquad v = (v_1, v_2), \qquad A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

is not nilpotent but polycyclic and metabelian. The subgroup of the affine group $ax + b$ generated by the affine transformations $u\colon x \mapsto x + 1$ and $v\colon x \mapsto 2x$ is metabelian but not polycyclic.

We will also need the notion of amenable groups (see, e.g., [35]). A finitely generated group G is *amenable* if it admits an invariant mean, that is, a continuous functional ν defined on the space B of all bounded functions on G such that $\nu(f) \geq 0$ if $f \geq 0$, $\nu(1) = 1$ and $\nu(f_x) = \nu(f)$ where $f_x\colon y \mapsto f(xy)$, $f \in B$, $x \in G$. A nonamenable group is a group which is not amenable.

We now proceed to recall some deep results concerning volume growth and its relation to the algebraic structure of groups. The paper [17] gives a short informative overview.

Volume growth. Let G be a group generated by a symmetric finite set S. Let $V(n)$ be the number of elements of G that can be written as a product of length at most n of elements of S (the empty product is equal to the neutral element). Thus $V(n)$ is the cardinality of the set $\{e\} \cup S \cup S^2 \cup \cdots \cup S^n$. The function V is called the volume growth function. For a book treatment containing a good bibliography, see [23].

1. V grows at most exponentially (folklore). Nonamenable groups have exponential growth (Følner [16] and [35], (6.8)).
2. If G is nilpotent or contains a nilpotent subgroup of finite index, there exists an integer d such that $V(n) \approx n^d$ (Dixmier [10], Guivarc'h [21] and Bass [2]).
3. If G is solvable, either G contains a nilpotent subgroup of finite index or $V(n) \approx e^n$ (Milnor [33] and Wolf [57]).
4. If there exist $A, C > 0$ and an increasing sequence n_i of integers such that $V(n_i) \leq Cn_i^A$, then G contains a nilpotent subgroup of finite index (Gromov [19]; see also Van den Dries and Wilkie [47]).
5. There exist finitely generated groups such that V grows faster than any polynomial but such that $V(n)^{1/n}$ tends to 1 as n tends to ∞. That is, there exist groups having intermediate volume growth (Grigorchuk [17]).

Next we recall what is known concerning simple random walks on a group G in relation to the volume growth and the algebraic structure.

Random walk behavior. Let G be a group generated by a symmetric finite set S. Let $\phi(n)$ be the probability that the simple random walk returns to its starting point after $2n$ steps. We will refer to ϕ as the return-probability function of (G, S). That is, $\phi(n)$ is the value at the neutral element $e \in G$ of the $2n$-fold convolution $p^{(2n)}$ where p is the probability measure $p = \frac{1}{\#S}\mathbf{1}_S$ on G. Except for the important early contribution of Kesten, most of the results below are due to Varopoulos. See [52] and [54]–[56].

1. G is nonamenable if and only if $\phi(n) \approx \exp(-n)$ (Kesten [29] and [30]).
2. If $V(n) \approx n^d$, then $\phi(n) \approx n^{-d/2}$ (Varopoulos [50]; see also [25] and [54]).
3. If $V(n) \geq c_1 \exp(c_2 n^{\alpha})$ for some $c_i, c_2 > 0$ and $0 < \alpha \leq 1$, then $\phi(n) \leq C \exp(-cn^{\alpha/(\alpha+2)})$ (Varopoulos [53]; see also [25] and [54]).
4. If G contains a polycyclic subgroup of finite index having exponential growth, then $\phi(n) \approx \exp(-n^{1/3})$ (Varopoulos [53] and Alexopoulos [1]; see also [25] and [39]).

REMARK. The reason for defining ϕ as the probability of return at time $2n$ and not n is to avoid obvious parity problems.

1.2. *Classical versus exotic random walk behaviors.* It is well known that subgroups of Lie groups have, to some extent, a simpler structure than general groups. In this direction we recall the following crucial results.

Structure of discrete linear groups. Let Γ be a discrete subgroup of a Lie group having finitely many connected components (here discrete refers to the topology induced on the subgroup by the topology of the ambient group). Then either Γ is nonamenable or Γ is amenable and then it must contain a polycyclic subgroup of finite index. In particular, in the second alternative, Γ must be finitely generated and its volume growth V must either be of exponential type $V(n) \approx \exp(n)$ or of polynomial type $V(n) \approx n^d$ for some integer d. This follows from Tits [46] and Mostow [34]. See also [57]. An exposition can be found in [39].

This implies, for instance, that intermediate volume growth cannot appear among discrete subgroups of Lie groups. It also excludes many solvable groups, for example, all those containing subgroups that are not finitely generated.

Return probability behaviors for discrete linear groups. From the results described above, three behaviors of the return-probability function ϕ emerge as the only possible behaviors for finitely generated discrete subgroups of Lie groups having finitely many connected components:

1. Polynomial behavior: $\phi(n) \approx n^{-d/2}$ for some integer d. This happens exactly if G contains a nilpotent subgroup of finite index.
2. $\exp(-n^{1/3})$ behavior: $\phi(n) \approx \exp(-n^{1/3})$. For discrete subgroups of Lie groups having finitely many connected components, this is the case if and only

if G has exponential growth and is amenable. This behavior also appears in some other examples that are not discrete subgroups of Lie groups.

3. Exponential behavior: $\phi(n) \approx \exp(-n)$. This happens exactly if G is non-amenable.

We refer to these three behaviors as the classical behaviors.

For solvable groups that are discrete subgroups of some Lie group having finitely many components, only the first two behaviors above can arise since solvable groups are always amenable. In this case, the behavior of the return probability function ϕ can be characterized in terms of the volume growth. Namely, $\phi(n) \approx n^{-d/2}$ if and only if $V(n) \approx n^d$ whereas $\phi(n) \approx \exp(-n^{1/3})$ if and only if $V(n) \approx \exp(n)$, and these are the only possible behaviors. One of the aims of this paper is to exhibit solvable groups having an exotic behavior, that is, a behavior that is different from the polynomial and $\exp(-n^{1/3})$ behaviors above. Recall that a group is metabelian if the commutator subgroup $[G, G]$ is abelian. Among other results, we will prove the following theorem.

THEOREM 1.1. (i) *For any finitely generated metabelian group, there exists* $\varepsilon \in (0, 1)$ *such that*

$$\phi(n) \geq \exp(-c_1 n^{1-\varepsilon}) \qquad \textit{for } n \textit{ large enough.}$$

(ii) *For each small* $\delta > 0$, *there exists a finitely presented metabelian group such that*

$$\phi(n) \leq \exp(-c_2 n^{1-\delta}) \qquad \textit{for } n \textit{ large enough.}$$

(iii) *There exists a finitely generated solvable group* (*not metabelian*) *for which for any* $\delta \in (0, 1)$ *there exists* c_δ *such that*

$$\phi(n) \leq \exp(-c_\delta n^{1-\delta}) \qquad \textit{for } n \textit{ large enough.}$$

More concrete examples and other behaviors will be described below. See, for example, Theorem 3.11. One thing that this paper demonstrates is that among solvable groups having exponential volume growth one finds a wealth of different behaviors of the return probability ϕ.

1.3. *Invariance by quasi-isometry.* Let S_i, $i = 1, 2$, be two finite symmetric generating sets of a group G. Let ϕ_i be the corresponding return-probability functions. Is it true that $\phi_1(n) \approx \phi_2(n)$? In full generality, a positive answer has only been obtained recently in [40]. See also the exposition in [56].

Recall that the Cayley graph (G, S) of a finitely generated group equipped with a symmetric finite generating set S is the graph with vertex set G and an edge from x to y if and only if $y = xs$ for some $s \in S$. For any two x, y, the graph distance $d(x, y)$ between x and y is the least number of edges that must be used to go from x to y.

Let (G_i, S_i) be Cayley graphs of two finitely generated groups. A map $\psi\colon G_1 \to G_2$ is a quasi-isometry if there exists a constant C such that:

1. $\forall x, y \in G_1$, $C^{-1}d_1(x, y) - C \le d_2(\psi(x), \psi(y)) \le Cd_1(x, y) + C$.
2. Any $z \in G_2$ is at distance at most C for $\psi(G_1)$.

One says that G_1, G_2 are quasi-isometric if there exists a quasi-isometry $\psi\colon G_1 \to G_2$. This defines an equivalence relation. See, for example, [5].

For instance, G_1 and G_2 are quasi-isometric if G_1 is a subgroup of finite index in G_2 or if G_1 is the quotient of G_2 by a finite normal subgroup. Also, the Cayley graphs (G, S_1), (G, S_2) of a group G with respect to two different finite symmetric generating sets are quasi-isometric.

The next two results show that, up to the equivalence relation $\approx$, the behavior of the probability of return of random walk is an invariant of quasi-isometry of the group G. Of course, for this to be true, some restrictions such as finite support and nondegeneracy must be imposed on random walks.

THEOREM 1.2 [40]. *Let (G_i, S_i), $i = 1, 2$, be two quasi-isometric Cayley graphs. Then the simple random walks on these two Cayley graphs have comparable return-probability functions, that is, $\phi_1(n) \approx \phi_2(n)$.*

THEOREM 1.3 [40]. *Let (G, S) be a Cayley graph with return-probability function ϕ. Let q be any symmetric* [*i.e.*, $q(x) = q(x^{-1})$], *finitely supported probability measure on G whose support generates G. Then*

$$q^{(2n)}(e) \approx \phi(n).$$

The condition that q has finite support can be relaxed to $\sum_{x\in G} |x|^2 q(x) < \infty$, where $|x|$ is the graph distance on (G, S) (see [40] and [51]).

Later we will need the following complementary result.

THEOREM 1.4 [40]. *Let (G, S) be a Cayley graph with return-probability function ϕ. Let q be any symmetric, finitely supported probability measure on a quotient H of G* (*the support of q does not necessarily generates H*). *Then there exist $C, c > 0$ such that*

$$\forall n, \quad \phi([cn]) \le Cq^{(2n)}(e).$$

2. Wreath products.

2.1. *Construction.* For the general definition of wreath products as permutation groups, we refer, for instance, to [43]. Let H, K be two finitely generated groups. It will be convenient to denote the neutral element of K by o and the neutral element of H by 0, although we are not assuming that H or K are Abelian. Consider the algebraic direct sum

$$\mathbf{K}_H = \sum_{h\in H} K_h$$

of a countable number of copies of K indexed by H. Thus $\mathbf{K}_H$ is the set of all H-indexed sequences with finitely many nontrivial entries, where trivial means equal the neutral element o in K. An element of $\mathbf{K}_H$ can also be interpreted as a function $f\colon\ H \to K$ having finite support $S_f = \{h \in H : f(h) \neq o\}$. It will be convenient to use this interpretation. Thus

$$\mathbf{K}_H = \{f \in \mathcal{F}(H, K) : \#S_f < \infty\}.$$

Denote the neutral element of $\mathbf{K}_H$ by $\mathbf{o}$. The product in $\mathbf{K}_H$ is pointwise multiplication in K. Note that H acts on $\mathbf{K}_H$ by translation with the action given by

$$\tau_h f(\ell) = f(h^{-1}\ell).$$

For any $k \in K$, we denote by $\mathbf{1}_k \in \mathbf{K}_H$ the function $\mathbf{1}_k\colon\ H \to K$ equal to k at 0 and equal to o elsewhere.

We define the wreath product $K \wr H$ to be the semidirect product

$$K \wr H = \mathbf{K}_H \rtimes_\tau H$$

of $\mathbf{K}_H$ by H. Thus an element g of $K \wr H$ is a pair $g = (f, h)$, $f \in \mathbf{K}$, $h \in H$, and the product law is given by

$$(f, h)(f', h') = (f\tau_h f', hh'). \tag{2.1}$$

This definition coincides with the classical definition of wreath products as permutation groups if we let K and H act on themselves by translation. However, there is a subtle difference that appears when one wants to iterate this construction. In the classical theory the operation $\wr$ is associative, that is, $(K \wr H) \wr L \cong K \wr (H \wr L)$ (see [43], 1.6.4), whereas this is not true at all in our notation. The reason is that when iterating the construction above we understand $K \wr H$ as acting on itself by translation instead of merely acting on $K \times H$ which would be the right interpretation in the classical notation.

Note that $K \wr H$ is finitely generated. For instance, if S_K, S_H are finite symmetric generating sets for K, H, respectively, then

$$\{(\mathbf{1}_k, 0) : k \in S_K\} \cup \{(\mathbf{o}, h) : h \in S_H\}$$

is a finite symmetric generating set of $K \wr H$.

Examples of wreath products as above have recently appeared in the literature under the name lamplighter groups. Let us explain this terminology. There is a lamplighter walking on H. Its position is thus an element h of H. Above each element of H is a lamp with variable intensity indicated by an element of K. Only finitely many lamps can be turned on (i.e., $\neq o$) at any given time. The lamps form a scenery or configuration that is conveniently described by a function $f \in \mathbf{K}_H$. An element (f, h) of $K \wr H$ thus describes a scenery and the position of the lamplighter. Right multiplication by $(\mathbf{o}, \ell)$ moves the lamplighter to $h\ell$ without changing the scenery. Right multiplication by $(\mathbf{1}_k, 0)$ changes the intensity of the light at the position h of the lamplighter from its current strength $f(h)$ to a new strength $f(h)k$, leaving the other lights unchanged. For instance, the group $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ can be pictured as an infinite street along which walks a lamplighter turning lights on and off as he passes by.

2.2. *Random walk.* Let μ be a probability measure on H and let ν be a probability measure on K. As K, H are naturally embedded in $K \wr H$ through the homomorphisms

$$k \mapsto \underline{k} = (\mathbf{1}_k, 0), \qquad h \mapsto \underline{h} = (\mathbf{o}, h), \tag{2.2}$$

we can view μ and ν as probability measures on $K \wr H$. For $h \in H$, set

$$\nu^h(k) = \nu(\underline{h}\,\underline{k}\,\underline{h}^{-1}).$$

The measure ν^h is a probability measure on $K \wr H$ which is supported on the subgroup

$$\underline{K}_h = \{(\mathbf{1}_k^h, 0) : k \in K\},$$

where

$$\mathbf{1}_k^h(\ell) = \begin{cases} o, & \text{if } \ell \neq h, \\ k, & \text{if } \ell = h. \end{cases}$$

Obviously, in $K \wr H = (\sum_{h \in H} K_h) \rtimes_\tau H$, $\underline{K}_h$ can be identified with K_h.

LEMMA 2.1. *For any measure ν on K and any $h, \ell \in H$, the measures ν^h, ν^ℓ commute under convolution in $K \wr H$. That is $\nu^h * \nu^\ell = \nu^\ell * \nu^h$.*

PROOF. This is obvious if $h = \ell$. If $h \neq \ell$, these measures are supported, respectively, on the subgroups $\underline{K}_h, \underline{K}_\ell$, and if $x \in \underline{K}_h$, $y \in \underline{K}_\ell$, then $xy = yx$. The result follows. $\square$

Our goal is to find a useful formula for the convolution powers $q^{(n)}$ of a certain measure on $K \wr H$. By Theorem 1.3, precisely which measure q we use is unimportant as long as it is symmetric, finitely supported and its support generates $K \wr H$. In [48] and [49], Varopoulos works with $\mu * \nu * \mu$ where μ, ν are appropriate probability measures on H, K, respectively. We will use Varopoulos's idea but work with $q = \nu * \mu * \nu$ which leads to slightly neater results.

Thus fix two measures μ, ν on H, K and interpret them as measures on $K \wr H$. For the time being, we do not assume anything about μ and ν. Set

$$q = \nu * \mu * \nu \tag{2.3}$$

and

$$q_n = \nu * q^{(n)} * \nu = \nu * (\nu * \mu * \nu)^{(n)} * \nu.$$

The reason for considering q_n will be apparent later on. Since

$$\varepsilon^2 q^{(n)}(e) \le q_n(e) \le q^{(n)}(e), \tag{2.4}$$

where $\varepsilon = \max_{k\in K}\{\nu(k)\}$, estimates on $q_n(e)$ translate easily into estimates on $q^{(n)}(e)$.

Let now (ζ_i) be a sequence of independent, identically distributed, K-valued random variables with law ν and let (ξ_i) be a sequence of independent, identically distributed, H-valued random variables of law μ. Let us also identify ζ_i (resp. ξ_i) with its image $\underline{\zeta}_i$ (resp. $\underline{\xi}_i$) in $K \wr H$; see (2.2). If $\mathbf{P}$ is the joint law of these two sequences, we have

$$q_n(x) = \mathbf{P}(\zeta_1\zeta_2\xi_1\zeta_3\zeta_4\xi_2\cdots\zeta_{2n-1}\zeta_{2n}\xi_n\zeta_{2n+1}\zeta_{2n+2} = x). \tag{2.5}$$

To simplify notation, let $\zeta_i' = \zeta_{2i-1}\zeta_{2i}$ so that (ζ_i') is a sequence of independent K-valued random variables with law $\nu_2 = \nu^{(2)}$. Then

$$q_n(x) = \mathbf{P}(\zeta_1'\xi_1\zeta_2'\xi_2\cdots\zeta_n'\xi_n\zeta_{n+1}' = x). \tag{2.6}$$

Next, for any $x, y \in K \wr H$, set $x^y = yxy^{-1}$. Consider the sequence of H-valued random variables given by

$$X_0 = 0, \qquad X_i = \xi_1\cdots\xi_i, \qquad i \ge 1.$$

We will again identify X_i with its image $\underline{X}_i$ in $K \wr H$. The product

$$\zeta_1'\xi_1\zeta_2'\xi_2\cdots\zeta_n'\xi_n\zeta_{n+1}'$$

can be written as

$$\zeta_1'^{X_0}\zeta_2'^{X_1}\cdots\zeta_n'^{X_{n-1}}\zeta_{n+1}'^{X_n}X_n.$$

Note that this product equals $(f,h) \in K \wr H$ if and only if

$$\zeta_1'^{X_0}\zeta_2'^{X_1}\cdots\zeta_n'^{X_{n-1}}\zeta_{n+1}'^{X_n} = f, \qquad X_n = h.$$

Thus, for $x = (f,h) \in K \wr H$, we get

$$\begin{aligned}
q_n(x) &= \mathbf{P}(\zeta_1'\xi_1\zeta_2'\xi_2\cdots\zeta_n'\xi_n\zeta_{n+1}' = x)\\
&= \mathbf{P}\big(\zeta_1'^{X_0}\zeta_2'^{X_1}\cdots\zeta_n'^{X_{n-1}}\zeta_{n+1}'^{X_n}X_n = (f,h)\big)\\
&= \mathbf{P}\big(\zeta_1'^{X_0}\zeta_2'^{X_1}\cdots\zeta_n'^{X_{n-1}}\zeta_{n+1}'^{X_n}X_n = (f,h);\ X_n = h\big)\\
&= \sum_{\substack{(h_i)_1^n\in H^n\\ h_1\cdots h_n = h}} \mathbf{P}\big(\zeta_1'\zeta_2'^{h_1}\zeta_3'^{h_1h_2}\cdots\zeta_n'^{h_1\cdots h_{n-1}}\zeta_{n+1}'^{h_1\cdots h_n} = f \mid (\xi_i)_1^n = (h_i)_1^n\big)\\
&\qquad\qquad \times \mathbf{P}\big((\xi_i)_1^n = (h_i)_1^n\big)\\
&= \sum_{\substack{(h_i)_1^n\in H^n\\ h_1\cdots h_n = h}} \nu_2 * \nu_2^{h_1} * \nu_2^{h_1h_2} * \cdots * \nu_2^{h_1\cdots h_{n-1}} * \nu_2^{h_1\cdots h_n}(f)\mathbf{P}\big((\xi_i)_1^n = (h_i)_1^n\big).
\end{aligned}$$

Here we are using the fact that all ν_2^ℓ have supports in the subgroup $\mathbf{K}_H$ so that we can compute their convolution in that subgroup.

Let $\mathbf{P}_\xi$ be the law of $\xi = (\xi_i)$ on H^∞ and let $\mathbf{E}_\xi$ be the corresponding expectation. We can then rewrite the formula above as

$$q_n\big((f,h)\big) = \mathbf{E}_\xi\Big(\nu_2^{X_0} * \nu_2^{X_1} * \cdots * \nu_2^{X_n}(f) \mid X_n = h\Big)\mathbf{P}_\xi(X_n = h). \tag{2.7}$$

Furthermore, observe that (X_n) is exactly the random walk on H driven by μ and started at 0. We need to introduce some notation concerning this walk on H. For any element $\ell \in H$, we define the integer-valued random variable $\theta_n(\ell)$ equal to the number of visits to ℓ until time n, counting the starting position as a visits to the starting point. That is,

$$\theta_n(\ell) = \#\{i : 0 \le i \le n,\ X_i = \ell\}.$$

Using Lemma 2.1, we find that

$$\begin{aligned} \nu_2^{X_0} * \cdots * \nu_2^{X_n}(f) &= \prod_{\ell \in H} \nu_2^{(\theta_n(\ell))}\big(f(\ell)\big) \\ &= \prod_{\ell \in H} \nu^{(2\theta_n(\ell))}\big(f(\ell)\big). \end{aligned} \tag{2.8}$$

Some comments are in order. The right-hand side is a product of numerical functions, not a convolution product. The measure ν is understood here as a measure on K. The right-hand side is written as an infinite product over all $\ell \in H$. However, $f(\ell) \ne o$ for finitely many $\ell \in H$ and $\theta_n(\ell) \ne 0$ for finitely many $\ell \in H$. As $\nu_2^{(0)}(o) = 1$ (by definition, $\nu_2^{(0)}$ is the Dirac mass at $o \in K$), it follows that all but finitely many terms of the products are equal to 1. In fact, there are at most $n+1$ terms that are not equal to 1. It may happen that some terms vanish [for instance, if $f(\ell) \ne o$ for some ℓ such that $\theta_n(\ell) = 0$]. In this case, the product also vanishes.

Using (2.8) in (2.7) yields

$$q_n\big((f,h)\big) = \mathbf{E}_\xi\left(\prod_{\ell \in H} \nu^{(2\theta_n(\ell))}\big(f(\ell)\big) \mid X_n = h\right)\mathbf{P}_\xi(X_n = h) \tag{2.9}$$

$$= \mathbf{E}_\xi\left(\prod_{\ell \in H} \nu^{(2\theta_n(\ell))}\big(f(\ell)\big)\mathbf{1}_{\{X_n = h\}}\right). \tag{2.10}$$

The same argument yields a similar formula for $q^{(n)}$. To write this formula, we need the following modification of the random variable $\theta_n(\ell)$. For each fixed $h \in H$, define

$$\theta_n^h(\ell) = \begin{cases} \theta_n(\ell), & \text{if } \ell \notin \{0,h\}, \\ \theta_n(\ell) - \frac{1}{2}, & \text{if } \ell \in \{0,h\},\ h \ne 0, \\ \theta_n(\ell) - 1, & \text{if } \ell = 0 = h. \end{cases}$$

Then we have

$$q^{(n)}\big((f,h)\big)=\mathbf{E}_\xi\bigg(\prod_{\ell\in H}\nu^{(2\theta_n^h(\ell))}\big(f(\ell)\big)\,\big|\,X_n=h\bigg)\mathbf{P}_\xi(X_n=h) \tag{2.11}$$

$$=\mathbf{E}_\xi\bigg(\prod_{\ell\in H}\nu^{(2\theta_n^h(\ell))}\big(f(\ell)\big)\mathbf{1}_{\{X_n=h\}}\bigg). \tag{2.12}$$

In practice, the difference between θ_n^h and θ_n is essentially irrelevant.

Some bounds on $q^{(n)}(e)$ immediately follow from (2.10).

THEOREM 2.2. *Let ν, μ be probability measures on K, H, respectively. Let $q=\nu*\mu*\nu$ on $K\wr H$ as in* (2.3). *Assume that*

$$\nu^{(2n)}(o)\le\exp\big(-F(n)\big)\quad\big[\textit{resp. }\nu^{(2n)}(o)\ge\exp\big(-F(n)\big)\big]$$

for some function F. Then

$$q^{(n)}(e)\le\varepsilon^{-2}\mathbf{E}_\xi\bigg(\exp\bigg(-\sum_{h\in H}F\big(\theta_n(h)\big)\bigg)\mathbf{1}_{\{X_n=0\}}\bigg)$$

$$\bigg[\textit{resp. } q^{(n)}(e)\ge\mathbf{E}_\xi\bigg(\exp\bigg(-\sum_{h\in H}F\big(\theta_n(h)\big)\bigg)\mathbf{1}_{\{X_n=0\}}\bigg)\bigg],$$

where $\varepsilon=\sup_{k\in K}\{\nu(k)\}$, $\xi=(\xi_i)$ is a sequence of H-valued iid random variables of law μ and $X_0=0$, $X_n=\xi_1\xi_2\cdots\xi_n$, $n\ge1$.

Obviously, (2.12) opens the door to refined asymptotics in some special cases. Such asymptotics will not be pursued here. See [42] for some results in this direction. Instead, we will focus on extracting from the formulas above some rough estimates giving the right order of magnitude for the return probability in various classes of examples.

3. Examples. In this section we keep the general notation introduced above. Namely, K, H are two finitely generated groups and we form the wreath product $K\wr H$. Given two probability measures ν, μ on K, H, we consider the probability $q=\nu*\mu*\nu$ on $K\wr H$. We let $\xi=(\xi)_1^\infty$ be a sequence of iid H-valued random variables of law μ and let $X_n=\xi_1\cdots\xi_n$ denote the associated random walk on H started at $0\in H$. We let $\mathbf{P}_\xi$ denote the law of ξ and let $\mathbf{P}_\xi^h$, $h\in H$, denote the law of the sequence $\xi(h)=(h\xi_1,\xi_2,\xi_3,\dots)$ which corresponds to the walk started at $h\in H$.

In H, we fix a finite symmetric generating set S_H and, for any $h\in H$, we let $|h|=d(0,h)$ be the distance between 0 and h in the Cayley graph (H,S_H). Thus $|h|$ is the minimal number of elements of S_H necessary to write h as a word with letters in S_H. We also set

$$B_H(r)=\{h\in H:|h|\le r\}$$

and

$$V_H(m) = \#B_H(m).$$

3.1. *The case where K is finite*: *number of visited points.* Assume that K is finite of cardinality κ. Choose ν to be the uniform distribution on K. Thus $\nu^{(n)} = \nu \equiv 1/\kappa$ for all $n \geq 1$. Let

$$N_n = \#\{X_i : i = 0, 1, 2, \dots, n\}$$

be the number of visited points up to time n for the walk of law μ on H. Thus

$$N_n = \#\{\ell \in H : \theta_n(\ell) > 0\}.$$

As $\nu^{(n)}$ is the uniform measure on K for any $n \geq 1$, formula (2.12) simplifies to

$$q^{(n)}\big((f, h)\big) = \mathbf{E}_\xi\big(e^{-(\log\kappa)N_n}\mathbf{1}_{\{X_n = h\}}\big). \tag{3.1}$$

In view of this, we define the function $\mathcal{L}_\mu(t, n)$ to be minus the logarithm of the Laplace transform of the number of visited points for the walk of law μ on H. That is,

$$\mathcal{L}_\mu(t, n) = -\log\big(\mathbf{E}_\xi\big(e^{-tN_n}\big)\big).$$

Note that $\mathcal{L}_\mu$ is a nondecreasing function of its two arguments.

THEOREM 3.1. *Assume that K is finite of cardinality $\kappa \geq 2$. Let ν be the uniform measure on K. For any symmetric probability measure μ on H with finite support, there exist constants $c_1, c_2 > 0$ such that the measure $q = \nu * \mu * \nu$ on $K \wr H$ satisfies*

$$\mathcal{L}_\mu(\log\kappa, 2n) \leq -\log q^{(2n)}(e) \leq c_1\mathcal{L}_\mu(c_2\log\kappa, n).$$

PROOF. The lower bound $-\log q^{(2n)}(e) \geq \mathcal{L}_\mu(\log\kappa, 2n)$ is obvious from (3.1). To prove the upper bound, we proceed as follows. Since μ is finitely supported, we can assume without loss of generality that its support is contained in S_H (this accounts for the fact that c_1, c_2 depend on μ).

First, for $n \geq 1$, we have

$$\begin{aligned}\mathbf{E}_\xi(e^{-(s+t)N_n}) &= \sum_{m=1}^{\infty} e^{-(s+t)m}\mathbf{P}_\xi(N_n = m)\\ &\leq \left(\sum_1^\infty e^{-2sm}\right)^{1/2}\left(\sum_1^\infty e^{-2tm}\big[\mathbf{P}_\xi(N_n = m)\big]^2\right)^{1/2}.\end{aligned}$$

Second, write

$$
\begin{aligned}
\big[\mathbf{P}_\xi(N_n=m)\big]^2 &= \Bigg(\sum_{h\in H}\mathbf{P}_\xi(N_n=m;\,X_n=h)\Bigg)^2\\
&= \Bigg(\sum_{h\in B_H(m)}\mathbf{P}_\xi(N_n=m;\,X_n=h)\Bigg)^2\\
&\le V_H(m)\sum_{h\in B_H(m)}\big[\mathbf{P}_\xi(N_n=m;\,X_n=h)\big]^2\\
&= V_H(m)\sum_{h\in B_H(m)}\mathbf{P}_\xi(N_n=m;\,X_n=h)\mathbf{P}^h_\xi(N_n=m;\,X_n=0)\\
&= V_H(m)\sum_{h\in B_H(m)}\mathbf{P}_\xi(N_n=m;\,X_n=h)\\
&\qquad\qquad\times\mathbf{P}_\xi(N_n^{2n}=m;\,X_{2n}=0\mid X_n=h)\\
&\le V_H(m)\mathbf{P}_\xi(N_{2n}\le 2m;\,X_{2n}=0),
\end{aligned}
$$

where N_n^{2n} denotes the number of visited points in the time interval $n\le s\le 2n$, that is, $N_n^{2n}=\#\{X_n,\dots,X_{2n}\}$. Note that we have used symmetry and the fact that N_n counts the starting point as visited to write $P_\xi(N_n=m;\,X_n=h)=P^h_\xi(N_n=m;\,X_n=0)$. For later use, note that, by the same token,

$$\big[\mathbf{P}_\xi(N_n\le m)\big]^2\le V_H(m)\mathbf{P}_\xi(N_{2n}\le 2m;\,X_{2n}=0). \tag{3.2}$$

Third, we have

$$
\begin{aligned}
q^{(2n)}(e) &= \sum_1^\infty e^{-(\log\kappa)m}\mathbf{P}_\xi(N_{2n}=m;\,X_{2n}=0)\\
&= (1-e^{-\log\kappa})\sum_1^\infty e^{-(\log\kappa)m}\mathbf{P}_\xi(N_{2n}\le m;\,X_{2n}=0) \qquad (3.3)\\
&\ge \tfrac12\sum_1^\infty e^{-2(\log\kappa)m}\mathbf{P}_\xi(N_{2n}\le 2m;\,X_{2n}=0).
\end{aligned}
$$

Now observe that $V_H(m)$ grows at most exponentially. Thus there exists a constant $c>0$ such that

$$
\begin{aligned}
q^{(2n)}(e) &\ge \frac12\sum_1^\infty e^{-2(\log\kappa)m}\mathbf{P}_\xi(N_{2n}\le 2m;\,X_{2n}=0)\\
&\ge \frac12\sum_1^\infty e^{-(c+2\log\kappa)m}\big[\mathbf{P}_\xi(N_n=m)\big]^2
\end{aligned}
$$

$$\geq \frac{1}{2}\left(\sum_{1}^{\infty} e^{-2cm}\right)^{-1}\left(\sum_{1}^{\infty} e^{-2(c+2\log\kappa)m}\mathbf{P}_{\xi}(N_n=m)\right)^{2}$$
$$\geq \frac{e^{c}-1}{2}\big[\mathbf{E}_{\xi}(e^{-2(c+2\log\kappa)N_n})\big]^{2}.$$

Thus

$$-\log q^{(2n)}(e) \leq \max\{0, \log(c/2)\} + 2\mathcal{L}_{\mu}(2(c+2\log\kappa), n).$$

This completes the proof of Theorem 3.1 □

COROLLARY 3.2. *Let H be a finitely generated group. Let μ_i, $i=1,2$, be symmetric probability measures on H, each with finite generating support. Set $\mathcal{L}_i=\mathcal{L}_{\mu_i}$, $i=1,2$. Then there are positive constants c_i such that*

$$c_1\mathcal{L}_1(c_2t, c_3n) \leq \mathcal{L}_2(t,n) \leq c_4\mathcal{L}_1(c_6t, c_7n)$$

for all t large enough and all n.

PROOF. Fix t large enough. Let ν be the normalized Haar measure on $K=\mathbb{Z}/k\mathbb{Z}$, where k is an integer such that $\log k \approx t$. Consider the measure $q_i = \nu * \mu_i * \nu$ on $K \wr H$. This measure has finite generating support and is symmetric. Theorem 1.3 applied to q_1, q_2 on $G = K \wr H$ implies that

$$q_1^{(2n)}(\mathrm{id}) \approx q_2^{(2n)}(\mathrm{id}).$$

Theorem 3.1 and the monotonicity of $\mathcal{L}_i$ then give the desired result. □

It is an open question whether or not the result above extends to quasi-isometric Cayley graphs.

It is not hard to check that $\mathcal{L}_{\mu}(t,n)$ is subadditive in its second argument. Thus, for any μ on a group G,

$$\lim_{n\to+\infty} \frac{\mathcal{L}_{\mu}(t,n)}{n} = c(t) \in [0,t]$$

exists. Moreover, by Jensen's inequality,

$$t \to \frac{\mathcal{L}_{\mu}(t,n)}{t}$$

is nonincreasing and takes values in $[0, n+1]$. It follows that

$$\lim_{t\to+\infty} \frac{c(t)}{t} = c_{\mu} \in [0,1] \tag{3.4}$$

exists.

THEOREM 3.3. *Let G be a finitely generated group. Then G is nonamenable if and only if there exists a symmetric probability measure μ whose support* (*not necessarily finite*) *generates G such that*

$$\lim_{n\to\infty}\frac{\mathcal{L}_\mu(t,n)}{n}>0$$

for some (*equivalently, for any*) $t>0$. *In particular, G is amenable if and only if $c_\mu=0$ for some* (*equivalently, for any*) *symmetric probability measure μ whose support generates G, where c_μ is defined by* (3.4).

PROOF. If G is nonamenable and the support of μ generates G, then, by Kesten's theorem, $\mu^{(2n)}(x)\le\rho^{2n}$ for some $0<\rho<1$ (see [29], [30] and [56]). As $V_H(n)$ grows exponentially, there exist two constants $a,\delta\in(0,1)$ such that, with probability at least $1-a^n$, a random walk of length $2n$ leaves the ball of radius δn. Thus, with probability at least $1-a^n$, a random walk of length $2n$ visits at least δn different points. Thus, for any $t>0$,

$$\lim_{n\to\infty}\frac{\mathcal{L}_\mu(t,n)}{n}>0.$$

Conversely, let μ be a symmetric probability measure on G such that $\lim_{n\to\infty}\mathcal{L}_\mu(t,n)/n>0$ for some $t>0$. Let F be a finite group of order κ such that $\log\kappa\ge t$. Consider the group $F\wr G$. By (3.1), there exists a probability q on $F\wr G$ such that

$$q^{(2n)}(e)\le\exp\bigl(-\mathcal{L}_\mu(\log\kappa,2n)\bigr)\le\exp\bigl(-\mathcal{L}_\mu(t,2n)\bigr).$$

By hypothesis, this implies that the spectral radius $\rho=\lim_{n\to\infty}(q^{(2n)}(e))^{1/2n}$ is strictly less than 1. Thus, by Kesten's theorem, $F\wr G$ must be nonamenable. But this implies that G itself is nonamenable because $G=[F\wr G]/[\sum_{g\in G}F_g]$ and $\sum_{g\in G}F_g$ is amenable [very generally, if G_0 is a normal subgroup of G and G_0 and G/G_0 are amenable, then G must be amenable; see, e.g., [35], (0.16)]. $\square$

3.2. *A theorem of Donsker and Varadhan.* Consider the case where $H=\mathbb{Z}^d$ and μ is the uniform distribution on the $2d$ standard unit vectors $\pm e_i$ in $\mathbb{Z}^d$. Thus (X_i) is the simple random walk on $\mathbb{Z}^d$ and we drop all reference to μ and ξ in our notation.

The number N_n of visited points by simple random walk up to time n on $\mathbb{Z}^d$ is an interesting random variable. Its mean is computed in [12] where estimates of the variance are also obtained. One has

$$\mathbf{E}(N_n)=\begin{cases}(8n/\pi)^{1/2}, & \text{if } d=1,\\ \pi n/\log n, & \text{if } d=2,\\ (1-R_d)n, & \text{if } d>3,\end{cases}$$

where R_d is the probability of eventual return to the starting point. A review of the subject is given in [26], Chapter 6. The quantity of interest to us, namely,

$\mathcal{L}_d(t,n) = -\log[\mathbf{E}(e^{-tN_n})]$, behaves differently from $tE(N_n)$. It is studied in a celebrated paper of Donsker and Varadhan [11] who proved the following theorem.

THEOREM 3.4 [11]. *For the simple random walk on $\mathbb{Z}^d$,*

$$\mathcal{L}_d(t,n) \sim c(d)t^{2/(d+2)}n^{d/(d+2)} \qquad \text{as } n \to \infty,$$

where $c(d) = 2^{-1}(d+2)\omega_d^{2/(d+2)}(\lambda_d/d)^{d/(d+2)}$, λ_d being the lowest eigenvalue of the Laplacian with Dirichlet boundary condition in the Euclidean ball of radius 1, *and $\omega_d = \pi^{d/2}/\Gamma(d/2+1)$ its volume.*

Together with Theorem 3.1, this yields the following corollaries.

THEOREM 3.5. *Let F be a finite group and set $G = F \wr \mathbb{Z}^d$. Then, for any symmetric probability measure q with finite support which generates G,*

$$q^{(2n)}(e) \approx \exp\bigl(-n^{d/(d+2)}\bigr).$$

PROOF. For the special measures q considered in Theorem 3.1, the result follows from the Donsker–Varadhan asymptotic and Theorem 3.1. By Theorems 1.2 and 1.3, the result extends to other measures as stated above. $\square$

REMARK. In [27], Kaimanovich and Vershik noted the elementary lower bound $\phi(n) \ge e^{-c_d n^{2d/(2d+1)}}$ for $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^d$.

By a result of Baumslag [3, 4], for each d, we can find a finitely presented metabelian group G_d which contains the metabelian group $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^d$ as a subgroup. Thus, by Theorems 1.4 and 3.5, we obtain assertion (ii) of Theorem 1.1.

THEOREM 3.6. *Let F be a finite group and let $G = F \wr H$ with $H = \mathbb{Z} \wr \mathbb{Z}$. Then, for any symmetric probability measure q with finite support which generates G and for any $\gamma \in (0,1)$, there are constants $C_{q,\gamma}, c_{q,\gamma} > 0$ such that*

$$\forall n = 1, 2, \dots, \qquad q^{(2n)}(e) \le C_{q,\gamma} \exp(-c_{q,\gamma} n^{\gamma}).$$

PROOF. It is easy to see that $H = \mathbb{Z} \wr \mathbb{Z}$ contains $\mathbb{Z}^d$ as a subgroup for all $d = 1, 2, \dots$. Thus the result follows from Theorems 3.5 and 1.4.

If F is abelian, the group G in Theorem 3.6 is obviously solvable (although not metabelian). This proves assertion (iii) of Theorem 1.1. $\square$

THEOREM 3.7. *Consider $H = \mathbb{Z} \wr \mathbb{Z}$ equipped with a finite symmetric generating set S. For the simple random walk on H, let N_n be the number of visited points in the first n steps. Then, for any $t > 0$ and for any $\gamma \in (0,1)$,*

$$\lim_{n \to \infty} -n^{-\gamma} \log \mathbf{E}\bigl(e^{-tN_n}\bigr) = \infty.$$

That is, for any $\gamma \in (0,1)$, $\mathbf{E}(e^{-tN_n})$ tends to 0 faster than $e^{-n^{\gamma}}$ as n tends to ∞.

It seems a worthwhile project to improve upon the two last results and obtain a more precise description of the behavior of ϕ on $F \wr (\mathbb{Z} \wr \mathbb{Z})$ and of $\mathbf{E}(e^{-tN_n})$ on the group $\mathbb{Z} \wr \mathbb{Z}$. Let us point out that $\mathbb{Z} \wr \mathbb{Z}$ can be realized as the subgroup (not a discrete subgroup) of the affine group of the line generated by the two affine transformations $u : x \mapsto x+1$, $v : x \mapsto \lambda x$ and their inverses where λ is any fixed real transcendental number.

3.3. *Elementary results for $K \wr \mathbb{Z}$.* The result of Donsker and Varadhan used above is not an easy result. The only case that can be given an elementary treatment is when $H = \mathbb{Z}$. In this case, one can also use simple arguments to study what happens when K is not finite.

We start by collecting results concerning the simple random walk on $\mathbb{Z}$. As above, we let N_n be the number of visited points up to time n. For each $i \in \mathbb{Z}$, $\theta_n(i)$ is the number of visits to i up to time n and

$$\theta_n = \max_i \{\theta_n(i)\}$$

is the number of visits to the most visited point.

The following estimate is well known (see, e.g., [37] and [49]):

$$\mathbf{P}(N_n \le m) \ge c \exp\left(-C \frac{n}{m^2}\right). \tag{3.5}$$

By (3.2), it follows that

$$\mathbf{P}(N_{2n} \le 2m;\, X_{2n} = 0) \ge c m^{-1} \exp\left(-2C \frac{n}{m^2}\right). \tag{3.6}$$

LEMMA 3.8. *For the simple random walk on $\mathbb{Z}$ and each fixed $t > 0$, we have*

$$\mathbf{E}\left(\exp\left(-t \sum_i \log\bigl(1 + \theta_{2n}(i)\bigr)\right) \mathbf{1}_{\{X_{2n}=0\}}\right) \ge c n^{-1/3} \exp\bigl(-C_t\, n^{1/3} (\log n)^{2/3}\bigr),$$

and, for $0 < \alpha < 1$,

$$\mathbf{E}\left(\exp\left(-t \sum_i \bigl[\theta_{2n}(i)\bigr]^\alpha\right) \mathbf{1}_{\{X_{2n}=0\}}\right) \ge c n^{-(1-\alpha)/(3-\alpha)} \exp\bigl(-C_t n^{(\alpha+1)/(3-\alpha)}\bigr).$$

PROOF. Let F be a nonnegative concave function such that $F(0) = 0$. Then

$$\begin{aligned} \sum_{i \in \mathbb{Z}} F\bigl(\theta_{2n}(i)\bigr) &= N_{2n} \sum_{\text{visited } i} \frac{F(\theta_{2n}(i))}{N_{2n}} \\ &\le N_{2n} F\left(\sum_{\text{visited } i} \frac{\theta_{2n}(i)}{N_{2n}}\right) \\ &= N_{2n} F\left(\frac{2n}{N_{2n}}\right). \end{aligned}$$

For $F(u) = \log(1+u)$, one checks that $u^{-1}F(u)$ is decreasing on $(0,1)$. Thus

$$\begin{aligned}\mathbf{E}\Big(\exp\Big(-t\sum_{\mathbb{Z}}\log\big(1+\theta_{2n}(i)\big)\Big)\mathbf{1}_{\{X_{2n}=0\}}\Big) \\ &\geq \mathbf{E}(\exp(-tN_{2n}\log(1+2n/N_{2n}))\mathbf{1}_{\{X_{2n}=0\}}) \\ &\geq \exp(-2tm\log(1+n/m))\mathbf{P}(N_{2n}\leq 2m;\, X_{2n}=0) \\ &\geq cm^{-1}\exp(-2tm\log(1+n/m) - Cn/m^2).\end{aligned}$$

This being true for all m, we can pick $m \approx n^{1/3}/(\log n)^{1/3}$ which gives

$$\mathbf{E}\Big(\exp\Big(-t\sum_{i}\log\big(1+\theta_{2n}(i)\big)\Big)\mathbf{1}_{\{X_{2n}=0\}}\Big) \geq c_1 n^{-1/3}\exp(-C_t n^{1/3}(\log n)^{2/3}),$$

as desired. If $F = u^\alpha$, $\alpha \in (0,1)$, we get

$$\begin{aligned}\mathbf{E}\Big(\exp\Big(-t\sum_{i}\theta_{2n}(i)^\alpha\Big)\mathbf{1}_{\{X_{2n}=0\}}\Big) &\geq \mathbf{E}(\exp(-t(2n)^\alpha N_{2n}^{1-\alpha})\mathbf{1}_{\{X_{2n}=0\}}) \\ &\geq \exp(-2tn^\alpha m^{1-\alpha})\mathbf{P}(N_{2n}\leq 2m;\, X_{2n}=0) \\ &\geq cm^{-1}\exp(-2tn^\alpha m^{1-\alpha} - Cn/m^2).\end{aligned}$$

Again, this being true for all m, we can pick $m = n^{(1-\alpha)/(3-\alpha)}$ which yields

$$\mathbf{E}\Big(\exp\Big(-t\sum_{i}\theta_{2n}(i)^\alpha\Big)\mathbf{1}_{\{X_{2n}=0\}}\Big) \geq cn^{-(1-\alpha)/(3-\alpha)}\exp\big(-C_t n^{(1+\alpha)/(3-\alpha)}\big). \quad \square$$

REMARK. The argument above yields $C_t = Ct$ for some constant C. Obviously, this can be improved, at least for (n,t) in a certain range. In particular, if $n \geq t$, one can take $C_t = Ct^{2/3}$ in the first inequality of Lemma 3.8 and $C_t = Ct^{2/(3-\alpha)}$ in the second inequality.

LEMMA 3.9. *For the simple random walk on* $\mathbb{Z}$,

$$\mathbf{P}(\theta_{2n} > r) \leq 2n^2\exp\Big(-\frac{r^2}{4n}\Big).$$

PROOF. By [15], 3.7, Theorem 4, the probability $p_{r,n}$ that the rth return to the origin occurs at epoch $2n$, $r \leq n$, is

$$\frac{r}{2n-r}\binom{2n-r}{n}2^{-2n+r} = 2^{-2n}\binom{2n}{n} \times \frac{r}{2n-r}2^r\frac{\binom{2n-r}{n}}{\binom{2n}{n}}.$$

Thus

$$\begin{aligned}
p_{r,n} &\le 2^r \frac{1}{\sqrt{2n}} \frac{(2n-r)!n!}{(n-r)!(2n)!} \\
&= 2^r \frac{1}{\sqrt{2n}} \frac{n \times (n-1) \times \cdots \times (n-r+1)}{(2n) \times (2n-1) \times \cdots \times (2n-r+1)} \\
&= \frac{1}{\sqrt{2n}} \frac{(1-1/n) \times \cdots \times (1-(r-1)/n)}{(1-1/(2n)) \times \cdots \times (1-(r-1)/(2n))}.
\end{aligned}$$

As $\log(1-2u) - \log(1-u) \le -u$ for $u \in (0, 1/2)$, we get

$$p_{r,n} \le \frac{1}{\sqrt{2n}} \exp\left(-\frac{r^2}{8n}\right).$$

Now the probability that, at epoch $2n$, more than $r \ge 2$ visits to the origin occurred is

$$\mathbf{P}\big(\theta_{2n}(0) > r\big) = \sum_{i=r}^{n} p_{r,2i} \le n \exp\left(-\frac{r^2}{8n}\right).$$

This upper bound is slightly crude but it suffices for our purpose.

For fixed $i = \pm 1, \pm 2, \dots, \pm n$, let τ_i be the time of the first visit at i and observe that

$$\begin{aligned}
\mathbf{P}\big(\theta_{2n}(i) > r\big) &= \sum_{\ell \le 2n} \mathbf{P}\big(\theta_{2n}(i) > r / \tau_i = \ell\big) \mathbf{P}(\tau_i = \ell) \\
&= \sum_{\ell \le 2n} \mathbf{P}\big(\theta_{2n-\ell}(0) > r\big) \mathbf{P}(\tau_i = \ell) \\
&\le \mathbf{P}\big(\theta_{2n}(0) > r\big).
\end{aligned}$$

Thus, for $r \ge 2$,

$$\begin{aligned}
\mathbf{P}(\theta_{2n} > r) &\le \sum_{i=-n}^{n} \mathbf{P}\big(\theta_{2n}(i) > r\big) \\
&\le (2n+1) \mathbf{P}\big(\theta_{2n}(0) > r\big) \\
&\le (2n+1)^2 \exp\left(-\frac{r^2}{8n}\right). \qquad \square
\end{aligned}$$

LEMMA 3.10. *For the simple random walk on $\mathbb{Z}$ and $tn^2 \ge 1$, we have*

$$\mathbf{E}\left(\exp\left(-t \sum_i \log\big(1 + \theta_{2n}(i)\big)\right)\right) \le C \exp\big(-ct^{2/3} n^{1/3} \big(\log(1+tn^2)\big)^{2/3}\big),$$

and, *for* $0 < \alpha < 1$,

$$\mathbf{E}\Big(\exp\Big(-t\sum_i [\theta_{2n}(i)]^\alpha\Big)\Big) \le C\exp\big(-ct^{2/(3-\alpha)}\,n^{(\alpha+1)/(3-\alpha)}\big).$$

PROOF. As $\sum_i \theta_{2n}(i) = 2n$, we have

$$\sum_i \log\big(1+\theta_{2n}(i)\big) = \sum_i \theta_{2n}(i)\frac{\log(1+\theta_{2n}(i))}{\theta_{2n}(i)} \ge \frac{2n\log(1+\theta_{2n})}{\theta_{2n}}$$

because $u \to u^{-1}\log(1+u)$ is decreasing on $(0,\infty)$. Similarly, for $\alpha \in (0,1)$,

$$\sum_i \theta_{2n}(i)^\alpha = \sum_i \theta_{2n}(i)\frac{1}{\theta_{2n}(i)^{1-\alpha}} \ge \frac{2n}{\theta_{2n}^{1-\alpha}}.$$

Thus

$$\begin{aligned}\mathbf{E}\Big(\exp\Big(-t\sum_i \log\big(1+\theta_{2n}(i)\big)\Big)\Big) &\le \mathbf{E}\big(\exp\big(-\big[2tn\log(1+\theta_{2n})\big]/\theta_{2n}\big)\big)\\ &= \sum_\ell \exp\big(-\big[2tn\log(1+\ell)\big]/\ell\big)\mathbf{P}(\theta_{2n}=\ell)\\ &\le Cn^2\sum_\ell \exp\big(-\big[2tn\log(1+\ell)\big]/\ell - \ell^2/8n\big).\end{aligned}$$

One easily checks that this sum is less than

$$C\exp\big(-ct^{2/3}n^{1/3}\big(\log(1+tn^2)\big)^{2/3}\big)$$

for t, n such that $tn^2 \ge 1$. Similarly, one finds

$$\mathbf{E}\Big(\exp\Big(-t\sum_i [\theta_{2n}(i)]^\alpha\Big)\Big) \le C\exp\big(-ct^{2/(3-\alpha)}n^{(\alpha+1)/(3-\alpha)}\big)$$

for $tn^2 \ge 1$.

We can now state our main result concerning random walks on $K \wr \mathbb{Z}$. The proof follows readily from the results above and Theorem 2.2. $\square$

THEOREM 3.11. *Let q be a symmetric probability measure on $K \wr \mathbb{Z}$ whose support is finite and generates $K \wr \mathbb{Z}$.*

(i) *Assume that K is an infinite finitely generated group having polynomial volume growth. Then*

$$q^{(2n)}(e) \approx \exp\big(-n^{1/3}(\log n)^{2/3}\big).$$

(ii) *Assume that K is a polycyclic group having exponential volume growth. Then*

$$q^{(2n)}(e) \approx \exp\big(-n^{1/2}\big).$$

(iii) *Assume that* K *is such that* $\phi(n) \le C\exp(-cn^{\alpha})$ [*resp.* $\phi(n) \ge c\exp(-Cn^{\alpha})$]. *Then*

$$q^{(2n)}(e) \le C_1 \exp\bigl(-c_1 n^{(1+\alpha)/(3-\alpha)}\bigr)$$
$$\bigl[\textit{resp. } q^{(2n)}(e) \ge c_1 \exp\bigl(-C_1 n^{(1+\alpha)/(3-\alpha)}\bigr)\bigr].$$

REMARK. Given two groups K and H, set $K \wr_0 H = K$ and $K \wr_i H = (K \wr_{i-1} H) \wr H$. Thus, for instance,

$$\mathbb{Z} \wr_2 \mathbb{Z} = (\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}, \qquad \mathbb{Z} \wr_3 \mathbb{Z} = \bigl[(\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}\bigr] \wr \mathbb{Z}.$$

By the same line of reasoning as above, one shows that $\mathbb{Z} \wr_2 \mathbb{Z}$ satisfies

$$\phi(n) \approx \exp\bigl(-n^{1/2}(\log n)^{1/2}\bigr)$$

and, more generally, for any group K having polynomial volume growth, $K \wr_d \mathbb{Z}$ satisfies

$$\phi(n) \approx \exp\bigl(-n^{d/(d+2)}(\log n)^{2/(d+2)}\bigr).$$

Similarly, if K is polycyclic, $K \wr_d \mathbb{Z}$ satisfies

$$\phi(n) \approx \exp\bigl(-n^{(d+1)/(d+3)}\bigr).$$

To see this, one simply checks by the method above that if the simple random walk on K satisfies $\phi_K(n) \approx \exp(-n^{\alpha}(\log n)^{\beta})$ with $\alpha \in (0, 1)$, then the random walk on $G = K \wr \mathbb{Z}$ satisfies

$$\phi_G(n) \approx \exp\bigl(-n^{(1+\alpha)/(3-\alpha)}(\log n)^{2\beta/(3-\alpha)}\bigr).$$

Then one solves the recurrence equations

$$\alpha_{i+1} = \frac{1+\alpha_i}{3-\alpha_i}, \qquad \beta_{i+1} = \frac{2\beta_i}{3-\alpha_i},$$

with the appropriate starting values ($\alpha_1 = 1/3$, $\beta_1 = 2/3$ when K has polynomial growth; $\alpha_0 = 1/3$, $\beta_0 = 0$ when K is polycyclic with exponential growth).

3.4. *The case when* H *has polynomial growth.* The case when H has polynomial growth should be studied in detail. In particular, we conjecture that a statement analogous to the theorem of Donsker and Varadhan holds true: if H has polynomial volume growth of order d, that is, $V(n) \approx n^d$, then, for any simple random walk on H, the number of visited point satisfies $-\log E(e^{-tN_n}) \approx n^{d/(d+2)}$ for each $t > 0$ as n tends to ∞. We hope to return to this problem elsewhere. More generally, one should study quantities such as $E(e^{-F(\theta_n(h))})$ for $F(t) = c\log(1+t)$ or $F(t) = ct^{\alpha}$, $\alpha \in (0, 1)$. This is beyond the scope of this paper, but we would like to record here some lower bounds.

We need the following interesting result.

THEOREM 3.12. *Let μ be a symmetric probability measure with finite support on a finitely generated group H having polynomial growth. Assume that the support S of μ generates H and contains e. Denote by $|h|$ the length of $h \in H$ with respect to this set of generators. Let $\xi = (\xi_i)$ be a sequence of iid H-valued random variables of law μ and set $X_i = \xi_1 \cdots \xi_i$, $Z_n = \sup_{1 \le i \le n}\{|X_i|\}$. Then there are positive constants c_1, c_2 such that, for all integers n, m,*

$$\mathbf{P}_\xi(Z_n \le m) \ge c_1 e^{-c_2 n/m^2}.$$

PROOF. When $H = \mathbb{Z}^d$ and S is the usual symmetric set of generators $\{\pm e_1, \dots, \pm e_d\}$, this is a well-known estimate (whose complete proof is difficult to find in the literature; see [1], [37] and [49]). One classical proof uses André's reflection principle, but this type of argument does not seem to generalize to the case of groups having polynomial growth. We outline a completely different proof based on analytical tools. Let us note that $n \mapsto \mathbf{P}_\xi(Z_n \le m)$ is a nonincreasing function of n. Thus it is enough to prove the desired estimate for all $n \ge m^2$ of the form $n = 4k$.

Let B denote the ball of radius m, that is, $B = \{h \in H : |h| \le m\}$. For functions f with support in B, let P^B be the sub-Markovian operator defined by

$$P_B f(x) = \mathbf{1}_B(x)[f * \mu](x).$$

This operator has kernel

$$p_B(x, y) = \mathbf{1}_B(x)\mathbf{1}_B(y)\mu(x^{-1}y).$$

This corresponds to performing the random walk of law μ with killing outside of B. Using a test function argument and the Dirichlet form associated to P_B, it is easy to see that

$$\|P_B\|_{2\to 2} \ge e^{-c_1/m^2}, \tag{3.7}$$

where $\|\cdot\|_{2\to 2}$ denotes the operator norm for linear operators acting on the finite-dimensional Hilbert space $\ell^2(B)$ (in fact, one can show that a similar upper bound also holds true but this is not needed here).

Next observe that

$$\mathbf{P}_\xi(Z_n \le m) = \sum_{y \in B} p_B^n(0, y),$$

where p_B^n is the iterated kernel of P_B defined by

$$p_B^n(x, y) = \sum_z p_B^{n-1}(x, z) p_B(z, y).$$

Indeed, both the left- and right-hand sides represent the probability of never leaving the ball B until time n. Obviously,

$$\sum_{y \in B} p_B^n(0, y) \ge \sum_{y \in B'} p_B^n(0, y),$$

where B' is the ball of radius $m/2$ around 0 in H. Now, for $y \in B'$, $n \geq m^2$, we can use a parabolic Harnack inequality satisfied by $(n, z) \mapsto p_B^{2n}(z, y)$ which gives the existence of a constant $c_2 > 0$ such that

$$\forall\, y \in B', \quad p_B^{4n}(0, y) \geq c_2 p_B^{2n}(y, y).$$

The necessary Harnack inequality (note that this is a very nontrivial inequality) can be found in [9]. With some work, it can also be obtained from the result of [25]. In any case, it follows that, for $n \geq m^2$,

$$\mathbf{P}_\xi(Z_{4n} \leq m) \geq c_2 \sum_{y \in B'} p_B^{2n}(y, y).$$

As $B' \subset B$, we have $p_{B'}^n(x, y) \leq p_B^n(x, y)$ for all $x, y \in B'$. Thus

$$\mathbf{P}_\xi(Z_{4n} \leq m) \geq c_2 \sum_{y \in B'} p_{B'}^{2n}(y, y).$$

The right-hand side is the trace of $P_{B'}^{2n}$ as an operator on $\ell^2(B')$. Thus the last inequality and the spectral inequality (3.7) yield

$$\mathbf{P}_\xi(Z_{4n} \leq m) \geq c_2 e^{-c_1 n/m^2}.$$

The desired inequality follows [observe that $\mathbf{P}_\xi(Z_n \leq m) = 1$ when $n < m$].

We need the following variant of Theorem 3.12. □

THEOREM 3.13. *Referring to the setting and notation of Theorem* 3.12, *we have*

$$\mathbf{P}_\xi(Z_n \leq m; X_n = 0) \geq c_2' V(m)^{-1} e^{-c_1' n/m^2},$$

where $V(m)$ is the cardinality of the ball of radius m in the Cayley graph (H, S).

PROOF. Observe that

$$\mathbf{P}_\xi(Z_n \leq m; X_n = 0) = p_B^n(0, 0)$$

(this is also a nonincreasing function of n) and that

$$p_B^{2n}(0, 0) = \sum_{z \in B} |p_B^n(0, z)|^2.$$

By Jensen's inequality,

$$p_B^{2n}(0, 0) \geq V(m)^{-1} \left(\sum_{z \in B} p_B^n(0, z) \right)^2.$$

Finally, by Theorem 3.12,

$$p_B^{2n}(0, 0) \geq c_2' V(m)^{-1} e^{-c_1' n/m^2},$$

as desired. □

Theorem 3.13 is the key to the following theorem, which is the main result of this section.

THEOREM 3.14. *Referring to the setting and notation of Theorem* 3.12, *assume that the volume growth of H is such that $V(m) \approx m^d$ for some d. Then, for each $t > 0$, there is a constant $c_t > 0$ such that*

$$-\log(\mathbf{E}_\xi(\exp(-tN_n)\mathbf{1}_{\{X_n=0\}})) \le c_t n^{d/(d+2)},$$

$$-\log\biggl(\mathbf{E}_\xi\biggl(\exp\biggl(-t\sum_{h\in H}\log(1+\theta_n(h))\biggr)\mathbf{1}_{\{X_n=0\}}\biggr)\biggr) \le c_t n^{d/(d+2)}(\log n)^{2/(d+2)},$$

$$-\log\biggl(\mathbf{E}_\xi\biggl(\exp\biggl(-t\sum_{h\in H}[\theta_n(h)]^\alpha\biggr)\mathbf{1}_{\{X_n=0\}}\biggr)\biggr) \le c_t n^\gamma,$$

with $\gamma = [2\alpha + d(1-\alpha)]/[2 + d(1-\alpha)]$.

PROOF. We only prove the second estimate (the other proofs are similar). Fix an integer m. We have (see the proof of Lemma 3.8)

$$\begin{aligned}
&\mathbf{E}_\xi\biggl(\exp\biggl(-t\sum_{h\in H}\log(1+\theta_n(h))\biggr)\mathbf{1}_{\{X_n=0\}}\biggr)\\
&\qquad\ge \mathbf{E}_\xi(\exp(-tN_n\log(1+n/N_n))\mathbf{1}_{\{X_n=0\}})\\
&\qquad\ge \exp(-tV(m)\log(1+n/V(m)))\mathbf{P}_\xi(Z_n\le m; X_n=0)\\
&\qquad\ge c_2' V(m)^{-1}\exp(-tV(m)\log(1+n/V(m)) - c_1' n/m^2).
\end{aligned}$$

Since m is arbitrary and $V(m) \approx m^d$, we can choose $m = [n/\log n]^{1/(d+2)}$. This yields

$$\mathbf{E}_\xi\biggl(\exp\biggl(-t\sum_{h\in H}\log(1+\theta_n(h))\biggr)\biggr) \ge c_3 n^{-d/(d+2)}\exp(-c_4 t n^{d/(d+2)}(\log n)^{2/(d+2)}).$$

For random walks on wreath products, Theorem 3.14 yields the following statement. □

THEOREM 3.15. *Let H be a finitely generated group having polynomial growth $V(m) \approx m^d$. Let K be a finitely generated group. Let μ, ν be symmetric probability measures with finite generating support on H and K, respectively, and consider the measure $q = \nu * \mu * \nu$ on $K \wr H$ as in* (2.3). *Then*

(i) *If K is finite*,

$$q^{(2n)}(e) \ge c_1 \exp(-c_2 n^{d/(d+2)}).$$

(ii) *If K has polynomial volume growth*,

$$q^{(2n)}(e) \geq c_1 \exp\bigl(-c_2 n^{d/(d+2)}[\log n]^{2/(d+2)}\bigr).$$

(iii) *If K contains a polycyclic group of finite index and has exponential volume growth*,

$$q^{(2n)}(e) \geq c_1 \exp\bigl(-c_2 n^{(d+1)/(d+3)}\bigr).$$

REMARKS. (i) It seems most probable that the lower bounds above are sharp. We hope to return to this elsewhere.

(ii) More examples can be treated. For instance, if $K = G \wr \mathbb{Z}$ with G polycyclic having exponential growth, we get

$$q^{(2n)}(e) \geq c_1 \exp\bigl(-c_2 n^{(d+2)/(d+4)}\bigr).$$

(iii) For a different proof of the case when K is finite, see [7].

The second statement in Theorem 3.15 gives a proof of the first statement in Theorem 1.1 asserting that, for any finitely generated metabelian group G, there exists $\varepsilon \in (0, 1)$ such that $\phi_G(n) \geq \exp(-cn^{1-\varepsilon})$. Indeed, according to a result of Magnus [2], any such group G embeds in a quotient W/N, where W is a wreath product of two finitely generated abelian groups. Thus assertion (i) of Theorem 1.1 follows from Theorems 1.4 and 3.15.

3.5. *The case when K is nonamenable.* We now make some observations concerning the case when K is nonamenable (and H is amenable).

THEOREM 3.16. *Let K be nonamenable, H amenable. Let ν, μ be symmetric probability measures on K, H, respectively. Assume that ν as spectral radius ρ, that is*,

$$\lim_{n\to\infty} \bigl[\nu^{(2n)}(o)\bigr]^{1/(2n)} = \rho.$$

Then the measure $q = \nu * \mu * \nu$ on $K \wr H$ has spectral radius $\rho(q) = \rho^2$.

PROOF. Start with the upper bound. It is well known that $\nu^{(n)}(o) \leq \rho^n$ [see, e.g., [56], Chapter 2, (8.1)]. Thus, by (2.2),

$$q^{(n)}(e) \leq \varepsilon^{-2} \mathbf{E}_\xi\bigl(\rho^{2\sum_{\ell\in H}\theta_n(h)}\bigr),$$

where $\varepsilon = \max_{h\in H}\{\mu(h)\}$. But $\sum_h \theta_n(h) = n$ and the desired upper bound, that is, $\rho(q) \leq \rho^2$, follows.

For the lower bound, we need to introduce some notation. Consider the function $\phi(t)$ defined as follows. For integer values of t,

$$\phi(t) = -\log\bigl[\rho^{-2t}\nu^{(2t)}(o)\bigr].$$

For other values of t, ϕ is obtained by linear interpolation. $\square$

CLAIM 1. The function ϕ is nonnegative, concave and $\phi(0) = 0$.

One easily checks that $\nu^{(2n)}(o)$ is nonincreasing (this uses positivity and symmetry and nothing else). As $\nu^{(0)}(o) = 1$, it follows that $\phi(0) = 0$ and that ϕ is nonnegative. Moreover, by a well-known application of the Cauchy–Schwarz inequality,

$$\begin{aligned}\big[\nu^{(2n)}(o)\big]^2 &= \Big(\sum_{k\in K} |\nu^{(n)}(k)|^2\Big)^2 = \langle \nu^{(n)} * \delta_o, \nu^{(n)} * \delta_o\rangle^2 \\ &= \langle \nu^{(n-1)} * \delta_o, \nu^{(n+1)} * \delta_o\rangle^2 \\ &\le \Big(\sum_k |\nu^{(n-1)}(k)|^2\Big)\Big(\sum_k |\nu^{(n+1)}(k)|^2\Big) \\ &= \nu^{(2n-2)}(o)\nu^{(2n+2)}(o).\end{aligned}$$

Here δ_o denotes the Dirac mass at o on K. From this, the concavity of ϕ follows.

Now, by Theorem 2.2 and the fact that $\sum_h \theta_{2n}(h) = 2n$,

$$\rho^{-4n} q^{(2n)}(e) \ge \mathbf{E}_\xi\Big(\exp\Big(-\sum_h \phi\big(2\theta_{2n}(h)\big)\Big)\mathbf{1}_{\{X_{2n}=0\}}\Big).$$

Using the claim above (see the proof of Lemma 3.8), this yields

$$\rho^{-4n} q^{(2n)}(e) \ge \mathbf{E}_\xi\big(e^{-N_{2n}\phi(2n/N_{2n})}\mathbf{1}_{\{X_{2n}=0\}}\big).$$

Now, by construction, we have

$$\lim_{n\to\infty} \frac{1}{n}\phi(n) = 0.$$

Thus, for any $\varepsilon > 0$, there exists an integer $A(\varepsilon)$ such that $\phi(n) < \varepsilon n$ for all $n \ge A(\varepsilon)$. Hence, for any $\varepsilon > 0$,

$$\rho^{-4n} q^{(2n)}(e) \ge e^{-2\varepsilon n}\mathbf{P}_\xi\big(2n/N_{2n} \ge A(\varepsilon); X_{2n} = 0\big). \tag{3.8}$$

CLAIM 2. For any symmetric measure μ on an amenable group H and any reals $A, \delta > 0$,

$$\mathbf{P}_\xi(2n/N_{2n} \ge A; X_{2n} = 0) \ge e^{-2\delta n}$$

for infinitely many integers n.

Indeed, assume that, for some $A, \delta > 0$, $\mathbf{P}_\xi(2n/N_{2n} \ge A; X_{2n} = 0) \le e^{-2\delta n}$ for all but finitely many n. Then let L be some finite group (e.g., $L = \mathbb{Z}/2\mathbb{Z}$) and consider the measure $\tilde{q} = \tilde{\nu} * \mu * \tilde{\nu}$ on $L \wr H$ where $\tilde{\nu}$ is the uniform measure on L as in Theorem 3.1. By (3.3), the measure $\tilde{q}$ on $L \wr H$ satisfies

$$\tilde{q}^{(2n)}(e) \le C_1\big[\mathbf{P}_\xi(2n/N_{2n} \ge A; X_{2n} = 0) + e^{-c_1 n/A}\big] \le C_2 e^{-c_2 n}$$

for all n large enough. But this contradicts the fact that H, and hence $L \wr H$, is amenable. This proves Claim 2.

Now fix $\varepsilon, \delta > 0$. Then, by (3.8) and Claim 2, we have

$$\rho^{-4n} q^{(2n)}(e) \geq e^{-2(\varepsilon+\delta)n}$$

for infinitely many n. Since $\varepsilon, \delta > 0$ are arbitrary, this shows that $\rho(q) \geq \rho^2$, as desired.

REMARKS. (i) A different proof of Theorem 3.16 is given in [58].

(ii) If one considers $p = \mu * \nu * \mu$ instead of $q = \nu * \mu * \nu$, one can show that $\rho(p) = \rho$ (this is actually the case treated in [58]).

(iii) In certain cases, more precise results can be obtained. For instance, assume that $K = F_d$ is the free group on d generators. Then, for any symmetric measure ν with finite generating support and spectral radius ρ, one has (see [55], Theorem 6.8, or [56], Corollary 17.8)

$$\nu^{(2n)}(o) \sim c(\nu) n^{-3/2} \rho^{2n}.$$

It follows from the results of Section 3.3 that if μ denotes the Bernoulli measure driving the simple random walk on $\mathbb{Z}$, then the measure $q = \nu * \mu * \nu$ on $F_d \wr \mathbb{Z}$ satisfies

$$\rho^{-4n} q^{(2n)}(e) \approx \exp\bigl(-n^{1/3} (\log n)^{2/3}\bigr).$$

Note that there are only few examples of nonamenable groups where such refined results are known (see [55], Section 6).

(iv) Let G be a nonamenable group and ν a symmetric measure with spectral radius ρ. Then, for any symmetric probability measure μ on $H = \mathbb{Z} \wr \mathbb{Z}$ having finite generating support, the measure $q = \nu * \mu * \nu$ on $G \wr H$ has spectral radius ρ^2 and satisfies

$$\lim_{n\to\infty} e^{n^{\gamma}} \rho^{-2n} q^{(n)}(e) = 0$$

for all $\gamma \in (0, 1)$.

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