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# BALLOT THEOREMS AND SOJOURN LAWS FOR STATIONARY PROCESSES

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The ballot theorem and the uniform law for sojourn times, both results known for cyclically stationary sequences and processes on a bounded index set, are here extended to infinite, stationary sequences and to stationary processes on $\mathbb{R}_+$. Our extensions contain all previously known versions as special cases.

**1. Introduction.** The classical ballot theorem, first noted by Bertrand (1887), is the elementary but remarkable fact that, if two candidates $A$ and $B$ in an election are getting the proportions $p$ and $1-p$ of the votes, the probability that $A$ will lead throughout the counting of ballots equals $(2p-1)\vee 0$. (The obvious underlying assumption is that the ballots are counted one by one in random order.) Extensions and new proofs have been provided by many authors, beginning with Barbier (1887) and André (1887). A modern discussion based on combinatorial arguments may be found in Section III.1 of Feller (1968), and a simple martingale argument appears in Section 7.4 of Chow and Teicher (1997).

The most recent progress in the area seems to be due to Takács (1967), who extended the mentioned result to cyclically stationary sequences. More generally, if $X$ is a nondecreasing process on $[0,1]$ with $X_0 = 0$ such that $X$ has singular paths and cyclically stationary increments, we have

$$P\{\sup_t (X_t - t) \le 0\} = E[(1 - X_1) \vee 0]. \tag{1.1}$$

The latter statement is slightly stronger than the one in Section 13 of Takács' book, but it follows by the same argument from his Theorem 1 of Section 1. Recall that a nondecreasing function $F$ is *singular* if its absolutely continuous component vanishes and that a process $X$ has *cyclically stationary increments* if the process $Y_t = X_{s+t} - X_s$ has the same distribution for every $s$, where $s+t$ is understood in the sense of addition modulo 1.

The mentioned result may be rephrased as a property of certain processes on $\mathbb{R}_+$ with stationary and *periodic* increments. In Section 2 we show that the periodicity assumption can be dropped, so that the statement remains true for any nondecreasing process $X$ on $\mathbb{R}_+$ with $X_0 = 0$ possessing stationary increments and singular paths. The extended result also contains a version for subordinators and their mixtures, given in Section 14 of Takács (1967). Rather than using the classical formulation in (1.1), we shall state our result

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in a more probabilistic form that reveals its true nature as a uniform law for the minimum of $t/X_t$.

Another class of results requiring cyclic stationarity are the uniform laws for various bridge-type processes on $[0, 1]$. Here the prototype is Lévy's (1939) uniform law for the standard Brownian bridge $B$, which states that the positive and negative sojourn times

$$\rho_\pm = \lambda\{t \in [0, 1]; \pm B_t > 0\}$$

are uniformly distributed on $[0, 1]$. [The latter distribution is henceforth denoted by $U(0, 1)$, and we use $\lambda$ to denote Lebesgue measure.] A uniform law is also known for the location of the maximum or minimum of $B$ and both results are closely related to the three celebrated arcsine laws for Brownian motion, also due to Lévy (1939). A modern discussion of those classical results appears in Chapter 11 of Kallenberg (1997).

The uniform laws for sojourn times remain valid with the same proof for any measurable process $X$ on $[0, 1]$ with cyclically stationary increments and $X_0 = X_1 = 0$, provided only that

$$\lambda\{t \in [0, 1];\ X_t = 0\} = 0 \quad \text{a.s.} \tag{1.2}$$

Similarly, the uniform law for the maximum remains true for any separable and cyclically stationary process on $[0, 1]$, whenever the maximum is a.s. unique. In two recent papers, by Fitzsimmons and Getoor (1995) and Knight (1996), the validity of these results is examined for suitable "Lévy bridges," that is, for the bridge-type processes obtained from an arbitrary Lévy process $X$ by conditioning on $X_1 = 0$ or subtraction of the linear drift term $tX_1$.

The uniform sojourn law may be restated as a property of cyclically stationary processes on $[0, 1]$, or, equivalently, of stationary and periodic processes on $\mathbb{R}_+$. In Section 3 we remove the periodicity assumption and establish a version for any stationary and measurable process on $\mathbb{R}_+$, subject only to a condition similar to (1.2). Even without the latter assumption, a uniform law holds after an appropriate randomization, which leads to corresponding bounds on the distributions of $\rho_\pm$.

We conclude with some general remarks on notational and other conventions. Throughout the paper, we assume that the underlying probability space $\Omega$ is rich enough to admit the introduction of an independent $U(0, 1)$ random variable, when required. (If this is not the case, we may replace $\Omega$ by the product space $\Omega \times [0, 1]$ with probability measure $P \otimes \lambda$.) For basic definitions associated with random measures and sets, we refer to Chapters 10 and 14 in Kallenberg (1997). Given any stationary random process, measure, or set $\xi$, we use $\mathscr{I}_\xi$ to denote the associated invariant $\sigma$-field [Kallenberg (1997), Chapter 9]. We also write $\perp\!\!\!\perp$ for independence and $1\{\cdots\}$ for the indicator function of the set within brackets. Finally, we adopt the notations $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{Z}_+ = \{0, 1, \ldots\}$, $\mathbb{Z}_n = \{0, \ldots, n-1\}$ and $\mathbb{R}_+ = [0, \infty)$.

**2. Ballot theorems.** Our main result may be stated most naturally in terms of random measures. Indeed, if $X$ is a nondecreasing, right-continuous process on $\mathbb{R}_+$ with $X_0 = 0$, there exists a unique random measure $\xi$ on $\mathbb{R}_+$ such that $\xi[0,t] = X_t$ for all $t \geq 0$, and we note that $\xi$ is stationary iff $X$ has stationary increments.

THEOREM 2.1. *Let $\xi$ be an a.s. singular, stationary random measure on $\mathbb{R}_+$ or $[0,1)$, and put $X_t = \xi[0,t]$ and $\rho = E[X_1|\mathscr{I}_\xi]$. Then there exists a $U(0,1)$ random variable $\sigma \perp\!\!\!\perp \rho$ such that*

$$\sup_{t>0} \frac{X_t}{t} = \frac{\rho}{\sigma} \quad a.s. \tag{2.1}$$

Already the previously known version for finite intervals has numerous interesting applications to queuing theory and other areas, as so amply testified by Takács (1967). With the present extension, we are widening the domain of validity to arbitrary stationary random measures that are either purely atomic or singular and diffuse. The latter case is more than a curiosity, as it applies in particular to many interesting additive functionals of stationary Markov processes. For example, we may think of the local time at a regular, nonsticky state of a positive-recurrent diffusion. [See, e.g. Chapters 19 and 20 in Kallenberg (1997).]

Before proceeding to a proof of Theorem 2.1, we need to justify our statement by showing that singularity is a measurable property of a random measure.

LEMMA 2.2. *Let $\mu = \mu_a + \mu_s$ be the Lebesgue decomposition of a Radon measure $\mu$ on $\mathbb{R}$. Then $\mu_a$ and $\mu_s$ are measurable functions of $\mu$. In particular, the set of singular Radon measures is measurable.*

PROOF. The martingale approach in Section XI.17 of Doob (1994) yields a product-measurable version of the Radon–Nikodym density $d\mu_a/d\lambda$. By Fubini's theorem, the mapping $\mu \mapsto \mu_a B$ is then measurable for every Borel set $B$, which implies the asserted measurability of $\mu_a$ and hence also of $\mu_s = \mu - \mu_a$. The last assertion follows since $\mu$ is singular iff $\mu_a = 0$.

The second statement can also be proved directly by a more elementary argument. Then note that $\mu$ is singular iff, for any bounded interval $I$ and constant $\varepsilon > 0$, there exist finitely many subintervals $I_1, \ldots, I_n$ such that $\lambda \bigcup_k I_k < \varepsilon$ and $\mu \bigcup_k I_k > \mu I - \varepsilon$. It is clearly enough to consider rational numbers $\varepsilon$ and intervals $I$ and $I_k$ with rational endpoints, which leaves us with a countable collection of measurable conditions. The asserted measurability then follows. $\square$

Our proof of Theorem 2.1 follows in relevant parts the corresponding argument in Section 2 of Takács (1967) and makes essential use of Lebesgue's differentiation theorem—the fact that any nondecreasing function $F$ is differentiable a.e. with a derivative $F'$ that agrees a.e. with the Radon-Nikodym derivative of the absolutely continuous component of $F$. For a discussion and proof of the latter result, see, for example, Section X.4 of Doob (1994).

PROOF OF THEOREM 2.1. It is enough to consider random measures on $\mathbb{R}_+$, since the result for $[0, 1)$ will then follow by a periodic continuation of $\xi$. Define

$$A_t = \inf_{s\geq t}(s - X_s), \quad \alpha_t = 1\{A_t = t - X_t\}, \qquad t \geq 0.$$

Consider first a realization of $X$ such that $A_0$ is finite. Then

$$0 \leq A_t - A_s \leq t - s, \qquad s < t,$$

which implies that $A$ is nondecreasing and absolutely continuous with an a.e. derivative $A'$. We shall prove that $A' = \alpha$ a.e.

Then fix a $t \geq 0$ with $\alpha_t = 0$. By the continuity of $A$ and the right-continuity and monotonicity of $X$, there exists some $\varepsilon > 0$ such that $A_s < s - X_s - \varepsilon$ and therefore $A_s = A_t$ whenever $|s - t| < \varepsilon$. Hence, $A'_t = 0$. Next consider a $t$ with $\alpha_t = 1$. Since $A'$ exists a.e. and $X' = 0$ a.e. by the singularity of $\xi$, we may assume that both conditions hold at $t$. We may also assume that $t$ is a cluster point of the set $D = \{s; \alpha_s = 1\}$, since the set of isolated points is at most countable. Choosing $t_1, t_2, \ldots \in D \setminus \{t\}$ with $t_n \to t$, we get

$$\frac{A_{t_n} - A_t}{t_n - t} = 1 - \frac{X_{t_n} - X_t}{t_n - t}, \qquad n \in \mathbb{N},$$

and as $n \to \infty$, it follows that $A'_t = 1$. Thus, we have indeed $A' = \alpha$ a.e.

Recalling that $A$ is absolutely continuous, we obtain a.s.,

$$\int_0^t \alpha_s ds = \int_0^t A'_s ds = A_t - A_0, \quad t \geq 0. \tag{2.2}$$

As $t \to \infty$, we have $X_t/t \to \rho$ a.s. by the pointwise ergodic theorem, and so $A_t/t \to (1-\rho) \vee 0$ a.s. Applying the same result to the left-hand side of (2.2), we get a.s.,

$$\begin{aligned} P[\sup\nolimits_{t>0}(X_t/t) \leq 1|\mathscr{I}_\xi] &= P[\sup\nolimits_{t\geq 0}(X_t - t) = 0|\mathscr{I}_\xi] \\ &= P[A_0 = 0|\mathscr{I}_\xi] \\ &= E[\alpha_0|\mathscr{I}_\xi] = (1-\rho) \vee 0. \end{aligned} \tag{2.3}$$

Now define

$$\sigma = \frac{\rho}{\sup_t(X_t/t)} \quad \text{on } \{0 < \rho < \infty\}$$

and put $\sigma = \vartheta$ otherwise, where $\vartheta$ is $U(0, 1)$ and independent of $X$. Relation (2.1) holds automatically on the set $\{0 < \rho < \infty\}$, and it is also true on the complement since $X_t \equiv 0$ a.s. when $\rho = 0$ and $X_t/t \to \infty$ a.s. when $\rho = \infty$. To verify the distributional claims, we may apply (2.3) to the process $rX/\rho$ for arbitrary $r \geq 0$ to get

$$P[\sigma \geq r|\rho] = P[r \sup\nolimits_t(X_t/t) \leq \rho|\rho] = (1-r) \vee 0 \quad \text{a.s.}$$

Since a similar relation holds trivially for $\vartheta$, we conclude that $\sigma$ is conditionally $U(0,1)$ given $\rho$. Thus, $\sigma$ is $U(0,1)$ and independent of $\rho$. $\square$

From Theorem 2.1 we may easily deduce the corresponding discrete-time result. Here (2.1) holds only with inequality and will be supplemented by a sharp relation of the classical type.

COROLLARY 2.3. *Let $\xi = (\xi_1, \xi_2, \dots)$ be a finite or infinite, stationary sequence of $\mathbb{R}_+$-valued random variables, and put $S_k = \sum_{j \le k} \xi_j$ and $\rho = E[\xi_1 | \mathscr{I}_\xi]$. Then there exists a $U(0,1)$ random variable $\sigma \perp\!\!\!\perp \rho$ such that*

$$\sup_{k>0} \frac{S_k}{k} \le \frac{\rho}{\sigma} \quad a.s.$$

*If the $\xi_k$ are $\mathbb{Z}_+$-valued, then also*

$$P[\sup_{k>0}(S_k - k) = -1 | \mathscr{I}_\xi] = (1-\rho) \vee 0 \quad a.s.$$

PROOF. Define $X_t = S_{[t+\tau]}$ where $\tau$ is $U(0,1)$ and independent of $(\xi_k)$. By Theorem 2.1 applied to $X$ or its periodic extension, there exists some $U(0,1)$ random variable $\sigma \perp\!\!\!\perp \rho$ such that a.s.

$$\sup_{k>0} \frac{S_k}{k} \le \sup_{t>0} \frac{S_{[t]}}{t} \le \sup_{t>0} \frac{X_t}{t} = \frac{\rho}{\sigma}.$$

If the $\xi_k$ are $\mathbb{Z}_+$-valued, then the same result yields a.s.,

$$\begin{aligned} P[\sup_{k>0}(S_k - k) = -1 | \mathscr{I}_\xi] &= P[\sup_{t\ge 0}(X_t - t) = 0 | \mathscr{I}_\xi] \\ &= P[\sup_{t>0}(X_t/t) \le 1 | \mathscr{I}_\xi] \\ &= P[\rho \le \sigma | \mathscr{I}_\xi] = (1-\rho) \vee 0. \end{aligned} \qquad \square$$

**3. Sojourn laws.** For any measurable process $X$ on $\mathbb{R}_+$ or $[0,1)$, the associated *sojourn times* $\Lambda_t^\pm$ above and below $X_0$ are given by

$$\Lambda_t^\pm = \lambda\{s < t;\ \pm(X_s - X_0) > 0\}, \qquad t \in \mathbb{R}_+ \text{ or } [0,1). \tag{3.1}$$

In discrete time, we use the same definition with $\lambda$ replaced by the counting measure on $\mathbb{Z}_+$ or $\mathbb{Z}_n$. Recall that a process $X$ on $[0,1)$ or $\mathbb{Z}_n$ is *(cyclically) stationary* if the shifted process $Y_t = X_{s+t}$ has the same distribution for every $s$, where $s+t$ means addition modulo 1 or $n$, respectively.

THEOREM 3.1. *For any stationary, measurable process $X$ on $\mathbb{R}_+$ or $\mathbb{Z}_+$, the ratios $\Lambda_t^\pm / t$ converge a.s. toward some limits $\rho_\pm$. Furthermore, if $\gamma$ is $U(0,1)$ and independent of $X$, the quantity*

$$\sigma = \rho_- + \gamma(1 - \rho_+ - \rho_-) \tag{3.2}$$

*is again $U(0,1)$. This remains true for any stationary and measurable process on $[0,1)$ or $\mathbb{Z}_n$, if we define $\rho_\pm$ as $\Lambda_1^\pm$ or $\Lambda_n^\pm/n$, respectively.*

Before proving this result, we note that (3.2) implies $\rho_- \le \sigma \le 1 - \rho_+$. In particular, $\rho_+$ and $\rho_-$ are both $U(0,1)$ when

$$\lim_{t\to\infty} t^{-1}\lambda\{s<t;\, X_0 = X_s\} = 0 \quad \text{a.s.},$$

or, for processes on $[0,1)$, when $\lambda\{s;\, X_0 = X_s\} = 0$ a.s. This clearly cannot occur for processes on $\mathbb{Z}_n$. If $X$ is a measurable process on $[0,1]$ with cyclically stationary increments, we may apply the theorem to the stationary process $Y_t = X_{\tau+t}$ where $\tau$ is $U(0,1)$ and independent of $X$. Noting that the associated random variables $\rho_\pm$ have the same distributions as $\lambda\{s;\, \pm X_s > 0\}$, we obtain the classical uniform laws.

Our proof of Theorem 3.1 will be based on the following lemma.

LEMMA 3.2. *Let the random variable $\tau$ and the random probability measure $\eta$ on $\mathbb{R}$ satisfy $P[\tau \in \cdot|\eta] = \eta$ a.s., and let $\gamma$ be $U(0,1)$ and independent of $(\tau,\eta)$. Then the quantity $\sigma = \eta(-\infty,\tau) + \gamma\,\eta\{\tau\}$ is again $U(0,1)$.*

PROOF. By conditioning on $\eta$, we may reduce to the case when $\eta$ is nonrandom and $P\circ\tau^{-1} = \eta$. Consider the distribution function $F(t) = \eta(-\infty,t]$ and its right-continuous inverse $F^{-1}(s) = \inf\{t;\, F(t) > s\}$, and let $\vartheta \perp\!\!\!\perp \gamma$ be $U(0,1)$. Then $\tilde\tau = F^{-1}(\vartheta)$ is again independent of $\gamma$ with distribution $\eta$, and so for convenience, we may assume that $\tilde\tau = \tau$. Next we define $\vartheta_\pm = F(F^{-1}(\vartheta)\pm)$ and note that $\sigma = \vartheta_- + \gamma(\vartheta_+ - \vartheta_-)$. Write $F(\mathbb{R})$ for the range of $F$ and introduce the connected components $I_k = (s_k, t_k)$ of the open set $(0,1)\setminus\overline{F(\mathbb{R})}$. For any $k$ we have a.s. $\vartheta_- = s_k$ iff $\vartheta \in I_k$, and then $\vartheta_+ = t_k$. Thus,

$$P[\vartheta \le t|\vartheta_- = s_k] = \frac{t-s_k}{t_k - s_k} = P[\sigma \le t|\vartheta_- = s_k], \qquad t \in I_k.$$

On the other hand, if $\vartheta_- \notin \{s_1, s_2, \ldots\}$, then a.s. $\vartheta \notin \bigcup_k [s_k, t_k]$, which implies $\sigma = \vartheta$. Hence, by combination, $P[\sigma \in \cdot|\vartheta_-] = P[\vartheta \in \cdot|\vartheta_-]$ a.s. and therefore $\sigma =_d \vartheta$. $\square$

PROOF OF THEOREM 3.1. We may restrict our attention to processes on $\mathbb{R}_+$ or $\mathbb{Z}_+$, as the results for $[0,1)$ or $\mathbb{Z}_n$ will then follow by periodic continuation or may be proved directly by similar but more elementary arguments. Let the empirical distributions $\eta_t$ and the mean occupation measure $\eta$ be given by

$$\eta_t B = t^{-1}\lambda\{s<t;\, X_s \in B\}, \qquad B \in \mathscr{B},\ t \ge 0,$$
$$\eta B = P[X_0 \in B|\mathscr{I}_X], \qquad B \in \mathscr{B}, \tag{3.3}$$

where $\mathscr{B}$ denotes the Borel $\sigma$-field in $\mathbb{R}$. By Tucker's (1959) extension of the Glivenko–Cantelli theorem, we have a.s.,

$$\lim_{t\to\infty} \sup_x |\eta_t(-\infty,x] - \eta(-\infty,x]| = 0,$$
$$\lim_{t\to\infty} \sup_x |\eta_t(-\infty,x) - \eta(-\infty,x)| = 0,$$

and so, as $t \to \infty$,

$$t^{-1}\Lambda_t^+ = \eta_t(X_0, \infty) \to \eta(X_0, \infty),$$
$$t^{-1}\Lambda_t^- = \eta_t(-\infty, X_0) \to \eta(-\infty, X_0).$$

Thus, the asserted convergence holds with a.s. limits

$$\rho_+ = \eta(X_0, \infty), \qquad \rho_- = \eta(-\infty, X_0).$$

Furthermore, (3.3) yields a.s. for any $B \in \mathscr{B}$,

$$P[X_0 \in B|\eta] = E[P[X_0 \in B|\mathscr{I}_X]|\eta] = E[\eta B|\eta] = \eta B,$$

which means that $\eta$ is a regular version of $P[X_0 \in \cdot|\eta]$. Since $\eta\{X_0\} = 1 - \rho_+ - \rho_-$, Lemma 3.2 shows that $\sigma$ is again $U(0, 1)$. □

From Theorem 3.1 we may easily deduce criteria for $\rho_+$ and $\rho_-$ to be $U(0, 1)$. The following statement translates immediately into similar critera for processes on $[0, 1]$ with cyclically stationary increments. The special case of processes on $[0, 1]$ with exchangeable increments was first recorded by Knight (1996).

Corollary 3.3. *Let $X$ be a stationary and measurable process on $\mathbb{R}_+$, $\mathbb{Z}_+$, $[0, 1)$ or $\mathbb{Z}_n$ with mean occupation measure $\eta$, and define $\rho_\pm$ as in Theorem 3.1. Then these statements are equivalent*:

(i) $\rho_-$ *(or $\rho_+$) is $U(0, 1)$.*
(ii) $\rho_+ + \rho_- = 1$ *a.s.*
(iii) $\eta$ *is a.s. diffuse.*

Proof. Let $\gamma$ and $\sigma$ be such as in Theorem 3.1. From (ii) we get $\rho_- = \sigma$ a.s., and (i) follows. Conversely, (i) yields

$$\tfrac{1}{2} = E\sigma = E\rho_- + E\gamma(1 - \rho_+ - \rho_-) = \tfrac{1}{2} + \tfrac{1}{2}E(1 - \rho_+ - \rho_-),$$

which implies (ii). Next, recall that $1 - \rho_+ - \rho_- = \eta\{X_0\}$ and $P[X_0 \in \cdot|\eta] = \eta$ a.s. Using the disintegration Theorem 5.4 in Kallenberg (1997), we obtain

$$\begin{aligned} E(1 - \rho_+ - \rho_-) &= E\eta\{X_0\} = E\,E[\eta\{X_0\}|\eta] \\ &= E\int \eta\{x\}\eta(dx) = E\textstyle\sum_x (\eta\{x\})^2, \end{aligned}$$

which shows that (ii) and (iii) are equivalent. □

For comparison, we consider the corresponding statements for maxima and minima. In the following result, we may think of $M$ as the set of absolute maxima of some underlying stationary process $X$ on $[0, 1]$. More precisely, let $X$ be a stationary, separable process on $I = [0, 1]$ and write $M$ for the set of points $t \in I$ such that $\sup_{s\in G} X_s = \sup_{s\in I} X_s$ for every neighborhood $G$ of $t$.

Proposition 3.4. *Let $M \neq \emptyset$ be a stationary, random, closed subset of $[0, 1]$, and define $\tau_1 = \inf M$ and $\tau_2 = \sup M$. Then $\tau_1 \le \sigma \le \tau_2$ for some $U(0, 1)$ random variable $\sigma$. Furthermore, $\tau_1$ (or $\tau_2$) is $U(0, 1)$ iff $\tau_1 = \tau_2$ a.s.*

PROOF. First consider any nonrandom, closed subset $F \neq \emptyset$ of the circle $T$, and let $\gamma$ be uniformly distributed on $T$. Given that $\gamma$ falls in a connected component $I$ of $F^c$, its conditional distribution is clearly uniform on $I$. The result translates immediately into a corresponding statement for closed subsets $F \neq \emptyset$ of $\mathbb{R}$ with period 1 and a $U(0,1)$ random variable $\gamma$. Equivalently, we may consider the location of the random set $F - \gamma$ relative to the origin.

In this form, the statement extends to $M - \gamma$ provided that $\gamma \perp\!\!\!\perp M$, and since $M - \gamma =_d M$ it remains true for $M$. More precisely, given the length of the random interval $(\tau_2 - 1, \tau_1)$, the location of the origin within the interval is uniformly distributed. Putting

$$\sigma = \frac{\tau_1}{1 + \tau_1 - \tau_2} \geq \tau_1 \tag{3.4}$$

whenever the denominator is positive, we may conclude that $\sigma$ is conditionally $U(0,1)$, given that $0 \notin M$. We also note that

$$\sigma = 1 - \frac{1 - \tau_2}{1 + \tau_1 - \tau_2} \leq 1 - (1 - \tau_2) = \tau_2. \tag{3.5}$$

If instead $0 \in M$, we may define $\sigma = \vartheta$ where $\vartheta$ is $U(0,1)$ and independent of $M$. Then $\sigma$ remains $U(0,1)$, unconditionally, and satisfies $\tau_1 \leq \sigma \leq \tau_2$.

To prove the last assertion, we note that the inequalities in (3.4) and (3.5) are strict when $0 < \tau_1 < \tau_2 < 1$, and also that trivially $\tau_1 < \sigma < \tau_2$ a.s. when $0 \in M$. Thus, if $P\{\tau_1 < \tau_2\} > 0$, then $E\tau_1 < E\sigma < E\tau_2$ and all three distributions are different. If instead $\tau_1 = \tau_2$ a.s., then $\tau_1 = \sigma = \tau_2$ a.s. and the three variables are $U(0,1)$. □

For processes $X$ on $[0,1]$ with exchangeable increments and $X_0 = 0$, Knight (1996) shows that the time $\tau_1$ of the first maximum has the same distribution as the positive sojourn time $\rho_+ = \lambda\{t; X_t > 0\}$, which extends a celebrated discrete-time result of Sparre-Andersen (1953, 1954) [see, e.g., Corollary 9.20 in Kallenberg (1997)]. Assuming in addition that $X_1 = 0$, we may conclude that $\tau_1 = \tau_2$ a.s. iff $1 - \rho_+ - \rho_- = 0$. Those results suggest that the pairs $(\tau_1, 1 - \tau_2)$ and $(\rho_-, \rho_+)$ might have the same distribution.

Rather surprisingly, the latter statement is false in general. For a simple counterexample, we may consider a process of the form

$$X_t = \sum_{k \leq n} b_k 1\{\sigma_k \leq t\}, \qquad t \in [0,1],$$

where $b_1 \leq \cdots \leq b_n$ are nonzero constants with sum 0 and $\sigma_1, \dots, \sigma_n$ are i.i.d. $U(0,1)$ random variables. Then $\tau_2 - \tau_1 = 1$ holds in particular when $\sigma_1 < \cdots < \sigma_n$, which occurs with the positive probability $1/n!$. On the other hand, we have $\rho_+ + \rho_- > 0$ a.s. since $\rho_+ + \rho_- = 0$ iff $X_t = 0$ a.e., which is clearly impossible unless $n = 0$.

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