

## A PARTICULAR CASE OF CORRELATION INEQUALITY FOR THE GAUSSIAN MEASURE

BY GILLES HARGÉ

*Université d'Evry*

Our purpose is to prove a particular case of a conjecture concerning the Gaussian measure of the intersection of two symmetric convex sets of  $\mathbb{R}^n$ . This conjecture states that the measure of the intersection is greater or equal to the product of the measures. In this paper, we prove the inequality when one of the two convex sets is a symmetric ellipsoid and the other one is simply symmetric. The general case is still open.

**1. Introduction.** This paper deals with the following conjecture. Let  $A$  and  $B$  be two symmetric convex sets; if  $\mu$  is a centered, Gaussian measure on  $\mathbb{R}^n$ , could we say that

$$(1) \quad \mu(A \cap B) \geq \mu(A)\mu(B)?$$

This formulation is due to Das Gupta, Eaton, Olkin, Perlman, Savage and Sobel [3]. Nevertheless, this question for some particular cases appeared before 1972 (see also [3] for the history of the problem). An important contribution is due to Khatri [6] and Šidák [10] who, in 1967, independently proved the result if  $B$  is of the following form:

$$B = \{x = (x_1, \dots, x_n), |x_1| \leq a\}$$

(Sugita [11] gives another proof of this).

In 1977, Pitt [7] proved the inequality (1) in the two-dimensional case. Recently, Schechtman, Schlumprecht and Zinn [9] have proved (1) if the two sets are centered ellipsoids.

In this paper, we prove (1) if  $A$  is an arbitrary symmetric convex set and  $B$  a centered ellipsoid,

$$B = \{x \in \mathbb{R}^n, \langle Cx, x \rangle \leq 1\}$$

( $C$  is a symmetric, nonnegative matrix,  $\langle, \rangle$  is the usual scalar product on  $\mathbb{R}^n$ ).

In particular, it contains the case studied in [6] and [10].

We shall use the following results. Recall that a function  $f: \mathbb{R}^n \rightarrow \mathbb{R}^+$  is called log-concave if for  $x, y \in \mathbb{R}^n$  and  $0 < \lambda < 1$ ,

$$f(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda f(y)^{1-\lambda}.$$

Prékopa [8] has proved the following.

---

Received December 1998; revised March 1999.

AMS 1991 subject classifications. Primary 28C20, 60E15.

Key words and phrases. Gaussian measure, correlation, log-concavity, semigroups.

THEOREM 1. *Let*

$$f: \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R} \text{ log-concave on } \mathbb{R}^{n+p}.$$

$$(x, y) \mapsto f(x, y)$$

*Then the function*

$$x \mapsto \int f(x, y) dy \text{ is log-concave on } \mathbb{R}^n.$$

Now, let  $C$  be a symmetric positive definite matrix. We denote by  $\mu_C$  the measure on  $\mathbb{R}^n$  which has the following density with respect to the Lebesgue measure:

$$x \mapsto \sqrt{\det C} \frac{\exp(-1/2\langle Cx, x \rangle)}{\sqrt{2\pi}^n}$$

(so  $\mu = \mu_I$ ).

Let

$$L_C(f) = \frac{1}{2}(\Delta f - \langle Cx, \nabla f \rangle),$$

where  $f$  is a function so that this expression makes sense.

Let  $E_{\mu_C}(f) = \int f d\mu_C$ , then

$$E_{\mu_C}(f L_C(g)) = -\frac{1}{2} E_{\mu_C}(\langle \nabla f, \nabla g \rangle).$$

If  $P_t^C(f)$  is the solution of

$$\frac{d}{dt}(P_t^C(f)) = L_C(P_t^C(f)),$$

$$P_0^C(f) = f,$$

then we can write the Mehler formula (see [4]):

$$P_t^C f(x) = \int f\left(\exp\left(-\frac{t}{2}C\right)x + (I - \exp(-tC))^{1/2}y\right) d\mu_C(y).$$

More exactly,  $P_t^C f(x) = [\Gamma(\exp(-t/2C))][f(C^{-1/2}\cdot)](C^{1/2}x)$  where  $\Gamma$  is given in [4] by

$$\Gamma(M)f(x) = \int f(M^*x + \sqrt{I - M^*M}y) d\mu(y),$$

where  $M$  is a matrix such that  $I - M^*M$  is a positive matrix.

If  $C = I$ , we obtain the Ornstein–Uhlenbeck semigroup.

Using Theorem 1, we see immediately that  $P_t^C f$  is log-concave if  $f$  is log-concave.

**2. Proof of the main result.**

**THEOREM 2.** *Let  $g: \mathbb{R}^n \rightarrow \mathbb{R}$  be a log-concave, even function and  $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a differentiable function such that  $f' \leq 0$  on  $\mathbb{R}^+$ .*

*Let  $C$  be a symmetric positive definite matrix. Then*

$$\int f(\langle C^{-1}x, x \rangle)g(x) d\mu(x) \geq \int f(\langle C^{-1}x, x \rangle) d\mu(x) \int g(x) d\mu(x).$$

**PROOF.** Let  $y = C^{-1/2}x$ . The inequality becomes

$$\int f(\|y\|^2)g(C^{1/2}y) d\mu_C(y) \geq \int f(\|y\|^2) d\mu_C(y) \int g(C^{1/2}y) d\mu_C(y).$$

We remark that  $\bar{g}(y) = g(C^{1/2}y)$  is a log-concave, even function. Let  $\varphi$  be the following function:

$$\varphi(t) = E_{\mu_C}(f(\|y\|^2)P_t^C(\bar{g})(y)).$$

We have

$$\lim_{t \rightarrow +\infty} \varphi(t) = E_{\mu_C}(f(\|y\|^2))E_{\mu_C}(\bar{g}) \quad \text{and} \quad \lim_{t \rightarrow 0} \varphi(t) = E_{\mu_C}(f(\|y\|^2)\bar{g}).$$

So, it is sufficient to show that  $\varphi' \leq 0$ .

$$\begin{aligned} \varphi'(t) &= E_{\mu_C}[f(\|y\|^2)L_C(P_t^C(\bar{g}))] \\ &= -\frac{1}{2} E_{\mu_C}[\langle \nabla(f(\|y\|^2)), \nabla(P_t^C(\bar{g})) \rangle] \\ &= -E_{\mu_C}[f'(\|y\|^2)\langle y, \nabla(P_t^C(\bar{g})) \rangle] \end{aligned}$$

The function  $y \mapsto P_t^C(\bar{g})(y)$  is a log-concave, even and positive function if  $g \neq 0$ . Then, it is of the form  $e^{-G}$  where  $G$  is a convex, even function. Moreover,  $G$  is arbitrarily often differentiable.

So, we have

$$\varphi'(t) = E_{\mu_C}[f'(\|y\|^2)\langle y, \nabla G(y) \rangle \exp(-G(y))].$$

Using the fact that  $G$  is convex and even, we can write

$$\nabla G(y) = 0 + \int_0^1 \text{Hess } G(ty) y dt.$$

Thus, we deduce that  $\langle \nabla G(y), y \rangle$  is positive, which achieves the proof of the theorem.  $\square$

**REMARK 1.** It is possible to prove the theorem without the change of variable used in the beginning of the proof. It is sufficient to choose the following semigroup, which is connected in a simple way to  $P_t^C$ :  $\tilde{P}_t^C = \Gamma(\exp(-\frac{t}{2}C))$  (with  $\Gamma$  given in [4]). The new proof which we obtain like this is similar to the previous one.

We deduce the following.

COROLLARY 3. *Let  $\nu$  be a centered Gaussian measure on  $\mathbb{R}^n$ . Let  $C$  be a symmetric nonnegative matrix and  $A$  a symmetric convex set on  $\mathbb{R}^n$ . Then*

$$\nu(\{\langle Cx, x \rangle \leq 1\} \cap A) \geq \nu(\{\langle Cx, x \rangle \leq 1\})\nu(A).$$

PROOF. If the covariance matrix of  $\nu$  is not invertible, it is possible to find an integer  $k < n$  and a centered Gaussian measure  $\tilde{\nu}$  on  $\mathbb{R}^k$  with an invertible covariance matrix, such that

$$\int_{\mathbb{R}^n} F d\nu = \int_{\mathbb{R}^k} F(P(x_1, \dots, x_k, 0, \dots, 0)^*) d\tilde{\nu}(x_1, \dots, x_k),$$

where  $P$  is a real unitary matrix.

So, it is enough to show the result when  $\nu$  has a density with respect to the Lebesgue measure. We have

$$1_{\langle Cx, x \rangle \leq 1} = 1_{]-\infty, 1]}(\langle Cx, x \rangle).$$

We approximate  $1_{]-\infty, 1]}$  with a decreasing differentiable function. (if  $C \neq 0$  then  $dx\{x, \langle Cx, x \rangle = 1\} = 0$ ). Then we approximate  $C$  with a symmetric positive definite matrix. Since  $1_A$  is a log-concave even function, Theorem 2 leads to the conclusion.  $\square$

REMARK 2.

(i) If we want to generalize this method to an arbitrary symmetric convex set, we need to show that a certain family of semigroups preserves the property of log-concavity. We deal with semigroups with a generator of the form

$$(2) \quad L(f) = \frac{1}{2} (\operatorname{div}(A(x)^{-1}\nabla f) - \langle A(x)^{-1}x, \nabla f \rangle),$$

where  $A(x)$  is a symmetric positive definite matrix. This question is still open.

The reason why it will be sufficient is the following. We would like to prove Theorem 2 with an even, strictly log-concave function  $e^{-F}$  instead of  $f(\langle C^{-1}x, x \rangle)$ . It is easy to show that  $\nabla F(x) = A(x)x$  where  $A(x)$  is a symmetric positive definite matrix. Let  $P_t$  be a semigroup with a generator given by (2) and define

$$\varphi(t) = \int e^{-F} P_t g d\mu.$$

We obtain

$$\begin{aligned} \varphi'(t) &= \int e^{-F} L P_t g d\mu = -\frac{1}{2} \int \langle A(x)^{-1}\nabla(e^{-F}), \nabla P_t g \rangle d\mu \\ &= \frac{1}{2} \int \langle x, \nabla P_t g \rangle e^{-F} d\mu. \end{aligned}$$

So,  $\varphi'(t) \leq 0$  if we could prove that  $P_t g$  is log-concave and even.

(ii) Inequalities like the one of the theorem and concerning other functions than log-concave functions have been proved in the papers of Bakry and Michel [1] and of Hu [5]. Proofs also use particular semigroups.

We could deduce from the corollary the following fact.

COROLLARY 4. *Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a nonnegative, decreasing, differentiable function.*

*Let  $C$  and  $D$  be a two-symmetric, nonnegative matrix. We assume that  $C$  is invertible. Let  $g$  be an even, log-concave function. Then*

$$\begin{aligned} & \int f(\langle Cx, x \rangle) \exp(-\frac{1}{2}\langle Dx, x \rangle) g(x) d\mu(x) \\ & \geq \int f(\langle Cx, x \rangle) \exp(-\frac{1}{2}\langle Dx, x \rangle) d\mu(x) \int g d\mu. \end{aligned}$$

PROOF.

$$\begin{aligned} & \int f(\langle Cx, x \rangle) \exp\left(-\frac{1}{2}\langle Dx, x \rangle\right) g(x) d\mu(x) \\ & = \int f(\langle \bar{C}y, y \rangle) g((I + D)^{-1/2} y) \frac{d\mu(y)}{\sqrt{\det(I + D)}} \\ & \qquad \qquad \qquad \text{with } \bar{C} = (I + D)^{-1/2} C (I + D)^{-1/2}, \end{aligned}$$

so

$$\begin{aligned} & \int f(\langle Cx, x \rangle) \exp\left(-\frac{1}{2}\langle Dx, x \rangle\right) g(x) d\mu(x) \\ & \geq \int f(\langle \bar{C}y, y \rangle) d\mu(y) \int g((I + D)^{-1/2} y) \frac{d\mu(y)}{\sqrt{\det(I + D)}} \\ & \geq \int f(\langle Cx, x \rangle) \exp\left(-\frac{1}{2}\langle Dx, x \rangle\right) d\mu(x) \\ & \quad \times \int g(x) \exp\left(-\frac{1}{2}\langle Dx, x \rangle\right) d\mu(x) \sqrt{\det(I + D)}. \end{aligned}$$

Using Theorem 2 for the second integral, we obtain the result.  $\square$

This corollary could be seen as a generalization of Theorem 2. The problem is to replace  $\exp(-\frac{1}{2}t)$  by an arbitrary decreasing function, which should allow us to show the conjecture for the intersection of two ellipsoids instead of  $B$ .

**3. Relation with Pitt’s proof.** We use notations of Theorem 2.

Pitt [7] introduces the function

$$\psi(t) = E_{\mu}(f(\langle C^{-1}x, x \rangle) P_t^I(g)).$$

We always have

$$\lim_{t \rightarrow \infty} \psi(t) = E_{\mu}(f(\langle C^{-1}x, x \rangle)) E_{\mu}(g),$$

$$\lim_{t \rightarrow 0} \psi(t) = E_{\mu}(f(\langle C^{-1}x, x \rangle)g)$$

and

$$\psi'(t) = -\frac{1}{2} E_\mu[\langle \nabla(f(\langle C^{-1}x, x \rangle)), \nabla P_t^I(g) \rangle]$$

So, it is enough to show that  $\psi$  is decreasing. The function  $\varphi$  of Theorem 2 is better than  $\psi$  because we can write  $\varphi'$  as an integral of a function which is negative everywhere. It is not the case for  $\psi$  and it explains the difficulty of Pitt's proof.

Pitt has remarked that it is sufficient to show for an arbitrary dimension the following inequality (which is nothing else but an expression of  $\psi'$ ):

(3) for all even and log-concave functions  $f$  and  $g$ :  $\int \langle \nabla f, \nabla g \rangle d\mu \geq 0$ .

This inequality seems to be stronger than conjecture (1). We will prove it in a particular case which unfortunately gives a result that is weaker than the one of Theorem 2.

**THEOREM 5.** *If  $f(x) = \exp(-\frac{1}{2}\langle Cx, x \rangle)$  with  $C$  a nonnegative, symmetric matrix and if  $g = e^{-G}$  where  $G$  is even and convex, then*

$$\int \langle \nabla f, \nabla g \rangle d\mu \geq 0.$$

**PROOF.** Define  $\bar{G}(x) = G((I + C)^{-1/2} x)$ . After using a change of variable, we obtain

$$\int \langle \nabla f, \nabla g \rangle d\mu = \alpha \int \langle Cx, \nabla \bar{G}(x) \rangle \exp\left(-\bar{G}(x) - \frac{\|x\|^2}{2}\right) dx,$$

where  $\alpha$  is a positive constant.

This is equal to

$$\beta \sum_{i,j} C_{i,j} E_\nu\left(x_i \frac{\partial \bar{G}}{\partial x_j}\right)$$

with  $\beta$  positive and  $d\nu(x) = \exp(-\bar{G}(x) - \|x\|^2/2) (dx/\lambda)$  ( $\lambda = \int \exp(-\bar{G}(x) - \|x\|^2/2) dx$ ).

Let  $M$  be the matrix

$$M_{i,j} = E_\nu\left(x_i \frac{\partial \bar{G}}{\partial x_j}\right).$$

We must show that  $\text{Tr}(CM) \geq 0$ . So, it is enough to show that  $M$  is symmetric and nonnegative.

Let

$$L = \frac{1}{2}(\Delta - \langle \nabla H, \nabla \rangle) \quad \text{with} \quad H(x) = \bar{G}(x) + \frac{\|x\|^2}{2}.$$

$H$  is even and convex. It is easy to see that  $L$  is symmetric with respect to  $\nu$  and

$$E_\nu(hLk) = -\frac{1}{2} E_\nu(\langle \nabla h, \nabla k \rangle).$$

We notice that

$$\begin{aligned} x_j + 2Lx_j = -\frac{\partial \tilde{G}}{\partial x_j} &\Rightarrow M_{i,j} = -E_\nu(x_i(x_j + 2Lx_j)) \\ &= -E_\nu(x_i x_j) + E_\nu(\langle \nabla x_i, \nabla x_j \rangle) \\ &= \delta_{ij} - E_\nu(x_i x_j). \end{aligned}$$

Consequently,  $M$  is symmetric. Let  $h = (h_1, \dots, h_n) \in \mathbb{R}^n$ :  $\sum_{i,j} h_i h_j M_{i,j} = \|h\|^2 - E_\nu(\langle h, x \rangle^2)$ . To prove  $M$  is nonnegative, it is sufficient to show

$$\forall h \in \mathbb{R}^n \setminus \{0\}, E_\nu\left(\left\langle \frac{h}{\|h\|}, x \right\rangle^2\right) \leq 1.$$

This is a particular case of a result of Brascamp and Lieb [2], Theorem 5.1, which is simple to prove in our case. In order to give a complete proof, we will demonstrate this result using Prekopa’s theorem. Let

$$k_1 = \frac{h}{\|h\|}.$$

We choose  $(k_2, \dots, k_n)$  such that  $(k_1, \dots, k_n)$  is an orthonormal basis of  $\mathbb{R}^n$ . Using a change of variable, we see that it is enough to prove

$$\int y_1^2 \exp\left(-\tilde{G}(Py) - \frac{\|y\|^2}{2}\right) dy \leq \int \exp\left(-\tilde{G}(Py) - \frac{\|y\|^2}{2}\right) dy,$$

where  $P$  is an unitary matrix.

$F(y) = \tilde{G}(Py)$  is even and convex. Consequently, the question is:

$$\int y_1^2 \exp\left(-F(y) - \frac{\|y\|^2}{2}\right) dy \leq \int \exp\left(-F(y) - \frac{\|y\|^2}{2}\right) dy?$$

We can write

$$\int y_1^2 \exp\left(-F(y) - \frac{\|y\|^2}{2}\right) dy = \int_{-\infty}^{+\infty} y_1^2 \exp(-K(y_1)) \exp\left(-\frac{y_1^2}{2}\right) dy_1$$

$$\text{with } \exp(-K(y_1)) = \int_{\mathbb{R}^{n-1}} \exp\left(-F(y_1, \bar{y}) - \frac{\|\bar{y}\|^2}{2}\right) d\bar{y}$$

and with Prékopa’s theorem,  $K$  is even and convex.

Let us make an integration by parts:

$$\begin{aligned} & \int_{-\infty}^{+\infty} y_1 \exp(-K(y_1)) y_1 \exp\left(-\frac{y_1^2}{2}\right) dy_1 \\ &= \int_{-\infty}^{+\infty} \exp\left(-\frac{y_1^2}{2}\right) (\exp(-K(y_1)) - y_1 K'(y_1) \exp(-K(y_1))) dy_1 \\ &= \int_{-\infty}^{+\infty} \exp\left(-\frac{y_1^2}{2}\right) \exp(-K(y_1)) dy_1 \\ &\quad - \int_{-\infty}^{+\infty} \exp\left(-\frac{y_1^2}{2}\right) y_1 K'(y_1) \exp(-K(y_1)) dy_1. \end{aligned}$$

$K$  is even and convex so  $y_1 K'(y_1) \geq 0$ , we deduce that

$$\begin{aligned} \int y_1^2 \exp\left(-F(y) - \frac{\|y\|^2}{2}\right) dy &\leq \int_{-\infty}^{+\infty} \exp\left(-\frac{y_1^2}{2}\right) \exp(-K(y_1)) dy_1 \\ &\leq \int \exp\left(-F(y) - \frac{\|y\|^2}{2}\right) dy. \end{aligned}$$

REMARK 3. One could ask if the following inequality is true. If  $C$  is a symmetric positive definite  $n \times n$  matrix, could we write

for all even and log-concave functions  $f$  and  $g$ ,

$$\int \langle \nabla f, \nabla g \rangle \exp\left(-\frac{1}{2} \langle Cx, x \rangle\right) dx \geq 0?$$

The answer is negative. Let

$$f(x) = \exp\left(-\frac{1}{2} \langle Ax, x \rangle\right) \quad \text{and} \quad g(x) = \exp\left(-\frac{1}{2} \langle Bx, x \rangle\right),$$

with  $A$  and  $B$  symmetric, nonnegative matrices; we obtain

$$\int \langle \nabla f, \nabla g \rangle \exp\left(-\frac{1}{2} \langle Cx, x \rangle\right) dx = \alpha \operatorname{Tr}((A + B + C)^{-1} AB) \quad \text{with } \alpha > 0.$$

In the two-dimensional case, we choose

$$A = \begin{pmatrix} 2 & \sqrt{\varepsilon} \\ \sqrt{\varepsilon} & 2\varepsilon \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -6\sqrt{\varepsilon} \\ -6\sqrt{\varepsilon} & 36\varepsilon \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}.$$

In this case,  $\operatorname{Tr}((A + B + C)^{-1} AB) = (1/131)(-1 + 174\varepsilon)$ .

Nevertheless, it is possible to deduce from Theorem 5 (or to prove it directly) that

$$\operatorname{Tr}((I + A + B)^{-1} AB) \geq 0 \quad \text{if } A \text{ and } B \text{ are symmetric and nonnegative } n \times n \text{ matrices.}$$



**4. Comparison of semigroups.** The method used in Theorem 2 allows us to obtain a result quite better than those of this theorem. The idea is to compare two particular semigroups.

Let  $G: \mathbb{R}^n \rightarrow \mathbb{R}$  be differentiable and let  $C$  be a symmetric, nonnegative matrix.  $(\Omega, \mathcal{F}, P)$  denotes a probability space with a filtration  $(\mathcal{F}_t)$ . Let  $B_t$  be a Brownian motion issued from 0 with values in  $\mathbb{R}^n$ .

Let  $X$  be the solution of the following stochastic differential equation:

$$dX_t^x = dB_t - \frac{1}{2}CX_t^x dt - \frac{1}{2}\nabla G(X_t^x) dt,$$

$$X_0^x = x.$$

We associate to  $X_t^x$  the semigroup

$$P_t(f)(x) = E(f(X_t^x)).$$

**THEOREM 6.** *If  $f$  is log-concave and even and if  $G(x) = u(\|x\|^2)$  where  $u$  is a differentiable, increasing and Lipschitz function then, for all nonnegative real  $t$ ,*

$$P_t f(x) \geq P_t^C f(x).$$

**PROOF.** First, we remark that the explosion time of  $X$  is equal to infinity. Let

$$G_n(x) = G(x) \quad \text{if } \|x\| \leq n,$$

$$G_n(x) = u(n^2) + 2u'(n^2)n(\|x\| - n) \quad \text{if } \|x\| \geq n.$$

For all  $x$ , we obtain  $\nabla G_n(x) = \lambda_n(x)x$  with  $\lambda_n \in \mathbb{R}^+$ . Moreover,  $\nabla G_n$  is bounded. Let

$$dX_t^n = dB_t - \frac{1}{2}CX_t^n dt - \frac{1}{2}\nabla G_n(X_t^n) dt,$$

$$X_0^n = x.$$

Define

$$P_t^n(f(x)) = E(f(X_t^n)).$$

We will use Girsanov's theorem for  $X^n$ . Let

$$B_t = \tilde{B}_t + \int_0^t \alpha_s^n ds,$$

where  $\alpha_t^n = \frac{1}{2}\nabla G_n(X_t^n)$ .

Define

$$D_t^n = \exp\left(\int_0^t \langle \alpha_s^n, dB_s \rangle - \frac{1}{2} \int_0^t \|\alpha_s^n\|^2 ds\right) = 1 + \int_0^t D_s^n \langle \alpha_s^n, dB_s \rangle.$$

$D_t^n$  is a martingale because  $\nabla G_n$  is bounded.

Define a new probability  $Q_n$  by

$$dQ_n|_{\mathcal{F}_t} = D_t^n dP|_{\mathcal{F}_t}.$$

$\tilde{B}$  is a  $Q_n$ -Brownian motion. We have

$$dX_t^n = d\tilde{B}_t - \frac{1}{2}CX_t^n dt,$$

$$X_0^n = x.$$

We can solve this equation and we obtain

$$X_t^n = \exp\left(-\frac{t}{2}C\right)x + \exp\left(-\frac{t}{2}C\right)\int_0^t \exp\left(\frac{s}{2}C\right)d\tilde{B}_s$$

$$= \exp\left(-\frac{t}{2}C\right)x + (I - \exp(-tC))^{1/2}C^{-1/2}N_t,$$

where  $N_t \sim N(0, I)$  under  $Q_n$ . Define

$$P_t^{Q_n}f(x) = E_{Q_n}(f(X_t^n)).$$

Consequently, we have

$$P_t^{Q_n}f(x) = P_t^Cf(x).$$

Moreover,

$$P_t^n f(x) = E(f(X_t^n)) = E_{Q_n}(f(X_t^n)(D_t^n)^{-1})$$

and

$$(D_t^n)^{-1} = 1 - \int_0^t (D_s^n)^{-1} \langle \alpha_s^n, d\tilde{B}_s \rangle \Rightarrow P_t^n f(x)$$

$$= P_t^{Q_n} f(x) - E_{Q_n}(f(X_t^n) \int_0^t (D_s^n)^{-1} \langle \alpha_s^n, d\tilde{B}_s \rangle).$$

Itô's formula applied to the function  $(s, y) \mapsto P_{t-s}^{Q_n}f(y)$  and to the semimartingale  $(s, X_s^n)$  gives

$$P_{t-s}^{Q_n}f(X_s^n) = P_t^{Q_n}f(x) + \int_0^s \langle \nabla P_{t-u}^{Q_n}f(X_u^n), d\tilde{B}_u \rangle \Rightarrow f(X_t^n)$$

$$= P_t^{Q_n}f(x) + \int_0^t \langle \nabla P_{t-u}^{Q_n}f(X_u^n), d\tilde{B}_u \rangle.$$

$(D_t^n)^{-1}$  is a martingale too, so

$$P_t^n f(x) = P_t^{Q_n} f(x) - \int_0^t E_{Q_n} \left( (D_s^n)^{-1} \langle \alpha_s^n, \nabla P_{t-s}^{Q_n} f(X_s^n) \rangle \right) ds.$$

Moreover, we have

$$\langle \alpha_s^n, \nabla P_{t-s}^{Q_n} f(X_s^n) \rangle = \frac{1}{2} \lambda_n(X_s^n) \langle X_s^n, \nabla P_{t-s}^C f(X_s^n) \rangle.$$

Since  $f$  is even and log-concave,  $P_{t-s}^C f$  is consequently even and log-concave too.

$$\langle X_s^n, \nabla P_{t-s}^C f(X_s^n) \rangle \leq 0.$$

Therefore,

$$P_t^n f(x) \geq P_t^C f(x).$$

Let  $T_n = \inf\{t, \|X_t\| \geq n\}$ . If  $t \leq T_n$  then  $X_t^n = X_t$ . Moreover,  $\lim_{n \rightarrow +\infty} T_n = +\infty$ . Moreover,  $f$  is bounded because it is an even, log-concave function. So we can use the dominated convergence theorem to obtain

$$\lim_{n \rightarrow +\infty} E(f(X_t^n)) = E(f(X_t)).$$

This achieves the proof.  $\square$

We deduce from Theorem 6 a new proof of Theorem 2:

Let us take notations of Theorem 2. Using the change of variable in the beginning of the proof of this theorem, we notice it is enough to show

$$\int f(\|y\|^2)g(C^{1/2}y) d\mu_C(y) \geq \int f(\|y\|^2) d\mu_C(y) \int g(C^{1/2}y) d\mu_C(y).$$

It is sufficient to assume that  $f$  is a strictly positive, decreasing and bounded function and that  $f'$  is also bounded. Therefore  $f(\|y\|^2) = \exp(-u(\|y\|^2))$  with  $u' \geq 0$  and  $u'$  bounded.

Define  $G(y) = u(\|y\|^2)$ .

Using Theorem 6, we obtain

$$P_t(g(C^{1/2}\cdot))(x) \geq P_t^C(g(C^{1/2}\cdot))(x).$$

Define a new measure  $\nu$  by

$$d\nu(x) = \frac{\exp(-\frac{1}{2}\langle Cx, x \rangle - G(x))}{\int \exp(-\frac{1}{2}\langle Cx, x \rangle - G(x)) dx} dx.$$

Consequently,

$$\int P_t(g(C^{1/2}\cdot))(x) d\nu(x) \geq \int P_t^C(g(C^{1/2}\cdot))(x) d\nu(x).$$

The infinitesimal generator of  $X$  is equal to

$$L = \frac{1}{2}(\Delta - \langle Cx + \nabla G, \nabla \rangle).$$

It is easy to show that  $L$  is symmetric with respect to  $\nu$  and  $P_t$  is symmetric with respect to  $\nu$ ; that is to say,

$$\int HP_t K d\nu = \int KP_t H d\nu.$$

Therefore

$$\int g(C^{1/2}x) d\nu(x) \geq \int P_t^C(g(C^{1/2}\cdot))(x) d\nu(x).$$

Let  $t$  go to infinity:

$$\begin{aligned} P_t^C(g(C^{1/2}\cdot))(x) &\rightarrow \int g(C^{1/2}y) d\mu_C(y) \Rightarrow \int g(C^{1/2}x)f(\|x\|^2) d\mu_C(x) \\ &\geq \int g(C^{1/2}x) d\mu_C(x) \int f(\|x\|^2) d\mu_C(x). \end{aligned}$$

**5. Moment inequalities.**

**THEOREM 7.** *Let  $\nu$  be a centered Gaussian measure on  $\mathbb{R}^n$ . Let  $f$  be an even, log-concave function and  $A$  a nonnegative symmetric matrix. Then*

$$\forall \alpha > 0, \quad \int |\langle Ax, x \rangle|^\alpha \frac{f d\nu}{\int f d\nu} \leq \int |\langle Ax, x \rangle|^\alpha d\nu.$$

**PROOF.** Let  $C_t = \{x, f(x) \geq t\}$ ,  $C_t$  is convex and symmetric. We have  $f(x) = \int_0^{+\infty} 1_{C_t}(x) dt$ . So it is enough to prove

$$\int |\langle Ax, x \rangle|^\alpha \frac{1_{C_t} d\nu}{\int 1_{C_t} d\nu} \leq \int |\langle Ax, x \rangle|^\alpha d\nu.$$

Define a new measure  $\tilde{\nu}$  by

$$d\tilde{\nu} = \frac{1_{C_t} d\nu}{\int 1_{C_t} d\nu}.$$

We obtain

$$\begin{aligned} \int |\langle Ax, x \rangle|^\alpha d\tilde{\nu} &= \int_0^{+\infty} \tilde{\nu}\left(|\langle Ax, x \rangle|^\alpha \geq u\right) du \\ &= \int_0^{+\infty} \frac{\nu\left(|\langle Ax, x \rangle| \geq u^{1/\alpha}, C_t\right)}{\nu(C_t)} du \\ &\leq \int_0^{+\infty} \nu\left(|\langle Ax, x \rangle| \geq u^{1/\alpha}\right) du \\ &\leq \int |\langle Ax, x \rangle|^\alpha d\nu. \end{aligned}$$

If we choose  $\langle Ax, x \rangle = \langle h, x \rangle^2$  with  $h \in \mathbb{R}^n$ , this theorem generalizes some of the results of Theorem 5.1 of [2]. We obtain in this way a comparison of the moments of a measure with an even, log-concave density with the moments of the Gaussian measure.  $\square$

**REFERENCES**

[1] BAKRY, D. and MICHEL, D. (1990). Sur les inégalités F.K.G. *Séminaire de Probabilités XXVI. Lecture Notes in Math.* **1526** 170–188. Springer, Berlin.  
 [2] BRASCAMP, H. J. and LIEB, E. H. (1976). On extensions of Brunn–Minkowski and Prékopa–Leindler theorems, including inequalities for log-concave functions, and with an application to the diffusion equation. *J. Funct. Anal.* **22** 366–389.

- [3] DAS GUPTA, S., EATON, M. L., OLKIN, I., PERLMAN, M., SAVAGE, L. J. and SOBEL, M. (1972). Inequalities and the probability content of convex regions for elliptically contoured distributions. *Proc. Sixth Berkeley Symp. Math. Statist. Probab.* **2** 241–264. Univ. California Press, Berkeley.
- [4] FEYEL, D. and DE LA PRADELLE A. (1994). Opérateurs linéaires gaussiens. *Potential Anal.* **3** 89–105.
- [5] HU, Y. (1997). Itô–Wiener chaos expansion with exact residual and correlation, variance inequalities. *J. Theory Probab.* **10** 835–848.
- [6] KHATRI, C. G. (1967). On certain inequalities for normal distributions and their applications to simultaneous confidence bounds. *Ann. Math. Statist.* **38** 1853–1867.
- [7] PITT, L. D. (1977). A Gaussian correlation inequality for symmetric convex sets. *Ann. Probab.* **5** 470–474.
- [8] PRÉKOPA, A. (1973). On logarithmic concave measure and functions. *Acta Sci. Math.* **34** 335–343.
- [9] SCHECHTMAN, G., SCHLUMPRECHT, T. and ZINN, J. (1998). On the Gaussian measure of the intersection of symmetric convex sets. *Ann. Probab.* **26** 346–357.
- [10] ŠIDÁK, Z. (1967). Rectangular confidence regions for the means of multivariate normal distributions. *J. Amer. Statist. Assoc.* **62** 626–633.
- [11] SUGITA, H. (1992). Various topologies in the Wiener space and Lévy stochastic area. *Probab. Theory Related Fields* **91** 283–296.

UNIVERSITÉ D'EVRY, ÉQUIPE D'ANALYSE  
ET DE PROBABILITÉS  
BD DES COQUIBUS  
91025 EVRY CEDEX  
FRANCE  
E-MAIL: harge@lami.univ-evry.fr