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THE ADJOINT PROCESS OF KILLED REFLECTED BROWNIAN MOTION IN A CONE AND APPLICATIONS

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Let X_t be reflected Brownian motion (RBM) in a cone with radially homogeneous reflection, killed upon reaching the vertex of the cone. We determine the adjoint process and use it to find the Martin boundary of the killed RBM together with all the corresponding positive harmonic functions. Then we can identify and prove uniqueness (up to positive scalar multiples) of the invariant measure for killed RBM and RBM without killing. Along the way, we prove the strong Feller property of the resolvent of RBM (no killing).

1. Introduction. Denote by S^{d-1} ($d \geq 3$) the unit sphere in $\mathbb{R}^d$. Let $\Omega \subseteq S^{d-1}$ be a domain such that $S^{d-1}\backslash\overline{\Omega}$ is nonempty and the boundary $\partial\Omega$ of Ω in S^{d-1} is C^∞. Suppose $G = \{r\omega\colon r > 0, \omega \in \Omega\}$ is an open cone with closure $\overline{G}$ and boundary ∂G. Consider a d-dimensional vector field $\mathbf{v}$ on $\partial G\backslash\{0\}$ that is C^∞ with $\mathbf{v}\cdot\mathbf{n} = 1$ for the unit inward normal $\mathbf{n}$ to $\partial G\backslash\{0\}$. We call $\mathbf{v}$ the *reflection field* and throughout this article we take $\mathbf{v}$ to be radially homogeneous:

$$\mathbf{v}(r\omega) = \mathbf{v}(\omega), \qquad r > 0,\ \omega \in \partial\Omega. \tag{1.1}$$

Kwon and Williams (1991) have completely answered the question of existence and uniqueness of reflected Brownian motion (RBM) in G with radially homogeneous reflection $\mathbf{v}$ at ∂G. One of many interesting properties is the possibility of the process hitting the vertex of the cone with positive probability. In this article we determine the adjoint process of killed RBM. Here we mean that RBM is killed upon first reaching the vertex of the cone. It turns out the adjoint is more or less another killed RBM with drift of order $1/r$ and the reflection field is obtained by reflecting the original field across the normal. See Theorem 4.1 below. Under certain circumstances the adjoint is a conditioned RBM; see Remark 3.3 below.

In the case $d = 2$ (a wedge with constant reflection on each side) it is possible to show the adjoint of killed RBM is another killed RBM where the reflection field is the original field reflected across the normal. The point is that no additional drift appears in the adjoint process as it does for higher dimensions.

The determination of the adjoint process is nontrivial because the state space is unbounded, the vertex is a singularity of the state space and the reflection field is singular at the vertex. There is an old paper of Nagasawa (1961)

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on adjoints of processes with reflection, where the state space is bounded with smooth boundary and the reflection is normal. He also provides some sufficient conditions for a candidate process to be the adjoint process in the case of smooth oblique reflection. While not directly applicable, we make use of these results together with deep results of Taira (1988) connecting pseudodifferential operators to processes with reflection.

One feature in our proof of existence of the adjoint process is the novel use of Bessel processes. The rest of our results are applications. First we compute the Martin boundary of killed RBM and identify the corresponding minimal harmonic functions. It turns out the cone of positive functions harmonic for killed RBM is two-dimensional. We obtain analogous results for the adjoint process. Our computation relies on the Martin boundary theory of Kunita and Watanabe (1965). In order to apply their theory, we prove that the resolvent of killed RBM has the strong Feller property in the transient case. As a corollary we prove (with one exceptional case) that the resolvent of RBM itself (no killing) is also strong Feller. This result is new and of interest in its own right. See Theorems 5.6 and 5.7 below. We are unable to handle the case where the RBM is not transient and does not hit 0.

In our next application we prove existence and uniqueness (up to positive scalar multiples) of invariant measures for RBM and killed RBM. We also identify the invariant measures. Williams (1985) and DeBlassie (1994) have proved the analogs of those results for RBM in a wedge with constant reflection at the sides. Our extension is nontrivial. The case in which RBM is killed more or less follows from our identification of the adjoint process and the Martin boundary theory for the adjoint process. When no killing occurs and the process is recurrent, matters are much more difficult. In the two-dimensional case, Williams (1985) gives a highly nontrivial argument that relies heavily on the dimension being two. Unfortunately, it is not clear how to extend this argument to the higher dimensional case. Intuitively, one expects the invariant measure to have a density (with respect to Lebesgue measure) and this density ought to be harmonic for the adjoint process. Using our determination of the minimal positive harmonic functions for the adjoint process, we can restrict attention to two possibilities. One of the candidates is the invariant density for the killed process so we can throw it out, and we are done. The major sticking point is showing that the density for the invariant measure is harmonic for the adjoint process. We overcome this problem by first showing the density is excessive for the adjoint process. Then we can use the Martin representation theorem to show that it is actually harmonic.

The paper is organized as follows. In Section 2 we introduce notation and collect known properties of RBM. We define our candidate adjoint process in Section 3 and show that it exists uniquely as the solution of a submartingale problem. In Section 4 we verify that this candidate is indeed the adjoint process. We also present the adjoint process explicitly for the case of a "circular cone." The hypotheses required by the Kunita–Watanabe theory of the Martin boundary are verified in Section 5, along with the strong Feller property of the resolvent of RBM. The Martin boundary and corresponding minimal harmonic functions for killed RBM are identified in Section 6. In Sections 7 and 8 we identify the invariant measures of killed RBM and RBM (and prove uniqueness up to positive scalar multiples). Finally, in the Appendix we collect facts about Bessel processes that we use in the paper.

2. Preliminaries. Throughout this article we will assume that the radially homogeneous reflection field has the form

$$\mathbf{v} = v_r \mathbf{e_r} + \mathbf{q} + \mathbf{n} \quad \text{on } \partial G\backslash\{0\},$$

where $v_r \in C^\infty(\partial G\backslash\{0\})$, $\mathbf{q} \in C^\infty(\partial G\backslash\{0\})$, $\mathbf{e_r}$ is the radial unit vector in $\mathbb{R}^d$ and $\mathbf{n}$ is the inward unit normal to $\partial G\backslash\{0\}$. By radial homogeneity, $\mathbf{q}(r\omega) = q(\omega)$ and $v_r(r\omega) = v_r(\omega)$.

Let $D \subseteq \mathbb{R}^d$ be a domain such that either $0 \notin \overline{D}$ and ∂D is C^∞ or $0 \in \partial D$ and $\partial D\backslash\{0\}$ is C^∞. Define $\Omega_D = C([0,\infty), \overline{D})$, where $\overline{D}$ is the closure of D in $\mathbb{R}^d$ and denote the coordinate mapping $\omega \in \Omega_D \to \omega(t)$ by $X_t(\omega)$. Let

$$\mathscr{M}_t = \sigma(X_s\colon\ 0 \le s \le t),$$
$$\mathscr{M} = \sigma(X_s\colon\ s \ge 0).$$

Suppose $\mathscr{A}$ and $\mathscr{L}$ are second- and first-order differential operators on $C^2(\overline{D})$ and $C^1(\partial D\backslash\{0\})$, respectively. A probability measure $\mathscr{P}_x$ on $(\Omega_D, \mathscr{M})$ *solves the* $(\mathscr{A}, \mathscr{L})$*-submartingale problem on* Ω_D, starting from $x \in \overline{D}$, if

$$\mathscr{P}_x(X_0 = x) = 1; \tag{2.1}$$

for each $f \in C_b^2(\overline{D}\backslash\{0\})$ that is constant on a neighborhood of 0 with $\mathscr{L}f \ge 0$ on $\partial D\backslash\{0\}$,

$$f(X_t) - \int_0^t (\mathscr{A}f)(X_s)\,ds \tag{2.2}$$

is a $\mathscr{P}_x$-submartingale;

$$E^{\mathscr{P}_x}\left[\int_0^\infty I_{\{0\}}(X_s)\,ds\right] = 0. \tag{2.3}$$

Note that in the case when $0 \notin \overline{D}$, we take the constancy requirement in (2.2) to be vacuous; also, in this case (2.3) is automatic. If $D = G$, $\mathscr{A} = \frac{1}{2}\Delta$ and $\mathscr{L} = \mathbf{v}\cdot\nabla$, then we call the process X under $\mathscr{P}_x$ *reflected Brownian motion* (RBM) *with reflection field* $\mathbf{v}$.

When $0 \in \partial D$, we will often need to consider the *absorbed process*, defined as follows. Let

$$\tau_0 = \tau_0(\omega) := \inf\{t \ge 0\colon\ \omega(t) = 0\}.$$

A probability measure $\mathscr{P}_x$ on $(\Omega_D, \mathscr{M})$ solves the $(\mathscr{A}, \mathscr{L})$*-submartingale problem* (starting at x) *with absorption* if

$$\mathscr{P}_x(X_0 = x) = 1; \tag{2.4}$$

for each $f \in C_b^2(\overline{D})$ constant on a neighborhood of 0 with $\mathscr{L} f \geq 0$ on $\partial D \backslash \{0\}$,

$$f(X(t \wedge \tau_0)) - \int_0^{t \wedge \tau_0} (\mathscr{A} f)(X_s)\, ds \tag{2.5}$$

is a $\mathscr{P}_x$-submartingale and

$$\mathscr{P}_x(X_t = 0 \text{ for } t \geq \tau_0) = 1. \tag{2.6}$$

We call X under $\mathscr{P}_x$ the *absorbed process*.

In the case of RBM, Kwon and Williams (1991) have obtained the following results.

THEOREM 2.1. (a) *There is a parameter α such that RBM exists uniquely if $\alpha < 2$. If $\alpha \geq 2$ then RBM does not exist and the absorbed process exists uniquely.*

(b) *If $0 < \alpha < 2$, then RBM hits the vertex in finite time almost surely. If $\alpha \leq 0$, then starting away from the vertex, RBM almost surely does not hit the vertex in finite time.*

In the terminology of Kwon (1992), α is called, the *coefficient of obliqueness*. Kwon and Williams (1991) also show existence of the following functions. For each α there is a function $\psi_\alpha \in C^2(\overline{\Omega})$ such that $\psi_\alpha > 0$ on $\overline{\Omega}$ for $\alpha \neq 0$, and the function $\Phi \in C^2(\overline{G} \backslash \{0\})$ defined for $r > 0$ and $\omega \in \overline{\Omega}$ by

$$\Phi(r\omega) = \begin{cases} r^\alpha \psi_\alpha(\omega), & \alpha \neq 0, \\ \ln r + \psi_\alpha(\omega), & \alpha = 0 \end{cases} \tag{2.7}$$

satisfies

$$\Delta \Phi = 0 \quad \text{in } \overline{G} \backslash \{0\}, \tag{2.8}$$

$$\mathbf{v} \cdot \nabla \Phi = 0 \quad \text{on } \partial G \backslash \{0\}. \tag{2.9}$$

Define

$$\Psi = \begin{cases} \Phi, & \alpha > 0, \\ e^\Phi, & \alpha = 0, \\ \Phi^{-1}, & \alpha < 0 \end{cases} \tag{2.10}$$

on $\overline{G} \backslash \{0\}$ and $\Psi(0) = 0$. Then Ψ is continuous on $\overline{G}$, $\Psi > 0$ on $\overline{G} \backslash \{0\}$, $\Psi \in C^2(\overline{G} \backslash \{0\})$ and $\mathbf{v} \cdot \nabla \Psi = 0$ on $\partial G \backslash \{0\}$. Since $\Psi(r\omega) = r^\beta h(\omega)$ where

$$\beta = \begin{cases} |\alpha|, & \alpha \neq 0, \\ 1, & \alpha = 0 \end{cases}$$

and $h > 0$ on $\overline{\Omega}$, we see Ψ can be used to measure distance to the vertex.

One can argue much like Williams (1985) and obtain the next theorem.

THEOREM 2.2. *If $\alpha < 0$, RBM is transient to ∞, and if $0 \leq \alpha < 2$, RBM is recurrent.*

The next result is from DeBlassie, Hobson, Housworth and Toby (1995).

THEOREM 2.3. *Define*

$$D = \begin{cases} \Phi^{1/\alpha}, & \text{if } \alpha \neq 0, \\ e^{\Phi}, & \text{if } \alpha = 0 \end{cases}$$

on $\overline{G}\setminus\{0\}$ *and set* $D(0) = 0$. *If* $\alpha < 2$, X *is RBM and*

$$\eta_t = \int_0^t I(X_s \neq 0)|\nabla D(X_s)|^2\, ds,$$

then the process $D(X(\eta_t^{-1}))$ *is a Bessel process with parameter* $2 - \alpha$.

If $\alpha \geq 2$ *and* X *is the absorbed process, then* $D(X(\eta_t^{-1}))$, $t \leq \eta(\tau_0)$ *is a Bessel process with parameter* $2 - \alpha$, *absorbed at* 0 *at the first hitting time* $\eta(\tau_0-)$ *of* $\{0\}$ *by the Bessel process.*

Moreover, for some positive constants c_1 *and* c_2,

$$\begin{aligned} &c_1|x| \leq D(x) \leq c_2|x|, \\ &\quad c_1 \leq \inf|\nabla D|^2 < \sup|\nabla D|^2 \leq c_2, \\ &c_1 t \leq \eta_t \leq c_2 t \end{aligned}$$

and a similar inequality holds for η_t^{-1}.

COROLLARY 2.4. *Let* $\alpha < 2$ *and suppose* Q_x *is the law of RBM starting from* $x \in \overline{G}$. *Denote its resolvent by* R_λ. *Let* Y_t *be a Bessel process with parameter* $\gamma = 2 - \alpha$ *and* E_y *expectation associated with* $Y_0 = y$. *Then for some positive constants* c_1, c_2, c_3, c_4, *for each* $\lambda \geq 0$ *and function* f *on* $[0, \infty)$,

$$\begin{aligned} c_1 E_y&\left[\int_0^\infty \exp(-c_2\lambda t) f(Y_t)\, dt\right] \\ &\leq R_\lambda(f \circ D)(x) \\ &\leq c_3 E_y\left[\int_0^\infty \exp(-c_4\lambda t) f(Y_t)\, dt\right], \qquad y = D(x), \end{aligned}$$

where D *is from Theorem* 2.3. *The constants are independent of* λ *and* f.

PROOF. By Theorem 2.3,

$$\begin{aligned} R_\lambda(f \circ D)(x) &= E^{Q_x}\left[\int_0^\infty e^{-\lambda t} f \circ D(X_t)\, dt\right] \\ &= E^{Q_x}\Bigg[\int_0^\infty \exp(-\lambda \eta_u^{-1}) f \circ D(X(\eta_u^{-1})) \\ &\qquad\qquad \times |\nabla D(X(\eta_u^{-1}))|^{-2} I(X(\eta_u^{-1}) \neq 0)\, du\Bigg] \\ &\leq c_3 E_y\left[\int_0^\infty \exp(-\lambda c_4 u) f(Y_u)\, du\right]. \end{aligned}$$

A similar argument gives the lower inequality. □

3. The adjoint of killed RBM: definition. In this section we prove existence and uniqueness of a process which, upon being killed at the vertex of G, is shown in Section 4 to be the adjoint of killed RBM. By killed RBM we mean the absorbed process killed upon first reaching the vertex. Our proof starts out mimicking the proof of Kwon and Williams (1991) for absorbed RBM. They use results of Lions and Sznitman (1984) to get existence and uniqueness up to the first exit time of larger and larger compact subsets of $\overline{G}\backslash\{0\}$. The next part of the proof is the hardest. One needs to show, more or less, that the exit point of the sets converges to 0. This is a subtle and difficult point. While we could use a proof like that of Kwon and Williams, we introduce a new proof using Bessel processes that is a bit shorter. Define

$$L^0 f = \frac{\partial f}{\partial n} - \operatorname{div}_{\partial G}(f[v_r \mathbf{e_r} + \mathbf{q}]), \tag{3.1}$$

where $\operatorname{div}_{\partial G}$ is the divergence on the manifold $\partial G\backslash\{0\}$.

REMARK. Recall $\Omega = G \cap S^{d-1}$. The inward unit normal to $\partial G\backslash\{0\}$ on $\partial\Omega$ is also the inward unit normal to $\partial\Omega$ in S^{d-1}. Hence we will abuse the notation $\mathbf{n}$, using it as a vector in $\mathbb{R}^n$ or in the tangent space to S^{d-1}. From context the meaning will be clear.

LEMMA 3.1. *There is a unique positive function* $\psi^0 \in C^2(\overline{\Omega})$ *such that*

$$\Delta_{S^{d-1}}\psi^0 = 0 \quad \text{in } \Omega, \tag{3.2}$$

$$(\mathbf{n} - \mathbf{q})\nabla_{S^{d-1}}\psi^0 - [\operatorname{div}_{\partial\Omega}\mathbf{q}]\psi^0 = 0 \quad \text{on} \partial\Omega, \tag{3.3}$$

$$\int_\Omega \psi^0 \, d\Theta = 1, \tag{3.4}$$

where $\Delta_{S^{d-1}}$ *is the Laplace–Beltrami operator on* S^{d-1}, $\nabla_{S^{d-1}}$ *is the tangential gradient on* S^{d-1}, $\operatorname{div}_{\partial\Omega}$ *is the divergence operator on the manifold* $\partial\Omega$ *and* $d\Theta$ *is surface measure on* S^{d-1}.

In particular, the function defined for $r > 0$ *and* $\omega \in \overline{\Omega}$ *by*

$$\Phi^0(r\omega) = r^{2-d}\psi^0(\omega)$$

is a solution in $C^2(\overline{G}\backslash\{0\})$ *of*

$$\Delta\Phi^0 = 0 \quad \text{in } G, \tag{3.5}$$

$$L^0\Phi^0 = 0 \quad \text{on } \partial G\backslash\{0\}. \tag{3.6}$$

PROOF. The function ψ^0 satisfying (3.2), (3.3) is from Lemma 2.5 in Kwon and Williams (1991). That Φ^0 satisfies (3.5) is routine to check. In spherical coordinates $(r, \omega) \in (0, \infty) \times S^{d-1}$,

$$\nabla_{\partial G} = \mathbf{e_r}\frac{\partial}{\partial r} + \frac{1}{r}\nabla_{\partial\Omega},$$

where $\nabla_{\partial\Omega}$ is the gradient on the manifold $\partial\Omega$, and

$$\begin{aligned}\operatorname{div}_{\partial G}[v_r \mathbf{e_r} + \mathbf{q}] &= r^{2-d}\frac{\partial}{\partial r}[r^{d-2}v_r] + \frac{1}{r}\operatorname{div}_{\partial\Omega}\mathbf{q}\\ &= \frac{d-2}{r}v_r + \frac{1}{r}\operatorname{div}_{\partial\Omega}\mathbf{q}\end{aligned}$$

[cf. Helgason (1962), pages 386, 387]. We always use the Riemannian structure inherited from $\mathbb{R}^d$. It is now routine to check the condition (3.6) using that

$$\operatorname{div}_{\partial G}[f[v_r\mathbf{e_r} + \mathbf{q}]] = (v_r\mathbf{e_r} + \mathbf{q})\nabla_{\partial G} f + f\ \operatorname{div}_{\partial G}[v_r\mathbf{e_r} + \mathbf{q}]. \qquad \square$$

Now we can define the operator and boundary operator associated with the adjoint. Define

$$A^* = \tfrac{1}{2}\Delta + (\Phi^0)^{-1}\nabla\Phi^0 \cdot \nabla, \tag{3.7}$$

$$\mathbf{v}^* = -v_r\mathbf{e_r} + \mathbf{n} - \mathbf{q} \tag{3.8}$$

and

$$L^* = \mathbf{v}^* \cdot \nabla. \tag{3.9}$$

Then in spherical coordinates (r, ω),

$$A^* = \frac{1}{2}\left[\frac{\partial^2}{\partial r^2} + \frac{3-d}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}[\mathbf{b}(\omega)\cdot\nabla_{S^{d-1}} + \Delta_{S^{d-1}}]\right], \tag{3.10}$$

where

$$\mathbf{b}(\omega) = [\psi^0(\omega)]^{-1} 2\,\nabla_{S^{d-1}}\psi^0(\omega) \tag{3.11}$$

is bounded on $\overline{\Omega}$. The next theorem gives existence and uniqueness of the absorbed process associated with (A^*, L^*).

THEOREM 3.2. *For each $x \in \overline{G}$ there is a unique probability measure P_x^* on $(\Omega_G, \mathscr{M})$ that solves the (A^*, L^*)-submartingale problem with absorption, starting from x.*

REMARK 3.3. (i) If $L^*\Phi^0 \le 0$ [which is equivalent to $(d-2)v_r + \operatorname{div}_{\partial\Omega}\mathbf{q} \le 0$] then Φ^0 is superharmonic for the $(\frac{1}{2}\Delta, L^*)$-process, which is merely RBM with radially homogeneous reflection field $\mathbf{v}^*$. Thus in this case, the (A^*, L^*) process killed at $\{0\}$ is a conditioning of RBM with reflection $\mathbf{v}^*$, killed at $\{0\}$.

(ii) By uniqueness, P_x^* has the strong Markov property.

(iii) By uniqueness and the form (3.10) of A^*, P_x^* satisfies the following scaling property:

$$P_x^*(A) = P_{rx}^*(r^{-1}\omega(r^2\cdot) \in A), \qquad x \in \overline{G},\ r > 0,\ A \in \mathscr{M}.$$

The proof is much like that of Lemma 2.3 in Kwon and Williams (1991).

To prove the theorem, we need the following lemma.

LEMMA 3.4. *There is a real number* α^* *and a function* $\psi^* \in C^2(\overline{\Omega})$ *with* $\psi^* > 0$ *on* $\overline{\Omega}$ *such that the function* $\Phi^* \in C^2(\overline{G}\backslash\{0\})$ *defined for* $r > 0$ *and* $\omega \in \overline{\Omega}$ *by*

$$\Phi^*(r\omega) = \begin{cases} r^{\alpha^*}\psi^*(\omega), & \alpha^* \neq 0, \\ \ln\ r + \psi^*(\omega), & \alpha^* = 0 \end{cases}$$

satisfies

$$A^*\Phi^* = 0 \quad \textit{in } \overline{G}\backslash\{0\}, \tag{3.12}$$

$$L^*\Phi^* = 0 \quad \textit{on } \partial G\backslash\{0\}. \tag{3.13}$$

Moreover,

$$\alpha^* < 0 \quad \textit{if} \int_{\partial\Omega} v_r \psi^0 \, d\sigma < 2 - d,$$

$$\alpha^* = 0 \quad \textit{if} \int_{\partial\Omega} v_r \psi^0 \, d\sigma = 2 - d$$

and

$$\alpha^* > 0 \quad \textit{if} \int_{\partial\Omega} v_r \psi^0 \, d\sigma > 2 - d,$$

where ψ^0 *is from Lemma* 3.1 *and* $d\sigma$ *is surface measure on* $\partial\Omega$.

PROOF. Because of the form (3.10) of A^*, we can use the argument proving Lemma 2.4 in Kwon and Williams (1991) to obtain the following analogue:

For each $a \in \mathbb{R}$ there is a unique pair $(\tilde{\lambda}(a), \tilde{\psi}_a) \in \mathbb{R} \times C^2(\overline{\Omega})$ such that $\tilde{\psi}_a > 0$ on $\overline{\Omega}$, $\int_\Omega \tilde{\psi}_a \, d\Theta = 1$,

$$\Delta_{S^{d-1}}\tilde{\psi}_a + \mathbf{b} \cdot \nabla_{S^{d-1}}\tilde{\psi}_a + \tilde{\lambda}(a)\tilde{\psi}_a = 0 \quad \text{in } \Omega, \tag{3.14}$$

$$(\mathbf{n} - \mathbf{q}) \cdot \nabla_{S^{d-1}}\tilde{\psi}_a - a v_r \tilde{\psi}_a = 0 \quad \text{on } \partial\Omega. \tag{3.15}$$

The functions $a \to \tilde{\lambda}(a) \in \mathbb{R}$ and $a \to \tilde{\psi}_a \in C^2(\overline{\Omega})$ are real analytic. Also, $\tilde{\lambda}(a)$ is a concave function of a that is bounded above by the first eigenvalue of $\Delta_{S^{d-1}} + \mathbf{b} \cdot \nabla_{S^{d-1}}$ on $\overline{\Omega}$ with Dirichlet boundary conditions.

Then exactly as in the proof of Lemma 2.7 in Kwon and Williams (1991), if $\tilde{\lambda}'(0) \neq 2 - d$ there is a unique $\alpha^* \neq 0$ such that $\tilde{\lambda}(\alpha^*) = \alpha^*(\alpha^* + 2 - d)$ with $\alpha^* > 0$ if $\tilde{\lambda}'(0) > 2 - d$ and $\alpha^* < 0$ if $\tilde{\lambda}'(0) < 2 - d$. For $\tilde{\lambda}'(0) = 2 - d$, set $\alpha^* = 0$. Define $\psi^* = \tilde{\psi}_{\alpha^*}$ to get the desired function ψ^*.

All that remains is the characterization of the sign of α^*. First we show

$$\tilde{\lambda}(a) = \frac{a \int_{\partial\Omega} v_r \tilde{\psi}_a \psi^0 \, d\sigma}{\int_\Omega \tilde{\psi}_a \psi^0 \, d\Theta}, \tag{3.16}$$

where ψ^0 is from Lemma 3.1. Now

$$\begin{aligned}\tilde{\lambda}(a)\int_\Omega \tilde{\psi}_a\psi^0\,d\Theta &= -\int_\Omega \Delta_{S^{d-1}}(\psi^0\tilde{\psi}_a)\,d\Theta \quad \text{[by (3.2) and (3.14)]}\\ &= \int_{\partial\Omega}\frac{\partial}{\partial n}(\psi^0\tilde{\psi}_a)\,d\sigma \quad \text{(divergence theorem)}\\ &= \int_{\partial\Omega}\left[\psi^0\frac{\partial\tilde{\psi}_a}{\partial n}+\tilde{\psi}_a\frac{\partial\psi^0}{\partial n}\right]d\sigma.\end{aligned}$$

By (3.3) and (3.15) the latter is

$$\begin{aligned}&\int_{\partial\Omega}[\psi^0[\mathbf{q}\cdot\nabla_{S^{d-1}}\tilde{\psi}_a + av_r\tilde{\psi}_a]+\tilde{\psi}_a[\mathbf{q}\cdot\nabla_{S^{d-1}}\psi^0+\psi^0\ \operatorname{div}_{\partial\Omega}\mathbf{q}]]\,d\sigma\\ &\quad=\int_{\partial\Omega}[av_r\tilde{\psi}_a\psi^0+\operatorname{div}_{\partial\Omega}(\mathbf{q}\psi^0\tilde{\psi}_a)]d\,\sigma\\ &\quad= a\int_{\partial\Omega}v_r\tilde{\psi}_a\psi^0\,d\sigma\end{aligned}$$

by the divergence theorem on the manifold $\partial\Omega$. Thus we get (3.16). By differentiability properties of $\tilde{\psi}_a$,

$$\tilde{\lambda}'(0)=\frac{\int_{\partial\Omega}v_r\tilde{\psi}_0\psi^0\,d\sigma}{\int_\Omega\tilde{\psi}_0\psi^0\,d\Theta}. \tag{3.17}$$

However, by uniqueness, $(\tilde{\lambda}(0),\tilde{\psi}_0)=\big(0,\big[\int_\Omega d\Theta\big]^{-1}\big)$; hence by (3.4),

$$\tilde{\lambda}'(0)=\int_{\partial\Omega}v_r\psi^0\,d\sigma. \tag{3.18}$$

The statement about the sign of α^* follows. □

REMARK. Recall that α is the coefficient of obliqueness for RBM. Kwon and Williams (1991) show

$$\begin{aligned}\alpha>0 &\quad \text{if } -\int_{\partial\Omega}v_r\psi^0\,d\sigma > d-2,\\ \alpha=0 &\quad \text{if } -\int_{\partial\Omega}v_r\psi^0\,d\sigma = d-2,\\ \alpha<0 &\quad \text{if } -\int_{\partial\Omega}v_r\psi^0\,d\sigma < d-2\end{aligned}$$

(see their Lemmas 2.7 and 2.6). Hence we see $\alpha=0$ iff $\alpha^*=0$, $\alpha^*<0$ iff $\alpha>0$ and $\alpha^*>0$ iff $\alpha<0$.

PROOF OF THEOREM 3.2. The proof of uniqueness is like that in Theorem 2.1 of Kwon and Williams (1991). We concentrate on existence. With α^* and

Φ^* from Lemma 3.4, set

$$\Psi^* = \begin{cases} \Phi^*, & \alpha^* > 0, \\ e^{\Phi^*}, & \alpha^* = 0, \\ (\Phi^*)^{-1}, & \alpha^* < 0 \end{cases}$$

on $\overline{G}\backslash\{0\}$ and set $\Psi^*(0) = 0$. Then $\Psi^* \in C(\overline{G}) \cap C^2(\overline{G}\backslash\{0\})$ and $\Psi^* > 0$ on $\overline{G}\backslash\{0\}$. Also, $\Psi^*(r\omega)$ is of the form $r^\beta h(\omega)$ where $\beta = |\alpha^*|$ if $\alpha^* \neq 0$, $\beta = 1$ if $\alpha^* = 0$ and $h > 0$ on $\overline{\Omega}$. We use Ψ^* to measure distance to the origin. Define

$$\sigma_m(\omega) = \inf\{t \geq 0\colon\ \Psi^*(\omega(t)) \notin (m^{-1}, m)\}.$$

By work of Lions and Sznitman (1984), on some filtered space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$, there is a unique pair (X^*, ℓ^*) such that for some explosion time e,

$$X^*(t) = x + B(t) + \int_0^t \frac{\nabla\Phi^0}{\Phi^0}(X^*(s))\, ds + \int_0^t \mathbf{v}^*(X^*(s))\, d\ell^*(s), \qquad t < e,$$

where $X^*(t) \in \overline{G}\backslash\{0\}$ for $t < e$, $\ell^*(\cdot)$ is a nondecreasing real valued process with

$$\ell^*(t) = \int_0^t I_{\partial G\backslash\{0\}}(X^*(s))\, d\ell^*(s), \qquad t < e$$

and $B(\cdot)$ is two-dimensional Brownian motion. Moreover,

$$e = \lim_{m\to\infty} \sigma_m(X^*).$$

This is precisely the argument given in Kwon and Williams (1991). As there, existence follows once we show

(3.19) $$\lim_{t\uparrow e} X^*(t) = 0 \quad \text{on } \{e < \infty\}.$$

Here we depart from their approach and prove (3.19) in a new way.

Define

(3.20) $$D^* = \begin{cases} (\Phi^*)^{1/\alpha^*}, & \alpha^* \neq 0, \\ \exp(\Phi^*), & \alpha^* = 0 \end{cases}$$

on $\overline{G}\backslash\{0\}$ and $D^*(0) = 0$. Then

(3.21) $$|\nabla D^*(r\omega)|^2 = \tilde{h}(\omega),$$

where

(3.22) $$\tilde{h}(\omega) = \begin{cases} (\alpha^*)^{-2}[(\alpha^*)^2(\psi^*)^2 + |\nabla_{S^{d-1}}\psi^*|^2](\psi^*)^{2/\alpha^*-2}, & \alpha^* \neq 0, \\ [1 + |\nabla_{S^{d-1}}\psi^*|^2]e^{2\psi^*}, & \alpha^* = 0 \end{cases}$$

and

(3.23) $$0 < \inf\, \tilde{h} \leq \sup\, \tilde{h} < \infty.$$

Also,

$$\mathbf{v}^*\nabla D^* = 0 \quad \text{on } \partial G\backslash\{0\},$$
$$A^* D^* = \frac{1-\alpha^*}{2D^*}\tilde{h} \quad \text{on } G.$$

Define

$$\tilde{\eta}_t = \int_0^t \tilde{h}(X_s^*/|X_s^*|)\,ds, \qquad t < e. \tag{3.24}$$

Then $t \in [0, e) \to \tilde{\eta}_t$ is continuous and strictly increasing with a continuous strictly increasing inverse $\tilde{\eta}_t^{-1}$: $[0, \eta_{e^-}) \to [0, e)$. By Itô's formula, for $t \le \sigma_m$,

$$dD^*(X_t^*) = dM_t + (A^* D)(X_t^*)\,dt,$$

where

$$M_t = \int_0^t \nabla D^*(X_s^*)\,dB_s, \qquad t < e.$$

Since for $t < e$,

$$\begin{aligned}\langle M, M\rangle_t &= \int_0^t |\nabla D^*(X_s^*)|^2\,ds\\ &= \int_0^t \tilde{h}(X_s^*/|X_s^*|)\,ds\\ &= \tilde{\eta}_t,\end{aligned}$$

it follows for some one-dimensional Brownian motion $\overline{B}(t)$, for $t \le \tilde{\eta}(\sigma_m)$,

$$\begin{aligned}D^*(X^*(\tilde{\eta}_t^{-1})) &= D^*(x) + \overline{B}_t + \int_0^{\tilde{\eta}^{-1}(t)} (A^* D^*)(X_s^*)\,ds\\ &= D^*(x) + \overline{B}_t + \int_0^{\tilde{\eta}^{-1}(t)} \frac{1-\alpha^*}{2D^*(X_s^*)}\tilde{h}(X_s^*/|X_s^*|)\,ds\\ &= D^*(x) + \overline{B}_t + \int_0^t \frac{1-\alpha^*}{2D^*(X^*(\tilde{\eta}^{-1}(u)))}\,du.\end{aligned} \tag{3.25}$$

Hence for $t < \tilde{\eta}(e^-)$,

$$Y_t = D^*(X^*(\tilde{\eta}_t^{-1})) \tag{3.26}$$

is a Bessel process with parameter $2 - \alpha^*$.

By definition of σ_m and that $D^* = (\Psi^*)^{1/|\alpha^*|}$ for $\alpha^* \neq 0$ and $D^* = \Psi^*$ for $\alpha^* = 0$,

$$\tilde{\eta}(\sigma_m) = \begin{cases} \inf\{t \ge 0\colon Y_t \notin (m^{-1/|\alpha^*|}, m^{1/|\alpha^*|})\}, & \alpha^* \neq 0,\\ \inf\{t \ge 0\colon Y_t \notin (m^{-1}, m)\}, & \alpha^* = 0.\end{cases}$$

Thus $\tilde{\eta}(\sigma_m)$ converges to the first time T_0 that Y_t hits 0, whence $\tilde{\eta}(e^-)$ is the latter. We end up with

$$\tilde{\eta}(e^-) = \lim_{m\to\infty} \tilde{\eta}(\sigma_m) = T_0, \tag{3.27}$$

$$\lim_{u\to\tilde{\eta}(e^-)} Y_u = 0 \quad \text{on } \{\tilde{\eta}(e^-) < \infty\} \tag{3.28}$$

and this is equivalent to (3.19). □

We collect some facts to use below from the last proof.

LEMMA 3.5. *Let D^* and α^* be from* (3.20) *and Lemma* 3.4, *respectively. For $\tilde{h} = |\nabla D^*|^2$ from* (3.21) *define*

$$\tilde{\eta}_t = \int_0^t \tilde{h}(X_s/|X_s|)\, ds, \qquad t < \tau_0(X).$$

Then for any $x \in \overline{G}\backslash\{0\}$, under P_x^*,

$$Y_t = D^*(X(\tilde{\eta}_t^{-1})), \qquad t \le \tilde{\eta}(\tau_0(X)-)$$

is a Bessel process with parameter $2 - \alpha^*$ up to time $T_0 = \inf\{t \ge 0\colon Y_t = 0\}$ and

$$\tilde{\eta}(\tau_0(X)-) = T_0.$$

Moreover, for some positive constants c_1 and c_2,

$$c_1 t \le \tilde{\eta}_t \le c_2 t, \qquad t \le \tau_0(X). \tag{3.29}$$

NOTE. Our convention henceforth is to let E_y denote expectation associated with $Y_0 = y$.

COROLLARY 3.6. (a) *Let D^* and α^* be from* (3.20) *and Lemma* 3.4, *respectively. Let Y_t be a Bessel process with parameter $\gamma = 2 - \alpha^*$* (*absorbed at* 0 *if $\gamma \le 0$*) *and E_y expectation associated with $Y_0 = y$. Then for some positive constants c_1, c_2, c_3 and c_4, for each $\lambda \ge 0$ and nonnegative f on $[0,\infty)$, with $y = D^*(x)$,*

$$c_1 E_y\left[\int_0^{T_0} \exp(-c_2\lambda t) f(Y_t)\, dt\right] \le E^{P_x^*}\left[\int_0^{\tau_0} e^{-\lambda t} f\circ D^*(X_t)\, dt\right] \le c_3 E_y\left[\int_0^{T_0} \exp(-c_4\lambda t) f(Y_t)\, dt\right].$$

The constants $c_1 - c_4$ are independent of d and f.

(b) *The same conclusion holds with P_x^*, α^* and D^* replaced by P_x, α and D, respectively, D from Theorem* 2.3.

(c) $P_x^*(\tau_0 < \infty) = 0$ *if $\alpha^* \le 0$ and $= 1$ if $\alpha^* > 0$.*

The proof is exactly like the proof of Corollary 2.4, using Lemma 3.5 in place of Theorem 2.3.

4. The adjoint of killed RBM: Proof. In this section we prove the process defined in Section 3 when killed upon reaching $\{0\}$ is the adjoint of killed RBM (again, killed upon reaching $\{0\}$). The plan of action is as follows. By the deep results of Taira (1988) on diffusions in smooth bounded domains with reflection, a theorem of Nagasawa (1961) yields the adjoint of RBM in such a domain. We apply this to smooth domains D increasing to G such that the reflection field on $\partial D \cap \partial G$ coincides with $\mathbf{v}$. We kill the RBM and its adjoint at $\partial D \backslash (\partial D \cap \partial G)$ and show they are adjoint to one another. One then wants to let $D \uparrow G$.

The big problem is the fact that the killed RBM and its adjoint on D are adjoint to one another with respect to a measure that depends on D, and the adjoint itself depends on D. Our way around this is to use conditioned diffusions.

For each $x \in \overline{G}$, let P_x and P_x^* denote the unique solutions of the $\left(\frac{1}{2}\Delta, \mathbf{v} \cdot \nabla\right)$ and (A^*, L^*)-submartingale problems with absorption, respectively, starting from x. The existence and uniqueness of the former is from Kwon and Williams (1991), the latter from Theorem 3.2. Define the corresponding semigroups for the killed processes as follows. For $x \in \overline{G} \backslash \{0\}$,

$$T_t f(x) = E^{P_x}[f(X_t) I_{\tau_0 > t}], \tag{4.1}$$

$$T_t^* f(x) = E^{P_x^*}[f(X_t) I_{\tau_0 > t}], \tag{4.2}$$

where f is measurable with compact support in $\overline{G} \backslash \{0\}$. Note the state space of the killed processes is $\overline{G} \backslash \{0\}$.

THEOREM 4.1. *The* $\left(\frac{1}{2}\Delta, \mathbf{v} \cdot \nabla\right)$- *and* (A^*, L^*)*-processes killed upon reaching* $\{0\}$ *are adjoint to one another with respect to the measure* $\Phi^0(x)\,dx$ *on* $\overline{G} \backslash \{0\}$, *where* Φ^0 *is from Lemma* 3.1. *More precisely, for bounded measurable* f *and* g *with compact support in* $\overline{G} \backslash \{0\}$,

$$\int g(T_t f) \Phi^0 \, dx = \int f(T_t^* g) \Phi^0 \, dx.$$

The proof is given through a sequence of lemmas. Consider any set D with smooth boundary and compact closure $\overline{D}$ such that

$$D \subseteq \overline{D} \subseteq \overline{G} \backslash \{0\}, \qquad \partial D \cap \partial G \neq 0.$$

One should view D as a truncation of G by two concentric balls centered at 0, then "smoothed" at the "edges." Set

$$\tau_D := \inf\{t \geq 0\colon\ X_t \in \overline{\partial D \backslash \partial D \cap \partial G}\}$$

and consider the killed semigroups

$$T_t^D f(x) = E^{P_x}[f(X_t) I_{\tau_D > t}],$$
$$T_t^{*D} f(x) = E^{P_x^*}[f(X_t) I_{\tau_D > t}].$$

Clearly, once we show that

$$\int_D g(T_t^D f)\Phi^0\,dx = \int_D f(T_t^{*D} g)\Phi^0\,dx \tag{4.3a}$$

for f and g continuous with compact support in $\overline{D}\backslash(\overline{G}\cap\partial D)$, upon letting $D \uparrow G$ we get the conclusion of Theorem 4.1.

Let $\mathbf{v_D}$ be a smooth nontangential vector field on ∂D that coincides with $\mathbf{v}$ on $\partial D \cap \partial G$. Write

$$\mathbf{v}_D = \mathbf{n}_D + \mathbf{q}_D,$$

where $\mathbf{n}_D$ is the unit inward normal to ∂D and $\mathbf{q}_D$ is a vector field in the tangent bundle to ∂D. Writing $\partial/\partial n$ for the normal derivative $\mathbf{n}_D \cdot \nabla$, $\nabla_{\partial D}$ for the tangential gradient operator on ∂D and $\operatorname{div}_{\partial D}$ for the divergence operator on the manifold ∂D, on $C^1(\partial D)$ set

$$\begin{aligned} L_D &= \mathbf{v}_D \cdot \nabla \\ &= \frac{\partial}{\partial n} + \mathbf{q}_D \cdot \nabla_{\partial D} \end{aligned} \tag{4.3b}$$

and

$$L_D^0 = \frac{\partial}{\partial n} - \mathbf{q}_D \cdot \nabla_{\partial D} - \operatorname{div}_{\partial D}\mathbf{q}_D. \tag{4.4}$$

The next lemma can be proved like Lemma 2.5 in Kwon and Williams (1991).

LEMMA 4.2. *There is a function* $\varphi_D \in C^2\overline{(D)}$ *such that* $\varphi_D > 0$ *on* $\overline{D}$, $\int_D \varphi_D\,dx = 1$ *and*

$$\Delta\varphi_D = 0 \quad \text{in } D, \tag{4.5}$$

$$L_D^0\varphi_D = 0 \quad \text{on } \partial D. \tag{4.6}$$

Next define

$$A_D^* = \frac{1}{2}\Delta + \frac{\nabla\varphi_D}{\varphi_D}\cdot\nabla \quad \text{on } C^2(\overline{D}),$$

$$L_D^* = \frac{\partial}{\partial n} - \mathbf{q}_D \cdot \nabla_{\partial D} \quad \text{on } C^1(\partial D).$$

Let Q_x and Q_x^* be the unique solutions to the $\left(\frac{1}{2}\Delta, L_D\right)$- and (A_D^*, L_D^*)-submartingale problems, respectively, on $(\Omega_D, \mathscr{M})$, starting from $x \in \overline{D}$. Existence and uniqueness follows from the work of Stroock and Varadhan (1971). Denote by J_t and J_t^* the corresponding semigroups. For $x \in \overline{D}$,

$$J_t f(x) = E^{Q_x} f(X_t),$$

$$J_t^* f(x) = E^{Q_x^*} f(X_t),$$

where f is bounded and measurable on $\overline{D}$.

LEMMA 4.3. *The semigroups J_t and J_t^* are Feller* [*i.e., map $C(\overline{D})$ into $C(\overline{D})$*] *and are adjoint to one another with respect to the measure $\varphi_D\,dx$,*

$$\int_D g(J_t f)\varphi_D\,dx = \int_D f(J_t^* g)\varphi_D\,dx$$

for all bounded measurable f and g on $\overline{D}$.

PROOF. By Theorem 10.1.1 in Taira (1988), there exist Feller semigroups $\mathscr{J}_t$ and $\mathscr{J}_t^*$ on $C(\overline{D})$ whose infinitesimal generators $\mathscr{U}$ and $\mathscr{U}^*$, respectively, are characterized as follows.

(i) The domains of $\mathscr{U}$ and $\mathscr{U}^*$ are, respectively,

$$\mathscr{D}(\mathscr{U}) = \{u \in C(\overline{D}):\ \Delta u \in C(\overline{D}),\ L_D u = 0 \text{ on } \partial D\},$$

$$\mathscr{D}(\mathscr{U}^*) = \{u \in C(\overline{D}):\ A_D^* u \in C(\overline{D}),\ L_D^* u = 0 \text{ on } \partial D\}.$$

(ii) $\mathscr{U} = \frac{1}{2}\Delta$ on $\mathscr{D}(\mathscr{U})$ and $\mathscr{U}^* = A_D^*$ on $\mathscr{D}(\mathscr{U}^*)$.

Here Δu, $L_D u$, $A_D^* u$ and L_D^* are defined in the weak sense described by Taira. Moreover, the generator $\mathscr{U}$ coincides with the minimal closed extension in $C(\overline{D})$ of the restriction of $\frac{1}{2}\Delta$ to $\{u \in C^2(\overline{D}):\ L_D u = 0 \text{ on } \partial D\}$. An analogous statement holds for $\mathscr{U}^*$. The corresponding resolvents map $C^\infty(\overline{D})$ into itself.

Hence by Section 5 in Nagasawa (1961), $\mathscr{J}_t$ and $\mathscr{J}_t^*$ are adjoint to one another with respect to the measure $\varphi_D\,dx$. To prove the lemma, it suffices to show $J_t = \mathscr{J}_t$ and $J_t^* = \mathscr{J}_t^*$. We verify the former, omitting the similar proof of the latter.

Denote the resolvents of J_t and $\mathscr{J}_t$ by R_α and $\mathscr{R}_\alpha$ respectively. It is enough to check $R_\alpha = \mathscr{R}_\alpha$ on $C^\infty(\overline{D})$. Since $\mathscr{R}_\alpha f \in C^\infty(\overline{D})$ for $f \in C^\infty(\overline{D})$, $\mathscr{U}(\mathscr{R}_\alpha f) = \frac{1}{2}\Delta\mathscr{R}_\alpha f$ and $L_D(\mathscr{R}_\alpha f) = 0$ on ∂D. By the submartingale property,

$$E^{Q_x}[(\mathscr{R}_\alpha f)(X_t)] = \mathscr{R}_\alpha f(x) + E^{Q_x}\left[\int_0^t \tfrac{1}{2}\Delta(\mathscr{R}_\alpha f)(X_s)\,ds\right].$$

Integration of both sides against $e^{-\alpha t}\,dt$ from 0 to ∞ gives

$$\begin{aligned} R_\alpha(\mathscr{R}_\alpha f) &= \frac{1}{\alpha}\mathscr{R}_\alpha f + \frac{1}{\alpha}R_\alpha\left(\frac{1}{2}\Delta(\mathscr{R}_\alpha f)\right) \\ &= \frac{1}{\alpha}\mathscr{R}_\alpha f + \frac{1}{\alpha}R_\alpha(\mathscr{U}(\mathscr{R}_\alpha f)) \\ &= \frac{1}{\alpha}\mathscr{R}_\alpha f + \frac{1}{\alpha}R_\alpha(\alpha\mathscr{R}_\alpha f - f) \\ &= \frac{1}{\alpha}\mathscr{R}_\alpha f + R_\alpha(\mathscr{R}_\alpha f) - \frac{1}{\alpha}R_\alpha f. \end{aligned}$$

Hence $\mathscr{R}_\alpha f = R_\alpha f$ for $f \in C^\infty(\overline{D})$, as desired. □

For typographical convenience, define

$$C_0 := C_0(\overline{D}\backslash\overline{\partial D\backslash\partial D\cap\partial G}).$$

Form the killed semigroups as follows. For $f \in C_0$,

$$J_t^D f(x) = E^{Q_x}[f(X_t)I_{\tau_D>t}],$$
$$J_t^{*D} f(x) = E^{Q_x^*}[f(X_t)I_{\tau_D>t}],$$

where $\tau_D = \inf\{t \geq 0\colon\ X_t \in \overline{\partial D\backslash\partial D\cap\partial G}\}$ as above. Clearly, since the reflection fields agree on $\partial D\cap\partial G$,

$$J_t^D f = T_t^D f, \qquad f \in C_0. \tag{4.7}$$

LEMMA 4.4. *The killed semigroups J_t^D and J_t^{*D} are adjoint to one another with respect to the measure $\varphi_D\,dx$. For all $f, g \in C_0$,*

$$\int_D g(J_t^D f)\varphi_D\,dx = \int_D f(J_t^{*D} g)\varphi_D\,dx. \tag{4.8}$$

PROOF. [After Port and Stone (1978), proof of Theorem 4.3 on page 37.] It is enough to prove (4.8) for $f = I_A$, $g = I_B$ where A and B are sets whose closures are subsets of $\overline{D}\backslash(\overline{\partial D\backslash(\partial D\cap\partial G)})$. Set $I_n = \{2^{-n}mt\colon m = 1,\dots,2^n\}$ and $W = \overline{D}\backslash\overline{\partial D\backslash\partial D\cap\partial G}$. Consider any open set H in $\overline{D}$ containing $\overline{\partial D\backslash\partial D\cap\partial G}$ with $A\cap H = \emptyset = B\cap H$ and let

$$T_H = \inf\{t\geq 0\colon\ X_t \in H\}.$$

Then by the strong Markov property, Lemma 4.3 and that $I_{H^c}I_A = I_A$, $I_{H^c}I_B = I_B$,

$$\begin{aligned}
&\int_B Q_x(T_H > t,\ X_t\in A)\varphi_D(x)\,dx\\
&\quad= \int_B Q_x(X_s\notin H \text{ for } 0 < s\leq t,\ X_t\in A)\varphi_D(x)\,dx\\
&\quad= \lim_{n\to\infty}\int_B Q_x(X_s\notin H \text{ for } s\in I_n,\ X_t\in A)\varphi_D(x)\,dx\\
&\quad= \lim_{n\to\infty}\int_D I_B(x)I_{H^c}(x)[(J_{t/2^n}I_{H^c})^{2^n}I_A](x)\varphi_D(x)\,dx\\
&\quad= \lim_{n\to\infty}\int_D I_A(x)I_{H^c}(x)[(J^*_{t/2^n}I_{H^c})^{2^n}I_B](x)\varphi_D(x)\,dx\\
&\quad= \int_A Q_x^*(T_H > t, X_t\in B)\varphi_D(x)\,dx.
\end{aligned}$$

Here the 2^n power is of the *operator* $J_{t/2^n}I_{H^c}$. Letting $H\downarrow\overline{\partial D\backslash\partial D\cap\partial G}$ we have $\{T_H > t\}\uparrow\{\tau_D > t\}$ a.s. Q_x^* for $x\in A$ and a.s. Q_x for $x\in B$ and consequently,

$$\int_B Q_x(\tau_D > t,\ X_t\in A)\varphi_D(x)\,dx = \int_A Q_x^*(\tau_D > t,\ X_t\in B)\varphi_D(x)\,dx.$$

This is (4.8) with $f = I_A$ and $g = I_B$. □

The key step in our proof of (4.3a) is the following lemma.

LEMMA 4.5. *For any* $f \in C_0$, *for* $\psi = \Phi^0 \varphi_D^{-1}$, *where* Φ^0 *is from Lemma* 3.1 *and* φ_D *is from Lemma* 4.2,

$$\psi^{-1} J_t^{*D}(\psi f) = T_t^{*D} f.$$

Before proving this, we show how it yields (4.3a) (which in turn yields Theorem 4.1). Let f and g be continuous with compact support in $\overline{G}\backslash\{0\}$. By choosing D as above with $\operatorname{supp} f \cup \operatorname{supp} g \subseteq D$, we have $f, g \in C_0$ and

$$\begin{aligned}
\int_D g(T_t^D f)\Phi^0 dx &= \int_D g(J_t^D f)\Phi^0\, dx \quad \text{[by (4.7)]}\\
&= \int_D (g\psi)(J_t^D f)\varphi_D\, dx\\
&= \int_D f(J_t^{*D}(g\psi))\varphi_D\, dx \quad \text{(Lemma 4.4)}\\
&= \int_D (\psi f)(T_t^{*D} g)\varphi_D\, dx \quad \text{(Lemma 4.5)}\\
&= \int_D f(T_t^{*D} g)\Phi^0\, dx,
\end{aligned}$$

which is (4.3a).

PROOF OF LEMMA 4.5. Using (3.5), (3.6), (4.5), (4.6) and that $\mathbf{v} = \mathbf{v_D}$ on $\partial G \cap \partial D$, it is routine to check

$$A_D^* \psi = 0 \quad \text{on } \overline{D}, \tag{4.9}$$

$$L_D^* \psi = 0 \quad \text{on } \partial G \cap \partial D. \tag{4.10}$$

Then by the submartingale property and optional stopping,

$$E^{Q_x^*}[\psi(X(t \wedge \tau_D))] = \psi(x). \tag{4.11}$$

We now show ψ is excessive for the process with state space $S = \overline{D}\backslash \overline{\partial D \backslash \partial G \cap \partial D}$ and transition function J_t^{*D},

$$\begin{aligned}
J_t^{*D}\psi &\le \psi \quad \text{on } S,\\
\lim_{t\to 0} J_t^{*D}\psi &= \psi \quad \text{on } S.
\end{aligned}$$

Indeed, by (4.11), for $x \in S$,

$$\begin{aligned}
J_t^{*D}\psi(x) &= E^{Q_x^*}[\psi(X_t) I_{\tau_0 > t}]\\
&= E^{Q_x^*}[\psi(X_{t\wedge\tau_D}) I_{\tau_D > t}]\\
&\le E^{Q_x^*}[\psi(X_{t\wedge\tau_D})]\\
&= \psi(x).
\end{aligned}$$

By path continuity and dominated convergence, for $x \in S$,

$$\begin{aligned}\lim_{t\to 0} J_t^{*D}\psi(x) &= \lim_{t\to 0} E^{Q_x^*}[\psi(X_t) I_{\tau_D > t}]\\ &= \psi(x) Q_x^*(\tau_D > 0)\\ &= \psi(x).\end{aligned}$$

Hence $\psi^{-1} J_t^{*D}(\psi\cdot)$ can be viewed as the transition function of a conditioned process, the ψ-path of the process with transition function J_t^{*D}. As such, it is continuous up to its lifetime and its law $\mathscr{Q}_x$ on $(\Omega_D, \mathscr{M})$ is characterized by

$$\mathscr{Q}_x(A) = \psi^{-1}(x) E^{Q_x^*}[\psi(X_t) I_{\tau_D > t} I_A], \qquad A \in \mathscr{M}_t \tag{4.12}$$

[see Doob (1984), 2.VI.13].

Let $D_n = \{x \in \overline{D}\colon d(x, \overline{\partial D \backslash \partial D \cap \partial G}) > 1/n\}$. We show the law of $X(\cdot \wedge \tau_{D_n})$ under $\mathscr{Q}_x$ solves the (A^*, L^*)-submartingale problem stopped at τ_{D_n}. The latter problem is to find a probability measure $\mathscr{P}_x$ on $(\Omega_D, \mathscr{M})$ such that

$$\mathscr{P}_x(X_0 = x) = 1; \tag{4.13}$$

for each $f \in C_0^2(\overline{D}_n \backslash \overline{\partial D_n \backslash \partial G \cap \partial D_n})$ with $L^* f \geq 0$ on $\partial G \cap \partial D_n$,

$$f(X(t \wedge \tau_{D_n})) - \int_0^{t\wedge\tau_{D_n}} (A^* f)(X_s)\, ds \tag{4.14}$$

is a $\mathscr{P}_x$-submartingale;

$$\mathscr{P}_x(X_t = X(\tau_{D_n}) \text{ for } t \geq \tau_{D_n}) = 1. \tag{4.15}$$

Since (4.13) and (4.15) are obvious when $\mathscr{P}_x$ is the law of $X(\cdot \wedge \tau_{D_n})$ under $\mathscr{Q}_x$, we concentrate on (4.14). For this, by (4.12), for any $0 \leq s \leq u$, $B \in \mathscr{M}_s$ and $g \in C_0(\overline{D}_n \backslash \overline{\partial D_n \backslash \partial G \cap \partial D_n})$, we have

$$E^{\mathscr{Q}_x}[g(X_u) I_{u < \tau_{D_n}} I_B] = \psi^{-1}(x) E^{Q_x^*}[(\psi g)(X_u) I_{u<\tau_{D_n}} I_B]. \tag{4.16}$$

Then for f as in (4.14), $0 \leq s < t$ and $B \in \mathscr{M}_s$,

$$\begin{aligned}
&E^{\mathscr{Q}_x}\left[f(X(t\wedge\tau_{D_n})) - f(X(s\wedge\tau_{D_n})) - \int_{s\wedge\tau_{D_n}}^{t\wedge\tau_{D_n}} (A^* f)(X_u)\, du\right] I_B\\
&\quad = E^{\mathscr{Q}_x}\left[f(X_t) I_{\tau_{D_n} > t} - f(X_s) I_{\tau_{D_n} > s} - \int_s^t (A^* f)(X_u) I_{\tau_{D_n} > u}\, du\right] I_B\\
&\quad = \psi^{-1}(x) E^{Q_x^*}[(\psi f)(X_t) I_{\tau_{D_n} > t} - (\psi f)(X_s) I_{\tau_{D_n} > s}] I_B\\
&\qquad - \int_s^t \psi^{-1}(x) E^{Q_x^*}[(\psi A^* f)(X_u) I_{\tau_{D_n} > u} I_B]\, du\\
&\quad = \psi^{-1}(x) E^{Q_x^*}\Bigg[(\psi f)(X(t\wedge\tau_{D_n})) - (\psi f)(X(s\wedge\tau_{D_n}))\\
&\qquad\qquad - \int_{s\wedge\tau_{D_n}}^{t\wedge\tau_{D_n}} (\psi A^* f)(X_u)\, du\Bigg] I_B.
\end{aligned} \tag{4.17}$$

Routine computations using (4.9) and (4.10) yield

$$A_D^*(\psi f) = \psi A^* f \quad \text{in } D,$$
$$L_D^*(\psi f) = \psi L_D^* f \quad \text{on } \partial G \cap \partial D_n.$$

Also, since $\mathbf{v}$ and $\mathbf{v_D}$ agree on $\partial G \cap \partial D$,

$$L_D^*(\psi f) = \psi L_D^* f = \psi L^* f \ge 0 \quad \text{on } \partial G \cap \partial D_n.$$

Thus the right-hand side of (4.17) is equal to

$$\psi^{-1}(x) E^{Q_x^*}\biggl[(\psi f)(X(t \wedge \tau_{D_n})) - (\psi f)(X(s \wedge \tau_{D_n})) - \int_{s\wedge\tau_{D_n}}^{t\wedge\tau_{D_n}} (A_D^*(\psi f))(X_u)\,du\biggr] I_B \ge 0,$$

by the submartingale property of Q_x^* and optional stopping. We have proved (4.14) for $\mathscr{P}_x$ being the law of $X(\cdot \wedge \tau_{D_n})$ under $\mathscr{Q}_x$.

On the other hand, P_x^* is the unique solution of the (A^*, L^*)-submartingale problem with absorption. Hence the (A^*, L^*)-submartingale problem stopped at τ_{D_n} has a unique solution, namely the law of $X(\cdot \wedge \tau_{D_n})$ under P_x^*. Hence for any $f \in C_0$, for n so large that $f = 0$ on $\partial D_n \backslash \overline{\partial G \cap \partial D_n}$, we have

$$\begin{aligned} E^{P_x^*}[f(X_t) I_{\tau_{D_n} > t}] &= E^{P_x^*}[f(X_{t\wedge\tau_{D_n}})] \\ &= E^{\mathscr{Q}_x}[f(X_{t\wedge\tau_{D_n}})] \\ &= E^{\mathscr{Q}_x}[f(X_t) I_{\tau_{D_n} > t}] \\ &= \psi^{-1}(x) E^{Q_x^*}[(\psi f)(X_t) I_{\tau_{D_n} > t}] \quad [\text{by (4.12)}]. \end{aligned}$$

Upon letting $n \to \infty$ we get

$$\begin{aligned} T_t^{*D} f(x) &= E^{P_x^*}[f(X_t) I_{\tau_D > t}] \\ &= \psi^{-1}(x) E^{Q_x^*}[(\psi f)(X_t) I_{\tau_D > t}] \\ &= \psi^{-1}(x) J_t^{*D}(\psi f)(x), \end{aligned}$$

precisely the conclusion of Lemma 4.5. □

EXAMPLE 4.6. Let G be the circular cone with angle $\xi \in (0, \pi)$. More precisely,

$$G = \{x \in \mathbb{R}^d \backslash \{0\}: \varphi(x) < \xi\},$$

where $\varphi(x)$ is the colatitude, namely the angle between x and the positive x_d-axis. Assume the reflection field takes the form

$$v = \beta \mathbf{e_r} + \mathbf{n},$$

where $\beta \in \mathbb{R}$. In this situation, it is known [Kwon and Williams (1991)] that

$$\alpha > 0 \text{ iff } \beta < (2-d)(\sin\xi)^{2-d}\int_0^{\xi}(\sin\theta)^{d-2}\,d\theta$$

and

$$\alpha \geq 2 \text{ iff } \xi < \cos^{-1}\left(\frac{1}{\sqrt{d}}\right) \quad \text{and} \quad \beta \leq -\frac{d\sin\xi\cos\xi}{d\cos^2\xi - 1}.$$

It is easy to see ψ^0 from Lemma 3.1 is simply a constant and so Φ^0 there is

$$\Phi^0(r\omega) = cr^{2-d}.$$

Then in spherical coordinates,

$$A^* = \frac{1}{2}\left[\frac{\partial^2}{\partial r^2} + \frac{3-d}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^{d-1}}\right],$$

$$L^* = r^{-1}\mathbf{n}\cdot\nabla_{S^{d-1}} - \beta\frac{\partial}{\partial r}.$$

5. The strong Feller property for the resolvents. Let α be the coefficient of obliqueness associated with the reflection field $\mathbf{v}$. By Theorems 2.1(b) and 2.2, killed RBM is transient when $\alpha \neq 0$ (if $\alpha < 0$ it is transient to ∞ whereas if $\alpha > 0$ it is transient to 0). In this section we prove the strong Feller property for the resolvents of killed RBM and its adjoint when $\alpha \neq 0$. As a corollary, we obtain the strong Feller property for the resolvent of RBM in the case $\alpha \neq 0$—no killing.

The strong Feller property for the resolvents of killed RBM and its adjoint is required in order to apply the Martin boundary theory of Kunita and Watanabe (1965), as done in the next section.

Our proof requires the existence of the 0-resolvent (equivalently, Green's function). In the case $\alpha = 0$, the latter does not exist and our proof breaks down. Hence the restriction $\alpha \neq 0$.

Write the resolvents for killed RBM and its adjoint as

$$R_\lambda^0 f(x) = E^{P_x}\left[\int_0^{\tau_0} e^{-\lambda t} f(X_t)\,dt\right], \tag{5.1}$$

$$R_\lambda^{0*} f(x) = E^{P_x^*}\left[\int_0^{\tau_0} e^{-\lambda t} f(X_t)\,dt\right]. \tag{5.2}$$

The key result to obtaining the strong Feller property is the next theorem.

THEOREM 5.1. *Assume the coefficient of obliqueness* $\alpha \neq 0$. *Then for each bounded measurable function* f *with compact support in* $\overline{G}\backslash\{0\}$ *and* $\lambda \geq 0$, $R_\lambda^0 f$ *and* $R_\lambda^{0*} f$ *are continuous on* $\overline{G}\backslash\{0\}$. □

We will concentrate on $R_\lambda^{0*} f$, the proof for $R_\lambda^0 f$ being similar. First we derive some preliminary results for the proof. Recall from the remark after the proof of Lemma 3.4 that $\alpha \neq 0$ iff $\alpha^* \neq 0$.

LEMMA 5.2. *Let* α^* *be from Lemma* 3.4 *and* D^* *from* (3.20). *If* $\alpha \neq 0$, *then for any bounded nonnegative measurable* f *with compact support in* $\overline{G}\backslash\{0\}$,

$$E^{P_x^*}\left[\int_0^{\tau_0} f(X_t)\,dt\right] \text{ is bounded on } x \in \overline{G}\backslash\{0\},$$

$$E^{P_x^*}\left[\int_0^{\tau_0} f(X_t)\,dt\right] \le c[D^*(x)]^{\alpha^*}, \qquad x \in \overline{G}\backslash\{0\}.$$

PROOF. Given f bounded, nonnegative and measurable with compact support in $\overline{G}\backslash\{0\}$, choose $0 < a < b$ such that $f(x) = 0$ for $D^*(x) \notin (a, b)$. Then for some constant $c > 0$,

$$f(x) \le cI_{(a,b)} \circ D^*(x).$$

Writing $y = D^*(x)$, by Corollary 3.6(a),

$$E^{P_x^*}\left[\int_0^{\tau_0} f(X_t)\,dt\right] \le \tilde{c}E_y\left[\int_0^{T_0} I_{(a,b)}(Y_t)\,dt\right]. \tag{5.3}$$

Using this and Lemma A.1 (a) from the Appendix, we get the desired conclusion. □

REMARK 5.3. By Corollary 3.4(b), the analog of Lemma 5.2 for D, P_x and α is valid.

The proof of the next result is deferred to the end of the section. Here

$$\tau_\varepsilon := \inf\{t \ge 0\colon D^*(X_t) = \varepsilon\}. \tag{5.4}$$

THEOREM 5.4. *For any bounded measurable* f *with compact support in* $\overline{G}\backslash\{0\}$,

$$E^{P_x^*}\left[\int_0^{\tau_\varepsilon} f(X_t)\,dt\right]$$

is continuous on $\overline{G} \cap \{x\colon\ D^*(x) > \varepsilon\}$ *if* $\alpha^* > 0$, *on* $\overline{G} \cap \{x\colon\ 0 < D^*(x) < \varepsilon\}$ *if* $\alpha^* < 0$.

By the resolvent equation and Lemma 5.2,

$$R_\lambda^{0*} = R_0^{0*} - \lambda R_\lambda^{0*} R_0^{0*} \tag{5.5}$$

on the set of bounded measurable functions with compact support in $\overline{G}\backslash\{0\}$.

LEMMA 5.5. *Let H be bounded and continuous on $\overline{G}\backslash\{0\}$. If $\alpha^* > 0$ and* $\lim_{x\to 0} H(x) = 0$, *then for any $\lambda > 0$, $R_\lambda^{0*}H$ is continuous on $\overline{G}\backslash\{0\}$. If $\alpha^* < 0$, then for any $\lambda > 0$, $R_\lambda^{0*}H$ is continuous on $\overline{G}\backslash\{0\}$.*

PROOF. By uniqueness, the law P_x^* of the absorbed process has the Feller property,

$$E^{P_x^*}[F(\omega)] \text{ is continuous as a function of } x \in \overline{G}$$

for any bounded continuous F on Ω_G.

If $\alpha^* > 0$, then for any $\lambda > 0$ and $H \in C_b(\overline{G}\backslash\{0\})$ with $\lim_{x\to 0} H(x) = 0$, the function

$$\omega \in \Omega_G \to \int_0^\infty e^{-\lambda t} H(\omega_t)\, dt$$

is bounded and continuous. Hence, setting $H(0) = 0$ (under P_x^* there is absorption at the origin),

$$\begin{aligned} R_\lambda^{0*}H(x) &= E^{P_x^*}\left[\int_0^{\tau_0} e^{-\lambda t} H(X_t)\, dt\right] \\ &= E^{P_x^*}\left[\int_0^{\infty} e^{-\lambda t} H(X_t)\, dt\right] \end{aligned}$$

is continuous on $\overline{G}\backslash\{0\}$.

If $\alpha^* < 0$, then for any $\lambda > 0$ and $H \in C_b(\overline{G}\backslash\{0\})$,

$$\omega \in \Omega_G \to \int_0^\infty e^{-\lambda t} H(\omega_t)\, dt$$

is continuous on the set $S_0 = \{\omega \in \Omega_G\colon \omega \text{ never hits } \{0\}\}$. But $P_x^*(S_0) = 1$ for $x \neq 0$, so by an extension of the continuous mapping theorem [see Theorem 5.1 in Billingsley (1968)],

$$R_\lambda^{0*}H(x) = E^{P_x^*}\left[\int_0^{\infty} e^{-\lambda t} H(X_t)\, dt\right]$$

is continuous on $\overline{G}\backslash\{0\}$. □

By Lemma 5.2, for any bounded measurable f with compact support in $\overline{G}\backslash\{0\}$, $R_0^{0*}f$ is bounded on $\overline{G}\backslash\{0\}$, and if $\alpha^* > 0$, $R_0^{0*}f(x) \to 0$ as $x \to 0$ in $\overline{G}\backslash\{0\}$. Hence once we show $R_0^{0*}f$ is continuous on $\overline{G}\backslash\{0\}$, by (5.5) and Lemma 5.5, it follows that $R_\lambda^{0*}f$ is continuous on $\overline{G}\backslash\{0\}$ for any $\lambda > 0$. This gives Theorem 5.1.

To this end, by the strong Markov property, for $0 < D^*(x) < \varepsilon$ if $\alpha^* < 0$ and for $\varepsilon < D^*(x)$ if $\alpha^* > 0$,

$$\begin{aligned}\left|R_0^{0*}f(x) - E^{P_x^*}\left[\int_0^{\tau_\varepsilon} f(X_t)\,dt\right]\right| &= \left|E^{P_x^*}\left[\int_{\tau_\varepsilon}^{\tau_0} f(X_t)\,dt\right]\right| \\ &= \left|E^{P_x^*}\left[E^{P^*_{X(\tau_\varepsilon)}}\left[\int_0^{\tau_0} f(X_t)\,dt\right]\right]\right| \\ &\le cE^{P_x^*}[D^*(X(\tau_\varepsilon))^{\alpha^*}] \quad \text{(by Lemma 5.2)} \\ &= c\varepsilon^{\alpha^*} \quad \text{[by (5.4)]}.\end{aligned}$$

It follows that

$$E^{P_x^*}\left[\int_0^{\tau_\varepsilon} f(X_t)\,dt\right] \to R_0^{0*}f(x)$$

uniformly on compact sets in $\overline{G}\backslash\{0\}$ as

$$\begin{aligned}&\varepsilon \to 0 \quad \text{if } \alpha^* > 0,\\ &\varepsilon \to \infty \quad \text{if } \alpha^* < 0.\end{aligned}$$

Then by Theorem 5.4, $R_0^{0*}f$ is continuous on $\overline{G}\backslash\{0\}$, as desired. This completes the proof of Theorem 5.1. □

Now we extend Theorem 5.1 to bounded measurable functions for $\lambda > 0$.

THEOREM 5.6. *Let $\alpha \neq 0$. Then for each $\lambda > 0$ the resolvents R_λ^{0*} and R_λ^0 are strong Feller: Both map bounded measurable functions on $\overline{G}\backslash\{0\}$ into bounded continuous functions on $\overline{G}\backslash\{0\}$.*

PROOF. Again, we concentrate on R_λ^{0*}, the proof for R_λ^0 being similar. Let f be bounded and measurable on $\overline{G}\backslash\{0\}$. Clearly $R_\lambda^{0*}f$ is bounded on $\overline{G}\backslash\{0\}$, so we need only verify continuity. For this, consider any $0 < a < b$ and recall $\alpha \neq 0$ iff $\alpha* \neq 0$.

With D^* from (3.20), by Corollary 3.6 there are positive constants c_3 and c_4 such that

$$\begin{aligned}&\left|R_\lambda^{0*}(f(I_{(a,b)} \circ D^*))(x) - R_\lambda^{0*}f(x)\right| \\ &\quad \le [\sup|f|]E^{P_x^*}\left[\int_0^{\tau_0} e^{-\lambda t}[I_{[0,a]}(D^*(X_t)) + I_{[b,\infty)}(D^*(X_t))]\,dt\right] \\ &\quad \le [\sup|f|]c_3E_y\left[\int_0^{T_0} \exp(-c_4\lambda u)[I_{[0,a]}(Y_u) + I_{[b,\infty)}(Y_u)]\,du\right],\end{aligned}$$

where $y = D^*(x)$. By Lemma A.3 of the Appendix, the latter expectation converges to 0 uniformly in y on compact subsets of $(0,\infty)$ as $a \to 0$ and

$b \to \infty$. In particular,

$$R_\lambda^{0*}\big(f(I_{(a,b)} \circ D^*)\big) \to R_\lambda^{0*} f$$

uniformly on compact subsets of $\overline{G}\backslash\{0\}$ as $a \to 0$ and $b \to \infty$. The support of $fI_{(a,b)} \circ D^*$ is compact, so by Theorem 5.1, $R_\lambda^{0*}(fI_{(a,b)} \circ D^*)$ is continuous on $\overline{G}\backslash\{0\}$. Hence by the uniform convergence above, $R_\lambda^{0*} f$ is continuous on $\overline{G}\backslash\{0\}$ as claimed. □

Finally, we derive the strong Feller property of the resolvent of RBM (no killing).

THEOREM 5.7. *For $\lambda > 0$ let R_λ denote the resolvent for RBM with coefficient of obliqueness $0 \neq \alpha < 2$. For any bounded measurable function f on $\overline{G}$, $R_\lambda f$ is continuous on $\overline{G}$.*

REMARK. Recall by the results of Kwon and Williams (1991) that RBM does not exist if $\alpha \geq 2$.

PROOF. First assume $0 < \alpha < 2$. As before, denote by Q_x the law of RBM starting from $x \in \overline{G}$. Then for any bounded measurable f on $\overline{G}$, by the strong Markov property,

$$\begin{aligned} R_\lambda f(x) &= E^{Q_x}\left[\int_0^\infty e^{-\lambda t} f(X_t)\,dt\right] \\ &= E^{Q_x}\left[\left(\int_0^{\tau_0} + \int_{\tau_0}^\infty\right)(e^{-\lambda t} f(X_t)\,dt)\right] \\ &= E^{P_x}\left[\int_0^{\tau_0} e^{-\lambda t} f(X_t)\,dt\right] + E^{Q_x}\left[\exp(-\lambda\tau_0) E^{Q_0}\left[\int_0^\infty e^{-\lambda t} f(X_t)\,dt\right]\right] \\ &= R_\lambda^0 f(x) + R_\lambda f(0) E^{P_x}[\exp(-\lambda\tau_0)] \\ &= R_\lambda^0 f(x) + R_\lambda f(0)[1 - \lambda R_\lambda^0 1(x)], \end{aligned} \tag{5.6}$$

since

$$\lambda R_\lambda^0 1(x) = \lambda E^{P_x}\left[\int_0^{\tau_0} e^{-\lambda t}\,dt\right] = 1 - E^{P_x}[\exp(-\lambda\tau_0)]. \tag{5.7}$$

Hence by Theorem 5.6, $R_\lambda f$ is continuous on $\overline{G}\backslash\{0\}$. To finish, we need to show continuity at 0. Since $|R_\lambda^0 f| \leq [\sup|f|] R_\lambda^0 1$, by (5.6) it is enough to show that

$$R_\lambda^0 1(x) \to 0 \quad \text{as } x \to 0, \quad x \in \overline{G}\backslash\{0\}.$$

But this is easy. We have for $y = D(x)$, by Corollary 3.6(b),

$$\begin{aligned} R^0_\lambda 1(x) &= E^{P_x}\left[\int_0^{\tau_0} e^{-\lambda t}\,dt\right] \\ &\le c_5 E_y\left[\int_0^{T_0} \exp(-\lambda c_6 u)\,du\right] \\ &\to 0 \quad \text{as } y \to 0 \text{ (hence as } x \to 0), \end{aligned}$$

by Lemma A.4, since the parameter of the Bessel process Y is $\gamma = 2-\alpha \in (0, 2)$.

Next consider $\alpha < 0$. For $x \in \overline{G}\backslash\{0\}$, since $\alpha < 0$, $P_x = Q_x$ and $\tau_0 = \infty$ a.s. Hence $R_\lambda f = R^0_\lambda f$ is continuous on $\overline{G}\backslash\{0\}$ for any bounded measurable f on $\overline{G}$, by Theorem 5.6.

For the moment, assume for any bounded measurable h on $\overline{G}$ with compact support in $\overline{G}\backslash\{0\}$,

$$\lim_{x\to 0} R_\lambda h(x) = R_\lambda h(0). \tag{5.8}$$

Then $R_\lambda h$ is continuous on $\overline{G}$. In particular, for bounded measurable f on $\overline{G}$ and $0 < a < b$, $R_\lambda(f(I_{[a,b]} \circ D))$ is continuous on $\overline{G}$. Moreover, for $y = D(x)$, using Corollary 2.4,

$$\begin{aligned} &\left|R_\lambda f(x) - R_\lambda[f(I_{[a,b]} \circ D)](x)\right| \\ &\qquad \le [\sup|f|]E^{Q_x}\left[\int_0^\infty e^{-\lambda t}[I_{[0,a]} \circ D(X_t) + I_{[b,\infty)} \circ D(X_t)]\,dt\right] \\ &\qquad \le c_7[\sup|f|]E_y\left[\int_0^\infty \exp(-\lambda c_8 t)[I_{[0,a]}(Y_t) + I_{[b,\infty)}(Y_t)]\,dt\right]. \end{aligned}$$

Since $\alpha < 0$, the parameter of Y_t is $\gamma = 2 - \alpha > 2$, so by the last part of Lemma A.3, the latter goes to 0 as $a \to 0$, $b \to \infty$ uniformly for y in sets of the form $[0, A]$. Hence the continuous functions $R_\lambda[f(I_{[a,b]} \circ D)]$ on $\overline{G}$ converge uniformly to $R_\lambda f$ as $a \to 0$, $b \to \infty$ uniformly on compact sets in $\overline{G}$. This gives the conclusion of the theorem when $\alpha < 0$.

All that remains is verification of (5.8). For D as in Theorem 2.3, write

$$\tau'_\delta = \inf\{t \ge 0\colon D(X_t) = \delta\},$$

where $\delta > 0$ is chosen to satisfy

$$\{x\colon D(x) < \delta\} \cap \operatorname{supp} h = \emptyset.$$

Then by the strong Markov property, since h vanishes on $\{x\colon\ D(x) < \delta\}$, for $D(x) < \delta$ we have

$$\begin{aligned} R_\lambda h(x) &= E^{Q_x}\left[\left(\int_0^{\tau'_\delta} + \int_{\tau'_\delta}^\infty\right)(e^{-\lambda t}h(X_t)\,dt)\right] \\ &= 0 + E^{Q_x}\left[\exp(-\lambda\tau'_\delta)E^{Q_{X(\tau'_\delta)}}\left[\int_0^\infty e^{-\lambda t}h(X_t)\,dt\right]\right] \\ &= E^{Q_x}[\exp(-\lambda\tau'_\delta)(R^0_\lambda h)(X_{\tau'_\delta})], \end{aligned}$$

once again using that $Q_x = P_x$ for $x \in \overline{G}\backslash\{0\}$. Now by Theorem 5.6, $R^0_\lambda h$ is continuous on $\overline{G}\backslash\{0\}$, and it is a simple matter to show the set of discontinuities of the functional

$$\omega \in \Omega_G \to \exp\{-\lambda\tau'_\delta(\omega)\}(R^0_\lambda h)(\omega(\tau'_\delta(\omega)))$$

has Q_0 measure 0. By an extension of the continuous mapping theorem [Billingsley (1968), Theorem 5.1], since $Q_x \to Q_0$ in law as $x \to 0$,

$$\begin{aligned} \lim_{x\to 0} R_\lambda h(x) &= \lim_{x\to 0} E^{Q_x}[\exp(-\lambda\tau'_\delta)(R^0_\lambda h)(X_{\tau'_\delta})] \\ &= E^{Q_0}[\exp(-\lambda\tau'_\delta)(R^0_\lambda h)(X_{\tau'_\delta})] \\ &= R_\lambda h(0). \end{aligned}$$

This gives (5.8) and completes the proof of Theorem 5.7. □

We close this section with the promised proof of Theorem 5.4.

Proof of Theorem 5.4. Let f be bounded and measurable with compact support in $\overline{G}\backslash\{0\}$. Write

$$h(x) = E^{P^*_x}\left[\int_0^{\tau_\varepsilon} f(X_t)\,dt\right],$$

where $x \in \overline{G} \cap \{x\colon D^*(x) > \varepsilon\}$ if $\alpha^* > 0$ and $x \in \overline{G} \cap \{x\colon\ 0 < D^*(x) < \varepsilon\}$ if $\alpha^* < 0$. Fix $x_0 \in$ domain (h). It suffices to show h is continuous at x_0. The key observation is for $r > 0$ small enough, say $0 < r < r_0$, and

$$\eta_r = \inf\{t \ge 0\colon |X_t - x_0| \ge r\},$$

by the strong Markov property,

$$h(x) = E^{P^*_x}[h(X(\eta_r))] + E^{P^*_x}\left[\int_0^{\eta_r} f(X_t)\,dt\right], \qquad |x - x_0| < r/4,\ x \in \overline{G}. \tag{5.9}$$

By making r_0 smaller if necessary, the coefficients of A^* on $\overline{G}\cap\{x\colon\ |x-x_0| < r_0/4\}$ are well behaved and bounded. Since A^* is half the Laplacian plus a drift term, by the Cameron–Martin–Girsanov Theorem [see Stroock and Varadhan (1971)], the analog of Lemma 3.3 in Kwon and Williams (1991) is true for P^*_x.

In particular, there is $\kappa > 0$ (independent of r) and $\gamma \in (0, r_0)$ such that for $0 < r \le \gamma$,

$$P_x^*(X(\eta_r) \in A) > \kappa, \qquad x \in \overline{G},\ |x - x_0| \le r/4$$

whenever $A \subseteq \overline{G} \cap \partial B_r(x_0)$ such that $|A| \ge \frac{1}{2}|\overline{G} \cap \partial B_r(x_0)|$. Here $B_r(x_0) = \{x\colon |x - x_0| < r\}$ and $|\cdot|$ is surface measure on $\partial B_r(x_0)$. Then much like the first part of the proof of Theorem 3.2 in Kwon and Williams (1991), for

$$\operatorname{osc}(r) = \sup\{|h(x) - h(y)|\colon\ x, y \in \overline{G} \cap \overline{B_r(x_0)}\}$$

and

$$g(x) = E^{P_x^*}\left[\int_0^{\eta_r} f(X_t)\,dt\right],$$

(5.9) leads to

$$\operatorname{osc}\left(\frac{r}{4}\right) \le \left(1 - \frac{\kappa}{2}\right) \operatorname{osc}(r) + \sup\{|g(x) - g(y)|\colon\ x, y \in \overline{G} \cap \overline{B_{r/4}(x_0)}\}, \qquad 0 < r \le \gamma.$$

Moreover, for $x, y \in \overline{G} \cap \overline{B_{r/4}(x_0)}$,

$$\begin{aligned} |g(x) - g(y)| &\le [\sup|f|][E^{P_x^*}\eta_r + E^{P_y^*}\eta_r] \\ &\le 2[\sup|f|]\sigma(r) \end{aligned}$$

where

$$\sigma(r) = \sup\{E^{P_x^*}\eta_r\colon\ x \in \overline{G} \cap B_{r/4}(x_0)\}$$

is nondecreasing for $r \in (0, \gamma]$. Hence for $C = 2\sup|f|$,

$$\operatorname{osc}\left(\frac{r}{4}\right) \le \left(1 - \frac{\kappa}{2}\right) \operatorname{osc}(r) + C\sigma(r), \qquad 0 < r \le \gamma.$$

By Lemma 8.23 in Gilbarg and Trudinger (1983), for any $0 < \mu < 1$ there are positive constants c_1 and a such that

$$\operatorname{osc}(r) \le c_1\left[\left(\frac{r}{\gamma}\right)^a \operatorname{osc}(\gamma) + \sigma(r^\mu \gamma^{1-\mu})\right], \qquad 0 < r \le \gamma.$$

Continuity of h at x_0 will follow once we show $\sigma(r) \to 0$ as $r \to 0$.

To this end, first assume $x_0 \in G$. By (3.7) we can choose $r_1 > 0$ so small that $\overline{B_{r_1}(x_0)} \subseteq G$ and

$$A^*|x - x_0|^2 \ge \frac{d}{2} \quad \text{if } x \in B_{r_1}(x_0).$$

Then by the submartingale property and optional stopping, for $r < r_1$,

$$E^{P_x^*}[|X(t \wedge \eta_r) - x_0|^2] \ge \frac{d}{2} E^{P_x^*}[\eta_r \wedge t], \qquad x \in B_{r/4}(x_0).$$

Letting $t \to \infty$,

$$r^2 \geq \frac{d}{2} E^{P_x^*}[\eta_r], \qquad x \in B_{r/4}(x_0).$$

Clearly in this case, $\sigma(r) \to 0$ as $r \to 0$.

For $x_0 \in \partial G \backslash \{0\}$, by Lemma 2.7 in Stroock and Varadhan (1971), for some constants C and r_1 both independent of x and r,

$$E^{P_x^*}[\eta_r] \leq Cr, \qquad r < r_1,\ x \in B_{r/4}(x_0).$$

That $\sigma(r) \to 0$ as $r \to 0$ is an immediate consequence. □

6. The Martin boundary for killed RBM. In this section we apply the Martin boundary theory developed in Kunita and Watanabe (1965). Verification of their hypotheses is mostly routine with the exception of the requirement that both R_λ^0 and R_λ^{0*} for $\lambda \geq 0$ map bounded measurable functions with compact support in $\overline{G} \backslash \{0\}$ to continuous functions on $\overline{G} \backslash \{0\}$. However, we have done this in Theorem 5.1. Here is a synopsis of the consequences of the Martin boundary theory relevant to us.

For $\lambda \geq 0$, the resolvents R_λ^0 and R_λ^{0*} have kernels (with respect to the measure $\Phi^0(x)\,dx$, Φ^0 from Lemma 3.1) we denote by $G_\lambda(x, y)$ and $G_\lambda^*(x, y)$, respectively, and

$$G_\lambda(x, y) = G_\lambda^*(y, x).$$

When $\lambda = 0$ we will drop the subscript in G_0 or G_0^*. Fix $g \in C_0^\infty(\overline{G} \backslash \{0\})$ and define the measure

$$\nu(dx) = g(x)\,dx$$

on the Borel sets of $\overline{G} \backslash \{0\}$. Since the function

$$\begin{aligned}(\nu G)(y) &:= \int G(x, y)\nu(dx) \\ &= \int G^*(y, x) g(x)\,dx \\ &= R_0^{0*} g(y)\end{aligned}$$

is positive and continuous on $\overline{G} \backslash \{0\}$ (by Theorem 5.1), ν is a reference measure. The *Martin kernel* is

$$\kappa(x, y) = \frac{G(x, y)}{\nu G(y)}, \qquad x, y \in \overline{G} \backslash \{0\}.$$

The Martin boundary of $\overline{G} \backslash \{0\}$, written $(\overline{G} \backslash \{0\})'$ is characterized by the following four properties [Theorem 3 on page 509 of Kunita and Watanabe (1965)]. Writing $S = \overline{G} \backslash \{0\}$:

1. $S \cup S'$ is a compact metric space.
2. S is dense and open in $S \cup S'$ and its relative topology coincides with its original topology.

3. To each $\eta \in S'$ corresponds an excessive function $\kappa(\cdot, \eta)$ on S, and if $\eta \neq \eta'$, then for some $x \in S$, $\kappa(x, \eta) \neq \kappa(x, \eta')$.
4. If $\eta \in S'$ and $y \to \eta$ in the topology of $S \cup S'$ with $y \in S$, then for each $f \in C_0(S)$,

$$\int f(x)\kappa(x, y)\, dx \to \int f(x)\kappa(x, \eta)\, dx.$$

THEOREM 6.1. *Suppose the coefficient of obliqueness* α *is not* 0. *Then the Martin boundary of* $\overline{G}\backslash\{0\}$ (*with respect to RBM*) *consists of two points* $\{0, \infty\}$.

REMARK. If $\alpha = 0$ the RBM is recurrent and started away from 0, never hits 0. Hence the transience required for the Kunita–Watanabe theory does not hold. It is possible to show the cone of positive harmonic functions in this case is one-dimensional. See Theorem 6.5 below.

PROOF. Consider any $f \in C_0(S)$. We must examine

$$\int f(x)\kappa(x, y)\, dx \tag{6.1}$$

as $y \to 0$ or $y \to \infty$ and show in either case that the limit exists independent of the means of approach. We concentrate on $y \to \infty$ since the case $y \to 0$ is similar (and easier!).

Our main tool is a technique of Bass and Pardoux (1987) which involves the Krein–Rutman theorem. By definition,

$$\begin{aligned} \int f(x)\kappa(x, y)\, dx &= \frac{\int G(x, y) f(x)\, dx}{\nu G(y)} \\ &= \frac{R_0^{0*} f(y)}{R_0^{0*} g(y)}. \end{aligned} \tag{6.2}$$

For any $\gamma > 0$ and $\varepsilon > 0$, define

$$\sigma_\varepsilon = \sigma_\varepsilon(\omega) := \inf\{t \geq 0\colon |\omega_t| = \varepsilon\},$$
$$\mathscr{Q}_\gamma(x, dy) := P_x^*(\gamma^{-1} X(\sigma_\gamma) \in dy,\ \sigma_\gamma < \infty), \qquad x, y \in \overline{\Omega}$$

(recall $\overline{\Omega} = \overline{G} \cap S^{d-1}$). By Remark 3.3, parts (ii) and (iii), exactly as in Proposition 5.3 and formula (5.4) of Bass and Pardoux (1987),

$$\mathscr{Q}_\gamma^n\left(\frac{x}{|x|}, dy\right) = P_x^*((r\gamma^n)^{-1} X(\sigma_{r\gamma^n}) \in dy, \sigma(r\gamma^n) < \infty), \qquad |x| = r. \tag{6.3}$$

Moreover, the proofs of Theorems 3.2 and 3.3 in Kwon and Williams (1991) show $\mathscr{Q}_\gamma\colon\ C(\overline{\Omega}) \to C(\overline{\Omega})$ is compact and strongly positive on $K = \{f \in C(\overline{\Omega})\colon\ f \geq 0 \text{ on } \overline{\Omega}\}$. The only source of concern is the use of their Lemma 3.3 and Theorem 3.1. But these results are essentially "local away from 0" properties of RBM in $\overline{G}\backslash\{0\}$. Since the process associated with (A^*, L^*) (i.e., the coordinate process under P_x^*) is locally RBM with locally bounded drift, by the Cameron–Martin–Girsanov theorem [cf. Stroock and Varadhan (1971)], the results hold in the present context too.

Then we can apply the Krein–Rutman theorem [as in Bass and Pardoux (1987), page 566] to get the following. There is an eigenvalue $\rho(\gamma) = \rho_\gamma \in (0, \infty)$, an associated eigenfunction φ_γ continuous and positive on $\overline{\Omega}$, and a positive linear functional Φ_γ: $C(\overline{\Omega}) \to \mathbb{R}$ such that for each $c > 0$,

$$\sup\Big\{\sup_{\overline{\Omega}} |\rho_\gamma^{-n} \mathscr{Q}_\gamma^n F - \Phi_\gamma(F)\varphi_\gamma|\colon F \in C(\overline{\Omega}), \sup_{\overline{\Omega}} |F| \le C\Big\} \to 0 \tag{6.4}$$

as $n \to \infty$.

By the Riesz representation theorem, there is a finite measure μ_γ on $\overline{\Omega}$ such that

$$\Phi_\gamma(F) = \int F(y)\mu_\gamma(dy), \qquad F \in C(\overline{\Omega}). \tag{6.5}$$

Setting

$$\tilde{\mu}_\gamma = \mu_\gamma(\overline{\Omega})^{-1}\mu_\gamma, \tag{6.6}$$

by compactness of $\overline{\Omega}$, the family of probability measures $\{\tilde{\mu}_\gamma\colon \gamma = 1 - 1/n, n = 2, 3, \ldots\}$ is tight. Let $\tilde{\mu}_{\gamma_n}$ be a convergent subsequence and call the limit μ,

$$\tilde{\mu}_{\gamma_n} \to_{\mathscr{L}} \mu, \quad \text{where } \gamma_n \uparrow 1. \tag{6.7}$$

Choose $M > 0$ so large that

$$\operatorname{supp} f \cup \operatorname{supp} g \subseteq \{x\colon |x| \le M\}.$$

For each $y \in S$, define

$$h(y) = R_0^{0*} f(y), \tag{6.8}$$

$$j(y) = R_0^{0*} g(y). \tag{6.9}$$

By (6.2),

$$\int f(x)\kappa(x, y)\, dx = \frac{h(y)}{j(y)}. \tag{6.10}$$

Consider any sequence of points $y_m \in S$ with $|y_m| \to \infty$ as $m \to \infty$.

Let $\varepsilon > 0$ be given. By continuity of h and j (Theorem 5.1), choose $\delta = \delta(\varepsilon) > 0$ such that

$$|h(x) - h(y)| < \varepsilon \text{ for } |x|, |y| \in [M, 2M], |x - y| < \delta M \text{ and } x, y \in S \tag{6.11}$$

with a similar statement holding for j. Then choose $N(\varepsilon)$ such that

$$\gamma_n \in \left(\frac{1}{2} \vee \frac{1}{1+\delta}, 1\right), \qquad n \ge N(\varepsilon). \tag{6.12}$$

For $n \ge N(\varepsilon)$ and $m \ge 1$ choose integers $p_m = p_m(n)$ such that

$$|y_m| \in (\gamma_n^{-p_m} M, \gamma_n^{-1-p_m} M]. \tag{6.13}$$

Then $p_m \to \infty$ as $m \to \infty$ and for any $y \in \overline{\Omega}$,

$$\big|h(|y_m|\gamma_n^{p_m} y) - h(My)\big| < \varepsilon, \qquad m \ge 1,\ n \ge N(\varepsilon) \tag{6.14}$$

by (6.11)–(6.13). A similar statement holds for j.

Given $|y| \ge \widetilde{M} > M$, by the strong Markov property and choice of M,

$$\begin{aligned} h(y) &= E^{P_y^*}\left[\int_0^{\tau_0} f(X_t)\,dt\right] \\ &= E^{P_y^*}\left[\int_0^{\tau_0} f(X_t)\,dt I_{\sigma(\widetilde{M})<\infty}\right] \\ &= E^{P_y^*}\left[\int_{\sigma(\tilde{M})}^{\tau_0} f(X_t)\,dt I_{\sigma(\tilde{M})<\infty}\right] \\ &= E^{P_y^*}[h(X(\sigma(\widetilde{M})))I_{\sigma(\widetilde{M})<\infty}]. \end{aligned} \tag{6.15}$$

Write

$$\begin{aligned} H_{m,n}(y) &= h(|y_m|\gamma_n^{p_m} y), \\ J_{m,n}(y) &= j(|y_m|\gamma_n^{p_m} y). \end{aligned} \tag{6.16}$$

Then since $\gamma_n^{-p_m} M < |y_m|$, by (6.15) with $\widetilde{M} = |y_m|\gamma_n^{p_m} < |y_m|$,

$$\begin{aligned} h(y_m) &= E^{P_{y_m}^*}[h(X(\sigma(|y_m|\gamma_n^{p_m})))I(\sigma(|y_m|\gamma_n^{p_m}) < \infty)] \\ &= \mathscr{Q}_{\gamma_n}^{p_m} H_{m,n}(y_m/|y_m|) \quad \text{[using (6.3)]}. \end{aligned} \tag{6.17}$$

Similarly,

$$j(y_m) = \mathscr{Q}_{\gamma_n}^{p_m} J_{m,n}(y_m/|y_m|) \tag{6.18}$$

By (6.10) and (6.4) we get

$$\begin{aligned} \int f(x)\kappa(x, y_m)\,dx &= \frac{h(y_m)}{j(y_m)} \\ &= \frac{[\rho(\gamma_n)]^{-p_m}\mathscr{Q}_{\gamma_n}^{p_m} H_{m,n}(y_m/|y_m|)}{[\rho(\gamma_n)]^{-p_m}\mathscr{Q}_{\gamma_n}^{p_m} J_{m,n}(y_m/|y_m|)} \\ &= \frac{\Phi_{\gamma_n}(H_{m,n})\varphi_{\gamma_n}(y_m/|y_m|) + \varepsilon_m}{\Phi_{\gamma_n}(J_{m,n})\varphi_{\gamma_n}(y_m/|y_m|) + \tilde{\varepsilon}_m}, \end{aligned} \tag{6.19}$$

where ε_m and $\tilde{\varepsilon}_m \to 0$ as $m \to \infty$, using that

$$\sup_{n,m,\overline{\Omega}} |H_{m,n}| \le \sup_{\overline{\Omega}} |h| < \infty \quad \text{and} \quad \sup_{n,m,\overline{\Omega}} |I_{m,n}| \le \sup_{\overline{\Omega}} |j| < \infty,$$

using Lemma 5.2.

Since $\inf \varphi_{\gamma_n} > 0$, $\sup_m \Phi_{\gamma_n}(J_{m,n}) < \infty$ and $\sup_m \Phi_{\gamma_n}(H_{m,n}) < \infty$, once we know

$$\inf_m \Phi_{\gamma_n}(J_{m,n}) > 0, \tag{6.20}$$

we can divide the numerator and denominator of (6.19) by $\varphi_{\gamma_n}(y_m/|y_m|)$ and let $m \to \infty$ to get

$$\limsup_{m\to\infty} \int f(x)\kappa(x, y_m)\,dx$$

$$(6.21) \qquad = \limsup_{m\to\infty} \frac{\Phi_{\gamma_n}(H_{m,n})}{\Phi_{\gamma_n}(I_{m,n})}, \qquad n \geq N(\varepsilon),$$

$$= \limsup_{m\to\infty} \frac{\int H_{m,n}(y)\tilde{\mu}_{\gamma_n}(dy)}{\int J_{m,n}(y)\tilde{\mu}_{\gamma_n}(dy)} \quad [\text{by } (6.5), (6.6)]),$$

with a similar statement holding for $\liminf_{m\to\infty}$. To this end, by the support theorem [cf. Kwon and Williams (1991), Theorem 3.1], which holds for us by the Cameron–Martin–Girsanov theorem [Stroock and Varadhan (1971)],

$$\begin{aligned}\inf_{m\geq 1,\, y\in\overline{\Omega}} j(|y_m|\gamma_n^{p_m} y) &\geq \inf\{j(z)\colon M \leq |z| \leq \gamma_n^{-1}M\}\\ &= \inf\{R_0^{0*}g(z)\colon\ M \leq |z| \leq \gamma_n^{-1}M\}\\ &> 0.\end{aligned}$$

Hence by (6.5) and (6.16),

$$\begin{aligned}\inf_m \Phi_{\gamma_n}(J_{m,n}) &= \inf_m \int J_{m,n}(y)\mu_{\gamma_n}(dy)\\ &= \inf_m \int j(|y_m|\gamma_n^{p_m} y)\mu_{\gamma_n}(dy)\\ &> 0,\end{aligned}$$

giving (6.20).

Now consider the right-hand side of (6.21). Writing

$$H(y) = h(My),$$
$$J(y) = y(My),$$

by (6.14) and (6.16),

$$\sup_{y\in\overline{\Omega}} |H_{m,n}(y) - H(y)| < \varepsilon, \qquad m \geq 1,\ n \geq N(\varepsilon).$$

Hence

$$\int H_{m,n}(y)\tilde{\mu}_{\gamma_n}(dy) = \int H(y)\tilde{\mu}_{\gamma_n}(dy) + O(\varepsilon), \qquad m \geq 1,\ n \geq N(\varepsilon),$$

where the constant in the $O(\varepsilon)$ can be chosen independent of $m \geq 1$ and $n \geq N(\varepsilon)$. A similar statement holds with $H_{m,n}$ and H replaced by $J_{m,n}$ and J, respectively.

Using this and (6.7),

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\limsup_{m\to\infty} \frac{\int H_{m,n}(y)\tilde{\mu}_{\gamma_n}(dy)}{\int J_{m,n}(y)\tilde{\mu}_{\gamma_n}(dy)} = \frac{\int H(y)\mu(dy)}{\int J(y)\mu(dy)},$$

with a similar statement holding for $\liminf_{m\to\infty}$. Now the left-hand side of (6.21) and its lim inf analog are independent of ε and $n \geq N(\varepsilon)$, so we end up with

$$\lim_{m\to\infty} \int f(x)\kappa(x, y_m)\,dx = \frac{\int H(y)\mu(dy)}{\int J(y)\mu(dy)}, \tag{6.22}$$

independent of $y_m \to \infty$. Thus the Martin boundary at ∞ consists of a single point, as claimed. □

REMARK. In (6.22) we have shown the limit on the left-hand side exists independent of the choice of $y_m \to \infty$. It appears that the limit depends on the choice of the sequence $\gamma_n \uparrow 1$ (through the measure μ). However, from (6.2), we know $\int f(x)\kappa(x, y_m)\,dx$ does not depend on γ_n. In particular, the *existence* of the limit in (6.22) shows in fact that the limit does not depend on γ_n.

The next theorem is the Martin representation theorem for functions that are *excessive* for killed RBM. A function h on $\overline{G}\backslash\{0\}$ is said to be excessive for killed RBM if it is measurable and nonnegative (allowing the value ∞),

$$E^{P_x}[h(X_t)I_{t<\tau_0}] \leq h(x), \qquad x \in \overline{G}\backslash\{0\},\ t \geq 0,$$

and

$$\lim_{t\to 0} E^{P_x}[h(X_t)I_{t<\tau_0}] = h(x).$$

We say h is *harmonic* for killed RBM if for each bounded open set $D \subseteq \overline{G}\backslash\{0\}$ and

$$\eta_D := \inf\{t \geq 0\colon\ X_t \notin D\},$$

we have

$$h(x) = E^{P_x}[h(X_{\eta_D})I_{\eta_D<\tau_0}], \qquad x \in \overline{G}\backslash\{0\}.$$

Recall $\nu(dx)$ is the measure $g(x)\,dx$ from the beginning of the section and $S = \overline{G}\backslash\{0\}$.

THEOREM 6.2. *For $\alpha \neq 0$, the class of excessive functions u in $L^1(\nu)$ is in one-to-one correspondence with the class of finite* (*Radon*) *measures μ on $S \cup S'$ through the integral formula*

$$u = \int_{S\cup S'} \kappa(\cdot, \eta)\mu(d\eta).$$

The total mass of μ lies on S' if and only if u is harmonic.

PROOF. A nonnegative function u harmonic for killed RBM is said to be *minimal* (or *extreme*) if whenever v is nonnegative and harmonic for killed RBM and $v \leq u$, then $v = cu$ for some constant $c > 0$. Let S_1' be the set of points $\eta \in S'$ such that $\kappa(\cdot, \eta)$ is minimal harmonic and such that $\int \kappa(x, \eta)\nu(dx) = 1$. Since $\nu G(y) < \infty$ for all $y \in S$ [which amounts to $S_r = S$ in Kunita and

Watanabe (1965)] by Proposition 13.1, Theorem 4 on page 513, and the note at the end of Section 11 in that reference, the conclusion of the present theorem is valid for S' replaced by S'_1. Then by Theorem 6.1 and definition of ν, once we show there are two linearly independent nonnegative harmonic functions (for killed RBM), it will follow that $S'_1 = S'$ and the conclusion of the theorem holds. By the submartingale property and optional stopping, 1 and Φ from (2.7) are nonnegative and harmonic for killed RBM. Clearly, they are linearly independent. □

If m is a Radon measure on $\overline{G}\backslash\{0\}$, define

$$R_0^0 m(x) = \int G(x, y) m(dy), \tag{6.23}$$

$$R_0^{0*} m(x) = \int G^*(x, y) m(dy), \tag{6.24}$$

where $G(x, y)$ and $G^*(x, y)$ are the 0-resolvent kernels defined at the beginning of the section. From the definition of the Martin kernel on $(\overline{G}\backslash\{0\}) \times (\overline{G}\backslash\{0\})$ we get the following corollary of Theorem 6.2.

COROLLARY 6.3. *For $\alpha \neq 0$, any excessive function $u \in L^1(\nu)$ can be written in the form*

$$u = c_1 + c_2\Phi + R_0^0 m$$

for nonnegative constants c_1 and c_2 and a Radon measure m on $\overline{G}\backslash\{0\}$. Moreover, c_1, c_2 and m are uniquely determined. Here Φ is from (2.7) *and R_0^0 is the* 0-*resolvent of the killed process.*

REMARK 6.4. Theorem 6.2 and Corollary 6.3 are true for the adjoint of killed RBM, where Φ, α and R_0^0 are replaced by Φ^*, α^* and R_0^{0*}.

The next theorem concerns the case $\alpha = 0$.

THEOREM 6.5. *Let $\alpha = 0$. The cone of positive harmonic functions for killed RBM consists of all positive constants.*

PROOF. Let h be positive and harmonic for killed RBM. Then by the proof of Theorem 3.2 in Kwon and Williams (1991), h is continuous on $\overline{G}\backslash\{0\}$. To get a contradiction, assume h is not constant. Then without loss of generality we can assume for some x and z in $\overline{G}\backslash\{0\}$,

$$0 < \varepsilon = h(z) < h(x) = 1. \tag{6.25}$$

Choose bounded open neighborhoods N and M (in $\overline{G}\backslash\{0\}$) of z and x, respectively, such that $\overline{N} \cap \overline{M} = \emptyset$, $h \leq \varepsilon + (1-\varepsilon)/4$ on $\overline{N}$ and $h \geq 1 - (1-\varepsilon)/4$

on $\overline{M}$. Define the sequence of stopping times

$$T_0 = 0,$$
$$\vdots$$
$$T_{2n+1} = \inf\{t \geq T_{2n}\colon X_t \in \overline{N}\},$$
$$T_{2n+2} = \inf\{t \geq T_{2n+1}\colon X_t \in \overline{M}\}.$$

For a closed set $B \subseteq \overline{G}\backslash\{0\}$, define the hitting time of B by

$$T_B = \inf\{t \geq 0\colon X_t \in B\}.$$

For m sufficiently large, so that $\overline{M} \cup \overline{N} \subseteq B_m(0)$, set

$$G_m = B_m(0)^c.$$

Then by the averaging property,

$$h(z) = E^{P_z}[h(X(T_{\overline{M}} \wedge T_{G_m}))]$$
$$\geq E^{P_z}[h(X(T_{\overline{M}}))I_{T_{\overline{M}} < T_{G_m}}].$$

Letting $m \to \infty$ and using recurrence,

$$h(z) \geq E^{P_z}[h(X(T_{\overline{M}}))].$$

Similarly, $h(x) \geq E^{P_x}[h(X(T_{\overline{N}}))]$. Then by the strong Markov property,

$$Y_n := h(X(T_n))$$

is a nonnegative L^1-bounded P_x-supermartingale, and as such converges almost surely. But clearly by the choice of M and N,

$$\liminf_{n\to\infty} Y_n \leq \varepsilon + \frac{1-\varepsilon}{4} < 1 - \frac{1-\varepsilon}{4} \leq \limsup_{n\to\infty} Y_n.$$

Hence h must be constant. □

7. Invariant measure for killed RBM. A nontrivial σ-finite measure μ on $(\overline{G}\backslash\{0\}$, Borels) is an *invariant measure* for killed RBM if for each nonnegative bounded measurable function h on $\overline{G}\backslash\{0\}$,

$$\int T_t h(x)\mu(dx) = \int h(x)\mu(dx) \quad \text{for all } t \geq 0.$$

Here T_t is the semigroup for killed RBM,

$$T_t h(x) = E^{P_x}[h(X_t)I_{\tau_0 > t}].$$

An invariant measure for killed RBM is *unique* iff its positive scalar multiples are the only invariant measures for killed RBM.

The main result of this section is the next theorem.

THEOREM 7.1. *Killed RBM has a unique invariant measure* μ. *If* $\alpha < 0$ *then* $\mu(dx) = \Phi^*(x)\Phi^0(x)\,dx$, *where* Φ^* *is from Lemma* 3.4 *and* Φ^0 *is from Lemma* 3.1. *If* $\alpha \geq 0$ *then* $\mu(dx) = \Phi^0(x)\,dx$.

Before giving the proof, we establish some preliminary results. As in Section 5, let R_λ^0 and R_λ^{0*} denote the resolvents of killed RBM and its adjoint, respectively. See (5.1) and (5.2). Recall α^* is from Lemma 3.4.

LEMMA 7.2. (a) *If* $\alpha^* > 0$ *then for each* $\lambda > 0$,

$$\lambda R_\lambda^{0*}\Phi^* = \Phi^*$$

$$\lambda R_\lambda^{0*}1 < 1.$$

(b) *If* $\alpha^* < 0$, *then for each* $\lambda > 0$,

$$\lambda R_\lambda^{0*}\Phi^* < \Phi^*$$

$$\lambda R_\lambda^{0*}1 = 1.$$

PROOF. Recall τ_ε from (5.4) and D^* from (3.20). By the submartingale property, (3.12), (3.13), stochastic calculus and optional stopping, for $\varepsilon < D^*(x) < M$,

$$\begin{aligned} &E^{P_x^*}[\exp(-\lambda[t\wedge\tau_\varepsilon\wedge\tau_M)]\Phi^*(X(t\wedge\tau_\varepsilon\wedge\tau_M))] \\ &\quad = \Phi^*(x) - \lambda E^{P_x^*}\left[\int_0^{t\wedge\tau_\varepsilon\wedge\tau_M} e^{-\lambda s}\Phi^*(X_s)\,ds\right]. \end{aligned}$$

Hence upon letting $t\to\infty$ and collecting expectations, using $\Phi^* = (D^*)^{\alpha^*}$,

$$\begin{aligned} \Phi^*(x) &= E^{P_x^*}[\exp(-\lambda\tau_\varepsilon)\Phi^*(X(\tau_\varepsilon))I_{\tau_\varepsilon<\tau_M}] \\ &\quad + E^{P_x^*}[\exp(-\lambda\tau_M)\Phi^*(X_{\tau_M})I_{\tau_M<\tau_\varepsilon}] \\ &\quad + \lambda E^{P_x^*}\left[\int_0^{\tau_\varepsilon\wedge\tau_M} e^{-\lambda s}\Phi^*(X_s)\,ds\right] \\ &= \varepsilon^{\alpha^*}E^{P_x^*}[\exp(-\lambda\tau_\varepsilon)I_{\tau_\varepsilon<\tau_M}] + M^{\alpha^*}E^{P_x^*}[\exp(-\lambda\tau_M)I_{\tau_M<\tau_\varepsilon}] \\ &\quad + \lambda E^{P_x^*}\left[\int_0^{\tau_\varepsilon\wedge\tau_M} e^{-\lambda s}\Phi^*(X_s)\,ds\right]. \end{aligned} \tag{7.1}$$

If $\alpha^* > 0$, let $\varepsilon \to 0$ to get for $0 < D^*(x) < M$,

$$\Phi^*(x) = M^{\alpha^*}E^{P_x^*}[\exp(-\lambda\tau_M)I_{\tau_M<\tau_0}] + \lambda E^{P_x^*}\left[\int_0^{\tau_0\wedge\tau_M} e^{-\lambda s}\Phi^*(X_s)\,ds\right].$$

That $\lambda R_\lambda^{0*}\Phi^* = \Phi^*$ will follow once we show that

$$\lim_{M\to\infty} M^{\alpha^*}E^{P_x^*}[\exp(-\lambda\tau_M)I_{\tau_M<\tau_0}] = 0.$$

By Lemma 3.5, for $y = D^*(x) \in (0, M)$, Y_t a Bessel process (absorbed upon hitting 0) with parameter $\gamma = 2 - \alpha^* < 2$ and $T_M := \inf\{t \geq 0\colon\ Y_t \geq M\}$, for some $c > 0$,

$$\begin{aligned} M^{\alpha^*} E^{P_x^*}[\exp(-\lambda\tau_M) I_{\tau_M<\tau_0}] &\leq M^{2-\gamma} E_y[\exp(-\lambda c T_M) I_{T_M<T_0}] \\ &= M^{2-\gamma} \frac{y^{1-\gamma/2} I_{1-\gamma/2}(\sqrt{2\lambda c}\, y)}{M^{1-\gamma/2} I_{1-\gamma/2}(\sqrt{2\lambda c}\, M)} \quad \text{(by Lemma A.5)} \\ &\to 0 \quad \text{as } M \to \infty \end{aligned}$$

by the asymptotic (A.8).

On the other hand, if $\alpha^* < 0$, then let $M \to \infty$ in (7.1) to get for $\varepsilon < D^*(x)$,

$$\Phi^*(x) = \varepsilon^{\alpha^*} E^{P_x^*}[\exp(-\lambda\tau_\varepsilon) I_{\tau_\varepsilon<\infty}] + \lambda E^{P_x^*}\left[\int_0^{\tau_\varepsilon} e^{-\lambda s}\Phi^*(X_s)\, ds\right].$$

Then $\lambda R_\lambda^{0*}\Phi^* < \Phi^*$ follows once we show that

$$\lim_{\varepsilon\to 0} \varepsilon^{\alpha^*} E^{P_x^*}[\exp(-\lambda\tau_\varepsilon)] > 0.$$

Using Lemma 3.5 and Lemma A.6, for $y = D^*(x) > \varepsilon$ and $\gamma = 2 - \alpha^* > 2$, for some $c > 0$,

$$\begin{aligned} \varepsilon^{\alpha^*} E^{P_x^*}[\exp(-\lambda\tau_\varepsilon)] &\geq \frac{y^{1-\gamma/2} K_{\gamma/2-1}(\sqrt{2\lambda c}\, y)}{\varepsilon^{\gamma/2-1} K_{\gamma/2-1}(\sqrt{2\lambda c}\, \varepsilon)} \\ &\to C(\lambda, \gamma) y^{1-\gamma/2} K_{\gamma/2-1}(\sqrt{2\lambda c}\, y) > 0 \quad \text{as } \varepsilon \to 0 \end{aligned}$$

by the asymptotic (A.7).

Thus we have verified the first halves of (a) and (b). Now for the second halves. We have

$$\begin{aligned} \lambda R_\lambda^{0*} 1(x) &= \lambda E^{P_x^*}\left[\int_0^{\tau_0} e^{-\lambda t} dt\right] \\ &= E^{P_x^*}[1 - \exp(-\lambda\tau_0)] \end{aligned}$$

and the desired conclusions follow immediately from Corollary 3.6 (c).

REMARK 7.3. The analogue Lemma 7.2 holds when α^* and Φ^* are replaced by α and Φ, respectively.

We will need the following version of the resolvent equation below.

PROPOSITION 7.4. *Let $\alpha \neq 0$. For any $\lambda > 0$ and Radon measure m on $\overline{G}\setminus\{0\}$,*

$$R_0^0 m - \lambda R_\lambda^0 R_0^0 m = R_\lambda^0 m \quad \text{on } \overline{G}\setminus\{0\}$$

and the analogue holds for R_0^{0*} and R_λ^{0*} when $\alpha^* \neq 0$.

PROOF. By Theorem 1 on page 496 in Kunita and Watanabe (1965), the following resolvent equation holds for the resolvent kernels (recall $R^0_\lambda h(x) = \int_{\overline{G}\setminus\{0\}} G_\lambda(x, y)h(y)\Phi^0(y)\,dy$). For $\lambda > \beta \geq 0$,

$$\begin{aligned} G_\beta(x, y) &= G_\lambda(x, y) + (\lambda - \beta)\int_{\overline{G}\setminus\{0\}} G_\beta(x, z)G_\lambda(z, y)\Phi^0(z)\,dz \\ &= G_\lambda(x, y) + (\lambda - \beta)\int_{\overline{G}\setminus\{0\}} G_\lambda(x, z)G_\beta(z, y)\Phi^0(z)\,dz. \end{aligned}$$

The proposition follows from this. The details are left to the reader. □

The next proposition describes a scaling property of the 0-resolvent kernel of the adjoint process.

PROPOSITION 7.5. *Let $\alpha^* \neq 0$. For any $\lambda > 0$ and $x, y \in \overline{G}\setminus\{0\}$,*

$$G^*(\lambda x, y) = G^*(x, y/\lambda).$$

PROOF. For any bounded measurable f with compact support in $\overline{G}\setminus\{0\}$,

$$\begin{aligned} \int G^*(\lambda x, y)f(y)\Phi^0(y)\,dy &= R_0^{0*}f(\lambda x) \\ &= E^{P^*_{\lambda x}}\left[\int_0^{\tau_0} f(X_t)\,dt\right] \\ &= E^{P^*_x}\left[\int_0^{\lambda^2\tau_0} f(\lambda X_{t/\lambda^2})\,dt\right] \\ &\qquad \text{(by scaling; see Remark 3.3)} \\ &= \lambda^2 E^{P^*_x}\left[\int_0^{\tau_0} f(\lambda X_u)\,du\right] \\ &= \lambda^2\int G^*(x, y)f(\lambda y)\Phi^0(y)\,dy \\ &= \lambda^2\int G^*\left(x, \frac{z}{\lambda}\right)f(z)\Phi^0\left(\frac{z}{\lambda}\right)\lambda^{-d}\,dz. \end{aligned}$$

The latter is

$$\int G^*\left(x, \frac{z}{\lambda}\right)f(z)\Phi^0(z)\,dz,$$

since $\Phi^0(z/\lambda) = \lambda^{d-2}\Phi^0(z)$ by Lemma 3.1. Hence for each $x \in \overline{G}\setminus\{0\}$ and $\lambda > 0$,

$$G^*(\lambda x, y) = G^*(x, y/\lambda) \quad \text{a.e. (in } y\text{)}.$$

Here a.e. is with respect to Lebesgue measure. The extension to all y follows from a standard argument [cf. top of page 496 in Kunita and Watanabe (1965)].

The final preliminary result we need is the next proposition. Here and in the sequel, "a.e." means a.e. with respect to Lebesgue measure

PROPOSITION 7.6. *Let $\alpha \neq 0$. Suppose f is nonnegative and measurable, and for all $\lambda > 0$,*

$$\lambda R^0_\lambda f \leq f \quad \text{a.e. on } \overline{G}\backslash\{0\}.$$

Then $\lambda R^0_\lambda f$ is increasing in λ and the limit is excessive for killed RBM. The analogue holds for R^{0*}_λ and the adjoint of killed RBM when $\alpha^* \neq 0$.

PROOF. For $\beta \leq \lambda$, by the resolvent equation,

$$\begin{aligned}
\beta R^0_\beta f &= \beta[R^0_\lambda f - (\beta - \lambda) R^0_\beta R^0_\lambda f] \\
&= \lambda R^0_\lambda f + (\beta - \lambda)[R^0_\lambda f - \beta R^0_\beta R^0_\lambda f] \\
&= \lambda R^0_\lambda f + (\beta - \lambda)[R^0_\lambda f - R^0_\lambda(\beta R^0_\beta f)] \\
&= \lambda R^0_\lambda f + (\beta - \lambda) R^0_\lambda[f - \beta R^0_\beta f] \\
&\leq \lambda R^0_\lambda f,
\end{aligned}$$

since $f - \beta R^0_\beta f \geq 0$ a.e., by hypothesis.

Now we prove Theorem 7.1. By the Remark after the proof of Lemma 3.4, we can replace $\alpha < 0$ by $\alpha^* > 0$ and $\alpha \geq 0$ by $\alpha^* \leq 0$. First consider existence. For convenience, on $\overline{G}\backslash\{0\}$, define

$$F = \begin{cases} 1, & \text{if } \alpha^* \leq 0, \\ \Phi^*, & \text{if } \alpha^* > 0 \end{cases} \tag{7.2}$$

and define $F(0) = 0$. If $\alpha^* = 0$ then by Corollary 3.6(c), $\lambda R^{0*}_\lambda 1 = 1$ on $\overline{G}\backslash\{0\}$. Then combined with Lemma 7.2 we have for any α^*,

$$\lambda R^{0*}_\lambda F = F \text{ on } \overline{G}\backslash\{0\}. \tag{7.3}$$

To show that

$$\mu(dx) = F(x)\Phi^0(x)\,dx$$

is an invariant measure for killed RBM, it is enough to check for each $\lambda > 0$ and bounded measurable h with compact support in $\overline{G}\backslash\{0\}$,

$$\int \lambda R^0_\lambda h(x)\mu(dx) = \int h\mu(dx).$$

However, by Theorem 4.1 and (7.3),

$$\begin{aligned}\int \lambda R^0_\lambda h(x)\mu(dx) &= \int \lambda R^0_\lambda h(x)F(x)\Phi^0(x)\,dx\\ &= \int \lambda R^{0*}_\lambda F(x)h(x)\Phi^0(x)\,dx\\ &= \int F(x)h(x)\Phi^0(x)\,dx\\ &= \int h(x)\mu(dx),\end{aligned}$$

as desired.

Now we prove uniqueness. Let $\tilde{\mu}$ be any invariant measure for killed RBM. It is a simple matter to check that $\tilde{\mu}$ is absolutely continuous with respect to Lebesgue measure on $\overline{G}\backslash\{0\}$; hence we can write

$$\tilde{\mu}(dx) = \varphi(x)\Phi^0(x)\,dx$$

for some nonnegative measurable function φ.

LEMMA 7.7. *It is no loss to assume φ is excessive for the adjoint process of killed RBM and for each $\lambda > 0$,*

$$\lambda R^{0*}_\lambda \varphi = \varphi \quad \textit{a.e. on } \overline{G}\backslash\{0\}.$$

PROOF. For any bounded measurable f with compact support in $\overline{G}\backslash\{0\}$, by Theorem 4.1,

$$\begin{aligned}\int f\varphi\Phi^0\,dx &= \int \lambda[R^0_\lambda f]\varphi\Phi^0\,dx\\ &= \int \lambda[R^{0*}_\lambda \varphi]f\Phi^0\,dx.\end{aligned}$$

Consequently,

$$\lambda R^{0*}_\lambda \varphi = \varphi \quad \text{a.e. on } \overline{G}\backslash\{0\} \tag{7.4}$$

Define

$$\varphi_1 = \lim_{\lambda\to\infty} \lambda R^{0*}_\lambda \varphi,$$

an excessive function for the adjoint of killed RBM by Proposition 7.6. In particular,

$$\lambda R^{0*}_\lambda \varphi_1 \le \varphi_1 \quad \text{a.e.}$$

Then a.e. on $\overline{G}\backslash\{0\}$,

$$\begin{aligned}\varphi &= \lambda R^{0*}_\lambda \varphi\\ &= \lambda R^{0*}_\lambda \varphi_1 \quad \text{since } \varphi = \varphi_1 \text{ a.e.}\\ &\le \varphi_1\\ &= \varphi.\end{aligned}$$

Hence

$$\lambda R_\lambda^{0*}\varphi_1 = \varphi_1 \quad \text{a.e. on } \overline{G}\backslash\{0\}.$$

To get the lemma, just replace φ by φ_1. □

First assume $\alpha^* \neq 0$. By Remark 6.4 there are nonnegative constants c_1 and c_2 together with a Radon measure m on $\overline{G}\backslash\{0\}$ such that

$$\varphi = c_1 + c_2\Phi^* + R_0^{0*}m. \tag{7.5}$$

Then a.e. on $\overline{G}\backslash\{0\}$ we have

$$\varphi = \lambda R_\lambda^{0*}\varphi = c_1\lambda R_\lambda^{0*}1 + c_2\lambda R_\lambda^{0*}\Phi^* + \lambda R_\lambda^{0*}R_0^{0*}m. \tag{7.6}$$

Equating (7.5) and (7.6) and collecting

$$\begin{aligned} c_1[1-\lambda R_\lambda^{0*}1] + c_2[\Phi^* - \lambda R_\lambda^{0*}\Phi^*] &= \lambda R_\lambda^{0*}R_0^{0*}m - R_0^{0*}m \\ &= -R_\lambda^{0*}m \quad \text{by Proposition 7.4.} \end{aligned} \tag{7.7}$$

By Lemma 7.2, both terms on the left-hand side of (7.7) are nonnegative, whereas the right-hand side is nonpositive. Therefore all the terms must be 0 (a.e. on $\overline{G}\backslash\{0\}$):

$$c_1[1-\lambda R_\lambda^{0*}1] = 0, \tag{7.8}$$

$$c_2[\Phi^* - \lambda R_\lambda^{0*}\Phi^*] = 0, \tag{7.9}$$

$$R_\lambda^{0*}m = 0. \tag{7.10}$$

By (7.10) and Proposition 7.4 on page 500 in Kunita and Watanabe (1965), $m = 0$. Thus

$$\varphi = c_1 + c_2\Phi^*.$$

If $\alpha^* < 0$, then by Lemma 7.2 and (7.9), $c_2 = 0$, yielding $\varphi = c_1$ as desired. If $\alpha^* > 0$ then by Lemma 7.2 and (7.8), $c_1 = 0$, giving $\varphi = c_2\Phi^*$ as needed. Thus the invariant measure for killed RBM is unique when $\alpha^* \neq 0$.

If $\alpha^* = 0$ then by the remark after the proof of Lemma 3.4, $\alpha = 0$ and in this case killed RBM is recurrent. By Azema, Kaplan-Duflo and Revuz (1967), there is a unique invariant measure and we are done. □

8. Invariant measure for RBM. An invariant measure for RBM is defined analogously to that for killed RBM, where now the measure is σ-finite on $\overline{G}$ equipped with its Borel sets.

THEOREM 8.1. *Assume $\alpha < 2$. Then RBM has a unique invariant measure μ. Moreover,*

$$\mu(A) = \begin{cases} \displaystyle\int_{A\backslash\{0\}} \Phi^*\Phi^0\,dx, & \alpha \neq 0, \\ \displaystyle\int_{A\backslash\{0\}} \Phi^0\,dx, & \alpha = 0, \end{cases}$$

where Φ^* and Φ^0 are from Lemmas 3.4 *and* 3.1, *respectively.*

We break the proof into several parts. First note it is easy to show if an invariant measure for RBM exists, then it is absolutely continuous with respect to Lebesgue measure on $\overline{G}$. Hence we can write the invariant measure in the form

$$\varphi(x)\Phi^0(x)\,dx \tag{8.1}$$

for some measurable φ on $\overline{G}$, where we take $\Phi^0(0) = 0$. As in Section 5, we use R_λ to denote the resolvent of RBM.

CASE 1. $\alpha < 0$. Then RBM started away from 0 never hits 0 (Theorem 2.1) and we see killed RBM is RBM when started away from 0. Then for $F = \Phi^*$ (since $\alpha^* > 0$) from (7.2) and any bounded measurable h with compact support in $\overline{G}$,

$$\begin{aligned}\int_{\overline{G}} \lambda[R_\lambda h]F\Phi^0\,dx &= \int_{\overline{G}\setminus\{0\}} \lambda[R_\lambda h]F\Phi^0\,dx\\ &= \int_{\overline{G}\setminus\{0\}} \lambda[R_\lambda^0 h]F\Phi^0\,dx\\ &= \int_{\overline{G}\setminus\{0\}} \lambda[R_\lambda^{0*} F]h\Phi^0\,dx\\ &= \int_{\overline{G}\setminus\{0\}} hF\Phi^0\,dx \quad \text{[by (7.3)]}.\end{aligned}$$

This will yield existence of an invariant measure for RBM once we verify that for any compact $K \subseteq \overline{G}$,

$$\int_{K\setminus\{0\}} F\Phi^0\,dx < \infty \tag{8.2}$$

(this is to verify σ-finiteness of $F\Phi^0\,dx$ on $\overline{G}$). But by (7.2) F is bounded on compact subsets of $\overline{G}$ and by Lemma 3.1, $\Phi^0(x) \le C|x|^{2-d}$, hence (8.2) holds.

As for uniqueness, let $\mu(dx)$ be an invariant measure for RBM. Then μ has the form (8.1) and for any bounded measurable h with compact support in $\overline{G}$,

$$\begin{aligned}\int_{\overline{G}\setminus\{0\}} \lambda[R_\lambda^0 h]\varphi\Phi^0 dx &= \int_{\overline{G}\setminus\{0\}} \lambda[R_\lambda h]\varphi\Phi^0 dx\\ &= \int_{\overline{G}} \lambda[R_\lambda h]\mu(dx)\\ &= \int_{\overline{G}} h\mu(dx)\\ &= \int_{\overline{G}\setminus\{0\}} h\varphi\Phi^0\,dx.\end{aligned}$$

In particular, $\varphi\Phi^0\,dx$ on $(\overline{G}\setminus\{0\}$, Borels) is an invariant measure for killed RBM. By Theorem 7.1 we must have $\varphi = F$ a.e., and μ has the asserted form.

CASE 2. $\alpha = 0$. In this case, by Theorem 2.2, RBM is recurrent. Hence by Azema, Kaplan-Duflo and Revuz (1967), RBM has a unique invariant measure μ. The argument above giving uniqueness can be used in this case to identify μ. Again, the key is starting away from 0, RBM and killed RBM coincide.

CASE 3. $\alpha > 0$. In this case RBM and killed RBM no longer coincide and the matter of identification is much trickier. Existence and uniqueness of an invariant measure μ follow from the work of Azema, Kaplan-Duflo and Revuz (1967) cited above, since in this case RBM is recurrent (Theorem 2.2). So what remains is identification. As pointed out above, we can assume μ has the form (8.1). Hence it suffices to show on $\overline{G}\backslash\{0\}$, for some constant $c > 0$,

$$\varphi = c\Phi^*. \tag{8.3}$$

LEMMA 8.2. *For any bounded measurable h with compact support in $\overline{G}$,*

$$R_\lambda h(x) = R_\lambda^0 h(x) + R_\lambda h(0)[1 - \lambda R_\lambda^0 1(x)], \qquad x \in \overline{G}\backslash\{0\}.$$

PROOF. As in section 5, denote the law of RBM starting from $x \in \overline{G}$ by Q_x. For $x \in \overline{G}\backslash\{0\}$, by the strong Markov property,

$$\begin{aligned} R_\lambda h(x) &= E^{Q_x}\left[\int_0^\infty e^{-\lambda t} h(X_t)\,dt\right] \\ &= E^{Q_x}\left[\int_0^{\tau_0} e^{-\lambda t} h(X_t)\,dt\right] + E^{Q_x}\left[\int_{\tau_0}^\infty e^{-\lambda t} h(X_t)\,dt\right] \\ &= R_\lambda^0 h(x) + E^{Q_x}\left[\exp(-\lambda\tau_0) E^{Q_0}\left[\int_0^\infty e^{-\lambda u} h(X_u)\,du\right]\right] \\ &= R_\lambda^0 h(x) + R_\lambda h(0) E^{Q_x}[\exp(-\lambda\tau_0)]. \end{aligned} \tag{8.4}$$

But

$$\begin{aligned} \lambda R_\lambda^0 1(x) &= \lambda E^{Q_x}\left[\int_0^{\tau_0} e^{-\lambda t} 1\,dt\right] \\ &= E^{Q_x}[1 - \exp(-\lambda\tau_0)]. \end{aligned}$$

Thus $E^{Q_x}[\exp(-\lambda\tau_0)] = 1 - \lambda R_\lambda^0 1(x)$, and using this in (8.4) yields the assertion of the lemma.

LEMMA 8.3. *For $\lambda > 0$,*

$$\lambda R_\lambda^{0*}\varphi \le \varphi \quad a.e. \tag{8.5}$$

$$\int_{\overline{G}\backslash\{0\}} [\varphi - \lambda R_\lambda^{0*}\varphi]\Phi^0\,dx = \int_{\overline{G}\backslash\{0\}} [1 - \lambda R_\lambda^0 1]\varphi\Phi^0\,dx < \infty. \tag{8.6}$$

REMARK 1. Here and throughout "a.e." means almost everywhere with respect to Lebesgue measure.

REMARK 2. Formula (8.5) is the source of difficulty. We do not know a priori that equality holds as we did in the killed case.

PROOF. By Lemma 8.2, for bounded measurable h with compact support in $\overline{G}$,

$$\begin{aligned}\infty > \int_{\overline{G}\backslash\{0\}} h\varphi\Phi^0 dx &= \int_{\overline{G}} h\mu(dx) \\ &= \int_{\overline{G}} \lambda[R_\lambda h]\mu(dx) \\ &= \int_{\overline{G}\backslash\{0\}} \lambda[R_\lambda h]\varphi\Phi^0 dx \\ &= \int_{\overline{G}\backslash\{0\}} \lambda[R_\lambda^0 h]\varphi\Phi^0\, dx + \lambda R_\lambda h(0) \\ &\qquad \times \int_{\overline{G}\backslash\{0\}} [1-\lambda R_\lambda^0 1]\varphi\Phi^0 dx.\end{aligned} \tag{8.7}$$

In particular,

$$\int_{\overline{G}\backslash\{0\}} [1-\lambda R_\lambda^0 1]\varphi\Phi^0 dx < \infty.$$

This gives half of (8.6). Also, for bounded measurable f with compact support in $\overline{G}\backslash\{0\}$,

$$\int_{\overline{G}\backslash\{0\}} \lambda[R_\lambda^0 h]\varphi f\Phi^0\, dx = \int_{\overline{G}\backslash\{0\}} \lambda[R_\lambda^{0*}(\varphi f)]h\Phi^0 dx.$$

Letting $f \uparrow 1$ on $\overline{G}\backslash\{0\}$ gives

$$\int_{\overline{G}\backslash\{0\}} \lambda[R_\lambda^0 h]\varphi\Phi^0\, dx = \int_{\overline{G}\backslash\{0\}} \lambda[R_\lambda^{0*}\varphi]h\Phi^0 dx.$$

Using this in (8.7) and collecting terms gives

$$\int_{\overline{G}\backslash\{0\}} [\varphi - \lambda R_\lambda^{0*}\varphi]h\Phi^0\, dx = \lambda R_\lambda h(0) \int_{\overline{G}\backslash\{0\}} [1-\lambda R_\lambda^0 1]\varphi\Phi^0\, dx. \tag{8.8}$$

By Remark 7.3, $1 - \lambda R_\lambda^0 1 \geq 0$ a.e., and so (8.8) yields (8.5). Letting $h \uparrow 1$ in (8.8) and noting

$$\begin{aligned}\lambda R_\lambda 1(0) &= \lambda E^{Q_0}\left[\int_0^\infty e^{-\lambda t} dt\right] \\ &= 1,\end{aligned}$$

we also get the other half of (8.6).

LEMMA 8.4. *It is no loss to assume φ is excessive for the adjoint of the killed RBM.*

PROOF. By Proposition 7.6,

$$\varphi_1 = \lim_{\lambda\to\infty} \lambda R_\lambda^{0*}\varphi$$

is excessive for the adjoint of the killed RBM and by (8.5),

$$\varphi_1 \le \varphi \quad \text{a.e.} \tag{8.9}$$

Letting $\lambda \to \infty$ in (8.8) gives

$$\int_{\overline{G}\setminus\{0\}} [\varphi - \varphi_1] h\Phi^0\, dx = \lim_{\lambda\to\infty} \lambda R_\lambda h(0)\left[\int_{\overline{G}\setminus\{0\}} [1 - \lambda R_\lambda^0 1]\varphi\Phi^0\, dx\right]. \tag{8.10}$$

Now by dominated convergence,

$$\begin{aligned} \lambda R_\lambda h(0) &= E^{Q_0}\left[\int_0^\infty \lambda e^{-\lambda t} h(X_t)\, dt\right] \\ &= E^{Q_0}\left[\int_0^\infty e^{-u} h(X_{u/\lambda})\, du\right] \\ &\to h(0) \text{ as } \lambda \to \infty. \end{aligned} \tag{8.11}$$

Also, by Remark 7.3 and Proposition 7.6, $1 - \lambda R_\lambda^0 1$ is decreasing in λ. Hence by Lemma 8.3,

$$\lim_{\lambda\to\infty} \int_{\overline{G}\setminus\{0\}} [1 - \lambda R_\lambda^0 1]\varphi\Phi^0 dx \le \int_{\overline{G}\setminus\{0\}} [1 - R_1^0 1]\varphi\Phi^0 dx < \infty.$$

Then for continuous h with compact support in $\overline{G}\setminus\{0\}$, using this and (8.11), we see (8.10) gives

$$\int_{\overline{G}\setminus\{0\}} [\varphi - \varphi_1] h\Phi^0 dx = 0.$$

Combined with (8.9), $\varphi = \varphi_1$ a.e. Hence we can replace φ by φ_1, and so it is no loss to assume φ is excessive for the adjoint of the killed RBM. □

By Lemma 8.4 and Remark 6.4, there are unique nonnegative constants c_1 and c_2 and a unique Radon measure on $\overline{G}\setminus\{0\}$ such that

$$\varphi = c_1 + c_2\Phi^* + R_0^{0*} m. \tag{8.12}$$

We will show $m = 0$ and then $c_1 = 0$.

LEMMA 8.5. *For some real number ρ, for some unbounded countable set $\Lambda \subseteq (1, \infty)$ and for some null* (*with respect to Lebesgue measure*) *set N*,

$$\varphi(\lambda x) = \lambda^\rho \varphi(x), \qquad \lambda \in \Lambda,\ x \in \overline{G}\setminus N.$$

PROOF. For any $\lambda > 0$ and bounded measurable f with compact support in $\overline{G}$,

$$\int_{\overline{G}\setminus\{0\}} E^{Q_x}[f(X_t)]\varphi(\lambda x)\Phi^0(x)\,dx = \int_{\overline{G}\setminus\{0\}} \big[E^{Q_{z/\lambda}}[f(X_t)]\big]\varphi(z)\Phi^0(z)\lambda^{-2}\,dz$$

[by Lemma 3.1, $\Phi^0(cz) = c^{2-d}\Phi^0(z)$ for $c \neq 0$, $z \neq 0$]

$$= \int_{\overline{G}\setminus\{0\}} [E^{Q_z}[f(\lambda^{-1}X_{\lambda^2 t})]]\varphi(z)\Phi^0(z)\lambda^{-2}\,dz$$

[by scaling–cf. Lemma 2.3 in Kwon and Williams (1991)]

$$= \int_{\overline{G}\setminus\{0\}} f(\lambda^{-1}z)\varphi(z)\Phi^0(z)\lambda^{-2}\,dz$$

$$= \int_{\overline{G}\setminus\{0\}} f(x)\varphi(\lambda x)\Phi^0(x)\,dx.$$

Hence $\varphi(\lambda x)\Phi^0(x)\,dx$ is an invariant measure. By uniqueness, for some $c(\lambda) > 0$,

$$\varphi(\lambda x) = c(\lambda)\varphi(x) \quad \text{a.e.,} \tag{8.13}$$

where the null set depends on λ.

Given any $\lambda, \mu > 0$, we have a.e.,

$$c(\mu\lambda)\varphi(x) = \varphi(\mu\lambda x) = c(\mu)\varphi(\lambda x) = c(\mu)c(\lambda)\varphi(x),$$

where the null set depends on μ and λ. Then

$$c(\mu\lambda) = c(\mu)c(\lambda).$$

Define

$$\tilde{c}(t) = \ln c(e^t).$$

Then $\tilde{c}(t+s) = \tilde{c}(t) + \tilde{c}(s)$. Hence for each positive integer n, $\tilde{c}(nt) = n\tilde{c}(t)$. Fix $t_1 > 0$ and set $\rho = \tilde{c}(t_1)/t_1$. Notice $n\tilde{c}\big(t_1/n\big) = \tilde{c}(t_1) = \rho t_1$, and this yields

$$\tilde{c}\left(\frac{t_1}{n}\right) = \rho\frac{t_1}{n}.$$

It follows that for any positive integer m,

$$\tilde{c}\left(\frac{m}{n}t_1\right) = \frac{m}{n}\rho t_1.$$

In particular,

$$\tilde{c}(t) = \rho t \quad \text{for} \quad t \in t_1\mathbf{Q}^+,$$

where $\mathbf{Q}^+$ is the set of positive rational numbers. From this we get

$$\begin{aligned} c(\lambda) &= \exp(\tilde{c}(\ln\lambda)) \\ &= \exp(\rho\ln\lambda), \quad \ln\lambda \in t_1\mathbf{Q}^+ \\ &= \lambda^\rho, \qquad \lambda \in e^{t_1\mathbf{Q}^+}. \end{aligned}$$

The desired conclusion follows from this and (8.13).

For any $\lambda > 0$, by Proposition 7.5, for $\tilde{m}_\lambda(A) = m(\lambda A)$,

$$\begin{aligned} R_0^{0*}m(\lambda x) &= \int_{\overline{G}\setminus\{0\}} G^*(\lambda x, y)m(dy) \\ &= \int_{\overline{G}\setminus\{0\}} G^*(x, y/\lambda)m(dy) \\ &= \int_{\overline{G}\setminus\{0\}} G^*(x, y)\tilde{m}_\lambda(dy) \\ &= R_0^{0*}\tilde{m}_\lambda(x). \end{aligned}$$

Using this in (8.12),

$$\begin{aligned} \varphi(\lambda x) &= c_1 + c_2\Phi^*(\lambda x) + R_0^{0*}m(\lambda x) \\ &= c_1 + c_2\lambda^{\alpha^*}\Phi^*(x) + R_0^{0*}\tilde{m}_\lambda(x), \end{aligned} \tag{8.14}$$

where we have also used that $\Phi^*(\lambda x) = \lambda^{\alpha^*}\Phi^*(x)$. But (8.12) also gives

$$\lambda^\rho\varphi(x) = \lambda^\rho c_1 + c_2\lambda^\rho\Phi^*(x) + \lambda^\rho R_0^{0*}m(x). \tag{8.15}$$

By Lemma 8.5 and the uniqueness of the representation (8.12), comparing (8.14) with (8.15), we must have for $\lambda \in \Lambda$,

$$c_1 = c_1\lambda^\rho, \tag{8.16}$$

$$c_2\lambda^{\alpha^*} = c_2\lambda^\rho, \tag{8.17}$$

$$\tilde{m}_\lambda = \lambda^\rho m. \tag{8.18}$$

Then (8.18) implies

$$m(\lambda A) = \lambda^\rho m(A), \qquad \lambda \in \Lambda. \tag{8.19}$$

LEMMA 8.6. $m \equiv 0$.

PROOF. By Proposition 7.4, for any bounded measurable h with compact support in $\overline{G}$ (recall G_λ^* is the resolvent kernel with respect to $\Phi^0\,dx$ defined at the beginning of Section 6),

$$\begin{aligned} \int_{\overline{G}\setminus\{0\}} [R_0^{0*}m - \lambda R_\lambda^{0*}R_0^{0*}m]h\Phi^0\,dx &= \int_{\overline{G}\setminus\{0\}} [R_\lambda^{0*}m]h\Phi^0\,dx \\ &= \int_{\overline{G}\setminus\{0\}} \left[\int_{\overline{G}\setminus\{0\}} G_\lambda^*(x, y)m(dy)\right]h(x)\Phi^0(x)\,dx \\ &= \int_{\overline{G}\setminus\{0\}} \left[\int_{\overline{G}\setminus\{0\}} G_\lambda(y, x)h(x)\Phi^0(x)dx\right]m(dy) \\ &= \int_{\overline{G}\setminus\{0\}} R_\lambda^0 h(y)m(dy). \end{aligned}$$

Hence by (8.8) and (8.12),

$$
\begin{aligned}
\lambda R_\lambda h(0)\left[\int_{\overline{G}\setminus\{0\}}[1-\lambda R_\lambda^0 1]\varphi\Phi^0\,dx\right] &= \int[\varphi-\lambda R_\lambda^{0*}\varphi]h\Phi^0\,dx\\
&= c_1\int[1-\lambda R_\lambda^{0*}1]h\Phi^0\,dx + c_2\int[\Phi^*-\lambda R_\lambda^{0*}\Phi^*]h\Phi^0\,dx\\
&\quad + \int_{\overline{G}\setminus\{0\}}[R_0^{0*}m-\lambda R_\lambda^{0*}R_0^{0*}m]h\Phi^0\,dx\\
&= c_1\int_{\overline{G}\setminus\{0\}}[1-\lambda R_\lambda^{0*}1]h\Phi^0\,dx + c_2\int_{\overline{G}\setminus\{0\}}[\Phi^*-\lambda R_\lambda^{0*}\Phi^*]h\Phi^0\,dx\\
&\quad + \int_{\overline{G}\setminus\{0\}}[R_\lambda^0 h](y)m(dy)
\end{aligned}
\tag{8.20}
$$

To get a contradiction, assume $m\neq 0$. By Lemma 7.2, the first two terms on the right-hand side of (8.20) are nonnegative, hence

$$
\lambda R_\lambda h(0)\int_{\overline{G}\setminus\{0\}}[1-\lambda R_\lambda^0 1]\varphi\Phi^0 dx \geq \int_{\overline{G}\setminus\{0\}} R_\lambda^0 h(y)m(dy).
\tag{8.21}
$$

By Remark 7.3, $1-\lambda R_\lambda^0 1\geq 0$, and so by Proposition 7.6,

$$
1-\lambda R_\lambda^0 1 \leq 1-R_1^0 1 \quad \text{for } \lambda\geq 1.
$$

Using this in (8.21) gives (after multiplying by $\lambda^{p/2}$)

$$
\begin{aligned}
&\lambda^{1+p/2}R_\lambda h(0)\int_{\overline{G}\setminus\{0\}}[1-R_1^0 1]\varphi\Phi^0\,dx\\
&\qquad \geq \lambda^{p/2}\int_{\overline{G}\setminus\{0\}} R_\lambda^0 h(y)m(dy), \quad \lambda\geq 1.
\end{aligned}
\tag{8.22}
$$

By monotone convergence, we can assume h is nonnegative with unbounded support. The plan is to pick h such that as $\lambda\to\infty$, the left-hand side of (8.22) is finite and the right-hand side is not. This contradiction will establish that $m\equiv 0$, as claimed.

To this end, choose $R>0$ such that

$$
m(|x|>R)>0.
$$

By Lemma A.7 in the Appendix and Corollary 3.6(b), for

$$
h(x)=D(x)^p
$$

with $p > (\alpha - 1)/2 \wedge 0$ such that $3p/2 + \rho - 1 > 0$, for $\lambda \in \Lambda$ we have for some constant c independent of $\lambda \geq 1$,

$$\begin{aligned}
\int_{\overline{G}\setminus\{0\}} R_\lambda^0 h(x) m(dx) &\geq c\lambda^{-1} \int I(|x| > \lambda R) D(x)^p m(dx) \\
&\geq c\lambda^{-1} \int I(|x| > \lambda R)|x|^p m(dx) \\
&\qquad (\text{since } c_1|x| \leq D(x) \leq c_2|x|) \\
&= c\lambda^{-1} \int I\left(\left|\frac{x}{\lambda}\right| > R\right)\left|\frac{x}{\lambda}\right|^p \lambda^p m(dx) \\
&\geq cR^p \lambda^{p-1} m(|x| > \lambda R) \\
&= cR^p \lambda^{p-1} \lambda^\rho m(|x| > R) \quad [\text{by } (8.19)].
\end{aligned}$$

Multiplying by $\lambda^{p/2}$ and letting $\lambda \to \infty$, $\lambda \in \Lambda$ gives

$$\liminf_{\lambda\to\infty,\lambda\in\Lambda} \lambda^{p/2} \int_{\overline{G}\setminus\{0\}} R_\lambda^0 h(x) m(dx) = \infty \quad \text{since } \frac{3p}{2} + \rho - 1 > 0. \tag{8.23}$$

On the other hand, for $\lambda \geq 1$,

$$\begin{aligned}
R_\lambda h(0) &\leq c_3 E_0\left[\int_0^\infty \exp(-c_4 \lambda t) Y_t^p \, dt\right] \quad (\text{Corollary 2.4}) \\
&\leq c\lambda^{-p/2-1} \quad (\text{Lemma A.8}),
\end{aligned}$$

where c_3, c_4 and c are independent of $\lambda \geq 1$. Multiplying by $\lambda^{1+p/2}$ and letting $\lambda \to \infty$, $\lambda \in \Lambda$ gives

$$\limsup_{\lambda\to\infty,\lambda\in\Lambda} \lambda^{1+p/2} R_\lambda h(0) < \infty.$$

Using this, (8.23) and (8.6), we see (8.22) yields a contradiction. Thus $m \equiv 0$.

Now we complete the proof of Theorem 8.1. Recall that this reduces to checking (8.3). By Lemma 8.6 and (8.12), $\varphi = c_1 + c_2\Phi^*$. Since c_1 and c_2 cannot be 0 simultaneously, by (8.16), (8.17) either

$$c_1 = 0, c_2 \neq 0 \quad \text{or} \tag{8.24}$$

$$c_1 \neq 0, c_2 = 0 \quad \text{or} \tag{8.25}$$

$$c_1 \neq 0, c_2 \neq 0, \quad 1 - \lambda^\rho = 0, \quad \lambda^{\alpha^*} - \lambda^\rho = 0 \quad \text{for } \lambda \in \Lambda. \tag{8.26}$$

If (8.25) holds, then $\varphi = c_1$ and (8.20) becomes

$$\lambda R_\lambda h(0)\left[\int_{\overline{G}\setminus\{0\}} [1 - \lambda R_\lambda^0 1] c_1 \Phi^0 \, dx\right] = c_1 \int_{\overline{G}\setminus\{0\}} [1 - \lambda R_\lambda^{0*} 1] h\Phi^0 \, dx.$$

Since $\alpha > 0$, by Remark 7.3 the left-hand side is nonzero. On the other hand, by the remark after the proof of Lemma 3.4, $\alpha^* < 0$. Then by Lemma 7.2, the right-hand side of the last equation is 0. Thus (8.25) cannot hold.

If (8.26) holds, then $\alpha^* = \rho = 0$ and by the remark after the proof of Lemma 3.4, $\alpha = 0$, a contradiction. Hence (8.26) does not hold either.

Thus we have shown (8.24) holds and so $\varphi = c_1\Phi^*$, giving (8.3) as needed. □

REMARK 8.7. In either case of RBM or killed RBM, by Theorems 7.1 and 8.1 and Example 4.6, explicit knowledge of the invariant measure in the case of the circular cone reduces to explicit determination of Φ^* from Lemma 3.4. However, this requires explicit solution of the problem

$$\Delta_{S^{d-1}}\psi^* + \alpha^*(\alpha^* + 2 - d)\psi^* = 0 \quad \text{in } \Omega,$$

$$\mathbf{n} \cdot \nabla_{S^{d-1}}\psi^* - \alpha^*\beta\psi^* = 0 \quad \text{on } \partial\Omega,$$

where ψ^* and α^* are to be found. Only in very special cases of this special case can this be done. See the last paragraph on page 752 in Kwon and Williams (1991).

APPENDIX

Bessel processes. For $\gamma > 0$ let

$$L_\gamma = \frac{1}{2}\left[\frac{d^2}{dx^2} + \frac{\gamma - 1}{x}\frac{d}{dx}\right]$$

with domain $\mathscr{D}(L_\gamma)$ consisting of all $f \in C_b^2([0, \infty))$ such that for some constants c and $0 < a_1 < a_2$,

$$f(x) = cx^2 \text{ for } x \in [0, a_1], \qquad f(x) = 0 \text{ for } x \in [a_2, \infty) \quad \text{and}$$
$$L_\gamma f(0) = c(\gamma - 1).$$

There exists a unique conservative diffusion process Y_t generated by L_γ which is called the Bessel process with parameter γ.

A more useful characterization of Y_t is the following: Y_t^2 is the (pathwise) unique solution of

$$\begin{aligned} dU_t &= 2[U_t \vee 0]^{1/2}\, dB_t + \gamma dt, \\ U_0 &= Y_0^2, \end{aligned} \tag{A.1}$$

where B_t is one-dimensional Brownian motion.

The transition density $p_\gamma(t, x, y)$ of Y_t is known,

$$p_\gamma(t, x, y) = \frac{\exp(-(x^2 + y^2)/2t)}{t(xy)^{\gamma/2-1}} y^{\gamma-1} I_{\gamma/2-1}\left(\frac{xy}{t}\right)$$

where

$$I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(x/2)^{2n}}{n!\Gamma(\nu + n + 1)}$$

is the modified Bessel function. All this can be found in Ikeda and Watanabe (1989) pages 237–240.

Using (A.1) is easy to see that a Bessel process Y_t is well-defined for *any* $\gamma \in \mathbb{R}$ up to the first time $\{0\}$ is hit, and Y_t has the following semimartingale representation up to $T_0 = \inf\{t \geq 0\colon\ Y_t = 0\}$. For some one-dimensional Brownian motion β_t,

$$dY_t = d\beta_t + \frac{\gamma - 1}{2Y_t}\,dt, \qquad t < T_0. \tag{A.2}$$

It is routine to show that starting away from 0 (i.e., $Y_0 \neq 0$),

$$\begin{aligned} T_0 &< \infty \quad \text{a.s. if } \gamma < 2,\\ T_0 &= \infty \quad \text{a.s. if } \gamma \geq 2. \end{aligned} \tag{A.3}$$

In what follows, E_y will denote expectation associated with $Y_0 = y$.

LEMMA A.1. (a) *Assume the parameter of Y_t is $\gamma \neq 2$. For $0 < a < b$ there is $C(a, b) > 0$ such that*

$$E_y\left[\int_0^{T_0} I_{(a,b)}(Y_u)\,du\right] \text{ is bounded for } y > 0 \tag{A.4}$$

and

$$E_y\left[\int_0^{T_0} I_{(a,b)}(Y_u)\,du\right] \leq C(a,b)y^{2-\gamma}, \qquad y > 0. \tag{A.5}$$

(b) *If $\gamma = 2$, the latter expectation is infinite for all $y > 0$.*
(c) *If $0 < \gamma \neq 2$, then $I_{(a,b)}$ can be replaced by $I_{[0,b)}$ in* (A.4) *and* (A.5).

PROOF. Let $\{L(t, x)\colon t \geq 0, x \geq 0\}$ be the local time process of Y_t,

$$\int_0^t f(Y_u)\,du = \int_0^\infty f(x)L(t,x)m(x)\,dx,$$

where $m(x)$ is the density of the speed measure with respect to Lebesgue measure,

$$m(x) = \begin{cases} \dfrac{2x^{\gamma-1}}{|\gamma - 2|}, & \gamma \neq 2\\ 2x, & \gamma = 2. \end{cases}$$

[see Borodin and Salminen (1996), pages 20 and 114].

For $T_\delta = \inf\{t \geq 0\colon\ Y_t = \delta\}$, the distribution of $m(x)L(T_\varepsilon \wedge T_m, x)$ is known. See Borodin and Salminen (1996), formula 3.3.2 on page 395 (for $\gamma > 2$) and formula 3.3.2 on page 303 (for $\gamma = 2$). The former is also valid for $\gamma < 2$

(using similar methods). Computing the expectation E_y of $m(x)L(T_\varepsilon \wedge T_m, x)$ and then letting $M \to \infty$ and $\delta \to 0$ in these formulas yields for $y > 0$,

$$\text{If } \gamma < 2 \text{ then } m(x)E_y(L(T_0, x)) = \begin{cases} \dfrac{2x}{2-\gamma}, & x \le y \\ \dfrac{2}{2-\gamma} x^{\gamma-1} y^{2-\gamma} & y < x; \end{cases}$$

$$\text{If } \gamma = 2 \text{ then } m(x)E_y(L(\infty, x)) = \infty \text{ and}$$

$$\text{If } \gamma > 2 \text{ then } m(x)E_y(L(\infty, x)) = \begin{cases} \dfrac{2}{\gamma-2} x^{\gamma-1} y^{2-\gamma}, & x \le y \\ \dfrac{2x}{\gamma-2}, & y < x. \end{cases}$$

Using the occupation formula given above, the desired conclusions follow easily. □

The differential equation

$$z^2 \frac{d^2 w}{dz^2} + z\frac{dw}{dz} - (z^2 + \nu^2)w = 0$$

has the modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$ as linearly independent solutions [see Abramowitz and Stegun (1972) page 374 ff]. Also, $I_\nu(z)$ and $K_\nu(z)$ are positive when $z > 0$ and $\nu > -1$. The following asymptotics will be used below.

$$I_\nu(z) \sim \left(\frac{z}{2}\right)^\nu \Gamma(\nu+1) \quad \text{as } z \to 0 \quad (\nu \ne -1, -2, \dots), \tag{A.6}$$

$$K_\nu(z) \sim \frac{1}{2}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu} \quad \text{as } z \to 0 \quad (\text{Re } \nu > 0), \tag{A.7}$$

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \quad \text{as } z \to \infty, \tag{A.8}$$

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \quad \text{as } z \to \infty. \tag{A.9}$$

Here $f \sim g$ means $f/g \to 1$.

LEMMA A.2. *Let $h \ge 0$ be bounded on $[0, \infty)$ and piecewise continuous on $(0, \infty)$. Then for $\nu = \left|\frac{\gamma}{2} - 1\right| \ne 0$ and*

$$\begin{aligned} f(y) &= \left[\int_y^\infty u^{\gamma/2} h(u) K_\nu(\sqrt{2\lambda} u)\, du\right] y^{1-\gamma/2} I_\nu(\sqrt{2\lambda} y) \\ &\quad + \left[\int_0^y u^{\gamma/2} h(u) I_\nu(\sqrt{2\lambda} u)\, du\right] y^{1-\gamma/2} K_\nu(\sqrt{2\lambda} y), \end{aligned} \tag{A.10}$$

$$E_y\left[\int_0^{T_0} e^{-\lambda t} h(Y_t)\, dt\right] = 2f(y), \qquad y > 0. \tag{A.11}$$

PROOF. First assume h is bounded and continuous on $[0, \infty)$. By (A.9), (A.6) and the growth conditions on h, the integrals in f are well defined. The Wronskian $W(K_\nu(z), I_\nu(z))$ is $1/z$, so it is a routine matter to verify that $f \in C^2((0,\infty))$ and satisfies

$$f''(y) + \frac{\gamma - 1}{y} f'(y) - 2\lambda f(y) = -h(y), \qquad y > 0.$$

By Itô's formula, stochastic calculus and optional stopping, for $0 < \varepsilon < y < M$,

$$E_y[\exp(-\lambda(t \wedge T_\varepsilon \wedge T_M)) f(Y(t \wedge T_\varepsilon \wedge T_M))] = f(y) - \tfrac{1}{2} E_y\left[\int_0^{t\wedge T_\varepsilon \wedge T_M} e^{-\lambda u} h(Y_u)\, du\right].$$

Letting $t \to \infty$ yields

$$E_y[\exp -\lambda T_\varepsilon f(\varepsilon) I_{T_\varepsilon < T_M}] + E_y[\exp(-\lambda T_M) f(M) I_{T_M < T_\varepsilon}] = f(y) - \tfrac{1}{2} E_y\left[\int_0^{T_\varepsilon \wedge T_M} e^{-\lambda u} h(Y_u)\, du\right].$$

By the asymptotics (A.6)–(A.9), f is bounded on $[0, \infty)$ and

$$\lim_{y\to 0} f(y) = 0 \quad \text{if } \gamma < 2. \tag{A.12}$$

As $\varepsilon \to 0$, $T_\varepsilon \to T_0$ and T_0 is ∞ if $\gamma > 2$, by (A.3). Then we can let $\varepsilon \to 0$ to get

$$E_y[\exp -\lambda T_M f(M) I_{T_M < T_0}] = f(y) - \tfrac{1}{2} E_y\left[\int_0^{T_0 \wedge T_M} e^{-\lambda u} h(Y_u)\, du\right].$$

Since $T_M \to \infty$ as $M \to \infty$, in either case $\gamma < 2$ or $\gamma > 2$, we get

$$f(y) = \tfrac{1}{2} E_y\left[\int_0^{T_0} e^{-\lambda u} h(Y_u)\, du\right],$$

as desired.

Next assume h is bounded on $[0,\infty)$ and piecewise continuous on $(0, \infty)$. The only trouble is when we use Itô's formula above. However, $f \in C^1(0, \infty)$ and the second derivative is piecewise continuous on $(0, \infty)$ and bounded near its discontinuities. Since Y_t spends zero Lebesgue time at singletons in $(0, \infty)$, a simple approximation argument shows our formula above obtained from Itô's formula is valid in this case too. The rest of the proof goes through. □

LEMMA A.3. *Assume the parameter γ of Y_t satisfies $\gamma < 2$ and let $\lambda > 0$. Then as $a \to 0$,*

$$E_y\left[\int_0^{T_0} e^{-\lambda t} I_{[0,a]}(Y_t)\, dt\right] \to 0 \tag{A.13}$$

uniformly for y in sets of the form $(0, A]$. Also, as $b \to \infty$,

$$E_y\left[\int_0^{T_0} e^{-\lambda t} I_{[b,\infty)}(Y_t)\,dt\right] \to 0 \tag{A.14}$$

uniformly for y in sets of the form $(0, A]$.

If $\gamma > 2$ we can replace T_0 by ∞ and $(0, A]$ by $[0, A]$.

PROOF. We prove (A.13) for $\gamma \neq 2$, then specialize to $\gamma > 2$. For $a > 0$ let

$$h = 2I_{[0,a]}$$

in (A.10). Then by Lemma A.2,

$$E_y\left[\int_0^{T_0} e^{-\lambda u} I_{[0,a]}(Y_u)\,du\right] = f(y). \tag{A.15}$$

Given $A > 0$, let $a < A \wedge 1$. Then for $y \in (0, a]$,

$$\begin{aligned} f(y) &= 2\left[\int_y^a u^{\gamma/2} K_\nu(\sqrt{2\lambda}u)\,du\right] y^{1-\gamma/2} I_\nu(\sqrt{2\lambda}y) \\ &\quad + 2\left[\int_0^y u^{\gamma/2} I_\nu(\sqrt{2\lambda}u)\,du\right] y^{1-\gamma/2} K_\nu(\sqrt{2\lambda}y). \end{aligned} \tag{A.16}$$

By making a smaller if necessary, by the asymptotics (A.6)–(A.7) we get

$$K_\nu(\sqrt{2\lambda}y) \le C y^{-\nu}, \qquad y \le a, \tag{A.17}$$

$$I_\nu(\sqrt{2\lambda}\,y) \le C y^{\nu}, \qquad y \le a. \tag{A.18}$$

Here and in what follows C can change from line to line but is independent of a. Then

$$\begin{aligned} \sup_{0<y\le a} f(y) &\le \sup_{0<y\le a} C\left[\left(\int_0^a u^{\gamma/2} u^{-\nu}\,du\right) y^{1-\gamma/2} y^{\nu} \right. \\ &\qquad \left. + \left(\int_0^y u^{\gamma/2} u^{\nu} du\right) y^{1-\gamma/2} y^{-\nu}\right] \\ &\le C a^{3/2}. \end{aligned} \tag{A.19}$$

On the other hand, for $y \in (a, A]$,

$$\begin{aligned} f(y) &= 2\left[\int_0^a u^{\gamma/2} I_\nu(\sqrt{2\lambda}u)\,du\right] y^{1-\gamma/2} K_\nu(\sqrt{2\lambda}y) \\ &\le 2\left[\int_0^a u^{\gamma/2} I_\nu(\sqrt{2\lambda}u)\,du\right] y^{1-\gamma/2} C y^{-\nu}, \end{aligned}$$

where we have used that $K_\nu(\sqrt{2\lambda}y)/y^{-\nu}$ is bounded on $(0, A]$. If $\gamma < 2$ then $\nu = 1 - \gamma/2$ and by (A.18) for $y \in (a, A]$ we get

$$\begin{aligned} f(y) &\le C\left[\int_0^a u^{\gamma/2} u^{1-\gamma/2} du\right] y^{1-\gamma/2} y^{-1+\gamma/2} \\ &= C a^2. \end{aligned}$$

If $\gamma > 2$ then $\nu = \gamma/2 - 1$ and by (A.18) for $y \in (a, A]$,

$$f(y) \le C\left[\int_0^a u^{\gamma/2} u^{\gamma/2-1}\, du\right] y^{1-\gamma/2} y^{-\gamma/2+1}$$
$$\le Ca^2.$$

In any case,

$$\sup_{a<y\le A} f(y) \le Ca^2. \tag{A.20}$$

Combining this with (A.19),

$$\sup_{0<y\le A} f(y) \le Ca^{3/2}.$$

The limit in (A.13) for $\gamma \ne 2$ is an immediate consequence, using (A.15).

Now specialize to $\gamma > 2$. Then $T_0 \equiv \infty$ and for $y > 0$, (A.15) becomes

$$E_y\left[\int_0^\infty e^{-\lambda u} I_{[0,a]}(Y_u)\, du\right] = f(y). \tag{A.21}$$

It is a routine matter to show the functional

$$\omega \to \int_0^\infty e^{-\lambda u} I_{[0,a]}(\omega_u)\, du$$

is continuous on the set

$$\left\{\omega: [0,\infty) \to [0,\infty): \omega \text{ is continuous and } \int_0^\infty I_{\{a\}}(\omega_u)\, du = 0\right\}.$$

The law of Y with $Y_0 = 0$ is supported on this set, so by an extension of the continuous mapping theorem [Billingsley (1968), Theorem 5.1],

$$\lim_{y\to 0} f(y) = \lim_{y\to 0} E_y\left[\int_0^\infty e^{-\lambda u} I_{[0,a]}(Y_u)\, du\right] = E_0\left[\int_0^\infty e^{-\lambda u} I_{[0,a]}(Y_u)\, du\right],$$

where we also use the fact that the law of Y with $Y_0 = y \ge 0$ being uniquely determined implies that the law of Y with $Y_0 = y$ converges to the law of Y with $Y_0 = 0$ if $y \to 0$. Hence by (A.16) and the asymptotics (A.6), (A.7),

$$E_0\left[\int_0^\infty e^{-\lambda u} I_{[0,a]}(Y_u)\, du\right] = C\int_0^a u^{\gamma/2} K_\nu(\sqrt{2\lambda}u)\, du$$
$$\le Ca^2.$$

Then we can replace T_0 and $(0, A]$ in (A.13) by ∞ and $[0, A]$, respectively. The proof of (A.14) and its extension for $\gamma > 2$ can be handled similarly. □

LEMMA A.4. *Assume the parameter γ of Y_t satisfies $0 < \gamma < 2$. Then for $\lambda > 0$,*

$$E_y\left[\int_0^{T_0} e^{-\lambda u}\, du\right] \to 0 \quad \textit{as } y \to 0.$$

PROOF. Taking $h \equiv 2$ in Lemma A.2, for the corresponding f in (A.10),

$$E_y\left[\int_0^{T_0} e^{-\lambda u}\, du\right] = f(y) \to 0 \quad \text{as } y \to 0$$

by (A.12). □

LEMMA A.5. *Assume the parameter γ of Y_t satisfies $\gamma < 2$. Then for any $\lambda > 0$ and $M > 0$,*

$$E_y[\exp(-\lambda T_M) I_{T_M < T_0}] = \frac{y^{1-(\gamma/2)} I_{1-(\gamma/2)}(\sqrt{2\lambda}\, y)}{M^{1-(\gamma/2)} I_{1-(\gamma/2)}(\sqrt{2\lambda}\, M)}, \qquad 0 < y < M.$$

PROOF. Direct substitution and the fact that

$$z^2 I_\nu''(z) + z I_\nu'(z) - (z^2 + \nu^2) I_\nu(z) = 0.$$

shows $h(y) = y^{1-(\gamma/2)} I_{1-(\gamma/2)}(\sqrt{2\lambda}\, y)$ satisfies

$$\frac{1}{2} h''(y) + \frac{\gamma - 1}{2y} h'(y) = \lambda h(y).$$

Since $\gamma < 2$, $h(0) = 0$ and it is clear that $h > 0$ on $(0, \infty)$. Hence by Itô's formula, stochastic calculus and optional stopping, for $\varepsilon < y < M$,

$$\frac{h(y)}{h(M)} = E_y[\exp(-\lambda(t \wedge T_\varepsilon \wedge T_M)) h(Y(t \wedge T_\varepsilon \wedge T_M))/h(M)].$$

Letting $t \to \infty$, then $\varepsilon \to 0$, using $h(0) = 0$, we get

$$\begin{aligned}\frac{h(y)}{h(M)} &= E_y[\exp(-\lambda(T_0 \wedge T_M)) I(T_M < T_0)] \\ &= E_y[\exp(-\lambda T_M) I_{T_M < T_0}],\end{aligned}$$

as desired. □

LEMMA A.6. *Assume the parameter of Y_t is $\gamma > 2$. Then for any $\lambda > 0$ and $\varepsilon > 0$,*

$$E_y[\exp(-\lambda T_\varepsilon)] = \frac{y^{1-(\gamma/2)} K_{\gamma/2-1}(\sqrt{2\lambda}\, y)}{\varepsilon^{1-(\gamma/2)} K_{\gamma/2-1}(\sqrt{2\lambda}\, \varepsilon)}, \qquad y > \varepsilon.$$

For the proof, see formula 2.0.1 on page 387 of Borodin and Salminen (1996).

LEMMA A.7. *Assume the parameter of Y_t is $\gamma > 0$. Given $k > 0$ and $p > (1 - \gamma)/2 \wedge 0$ there is a constant $c > 0$ such that*

$$E_y\left[\int_0^{T_0} e^{-\lambda t} |Y_t|^p dt\right] \ge c\lambda^{-1} y^p, \qquad \lambda \ge k,\ y \ge 1.$$

PROOF. By applying Lemma A.2 to truncations $|y|^p \wedge M$ and letting $M \to \infty$, we have

$$E_y\left[\int_0^{T_0} e^{-\lambda t} Y_t^p\,dt\right] = 2f(y), \tag{A.22}$$

where

$$f(y) = \left[\int_y^\infty u^{\gamma/2+p} K_\nu(\sqrt{2\lambda}u)\,du\right] y^{1-\gamma/2} I_\nu(\sqrt{2\lambda}y) + \left[\int_0^y u^{\gamma/2+p} I_\nu(\sqrt{2\lambda}u)\,du\right] y^{1-\gamma/2} K_\nu(\sqrt{2\lambda}y),$$

all integrals being finite by the asymptotics (A.6)–(A.9). Changing variables $s = \sqrt{2\lambda}u$ and using (A.8), (A.9), for some positive constants c_1 and c_2, for $\lambda \ge k$ and $y \ge 1$,

$$\begin{aligned}
f(y) &= (\sqrt{2\lambda})^{-\gamma/2-p-1}\left[\left[\int_{\sqrt{2\lambda}y}^\infty s^{\gamma/2+p} K_\nu(s)\,ds\right] y^{1-\gamma/2} I_\nu(\sqrt{2\lambda}y) + \left[\int_0^{\sqrt{2\lambda}y} s^{\gamma/2+p} I_\nu(s)\,ds\right] y^{1-\gamma/2} K_\nu(\sqrt{2\lambda}y)\right]\\
&= (\sqrt{2\lambda})^{-2-p}\left[\left[\int_{\sqrt{2\lambda}y}^\infty s^{\gamma/2+p} K_\nu(s)\,ds\right] (\sqrt{2\lambda}y)^{1-\gamma/2} I_\nu(\sqrt{2\lambda}y) + \left[\int_0^{\sqrt{2\lambda}y} s^{\gamma/2+p} I_\nu(s)\,ds\right] (\sqrt{2\lambda}y)^{1-\gamma/2} K_\nu(\sqrt{2\lambda}y)\right]\\
&\ge (\sqrt{2\lambda})^{-1-p}\left[c_1\left[\int_{\sqrt{2\lambda}y}^\infty s^{\gamma/2+p-1/2} e^{-s}\,ds\right] (\sqrt{2\lambda}y)^{(1-\gamma)/2} \exp(\sqrt{2\lambda}y) + c_2\left[\int_0^{\sqrt{2\lambda}\,y} s^{\gamma/2+p-1/2} e^{s}\,ds\right] (\sqrt{2\lambda}y)^{(1-\gamma)/2} \exp(-\sqrt{2\lambda})y\right].
\end{aligned}$$

Since

$$\lim_{z\to\infty} \frac{\int_z^\infty s^{\gamma/2+p-1/2} e^{-s}\,ds}{z^{\gamma/2+p-1/2} e^{-z}} = 1 \quad \text{and} \quad \lim_{z\to\infty} \frac{\int_0^z s^{\gamma/2+p-1/2} e^{s}\,ds}{z^{\gamma/2+p-1/2} e^{z}} = 1,$$

we end up with

$$f(y) \ge c\lambda^{-1} y^p \quad \text{for } \lambda \ge k \text{ and } y \ge 1,$$

where c is independent of such λ and y. Combined with (A.22), the conclusion of the lemma follows. □

LEMMA A.8. *Let $\gamma > 0$ and $p > 0$. Then for some constant $c > 0$,*

$$E_0\left[\int_0^\infty e^{-\lambda t} Y_t^p\,dt\right] = c\lambda^{-1-p/2} \quad \textit{for } \lambda \ge 1.$$

PROOF. From the expression at the beginning of this section for the transition density of the Bessel process and the asymptotic (A.6)

$$p_\gamma(t, 0, y) = c(\gamma) t^{-\gamma/2} y^{\gamma-1} \exp(-y^2/2t)$$

where $c(\gamma)$ is independent of y and t. Then by Fubini's theorem,

$$\begin{aligned} E_0\left[\int_0^\infty e^{-\lambda t} Y_t^p \, dt\right] &= \int_0^\infty e^{-\lambda t}\left[\int_0^\infty c(\gamma) t^{-\gamma/2} y^{\gamma-1} \exp(-y^2/2t) y^p \, dy\right] dt \\ &= c(\gamma) \int_0^\infty t^{p/2} e^{-\lambda t}\left[\int_0^\infty u^{\gamma-1+p} \exp(-u^2)\, du\right] dt \\ &= c(\gamma) \int_0^\infty t^{p/2} e^{-\lambda t} \, dt \\ &= c(\gamma) \lambda^{-1-p/2}, \end{aligned}$$

where $c(\gamma)$ can change from line to line, but is independent of y and t.

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