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A NEW PROOF THAT FOR THE CONTACT PROCESS ON HOMOGENEOUS TREES LOCAL SURVIVAL IMPLIES COMPLETE CONVERGENCE

By Marcia Salzano[1] and Roberto H. Schonmann[2]

University of California, Los Angeles

We provide a new proof, substantially simpler than Zhang's original one, that for the contact process on homogeneous trees, local survival implies complete convergence.

1. Introduction and results. In this paper we will consider the contact process with infection parameter $\lambda > 0$, on the homogeneous tree of degree $d+1$, denoted by $\mathbb{T}_d$. The case in which $d = 1$ corresponds to the linear chain $\mathbb{Z}$ and will not be considered here, so that we assume that $d \geq 2$. A great deal of attention has been given to contact processes on such trees since the pioneering paper by Pemantle (1992) appeared, and in Liggett (1997) the reader will find a survey of recent progress on these models. Interest stems to a great extent from the fact that there are two distinct critical points $0 < \lambda_1 < \lambda_2 < \infty$. To describe their roles we define the survival probability $\rho(\eta)$ as the probability that the contact process started from the configuration η never reaches the configuration with no particles. The recurrence probability $\beta(\eta)$ is defined as the probability that the contact process started from the configuration η will for every given site have particles at that site at arbitrarily large times (this is, of course, equivalent to requiring it to happen for one specific site of $\mathbb{T}_d$). The following then happens.

1. For $\lambda \leq \lambda_1$, $\rho(\eta) = 0$ for every configuration η with finitely many particles. One says in this case that the process dies out.
2. For $\lambda_1 < \lambda \leq \lambda_2$, $\rho(\eta) > 0$ for every configuration η with a finite and positive number of particles, but $\beta(\eta) = 0$ for every such configuration. One says in this case that the process survives globally but not locally.
3. For $\lambda > \lambda_2$, $\beta(\eta) > 0$ for every configuration η with a positive number of particles. One says in this case that the process survives locally.

Note that it is known that at λ_1 the system dies out, and that at λ_2 it survives globally but not locally. The second of these two facts was first proven by Zhang (1996), but a simpler argument appeared in Lalley and Sellke (1996). It is interesting to point out that Zhang's proof relied on the same machinery

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developed to prove the complete convergence theorem above λ_2, the result whose proof we simplify in the current paper.

Next we introduce some notation, mostly similar to that used in our recent study of contact processes on graphs, Salzano and Schonmann (1997). We suppose that the reader is familiar with the field of interacting particle systems. For those who need an introduction to it, Liggett (1985) is an excellent reference.

We will denote by $(\xi^{\eta}_{G;t}\colon t \geq 0)$ the contact process on the graph G starting from the configuration η. When there is no risk of confusion, G will be omitted in this notation. The set of vertices of the graph G will be denoted by $\mathscr{V}_G$.

We immerse the graph $\mathbb{Z}^+$ which has vertices $\{0, 1, 2, \ldots\}$ and edges connecting points which differ by 1 unit into $\mathbb{T}_d$, in an arbitrary fashion. This allows us to refer to the sites $0, 1, 2, \ldots$ of $\mathbb{T}_d$. The site 0 of $\mathbb{T}_d$ is called its root. An important subgraph of $\mathbb{T}_d$ is obtained from this tree by removing one of the neighbors of the root and defining it as the remaining connected component which contains the root. We will suppose that the removed vertex is not the vertex 1, so that the set of sites $\{0, 1, 2, \ldots\}$ is contained in the set of vertices of the new graph. This new graph will be denoted $\mathbb{T}^+_d$. We will measure the distance between sites in $\mathbb{T}_d$ by the length of the minimal path along neighboring sites which joins them. Then $B(x, N)$ will denote the ball of center $x \in \mathscr{V}_{\mathbb{T}_d}$ and radius N. We use also the abbreviation $B(N) = B(0, N)$.

The following quantity will play a major role in this paper, as it did in Liggett (1996b) and Lalley and Sellke (1996):

$$u_n = \mathbb{P}\Big(\xi^0_{\mathbb{T}_d;t}(n) = 1 \text{ for some } t \geq 0\Big).$$

From the inequality $u_{n+m} \geq u_n u_m$ it follows that

$$\lim_{n\to\infty} (u_n)^{1/n} = \alpha = \alpha(\lambda)$$

exists. [Our α was called ρ in Liggett (1996), but this conflicts with the standard use of ρ for the survival probability; it was called β in Lalley and Sellke (1996) but we used β for the recurrence probability in Salzano and Schonmann (1997) and therefore prefer to use a different notation here.]

In this paper we will focus on the following theorem, originally proven by Zhang (1996).

THEOREM 1 (Zhang). *For the contact process on $\mathbb{T}_d$, if $\lambda > \lambda_2$, then for any $\eta \in \{0, 1\}^{\mathscr{V}_{\mathbb{T}_d}}$,*

$$\xi^{\eta}_t \Rightarrow (1 - \rho(\eta))\delta_{\varnothing} + \rho(\eta)\bar{\nu}, \tag{1.1}$$

as $t \to \infty$, where $\delta_{\varnothing}$ is the measure concentrated on the configuration with no particles and $\bar{\nu}$ is the upper invariant measure.

The statement (1.1) is known as complete convergence. When it holds, it fully describes the ergodic behavior of the contact process.

Technically, the most difficult part of Zhang's proof of complete convergence for the contact process on $\mathbb{T}_d$ is his proof of Proposition 5 in his paper, which states that when $\lambda > \lambda_2$,

$$\inf_{t\geq 0} \mathbb{P}\Big(\xi^0_{\mathbb{T}_d^+;\, t}(0) = 1\Big) > 0. \tag{1.2}$$

After proving this, Zhang completes his proof of Theorem 1 by using a classical result in Griffeath (1978) which establishes a necessary and sufficient condition for complete convergence to hold based on the behavior of two independent copies of the contact process. In Salzano and Schonmann (1997) we provided a minor simplification of Zhang's proof of Theorem 1 in that we showed how the machinery which we developed in that paper can replace the result of Griffeath (1978) in proving complete convergence on $\mathbb{T}_d$, once (1.2) is available.

Here we will give a self-contained, relatively short and simple proof of the following result.

PROPOSITION 1. *If $\alpha(\lambda) > 1/\sqrt{d}$, then (1.2) holds.*

When $\lambda > \lambda_2$ we have $u_n \geq \beta(\{0\}) > 0$ for each n. Therefore $\alpha(\lambda) = 1$ and from Proposition 1 the estimate (1.2) follows. In this way Zhang's proof of Theorem 1 is greatly simplified. It is worth stressing that Zhang's renormalization procedure (see page 1418 and following in his paper) is not needed in our proof.

From Proposition 1, one also obtains the following immediate consequence, of a different nature.

THEOREM 2 (Lalley and Sellke). *If $\lambda < \lambda_2$ then $\alpha(\lambda) \leq 1/\sqrt{d}$.*

We end this introduction with a few comments which may be seen as a review of some points in Salzano and Schonmann (1997) and which are of relevance here.

The proof that local survival implies complete convergence presented here depends heavily on the structure of the graph $\mathbb{T}_d$; likewise does Zhang's original proof. For the contact process on $\mathbb{Z}^d$ the statement that local survival implies complete convergence is also true, as proved by Bezuidenhout and Grimmett (1990). It is natural to ask if for general graphs this statement is still true, but this was disproved (even for trees) in Salzano and Schonmann (1997). A problem left open in that paper and that seems extremely difficult but also very interesting is whether for homogeneous graphs (i.e., graphs such that any vertex can be mapped into any other vertex by means of a graph-automorphism) the statement is true or not.

An important consequence of complete convergence is that it implies that the set of extremal invariant distributions is $\mathscr{I}_e = \{\delta_\varnothing, \overline{\nu}\}$. It is worth reminding the reader that in Salzano and Schonmann (1997) there is a proof that for an arbitrary homogeneous graph local survival implies $\mathscr{I}_e = \{\delta_\varnothing, \overline{\nu}\}$.

This proof is substantially simpler than the proof presented here of complete convergence just for the homogeneous trees.

2. Proofs. Our proof of Proposition 1 will require the introduction of the standard graphical construction of the contact process. We recall next this graphical construction and some related basic notions. We associate each site $x \in \mathbb{T}_d$ with $d+2$ independent Poisson processes, a first one with rate 1 and $d+1$ others with rate λ. Make these Poisson processes also independent from site to site. For each x, let $\{T_n^{x,k}\colon n \geq 1\}$, $k = 0, 1, 2, \dots, d+1$ be the arrival times of these $d+2$ processes, respectively. For each x and $n \geq 1$ we write a δ mark at the point $(x, T_n^{x,0})$ and draw arrows from $(x, T_n^{x,k})$ to $(x^{(k)}, T_n^{x,k})$ for $k = 1, \dots, d+1$, where $x^{(k)}$, $k = 1, \dots, d+1$, are the neighbors of x, arranged in some arbitrary order. We say that there is a path from (x, s) to (y, t) if there is a sequence of times $s = s_0 < s_1 < \cdots < s_n < s_{n+1} = t$ and spatial locations $x = x_0, x_1, \dots, x_n = y$ so that for $i = 1, 2, \dots, n$ there is an arrow from x_{i-1} to x_i at time s_i and the vertical segments $\{x_i\} \times (s_i, s_{i+1})$ for $i = 0, 1, \dots, n$ do not contain any δ. We will say that the path is inside a set $D \subset \mathscr{V}_{\mathbb{T}_d} \times \mathbb{R}_+$ in case these vertical segments are all contained in D. Given two sets $A, B \subset \mathscr{V}_{\mathbb{T}_d} \times \mathbb{R}_+$, we will say that there is a path from A to B inside D if there is such a path from a point of A to a point of B.

If $\eta \subset \mathbb{T}_d$ and we define ξ_t^η as the indicator function of the set $\{x \in \mathbb{T}_d\colon$ there is a path from $\eta \times \{0\}$ to $(x, t)\}$, then $(\xi_t^\eta,\ t \geq 0)$ is a version of the contact process on $\mathbb{T}_d$.

We shall use the definition of an increasing event that was given by Bezuidenhout and Grimmett (1991). Briefly, an event E is said to be increasing if the following holds: for any realization of the graphical construction that is in E, every other realization obtained from it by the addition of arrows or the suppression of δ marks is also in E. The Harris–FKG inequality says that if E and F are both increasing events, then

$$P(E \cap F) \geq P(E)P(F).$$

The following object will be important in our proof of Proposition 1:

$$\begin{aligned} Y_{n,s} &= \mathbb{P}\big(\text{there is a path from } (0,0) \text{ to } (n,s) \text{ inside } \mathbb{T}_d^+ \times \mathbb{R}_+\big) \\ &= \mathbb{P}\Big(\xi^0_{\mathbb{T}_d^+;\, s}(n) = 1\Big). \end{aligned}$$

Lemma 1. *There exists a sequence* $(s(n))_{n\geq 1}$ *such that* $\lim_{n\to\infty}(Y_{n,s(n)})^{1/n} = \alpha$.

Proof. Clearly for any sequence $s(n)$,

$$\limsup_{n\to\infty}\big(Y_{n,s(n)}\big)^{1/n} \leq \limsup_{n\to\infty}(u_n)^{1/n} = \alpha.$$

This is not a very useful inequality. We are really after the complementary inequality, which we address next. Define

$$\begin{aligned} V_{m,k} &= \mathbb{P}\big(\text{there is a path from } (0,0) \text{ to } \{m\} \times \mathbb{R}_+ \text{ inside } B(k) \times [0,k]\big) \\ &= \mathbb{P}\Big(\xi^0_{B(k);\, t}(m) = 1 \text{ for some } t \leq k\Big). \end{aligned}$$

Clearly,

$$\lim_{k\to\infty} V_{m,k} = u_m. \tag{2.1}$$

Next define

$$\begin{aligned} W_{n,k} &= \mathbb{P}\big(\text{there is a path from } (0,0) \text{ to } \{n\}\times\mathbb{R}_+ \text{ inside } \mathbb{T}_d^+\times[0,kn]\big)\\ &= \mathbb{P}\Big(\xi^0_{\mathbb{T}_d^+;t}(n)=1 \text{ for some } t\le kn\Big). \end{aligned}$$

We will argue next that

$$W_{n,k} \ge C_{m,k}(V_{m,k})^{\lceil n/m\rceil}, \tag{2.2}$$

where $C_{m,k}$ is a positive quantity which does not depend on n. For this purpose set $I=\min\{i\in\{1,2,\dots\}: im>k\}=\min\{i\in\{1,2,\dots\}: B(im,k)\subset\mathbb{T}_d^+\}$. Consider now the sequence of sites $x_1=Im$, $x_2=(I+1)m$, $x_3=(I+2)m,\dots,x_J=\lceil n/m\rceil m$. By pasting together a sequence of paths and using the translation invariance of the graphical construction and the strong Markov property of the underlying Poisson processes, one readily obtains (2.2). In this argument, the first path is somewhat different from the others and it can go from $(0,0)$ to $(x_1,1)$ without exiting $\mathbb{T}_d^+\times\mathbb{R}_+$. The second path should go from $(x_1,1)$ to (x_2,T_2), for some random time T_2 which satisfies $T_2-1\le k$, and it should not exit $B(x_1,k)$. The third path should go from (x_2,T_2) to (x_3,T_3), for some random time T_3 which satisfies $T_3-T_2\le k$, and it should not exit $B(x_2,k)$. By now the way to choose the other paths should be clear. The reason for (2.2) should also be clear, with the factor $C_{m,k}>0$ being the probability of the existence of the first path in this construction.

Define

$$\begin{aligned} \overline{W}_{n,k} &= \max_{j=0,\dots,nk-1} \mathbb{P}\big(\text{there is a path from } (0,0)\\ &\qquad\qquad \text{to } \{n\}\times[j,j+1] \text{ inside } \mathbb{T}_d^+\times\mathbb{R}_+\big)\\ &= \max_{j=0,\dots,nk-1} \mathbb{P}\Big(\xi^0_{\mathbb{T}_d^+;t}(n)=1 \text{ for some } t\in[j,j+1]\Big). \end{aligned}$$

The event in this definition and the event that there is no death mark in $\{n\}\times[j,j+1]$ are both increasing; therefore, for a proper choice of $s(n)$,

$$Y_{n,s(n)} \ge e^{-1}\overline{W}_{n,k} \ge e^{-1}\frac{1}{kn}W_{n,k}. \tag{2.3}$$

Using now (2.3), (2.2), (2.1) and the definition of α,

$$\begin{aligned} \liminf_{n\to\infty}(Y_{n,s(n)})^{1/n} &= \liminf_{m\to\infty}\ \liminf_{k\to\infty}\ \liminf_{n\to\infty}(Y_{n,s(n)})^{1/n}\\ &\ge \liminf_{m\to\infty}\ \liminf_{k\to\infty}(V_{m,k})^{1/m}\\ &= \liminf_{m\to\infty}(u_m)^{1/m}\\ &= \alpha, \end{aligned}$$

completing the proof. □

PROOF OF PROPOSITION 1. Suppose that $\alpha(\lambda) > 1/\sqrt{d}$. Thanks to Lemma 1, we can take n and s such that

$$(Y_{n,s})^{1/n} = a > 1/\sqrt{d}. \tag{2.4}$$

We will show that for a proper choice of a positive integer l,

$$\inf_{i=0,1,2,\dots} \mathbb{P}\Big(\xi^0_{\mathbb{T}_d^+;\,2ils}(0) = 1\Big) > 0. \tag{2.5}$$

This clearly suffices for our purposes, since then

$$\inf_{t\geq 0} \mathbb{P}\Big(\xi^0_{\mathbb{T}_d^+;\,t}(0) = 1\Big) \geq e^{-2ls} \inf_{i=0,1,2,\dots} \mathbb{P}\Big(\xi^0_{\mathbb{T}_d^+;\,2ils}(0) = 1\Big) > 0,$$

by the same reasoning behind the first inequality in (2.3).

In order to prove (2.5), we consider now the following modification of the contact process on $\mathbb{T}_d^+$. Until time s we run the usual contact process on this graph started from a single particle at the origin. At time s we remove all particles except for those which are in $B(n)\backslash B(n-1)$; from this time on we keep the set $B(n-1)$ free of particles, but until time $2s$ we let the system evolve in the remaining vertices with the usual contact process rules. At time $2s$ we remove all particles except for those which are in $B(2n)\backslash B(2n-1)$; from this time on we keep the set $B(2n-1)$ free of particles, but until time $3s$ we let the system evolve in the remaining vertices with the usual contact process rules. The modification should now be clear. For each j, at time js we remove all particles except for those which are in $B(jn)\backslash B(jn-1)$; from this time on we keep the set $B(jn-1)$ free of particles, but until time $(j+1)s$ we let the system evolve in the remaining vertices with the usual contact process rules.

Let Z_j be the number of particles in this modified process at time js [all of them are in $B(jn)\backslash B(jn-1)$]. It is clear that $(Z_j)_{j=0,1,2,\dots}$ is a branching process with mean offspring number $d^n Y_{n,s} = (da)^n$. Since the offspring distribution has a finite support (namely $\{0, 1, \dots, d^n\}$), and in particular a finite second moment, it follows from standard branching-process theory [see, e.g., Example 4.3 in Section 4.4, page 254 of Durrett (1996)] that for some random variable X with mean $\mathbb{E}(X) = 1$,

$$\frac{Z_j}{(da)^{nj}} \to X \quad \text{a.s. as } j \to \infty.$$

In particular, there is $\varepsilon > 0$ such that

$$\mathbb{P}\bigg(Z_l \geq \frac{(da)^{nl}}{2}\bigg) \geq \varepsilon, \tag{2.6}$$

for all large enough l.

Choose now l large enough for (2.6) to hold, and also so that

$$\Big(1 - \frac{\varepsilon}{2} a^{nl}\Big)^{(da)^{nl}/2} \leq \frac{1}{2}. \tag{2.7}$$

This last requirement can be fulfilled because

$$\lim_{l\to\infty}\left(1-\frac{\varepsilon}{2}a^{nl}\right)^{(da)^{nl}/2} \le \lim_{l\to\infty}\exp\left(-\frac{\varepsilon}{2}a^{nl}\frac{(da)^{nl}}{2}\right) = \lim_{l\to\infty}\exp\left(-\frac{\varepsilon}{4}(da^2)^{nl}\right) = 0,$$

since $da^2 > 1$ by (2.4).

Define now

$$r_i = \mathbb{P}\Big(\xi^0_{\mathbb{T}^+_d;\,2ils}(0) = 1\Big).$$

We will show inductively in i that

$$r_i \ge \frac{\varepsilon}{2}a^{nl}, \tag{2.8}$$

verifying therefore the validity of (2.5).

For $i = 0$ inequality (2.8) is clearly true, and we will show now that if it is true for i it is true for $i+1$. From the Markov property and attractiveness of the contact process we have

$$\begin{aligned} r_{i+1} &\ge \mathbb{P}\Big(\xi^0_{\mathbb{T}^+_d;(2i+1)ls}(x) = 1 \text{ for some } x \in B(nl)\backslash B(nl-1)\Big)Y_{ln,\,ls} \\ &\ge \mathbb{P}\Big(\xi^0_{\mathbb{T}^+_d;(2i+1)ls}(x) = 1 \text{ for some } x \in B(nl)\backslash B(nl-1)\Big)(Y_{n,\,s})^l. \end{aligned}$$

It is clear that the contact process on $\mathbb{T}^+_d$ dominates the modified process introduced above, immediately before the definition of Z_j, in the usual sense that it has a particle at any space-time location in which this other one has a particle. The same is true if we consider a somewhat different modification of the contact process on $\mathbb{T}^+_d$, in which until time ls we make the same modifications as before, but from this time on we keep only the sites in $B(nl-1)$ free of particles, and we let the system evolve in the remaining vertices with the usual contact process rules. Using these observations, the last display above yields

$$r_{i+1} \ge \mathbb{P}\left(Z_l \ge \frac{(da)^{nl}}{2}\right)\big(1-(1-r_i)^{(da)^{nl}/2}\big)(Y_{n,\,s})^l \ge \frac{\varepsilon}{2}a^{nl},$$

where in the second inequality we are using (2.6), the induction hypothesis (2.8), (2.7) and (2.4). This completes the proof of Proposition 1. □

Acknowledgments. Theorem 2 was conjectured by Liggett (1996b) and proven by Lalley and Sellke (1996). While we were working on the current project and had already found a simplified proof of Theorem 1, we became aware of the paper Lalley and Sellke (1996). We then realized that using ideas in that paper we could further improve our proof and write it in a self-contained fashion which includes also Theorem 2 as a corollary. We thank Lalley and Sellke for sending a copy of their paper to UCLA and Tom Liggett for making us aware of it and also for his comments on the first version of our manuscript.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
LOS ANGELES, CALIFORNIA 90095
E-MAIL: msalzano@math.ucla.edu
rhs@math.ucla.edu