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WAVE PROPAGATION IN A LATTICE KPP EQUATION IN RANDOM MEDIA

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We extend a result of Freidlin and Gartner for KPP wave fronts to the case $d \geq 2$ for i.i.d. random media. We show a wave front propagation speed is attained for the discrete-space (lattice) KPP using a large deviation approach.

1. Introduction. The problem of wave front propagation in one-dimensional space homogeneous random media was considered in Gartner and Freidlin (1979), where they extend the results from the periodic case. See Freidlin (1985) for an exposition. Although the spaces used there were continuous, the result is similar if we consider lattice space equations. For an excellent account of analysis and modeling of front propagation in parabolic partial differential equations (PDEs) and their utility in applications, see Xin (1997).

In this paper we consider discrete-space KPP equations of the form

$$\frac{\partial u}{\partial t} = \tilde{\Delta} u + \xi(x) u(1-u), \qquad t > 0, x \in Z^d, \tag{1.1}$$
$$u(0, x) = 1_0(x),$$

where $u = u(t, x)$, denotes $\tilde{\Delta}$ the discrete Laplacian: $\tilde{\Delta} f(x) = (1/2d) \times \sum_{y:\, |y|=1} [f(x+y) - f(x)]$, and $1_0(x) = 1$ if $x = 0$ and equals 0 otherwise. The random field $\xi \equiv \{\xi(x)\colon x \in Z^d\}$ is supported on the probability space $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P})$. It is not difficult to treat the more general form of KPP nonlinearity. The above form is chosen for the simplicity of presentation. We further assume that the $\xi(x)$ are independent and identically distributed, bounded and nonnegative random variables, and we set

$$A = \operatorname{ess\,sup} \xi(0),$$

where the ess sup is with respect to $\hat{P}$. The case where $\operatorname{ess\,sup} \xi(0) = \infty$ is easy and is discussed in Section 5, Remark 5.4. The solution $u(t, x) = u(t, x; \hat{\omega})$ to (1.1) is a function of $t \geq 0$, $x \in Z^d$ and $\hat{\omega} \in \hat{\Omega}$.

In order to state the main result of this paper, we define some notation. We set for $z < -A$ and $e \in R^d \backslash \{0\}$,

$$\mu(z; e) = \lim_{t \to \infty} \frac{1}{t} \log E_{te}\left[\exp\left(\int_0^{\tau_0} (\xi(\eta_s) + z)\, ds \right) \right]. \tag{1.2}$$

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Here $\eta_.$ is the strong Markov process on Z^d that corresponds to the generator $\tilde{\Delta}$, a continuous time symmetric simple random walk, and τ_a is the first time η_t hits the site $a \in Z^d$. Probabilities and expectations with respect to the process $\eta_.$ will be denoted by P_x and E_x, respectively. The subscript x refers to the fact that the process η starts at x: $\eta_0 = x$. Since our functions evolve in Z^d, it is important that x be in Z^d, so by P_x and E_x we will mean to use $x = [x]$, the nearest lattice site to x, where there is some deterministic rule to break ties. Similarly, $u(t, x) = u(t, [x])$.

We will prove in Section 2 that if $z > -A$, then (1.2) is infinite under an assumption weaker than the i.i.d. assumption. It is an easy consequence of the subadditive ergodic theorem that this limit exists $\hat{P}$-a.s., and since our underlying random medium is ergodic, the limit $\mu(z; \mathrm{e})$ is nonrandom. The limit function $\mu(z; \mathrm{e})$ is also relevant in the study of Brownian motions in random potential and percolation processes. See Sznitman (1994) for an excellent account of some relations and further references.

Let $I(y; \mathrm{e})$ denote the Legendre transform of the function $\mu(z; \mathrm{e})$:

$$I(y; \mathrm{e}) = \sup_{z<-A} [yz - \mu(z; \mathrm{e})]. \tag{1.3}$$

The main result of this paper is

THEOREM 1. *Make assumptions as above and let* $\mathrm{e} \in R^d \backslash \{0\}$. *Then for any* $v > 0$, $\hat{P}$*-a.s.,*

$$\lim_{t\to\infty} \frac{1}{t} \log u(t, tv\mathrm{e}) = -\left[vI\left(\frac{1}{v}; \mathrm{e}\right) \vee 0 \right],$$

where $a \vee b = \max(a, b)$. *Moreover,* $I(1/v; \mathrm{e}) = 0$ *has a unique solution* $v = v_{\mathrm{e}}$ *which can be characterized as*

$$v_{\mathrm{e}} = \inf_{z<-A} \frac{z}{\mu(z; \mathrm{e})}. \tag{1.4}$$

REMARK 1.1. It should be mentioned that the i.i.d. assumption on ξ will only be used to satisfy a certain condition of Lemma 3 in Section 2. The results of the rest of the paper will be true if i.i.d. is replaced by the more general condition of ergodic stationary media of "purely" random type, by which we mean that $\hat{P}$ satisfies: For any $L > 0$ and $\varepsilon > 0$,

$$\hat{P}[\xi(x) > \operatorname{ess\,sup} \xi(0) - \varepsilon,\ \forall x \in \Lambda_L] > 0,$$

where Λ_L is the box centered at the origin of side width $2L + 1$.

REMARK 1.2. In view of this theorem we define v_{e} as the wave speed in the logarithmic sense. The variational formula of v_{e} is easily derived from the definition of I. The speed v_{e} is nonrandom as $\mu(z; \mathrm{e})$ is. We believe that the solution u goes to 1 ($\hat{P}$-a.s.) for $v < v_{\mathrm{e}}$ and do not work on its proof. Such a result should follow from analysis similar to that in Freidlin [(1985), page 521].

In Section 2 we summarize some basic properties of the functions μ and I and show that the uniquess of the wave speed is a consequence of the strict monotonicity of the function I in y.

In Section 3 we prove the upper bound. The trivial upper bound $u \leq 1$ follows from a comparison if we substitute the initial function in (1.1) by the function identically equal to 1.

In Section 4 we show the lower bound. A difficulty arises here that does not appear in the one-dimensional case studied in Gartner and Freidlin (1979). In that case, $\mu(z)$ is in fact a smooth function of z (this follows from the ergodic theorem), and we can easily apply the Gartner–Ellis theorem [e.g., Dembo and Zeitouni (1993)]. However, in the case of $d \geq 2$ we only know the function $\mu(z; \mathrm{e})$ is convex, so there needs to be more work to derive a useful large deviation lower bound. This is the content of Lemma 4, which is essentially Lemma 20 in Zerner (1997). In Zerner's proof he uses first passage percolation times to help prove a lower bound large deviation result. We basically illustrate the parts of his proof that we need without the use of first passage percolation times.

Note that Theorem 1 will follow from the properties of μ and I (Section 2), the upper bound inequality (Section 3, Theorem 2) and the lower bound inequality (Section 4, Theorem 3).

In Section 5 we conclude this article with some remarks.

2. Preliminaries.

We summarize some important properties of the functions μ and I in the following lemmas:

LEMMA 1. *Assume the medium is stationary, ergodic and purely random. If $z > -A$, then $E_x[\exp(\int_0^{\tau_0}(\xi(\eta_s) + z)\,ds)] = \infty$ for all $x \in Z^d \backslash \{0\}$; hence $\mu(z; \mathrm{e}) = \infty$ for all $\mathrm{e} \in R^d \backslash \{0\}$, $\hat{P}$-a.s.*

PROOF. From the pure randomness assumption there exists for each $\varepsilon > 0$, $\hat{P}$-a.s., a $c = c(\hat{\omega}) \in Z^d$ and a box $B = B_L$ of linear size $2L + 1$ centered at c such that $\xi(y) > A - \varepsilon$ for all $y \in B_L$ and $0 \notin B_L$. Suppose $z = -A + \delta$ for some positive δ. Choose L so large that the smallest eigenvalue $\lambda^{(L)}$ of $-\tilde{\Delta}$ associated with B_L with Dirichlet boundary satisfies $\lambda^{(L)} < \delta/3$, and take $\varepsilon = \delta/3$.

If $x \neq 0$ we show that the trajectories that hit $c = c(\hat{\omega})$ at time 1 before hitting 0, stay in B_L for a long time before returning to c, then in the last unit of time hit the site 0 from c, already make the expected value infinite,

$$
\begin{aligned}
&E_x\left[\exp\left(\int_0^{\tau_0}(\xi(\eta_s) + z)\,ds\right)\right] \\
&\quad \geq a_1 \exp((-A + \delta)) \\
&\qquad \times E_{c(\hat{\omega})}\left[\exp\left(\int_0^{s}(\xi(\eta_t) + z)\,dt\right); \eta_t \in B_L, t \in [0, s], \text{ and } \eta_s = c(\hat{\omega})\right] \\
&\qquad \times a_3 \exp((-A + \delta))
\end{aligned}
$$

$$\geq a_1 a_3 \exp(2(-A+\delta)) \cdot a_2 \exp(-\lambda^{(L)} s) \exp([(-A+\delta)+(A-\varepsilon)]s)$$
$$\geq a_1 a_2 a_3 \exp(2(-A+\delta)) \exp\left(-\frac{\delta}{3}s\right) \exp\left(\left(\delta - \frac{\delta}{3}\right)s\right)$$
$$= a_1 a_2 a_3 \exp(2(-A+\delta)) \exp\left(\frac{\delta}{3}s\right),$$

and since s is arbitrary, the expected value is infinite. Here a_1 is the positive probability that the particle starting at x hits c in a unit time before hitting the origin; a_2 and a_3 are constants having similar descriptions. □

LEMMA 2. *Let* $z < -A$ *and* $\mathrm{e} \in R^d \backslash \{0\}$.

(a) *The function* $\mu(z;\mathrm{e}) = \lim_{t\to\infty}(1/t)\log E_{t\mathrm{e}}[\exp(\int_0^{\tau_0}(\xi(\eta_s)+z)\,ds)] > -\infty$ *exists* $\hat{P}$*-a.s., is a convex function in* z, *and* $a\mu(z;\mathrm{e}) = \mu(z;a\mathrm{e})$ *for any* $a > 0$.

(b) $\mu(z;\mathrm{e})$ *is a concave function of* e.

(c) $\mu(z;\mathrm{e})$ *tends to* $-\infty$ *as* z *tends to* $-\infty$.

(d) *The Legendre transform* $I(y;\mathrm{e})$ *(see (1.3)) of* $\mu(z;\mathrm{e})$ *defined above is a convex function of both its arguments, is a strictly decreasing function of* $y > 0$ *and* $I(y;\mathrm{e})$ *tends to* ∞ *as* $y \downarrow 0$.

(e) $vI(1/v;\mathrm{e})$ *in Theorem 1 equals* $I(1;v\mathrm{e})$ *and is convex in* $\mathrm{f} = v\mathrm{e}$, *hence in* v *for fixed* e.

(f) *The solution* v_{e} *of* $I(1/v_{\mathrm{e}};\mathrm{e}) = 0$ *is unique.*

PROOF. If we define $\overline{\mu}(s,t;z) = \log E_{t\mathrm{e}}[\exp(\int_0^{\tau_{s\mathrm{e}}}(\xi(\eta_r)+z)\,dr)]$, then $\overline{\mu}$ is superadditive,

$$\overline{\mu}(s,t;z) \geq \overline{\mu}(s,u;z) + \overline{\mu}(u,t;z), \tag{2.1}$$

$\hat{P}$-a.s. for any $0 \leq s < u < t$. To see this, write

$$\begin{aligned}
&E_{t\mathrm{e}}\left[\exp\left(\int_0^{\tau_{s\mathrm{e}}}(\xi(\eta_r)+z)\,dr\right)\right]\\
&\quad= E_{t\mathrm{e}}\left[\exp\left(\int_0^{\tau_{s\mathrm{e}}}(\xi(\eta_r)+z)\,dr\right)\chi\{\tau_{u\mathrm{e}} \leq \tau_{s\mathrm{e}}\}\right]\\
&\qquad+ E_{t\mathrm{e}}\left[\exp\left(\int_0^{\tau_{s\mathrm{e}}}(\xi(\eta_r)+z)\,dr\right)\chi\{\tau_{u\mathrm{e}} > \tau_{s\mathrm{e}}\}\right]\\
&\quad\geq E_{t\mathrm{e}}\left[E_{t\mathrm{e}}\left[\exp\left(\int_0^{\tau_{u\mathrm{e}}}(\xi(\eta_r)+z)\,dr\right)\exp\left(\int_{\tau_{u\mathrm{e}}}^{\tau_{s\mathrm{e}}}(\xi(\eta_r)+z)\,dr\right)\right.\right.\\
&\qquad\qquad\qquad\left.\left.\times\,\chi\{\tau_{u\mathrm{e}} \leq \tau_{s\mathrm{e}}\}\,|\,\mathscr{F}_{\tau_{u\mathrm{e}}}\right]\right]\\
&\qquad+ E_{t\mathrm{e}}\left[\exp\left(\int_0^{\tau_{u\mathrm{e}}}(\xi(\eta_r)+z)\,dr\right)\chi\{\tau_{u\mathrm{e}} > \tau_{s\mathrm{e}}\}\right]
\end{aligned}$$

$$\geq E_{t\mathrm{e}}\left[\exp\left(\int_0^{\tau_{u\mathrm{e}}}(\xi(\eta_r)+z)\,dr\right)\chi\{\tau_{u\mathrm{e}} \leq \tau_{s\mathrm{e}}\}\right]E_{u\mathrm{e}} \tag{2.2}$$

$$
\begin{aligned}
&\times \left[\exp\left(\int_0^{\tau_{s\mathrm{e}}} (\xi(\eta_r)+z)\,dr\right)\right] \\
&+ E_{t\mathrm{e}}\left[\exp\left(\int_0^{\tau_{u\mathrm{e}}} (\xi(\eta_r)+z)\,dr\right)\chi\{\tau_{u\mathrm{e}} > \tau_{s\mathrm{e}}\}\right] \\
&\times E_{u\mathrm{e}}\left[\exp\left(\int_0^{\tau_{s\mathrm{e}}} (\xi(\eta_r)+z)\,dr\right]\right. \\
&= E_{t\mathrm{e}}\left[\exp\left(\int_0^{\tau_{u\mathrm{e}}} (\xi(\eta_r)+z)\,dr\right)\right] E_{u\mathrm{e}}\left[\exp\left(\int_0^{\tau_{s\mathrm{e}}} (\xi(\eta_r)+z)\,dr\right)\right],
\end{aligned}
$$

where we have used, in the second inequality, the strong Markov property of $\eta_.$ and the fact that $\xi(x)+z<0$. Here $\mathscr{F}_{\tau_a}$ denotes the σ-field of events prior to τ_a and $\chi\{B\}$ is the indicator function of the set B. We see that (2.1) follows after taking logs.

One can easily check that for any $\kappa > 0$,

$$\overline{\mu}(s,t;z) \ \geq\ K_z(\kappa,d)\cdot(t-s),$$

where $K_z(\kappa,d) \geq \kappa z - C$ for some constant $C = C(\kappa,d,A)$. If we let $\theta_{h\mathrm{e}}$ denote the space shift by $h\mathrm{e}$,

$$\theta_{h\mathrm{e}} f(\xi(\cdot)) = f(\xi(\cdot + h\mathrm{e})),$$

then $\overline{\mu}$ has the property

$$\theta_{h\mathrm{e}}\overline{\mu}(s,t;z) = \log E_{t\mathrm{e}}\left[\exp\left(\int_0^{\tau_{s\mathrm{e}}} (\xi(\eta_r + h\mathrm{e})+z)\,dr\right)\right] = \overline{\mu}(s+h,t+h;z)$$

for any h. Thus we can apply the subadditive ergodic theorem [Liggett (1985), Theorem VI.2.6] to the $\overline{\mu}$ to get, for each $z<-A$, the existence of the limit $\mu(z;\mathrm{e}) = \lim_{t\to\infty}(1/t)\overline{\mu}(0,t;z) > -\infty$ for $\hat{P}$-a.s. $\hat{\omega}$. Thus we immediately have this limit holding simultaneously for all rational z. Since the functions $\overline{\mu}$ are each convex in z, the limit $\lim_{t\to\infty}(1/t)\overline{\mu}(0,t;z)$ is also convex for all rational z. Thus the convexity of $\mu(z;\mathrm{e})$ in z follows from Rockafellar ((1970), Theorem 10.8], which also gives us that the limit holds simultaneously for all real z.

The assertion $a\mu(z;\mathrm{e}) = \mu(z;a\mathrm{e})$ immediately follows from the definition of $\mu(z;\mathrm{e})$. This proves (a).

In order to prove (b), we shall need a lemma which in a sense is a stronger version of the subadditive ergodic theorem. The subadditive ergodic theorem gives us the existence of a limit along single rays, where we will now need a result that gives us the existence along "parallel shifts." We now quote a lemma from Zerner [(1997), Lemma 14] in our notation:

LEMMA 3 (Zerner). *Assume that ξ is i.i.d. and $\xi(0)$ has finite second moment and $z<-A$. Then $\hat{P}$-a.s. for all $x, y \in Z^d$,*

$$\lim_{t\to\infty}\frac{1}{t}\log E_{tx}\left[\exp\left(\int_0^{\tau_{ty}} (\xi(\eta_s)+z)\,ds\right)\right] = \mu(z;x-y).$$

The idea of the proof is essentially to use the Borel–Cantelli lemma along subsequences and then use some known estimates to apply the usual subadditive ergodic theorem. As Zerner commented, this lemma applies to continuous, as well as discrete time, random walks. The correspondence of our notation with Zerner's (1997) is as follows. His $\lambda > 0$ is our $-(z+A)$; his function $\alpha_\lambda(\cdot)$ is our $-\mu(z;\cdot)$; and, finally, his $a_\lambda(x, y)$ is our $-\log E_x[\exp(\int_0^{\tau_y}(\xi(\eta_s) - A - \lambda)\, ds)]$, that is, $-\log E_x[\exp(\int_0^{\tau_y}(\xi(\eta_s)+z)\, ds)]$. We remind the reader that our random medium is bounded and so we satisfy the hypotheses of the lemma.

Zerner's work mainly deals with the asymptotic shape theorems associated with the (almost-sure) exponential decay rate of the Green's function. Much of his work centers on trying to prove "uniform" shape theorems for random walks and includes a large deviation analysis in the spirit of Sznitman (1994).

Now we prove the concavity of μ in e. Following essentially identical steps as in (2) above, if $0 < p < 1$, $q = 1-p$, we obtain

$$\begin{aligned}(2.3)\quad & E_{tp\mathrm{e}+tq\mathrm{f}}\left[\exp\left(\int_0^{\tau_0} \xi(\eta_s) + z\, ds\right)\right] \\ &\geq E_{tp\mathrm{e}+tq\mathrm{f}}\left[\exp\left(\int_0^{\tau_{tq\mathrm{f}}} \xi(\eta_s) + z\, ds\right)\right] E_{tq\mathrm{f}}\left[\exp\left(\int_0^{\tau_0} \xi(\eta_s) + z\, ds\right)\right]\end{aligned}$$

Now if we take logs on both sides above and use the above lemma, we obtain (b).

Since $\mu(z;\mathrm{e})$ is explicitly calculated for the case of constant media in Zerner [(1997), Theorem 21], (c) is readily proved by a simple comparison with constant media. For the purpose of proving (c), the following rough estimate also suffices. Let $x = (x_1, \dots, x_d)$ and set

$$\|x\| = \max_{i=1,\dots,d} |x_i|$$

and

$$\overline{u}(x,a) = \exp\left(-\|x\| \cosh^{-1}(1+da)\right), \qquad a > 0.$$

We claim that

$$E_x\left[\exp\left(\int_0^{\tau_0} (\xi(\eta_s) + z)\, ds\right)\right] \leq E_x[\exp((A+z)\tau_0)] \leq \overline{u}(x, -(A+z))$$

for $z < -A$, from which (c) follows easily. Our assertion will be proved by a comparison principle. We only need to check that the second term $\underline{u}(x,a) = E_x[\exp(-a\tau_0)]$ satisfies

$$(-\tilde{\Delta} + a)\underline{u}(x,a) = 0, \qquad x \in Z^d \backslash \{0\},$$
$$\underline{u}(0,a) = 1,$$

while $\overline{u}$ satifies the same equation with equality replaced by $\geq$. Both $\underline{u}$ and $\overline{u}$ tend to 0 as x tends to ∞. The differential inequality is actually equality for those $x \in Z^d$ with $2d-2$ neighbors of norm $\|x\|$, one neighbor of norm

$\|x\|-1$ and one neighbor of norm $\|x\|+1$. The following shows how we choose our factor $\cosh^{-1}(1+da)$:

$$\begin{aligned}\frac{(-\tilde{\Delta}+a)\overline{u}(x,a)}{\overline{u}(x,a)} &= 1-\frac{1}{2d}\cdot(2d-2)-\frac{1}{2d}\cdot\exp(\cosh^{-1}(1+da))\\ &\quad -\frac{1}{2d}\cdot\exp(-\cosh^{-1}(1+da))+a\\ &= \frac{1}{d}(1-\cosh(\cosh^{-1}(1+da))+da)=0.\end{aligned}$$

We omit the simple calculation for the other sites x where there is a strict differential inequality. This proves (c).

By part (b), $yz-\mu(z;\mathrm{e})$ is a convex function of e for each z. Hence the sup over all z is also a convex function. The convexity of I in y comes immediately from the definition. This proves the first part of (d).

Since I is the supremum of lines of negative slope, it is easy to see that $I(y;\mathrm{e})$ is nonincreasing. That $I(y;\mathrm{e})$ is in fact strictly decreasing follows from convexity and that $I(y;\mathrm{e})\downarrow-\infty$ as $y\uparrow\infty$. Now, from part (a) we know that $\mu(z;\mathrm{e})\geq z-C$ [recall the estimate after (2.2)]. Substituting this into the definition of I immediately yields the limit $-\infty$ as $y\uparrow\infty$.

We now show $I(0+;\mathrm{e})=\infty$. First of all, if $y<0$, then $I(y;\mathrm{e})=\infty$. Second, we have that

$$I(y;\mathrm{e})\geq yz-\mu(z;\mathrm{e})$$

for every $z<-A$. Given $M>0$, by part (c) we can choose $z<-A$ so that $\mu(z;\mathrm{e})<-M<0$. For this z we can take $y=M/-2z$ so that $I(y;\mathrm{e})\geq M/2$. Since M is arbitrary and I is decreasing in y we have shown that $\lim_{y\downarrow 0}I(y;\mathrm{e})=\infty$. This completes the proof of (d).

Now we prove (e). By the definition and the last part of (a), $vI(1/v;\mathrm{e})=\sup_{z<-A}[z-v\mu(z;\mathrm{e})]=\sup_{z<-A}[z-\mu(z;v\mathrm{e})]=I(1;v\mathrm{e})$. The last term is convex in $\mathrm{f}=v\mathrm{e}$ since it is the supremum of a family of convex functions in f by (b). This completes (e).

Since $I(y;\mathrm{e})$ is strictly decreasing, goes to ∞ as $y\downarrow 0$ and goes to $-\infty$ as $y\uparrow\infty$, we see that I crosses 0 at a unique point $1/v_{\mathrm{e}}$. The variational formula (1.4) follows easily. This completes (f) as well as the entire proof. □

3. Upper bound.

We first obtain an upper bound on the solution.

THEOREM 2. *For any $v>0$ and any $\mathrm{e}\in R^d$,*

$$\limsup_{t\to\infty}\frac{1}{t}\log\sup_{r\geq vt}u(t,r\mathrm{e})\leq-\left[vI\left(\frac{1}{v};\mathrm{e}\right)\vee 0\right],\qquad \hat{P}\text{-a.s.}$$

PROOF. By a simple comparison we see that the solution u is bounded by 1; thus u is logarithmically bounded by 0. If v is such that $I(1/v;\mathrm{e})\leq 0$, then there is nothing to prove, so we will assume that $I(1/v;\mathrm{e})>0$, that is,

$v > v_{\mathrm{e}}$. Let $r \geq vt$. Then by (1.1) and using the strong Markov form of the Feynman–Kac representation and the fact that $0 \leq u \leq 1$,

$$\begin{aligned} u(t, r\mathrm{e}) &= E_{r\mathrm{e}}\left\{u(t-\tau_0, 0)\exp\left(\int_0^{\tau_0} \xi(\eta_s)[1-u(t-s,\eta_s)]\,ds\right)\chi_{\{\tau_0\leq t\}}\right\} \\ &\leq E_{r\mathrm{e}}\left\{\exp\left(\int_0^{\tau_0} \xi(\eta_s)\,ds\right)\chi_{\{\tau_0\leq t\}}\right\} \\ &\leq \exp(-zt)E_{r\mathrm{e}}\left\{\exp\left(\int_0^{\tau_0}(\xi(\eta_s)+z)\,ds\right)\right\}. \end{aligned}$$

Here, of course, $z < -A$ [see display preceding (1.2)]. In the case when $z > -A$ we get a triviality since we know $u \leq 1$. Now, taking logs and dividing by t, we see that

$$\frac{1}{t}\log u(t, r\mathrm{e}) \leq -z + \frac{1}{t}\log E_{r\mathrm{e}}\left\{\exp\left(\int_0^{\tau_0}(\xi(\eta_s)+z)\,ds\right)\right\}.$$

Appealing to the fact that $\mu(z;\mathrm{e}) < 0$ whenever $z < -A$, we have for each $0 < a < 1$ the existence of $T > 0$ such that for $t > T$ (and $r > vt$),

$$\frac{1}{r}\log E_{r\mathrm{e}}\left\{\exp\left(\int_0^{\tau_0}(\xi(\eta_s)+z)\,ds\right)\right\} < a\mu(z;\mathrm{e}).$$

Thus,

$$\frac{1}{t}\sup_{r>vt} E_{r\mathrm{e}}\left\{\exp\left(\int_0^{\tau_0}(\xi(\eta_s)+z)\,ds\right)\right\} \leq \frac{1}{t}\sup_{r>vt} r\,a\mu(z;\mathrm{e}) \leq v\,a\mu(z;\mathrm{e}).$$

Letting $t \to \infty$ and then letting a tend to 1, we have

$$\begin{aligned} \limsup_{t\to\infty}\frac{1}{t}\log\sup_{r\geq vt} u(t, r\mathrm{e}) &\leq -z + v\,\mu(z;\mathrm{e}) \\ &= -v\left[\frac{1}{v}z - \mu(z;\mathrm{e})\right]. \end{aligned}$$

The result follows from taking the infimum over all z. □

REMARK 3.1. One can adapt the above argument to get the stronger result for any $v > 0$,

$$\limsup_{t\to\infty}\frac{1}{t}\log E_{vt\mathrm{e}}\left[\exp\left(\int_0^t \xi(\eta_s)\,ds\right)\chi_{\{\eta_t=0\}}\right] \leq -vI\left(\frac{1}{v};\mathrm{e}\right).$$

To see this, notice that $u(t-\tau_0, 0) \leq \exp(A(t-\tau_0))$, $\hat{P}$-a.s., so that for $z < 0$,

$$\begin{aligned} u(t, vt\mathrm{e}) &\leq E_{vt\mathrm{e}}\left[\exp\left(\int_0^{\tau_0} \xi(\eta_s)\,ds\right)\exp(A(t-\tau_0))\chi_{\{\tau_0\leq t\}}\right] \\ &= E_{vt\mathrm{e}}\left[\exp\left(\int_0^{\tau_0}(\xi(\eta_s)-A)\,ds\right)\exp(At)\chi_{\{\tau_0\leq t\}}\right] \\ &\leq \exp((A-z)t)E_{vt\mathrm{e}}\left[\exp\left(\int_0^{\tau_0}(\xi(\eta_s)-A+z)\,ds\right)\chi_{\{\tau_0\leq t\}}\right]. \end{aligned}$$

Now following as in the proof above, we obtain

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\log E_{vt\mathrm{e}}\bigg[\exp\bigg(\int_0^t \xi(\eta_s)\,ds\bigg)\chi_{\{\eta_t=0\}}\bigg]\\ \le -\sup_{z<0}\big[(z-A)-v\mu(z-A;\mathrm{e})\big]\\ = -vI\bigg(\frac{1}{v};\mathrm{e}\bigg).\end{aligned} \tag{3.1}$$

The above argument is essentially the one Sznitman [(1994), see (2.12)] used, although we found this out after we proved it.

4. Lower bound. In order to introduce the next lemma, we need some notation. For $z < 0$, let

$$\beta(z;\mathrm{e}) = \mu(z-A;\mathrm{e})$$

and let $\beta'_-(z;\mathrm{e})$ [resp., $\beta'_+(z;\mathrm{e})$] denote the left- (resp., right-) hand derivative in z of the function $\beta(z;\mathrm{e})$. The function β is convex in z and hence has a derivative everywhere outside of a countable set of points z; so $\beta'_-(z;\mathrm{e}) = \beta'_+(z;\mathrm{e})$ except for possibly a countable set of z values. Finally, let $Q_{t\mathrm{e},z}$ be the path probability measures defined by

$$Q_{t\mathrm{e},z}[\mathscr{A}] = \frac{E_{t\mathrm{e}}[\exp\{\int_0^{\tau_0}(\xi(\eta_s)-A+z)\,ds\};\mathscr{A}]}{E_{t\mathrm{e}}[\exp\{\int_0^{\tau_0}(\xi(\eta_s)-A+z)\,ds\}]},$$

where $E[X;\mathscr{A}]$ is the expectation of X restricted to the set $\mathscr{A}$. The following lemma plays an essential role in the proof of Theorem 2 (see below). In effect, this lemma saves us from having to prove regularity of the function $\beta(\cdot;\mathrm{e})$. Indeed, in the one-dimensional case, the function $\beta(\cdot;\mathrm{e})$ is actually continuously differentiable (and convex) so that a large deviation lower bound follows for the path measures $Q_{t\mathrm{e},z}$, thus simplifying the proof of Theorem 2 considerably. In the case of dimensions greater than 1, the smoothness of $\beta(\cdot;\mathrm{e})$ is not known. Zerner [(1997), Lemma 20] proved this lemma in the setting of discrete-time random walks. We now state his lemma for our situation.

LEMMA 4. *Fix* $\mathrm{e}\in R^d$. *Let* $z_0 < 0$ *be such that* $\beta'_-(z_0;\mathrm{e}) < \beta'_+(z_0;\mathrm{e})$ *and let* $\mathscr{I}$ *be any open subinterval of* $[\beta'_-(z_0;\mathrm{e}),\beta'_+(z_0;\mathrm{e})]$. *Then*

$$\liminf_{t\to\infty}\frac{1}{t}\log Q_{t\mathrm{e},z_0}\bigg[\frac{\tau_0}{t}\in\mathscr{I}\bigg] = 0,\qquad \hat{P}\text{-a.s.}$$

PROOF. We first state the following law of large numbers type result for the first hitting time τ_0:

$$\lim_{t\to\infty} Q_{t\mathrm{e},z}\bigg[\frac{\tau_0}{t}\in(C,B)\bigg] = 1 \tag{4.1}$$

if $C < \beta'_-(z;\mathrm{e}) \le \beta'_+(z;\mathrm{e}) < B$ (note that β'_- and β'_+ are allowed to be equal here). To see this just notice that for any $0 > \theta > z$,

$$\limsup_{t\to\infty} \frac{1}{t} \log Q_{t\mathrm{e},z}\left[\frac{\tau_0}{t} \ge B\right] \le \theta B + \limsup_{t\to\infty} \frac{1}{t} \log E^{Q_{t\mathrm{e},z}}[\exp(-\theta\tau_0)] \\ = \theta B + \beta(z-\theta;\mathrm{e}) - \beta(z;\mathrm{e}), \tag{4.2}$$

which is negative for θ sufficiently close to 0. We obtain a similar statement for the event $[\tau_0/t \le C]$, thus implying (4.1).

One essential subcase of the present lemma concerns the left end $\beta'_-(z_0;\mathrm{e})$: For any $\delta > 0$,

$$b \equiv \liminf_{t\to\infty} \frac{1}{t} \log Q_{t\mathrm{e},z_0}\left[\left|\frac{\tau_0}{t} - \beta'_-(z_0;\mathrm{e})\right| < \delta\right] = 0. \tag{4.3}$$

This can be proved using the tilted measure $Q_{t\mathrm{e},z}$ with z approaching z_0 from below and $\beta'_-(z;\mathrm{e}) = \beta'_+(z;\mathrm{e}) = \beta'(z;\mathrm{e})$. Indeed, from the definition of the tilted measures,

$$b \ge \liminf_{t\to\infty} \frac{1}{t} \log Q_{t\mathrm{e},z}\left[\left|\frac{\tau_0}{t} - \beta'_-(z_0;\mathrm{e})\right| < \delta\right] + \big(\beta(z;\mathrm{e}) - \beta(z_0;\mathrm{e})\big). \tag{4.4}$$

It follows from (4.1) and the fact that β'_- is left continuous that the first term of the right-hand side of (4.4) equals 0. The second term tends to 0 as $z \uparrow z_0$. Since we already know that $b \le 0$, (4.3) is proved. By the same kind of argument we have the counterpart of (4.3) for the right end $\beta'_+(z_0;\mathrm{e})$:

$$\liminf_{t\to\infty} \frac{1}{t} \log Q_{t\mathrm{e},z_0}\left[\left|\frac{\tau_0}{t} - \beta'_+(z_0;\mathrm{e})\right| < \delta\right] = 0. \tag{4.5}$$

Now that the random variable τ_0/t is understood near the two endpoints $\beta'_-(z_0;\mathrm{e})$ and $\beta'_+(z_0;\mathrm{e})$, we are ready to deal with any intermediate value $\rho\beta'_-(z_0;\mathrm{e}) + (1-\rho)\beta'_+(z_0;\mathrm{e})$ with $0 < \rho < 1$. What we need to prove is roughly the convexity of the large deviation rate function for τ_0/t. Such a "convexity" property follows from a simple consideration of having two different τ_0/t asymptotics at different time intervals. We now give the detail. We have the following containment:

$$\begin{aligned} &\left[\left|\frac{\tau_0}{t} - \Big(\rho\beta'_-(z_0;\mathrm{e}) + (1-\rho)\beta'_+(z_0;\mathrm{e})\Big)\right| < \delta\right] \\ &\quad\supset \left[\left|\frac{\tau_{\rho t\mathrm{e}}}{t} - (1-\rho)\beta'_+(z_0;\mathrm{e})\right| < (1-\rho)\delta,\right. \\ &\qquad\qquad \left.\left|\frac{\tau_0 - \tau_{\rho t\mathrm{e}}}{t} - \rho\beta'_-(z_0;\mathrm{e})\right| < \rho\delta\right] \\ &\quad= \left[\left|\frac{\tau_{\rho t\mathrm{e}}}{t(1-\rho)} - \beta'_+(z_0;\mathrm{e})\right| < \delta, \left|\frac{\tau_0 - \tau_{\rho t\mathrm{e}}}{t\rho} - \beta'_-(z_0;\mathrm{e})\right| < \delta\right]. \end{aligned} \tag{4.6}$$

Applying the strong Markov property to $\tau_{t\rho \mathrm{e}}$ and applying (4.3), with $t\rho$ replacing t, and (4.5) with $t(1-\rho)$ replacing t, then completes the proof of Lemma 4. □

REMARK 4.1. A closer look at (4.1) reveals that if $\beta'_-(z;\mathrm{e}) < \beta'_-(z_0;\mathrm{e})$ for all $z < z_0$, then the event in (4.4) can be replaced by the event

$$\left[\beta'_-(z_0;\mathrm{e}) - \delta < \frac{\tau_0}{t} < \beta'_-(z_0;\mathrm{e})\right],$$

so that the lemma still holds true in this case. Indeed, look at the set in (4.4). As long as $\beta'_-(z_0;\mathrm{e}) - \delta \equiv C < \beta'_-(z;\mathrm{e}) \le \beta'_+(z;\mathrm{e}) < B$ is satisfied, we can apply (4.1). Now, by the way our sequence of z is chosen, the middle inequality is actually equality $\beta'_-(z;\mathrm{e}) = \beta'_+(z;\mathrm{e})$. Furthermore, by assumption we can take $B = \beta'_-(z_0;\mathrm{e})$. This justifies the remark.

THEOREM 3. *Fix* $\mathrm{e} \in R^d$. *For any* $v > 0$ *we have* $\hat{P}$*-a.s.,*

$$\liminf_{t\to\infty} \frac{1}{t} \log u(t, vt\mathrm{e}) \ge -\left[vI\left(\frac{1}{v};\mathrm{e}\right) \vee 0\right].$$

PROOF. We start by first showing the following inequality for $v > 0$:

$$\liminf_{t\to\infty} \frac{1}{t} \log E_{vt\mathrm{e}}\left\{\exp\left(\int_0^t \xi(\eta_s)\,ds\right)\chi_{\{\eta_t=0\}}\right\} \ge -vI\left(\frac{1}{v};\mathrm{e}\right). \tag{4.7}$$

Set $z_0 = \inf\{z\colon \beta'_-(z;v\mathrm{e}) \ge 1\}$ and notice that $-vI(1/v;\mathrm{e}) = A - z_0 + v\beta(z_0;\mathrm{e})$. Now we prove (4.7) by proving the equivalent problem:

$$c \equiv \liminf_{t\to\infty} \frac{1}{t} \log \frac{E_{vt\mathrm{e}}\{\exp(\int_0^t(\xi(\eta_s) - A + z_0)\,ds)\chi_{\{\eta_t=0\}}\}}{E_{vt\mathrm{e}}\{\exp(\int_0^{\tau_0}(\xi(\eta_s) - A + z_0)\,ds)\}} \ge 0. \tag{4.8}$$

Let $\varepsilon > 0$. Then

$$\begin{aligned} c \ge{}& \liminf_{t\to\infty} \frac{1}{t} \log \\ &\times \frac{E_{vt\mathrm{e}}[\exp(\int_0^{\tau_0}(\xi(\eta_s) - A + z_0)\,ds; \eta_t = 0, 1-\varepsilon < \tau_0/t \le 1]\exp(\varepsilon t(-A+z_0))}{E_{vt\mathrm{e}}[\exp(\int_0^{\tau_0}(\xi(\eta_s) - A, +z_0)\,ds)]}. \end{aligned} \tag{4.9}$$

Now, by the way we have defined the path measures, (4.9) implies

$$c \ge \varepsilon(-A + z_0) + \liminf_{t\to\infty} \frac{1}{t} \log Q_{tv\mathrm{e},\, z_0}\left[\eta_t = 0, 1 - \varepsilon < \frac{\tau_0}{t} \le 1\right].$$

If we let $p(s, x)$ denote the time and space homogeneous transition density of the process $\eta_.$, applying the strong Markov property to the stopping time τ_0 gives

$$\begin{aligned} c \ge{}& \varepsilon(-A + z_0) + \liminf_{t\to\infty} \frac{1}{t} \log Q_{tv\mathrm{e},\, z_0}\left[1 - \varepsilon < \frac{\tau_0}{t} \le 1\right] \\ &+ \liminf_{t\to\infty} \frac{1}{t} \log E^{Q_{tv\mathrm{e}, z_0}}\left[p(t - \tau_0, 0) | 1 - \varepsilon < \frac{\tau_0}{t} \le 1\right]. \end{aligned} \tag{4.10}$$

Since $p(t-\tau_0, 0) \geq p(\varepsilon t, 0)$ and it is easily checked that $\liminf_{t\to\infty}(1/t) \log p(\varepsilon t, 0) = 0$, the last term on the right-hand side of (4.10) is 0. Now by the way z_0 was chosen and the left-continuity of β'_-, it is clear that either $\beta'_-(z_0; v\mathrm{e}) < 1$ or $\beta'_-(z; v\mathrm{e}) < \beta'_-(z_0; v\mathrm{e}) = 1$ for all $z < z_0$. In the first case, the hypotheses of Lemma 4 are fulfilled. In the second case, we appeal to Remark 4.1. Thus we can apply Lemma 4 to the second term on the right-hand side of (4.10) to see that this is 0. By letting ε tend to 0, (4.8) follows.

To complete the proof of Theorem 3 we will show that if $v > v_{\mathrm{e}}$, then

$$\liminf_{t\to\infty} \frac{1}{t} \log u(t, tv\mathrm{e}) \geq \liminf_{t\to\infty} \frac{1}{t} \log E_{vt\mathrm{e}}\left\{\exp\left(\int_0^t \xi(\eta_s)\, ds\right)\chi_{\{\eta_t=0\}}\right\} \tag{4.11}$$

and take care of $0 < v \leq v_{\mathrm{e}}$ later. The argument resembles that in Freidlin (1985), so we will briefly outline it, emphasizing the distinct difficulty due to the high dimension $d \geq 2$. The main idea is to look at the Feynman–Kac representation and use Theorem 1. We integrate only over those trajectories $\eta_{.}$ that stay in the region where the solution u is small. We state a lemma to make this more precise.

LEMMA 5. *Given* $\mathrm{e} \in R^d \backslash \{0\}$ *and* $v > v_{\mathrm{e}}$, *there exists a* $\delta > 0$ *such that for any* $h > 0$, *there is a* $T_{\delta, h} = T_{\delta, h, \mathrm{e}}(\hat{\omega})$ *with* $\hat{P}[T_{\delta, h} < \infty] = 1$ *and*

$$\sup_{t > T_{\delta, h},\, |\mathrm{f}-\mathrm{e}| < \delta} u(t, tv\mathrm{f}) \leq h.$$

In addition to Lemma 5 we will need:

LEMMA 6. *For any* $v > 0$ *and* $\mathrm{e} \in R^d$ *there exists a nonnegative function* J *on* R^d *such that*

$$\liminf_{s\to\infty} \frac{1}{s} \log P_{vs\mathrm{e}}[\eta_s = 0] \geq -J(v\mathrm{e}).$$

We postpone the proofs of Lemmas 5 and 6 until the end of the section.

Let us define the quantities

$$\sigma_L = \inf[s \geq 0\colon |\eta_s - \eta_0 + vs\mathrm{e}| \geq L],$$

$$H_{h,L}(s,t) = \log E_{tv\mathrm{e}}\left[\exp\left(\int_0^{t-s} \xi(\eta_r)[1-h]\, dr\right)\chi_{\{\eta_{t-s}=vs\mathrm{e},\, \sigma_L > t-s\}}\right],$$

$$H(s,t) = \log E_{tv\mathrm{e}}\left[\exp\left(\int_0^{t-s} \xi(\eta_r)\, dr\right)\chi_{\{\eta_{t-s}=vs\mathrm{e}\}}\right],$$

for $L > 0$.

Let $\varepsilon > 0$. From Lemma 5 we notice that (a) if $t > T_{\delta,h}/\varepsilon$, then $u(s, sv\mathrm{f}) \leq h$ for all $s \geq \varepsilon t$ and all f such that $|\mathrm{f}-\mathrm{e}| < \delta$ and (b) if $0 < L \leq T_{\delta, h} v\delta$, then the ball of radius L centered at $\varepsilon tv\mathrm{e}$ is contained in the cone $\{a\mathrm{f}\colon a \geq 0, |\mathrm{f}-\mathrm{e}| \leq \delta\}$; in fact, so are all the balls of radius L centered at $sv\mathrm{e}$, $s \geq \varepsilon t$.

We would like to remark at this point that Lemma 5 is not needed for the one-dimensional case, but is needed in higher dimensions.

Now let L be chosen as in (b) above. Then by using the strong Markov property (as in the proof of Theorem 2) and Lemma 5, we obtain

$$u(t, tv\mathrm{e}) \geq E_{tv\mathrm{e}}\left[\exp\left(\int_0^{t-\varepsilon t} \xi(\eta_s)[1-h]\, ds\right)\chi_{\{\eta_{t-\varepsilon t}=v\varepsilon t\mathrm{e},\, \sigma_L > t-\varepsilon t\}}\right] P_{v\varepsilon t\mathrm{e}}[\eta_{\varepsilon t}=0].$$

Now if we notice that the function $H_{h,L}$ is superadditive: $H_{h,L}(s,t) \geq H_{h,L}(s,r)+H_{h,L}(r,t)$ whenever $s<r<t$, and that it satisfies the hypotheses of the subadditive ergodic theorem, then by Lemma 6 we arrive at

$$\liminf_{t\to\infty} \frac{1}{t}\log u(t, tv\mathrm{e}) \geq (1-\varepsilon)\liminf_{t'\to\infty}\frac{1}{t'}H_{h,L}(0,t') - \varepsilon J(v\mathrm{e}),$$

where $t' = (1-\varepsilon)t$. Letting ε tend to 0, we have

$$\liminf_{t\to\infty} \frac{1}{t}\log u(t, tv\mathrm{e}) \geq \liminf_{t\to\infty}\frac{1}{t}H_{h,L}(0,t).$$

The function $(1/t)H_{h,L}(0,t)$ is monotone increasing in $t \uparrow \infty$. It also increases to $(1/t)H(0,t)$ as $h \downarrow 0$ and $L \uparrow \infty$. So if we take the limit on the right-hand side above as $h \downarrow 0$ and $L \uparrow \infty$, we can interchange limits to give

$$\liminf_{t\to\infty} \frac{1}{t}\log u(t, tv\mathrm{e}) \geq \liminf_{t\to\infty}\frac{1}{t}H(0,t).$$

Since this is exactly (4.11), this completes the proof in the case $v > v_\mathrm{e}$.

For $0 < v \leq v_\mathrm{e}$, it is easy to see the desired result

$$\liminf_{t\to\infty} \frac{1}{t}\log u(t, vt\mathrm{e}) \geq 0.$$

For all $\varepsilon > 0$, let us estimate the contribution from the trajectories that are in $vt\mathrm{e}$ at time $(1-(v/(v_\mathrm{e}+\varepsilon))t$ and, in the remaining time of length $(v/(v_\mathrm{e}+\varepsilon))t$, travel to the origin (thus, the speed is $v_\mathrm{e}+\varepsilon$). We have

$$\begin{aligned} u(t, vt\mathrm{e}) &= E_{vt\mathrm{e}}\left[\exp\left(\int_0^{(1-v/(v_\mathrm{e}+\varepsilon))t} \xi(\eta_s)[1-u(t-s,\eta_s)]\, ds\right)\right. \\ &\qquad\qquad \left.\times\, u\left(\frac{v}{v_\mathrm{e}+\varepsilon}t, \eta_{(1-v/(v_\mathrm{e}+\varepsilon))t}\right)\right] \\ &\geq P_{vt\mathrm{e}}\left[\eta_{(1-v/(v_\mathrm{e}+\varepsilon))t} = vt\mathrm{e}\right] u\left(\left(\frac{v}{v_\mathrm{e}+\varepsilon}\right)t, vt\mathrm{e}\right). \end{aligned}$$

Thus if we take logs on both sides and divide by t, the first term is negligible, while the second term falls into the case $v = v_{\mathrm{e}} + \varepsilon > v_{\mathrm{e}}$ which we just proved,

$$\begin{aligned}\liminf_{t\to\infty} \frac{1}{t} \log u\left(\frac{v}{v_{\mathrm{e}}+\varepsilon}t, vt\mathrm{e}\right) &= \frac{v}{v_{\mathrm{e}}+\varepsilon} \cdot \liminf_{s\to\infty} \frac{1}{s} \log u\big(s, (v_{\mathrm{e}}+\varepsilon)s\mathrm{e}\big) \\ &= -\frac{v}{v_{\mathrm{e}}+\varepsilon}(v_{\mathrm{e}}+\varepsilon)\cdot I\left(\frac{1}{v_{\mathrm{e}}+\varepsilon}; \mathrm{e}\right) \\ &= -vI\left(\frac{1}{v_{\mathrm{e}}+\varepsilon}; \mathrm{e}\right),\end{aligned}$$

and this tends to 0 as ε tends to 0. The proof of Theorem 3 is complete once we prove Lemmas 5 and 6.

PROOF OF LEMMA 5. We prove this result in the case $d = 2$. The case of higher dimensions follows analogously. Fix $\mathrm{e}' = a\mathrm{e} \in R^2$ with $0 < a < 1$ and $I(1/v; \mathrm{e}') > 0$. From Lemma 2(d), there exists $\mathrm{b} \in R^2$ with $I(1/v; \mathrm{b}) > 0$ and b is not collinear with e. Thus for $\mathrm{a} \equiv (\mathrm{e}' + \mathrm{b})/2$ we have $I(1/v; \mathrm{a}) > 0$. Let $1/4 < p < 3/4$ and $q = 1 - p$. We will explain this range for p later, but we will say that any neighborhood of 1/2 that is bounded away from 0 and 1 will suffice. Now, define $\mathrm{g} = \mathrm{g}(p)$ to be the point in R^2 such that $p\mathrm{g} + q\mathrm{b} = \mathrm{a}$, and notice that $\mathrm{g}(1/2) = \mathrm{e}'$. If we let w denote the solution to the linearized problem

$$\begin{aligned}\frac{\partial w}{\partial t} &= \tilde{\Delta} w + \xi(x) w, \qquad t > 0, x \in Z^d, \\ w(0, x) &= 1_0(x),\end{aligned} \tag{4.12}$$

then by the strong Markov property we have

$$\begin{aligned}w(t, tv\mathrm{a}) &\ge E_{tv\mathrm{a}}\left[\exp\left(\int_0^t \xi(\eta_s)\,ds\right) 1_0(\eta_t) 1_{ptv\mathrm{g}}(\eta_{qt})\right] \\ &= E_{tv\mathrm{a}}\left[E_{tv\mathrm{a}}\left[\exp\left(\int_0^{qt} \xi(\eta_s)\,ds\right)\exp\left(\int_{qt}^t \xi(\eta_s)\,ds\right)\right.\right. \\ &\qquad\qquad \left.\left. \times 1_0(\eta_t) 1_{ptv\mathrm{g}}(\eta_{qt}) | \mathscr{F}_{qt}\right]\right] \\ &= E_{tvq\mathrm{b}}\left[\exp\left(\int_0^{qt} \tilde{\xi}(\eta_s)\,ds\right) 1_0(\eta_{qt})\right] E_{tvp\mathrm{g}}\left[\exp\int_0^{pt} \xi(\eta_s)\,ds\Big) 1_0(\eta_{pt})\right] \\ &\equiv \tilde{w}(qt, qtv\mathrm{b}) \cdot w(pt, ptv\mathrm{g}),\end{aligned}$$

where $\tilde{\xi}(x) = \xi(x + ptv\mathrm{g})$ is the shifted media and

$$\tilde{w}(t, x) = E_x\left[\exp\left(\int_0^t \tilde{\xi}(\eta_s)\,ds\right) 1_0(\eta_t)\right].$$

Notice by comparing (4.12) with (1.1) we have $u \le w$. Now taking logs in the above equation, we see that

$$\frac{1}{t}\log w(pt, ptv\mathrm{g}) \le \frac{1}{t}\log w(t, tv\mathrm{a}) - \frac{1}{t}\log \tilde{w}(qt, qtv\mathrm{b}). \tag{4.13}$$

The idea here is that we are going to use (4.13) to uniformly control (in p) the behavior of the function w at the point g (and hence u at the point g) by controlling the behavior at the two points a and b only.

First, we assert that, based on Lemma 3, the last term in (4.13) behaves as the "unshifted" term. The assertion is proved at the end. In order to use the large time behavior of $\tilde{w}$, we naturally restrict q to stay away from 0. For example, take $q > 1/4$. Then for any $\varepsilon > 0$ we can find $T_1 = T_1(\hat{\omega})$ such that the rightmost term is greater than $-qvI(1/v;\mathrm{b}) - \varepsilon$ for $t > T_1$. Notice, by the way, that if we assume $q < 3/4$, we have for $t > T_1$ the rightmost term in (4.13) greater than $-(3v/4)I(1/v;\mathrm{b}) - \varepsilon$. This is good for us because we have bounded the last term on the right-hand side of (4.13) by a quantity which is independent of $1/4 < q < 3/4$ and, hence, of $1/4 < p < 3/4$.

Second, by Theorem 2, there exists T_2 such that the first term on the right-hand side of (4.13) is less than $-vI(1/v;\mathrm{a}) + \varepsilon$ for $t > T_2$. If we take $T = T_1 \vee T_2$ we will have for each $1/4 < p < 3/4$ that the left-hand side of (4.13) is less than 0 when $vI(1/v;\mathrm{a}) > (3v/4)I(1/v;\mathrm{b}) + 2\varepsilon$.

Now we can find a b such that

$$I\left(\frac{1}{v};\mathrm{a}\right) - \frac{3}{4}I\left(\frac{1}{v};\mathrm{b}\right) > 0.$$

We would like to remark at this point that if p does not stay away from 1, then we cannot guarantee the existence of such a b, Also note that although a depends on b, this does not cause any problems here.

We can take $\varepsilon > 0$ to be such that $vI(1/v;\mathrm{a}) - (3v/4)I(1/v;\mathrm{b}) > 2\varepsilon$. With this ε and the above remarks, it is easy to show that

$$\limsup_{t\to\infty} \frac{1}{t}\log \sup_{1/4<p<3/4} w(t, tv\mathrm{g}(p)) < 0.$$

Now using the argument in Theorem 2, we get

$$\limsup_{t\to\infty} \frac{1}{t}\log \sup_{k>1} \sup_{1/4<p<3/4} w(t, tvk\mathrm{g}(p)) < 0.$$

The set

$$\mathscr{G} = \left[k\mathrm{g}(p)\colon k > 1, 1/4 < p < 3/4\right]$$

is open and contains e since $\mathrm{g}(1/2) = \mathrm{e}' < \mathrm{e}$. Thus $\mathrm{e} \cap \partial\mathscr{G} = \varnothing$. Therefore, we can take $\delta = \min_{\mathrm{k}\in\partial\mathscr{G}} |\mathrm{e} - \mathrm{k}| > 0$. Then it is easy to see that given any $h > 0$

we can find a $T_{\delta,h} \geq T$ so that

$$h \geq \exp\left\{-T_{\delta,h}\left[vI\left(\frac{1}{v};\mathrm{a}\right) - \frac{3}{4}vI\left(\frac{1}{v};\mathrm{b}\right) - 2\varepsilon\right]\right\} \geq w(t, tv\mathrm{f}) \geq u(t, tv\mathrm{f})$$

for any $\mathrm{f} \in \mathscr{G}$ and any $t > T_{\delta,h}$.

Now we prove the assertion. Note that the solution w of (4.12) is the left-hand side of (4.7), and (4.7) with $\tilde{\xi}$ replacing ξ is exactly what we need. The proof of (4.7) is given in (4.8), (4.9) and (4.10), using Lemma 4. So Lemma 4 with the probability measure Q defined through $\tilde{\xi}$ needs to be established. This is easily done by using the same proof, together with Lemma 3. The proof is complete. □

PROOF OF LEMMA 6. This is actually quite easy. Let $X_1, X_2, X_3, \dots$ be independent random vectors which are uniformly distributed on $\{\pm\mathrm{e}_1, \dots, \pm\mathrm{e}_d\}$, the unit vectors in Z^d. Let N_t be a Poisson process of intensity 1 which is independent of the X_i. Then

$$\begin{aligned}\frac{1}{t}\log E_0[\exp(\theta\cdot\eta_t)] &= \frac{1}{t}\log E_0\left[E_0[\exp(\theta\cdot(X_1+\cdots+X_{N_t}))\big|N_t]\right]\\ &= \frac{1}{t}\log E_0\left[\left(E_0[\exp(\theta\cdot X_1)]\right)^{N_t}\right]\\ &= E_0[\exp(\theta\cdot X_1)] - 1\\ &\equiv K(\theta).\end{aligned}$$

This is a smooth strictly convex function of $\theta \in R^d$ and, hence, by the Gartner–Ellis theorem, its Legendre transform $J(\mathrm{f}) = \sup_{\theta\in R^d}[\mathrm{f}\cdot\theta - K(\theta)]$ is the large deviation rate function for the process η.

It is easy to see that $J(\mathrm{f}) < \infty$ for all $\mathrm{f} \in R^d$ and $J(\mathrm{f}) = 0$ if and only if $\mathrm{f} = 0$. In fact, an explicit formula for J can be obtained, but we do not need it. □

5. Some remarks.

REMARK 5.1. Let v_{e} be the wave speed in the direction e and let $\delta > 0$ be arbitrary. Let $C_{\mathrm{e},\delta}$ be the cone region

$$C_{\mathrm{e},\delta} = \{a\mathrm{f}\colon a > 0, |\mathrm{f} - \mathrm{e}| < \delta\}.$$

Then the wave speed remains unchanged if we decrease $\xi(x)$ for $x \notin C_{\mathrm{e},\delta}$. This is because both the upper and lower bounds remain effective. Clearly, the upper bound holds. From the proof of the lower bound in Section 4 we see that the trajectories "wandering out of the cone $C_{\mathrm{e},\delta}$" did not contribute at all. This assures us that the lower bound is still true.

REMARK 5.2. What happens to the wave speed v_{e} if we increase $\xi(x)$ for $x \notin C_{\mathrm{e},\delta}$? The speed v_{e} can get a real boost. For example, for $x \notin C_{\mathrm{e},\delta}$ let a large $B > 1$ be multiplied to the original $\xi(x)$: we have $B\xi(x)$ for $x \notin C_{e,\delta}$. In any direction b outside of $\overline{C_{e,\delta}}$, the closure of the cone, Remark 5.1 then implies that v_b is as in the case of $B\xi(x)$ for $x \in Z^d$ random media, which is as large as one desires (by choosing large B). It is now easy to see that the most favorable trajectory, in the case of large B, would be wandering out of the cone $C_{e,\delta}$ first and then take advantage of the enhanced (by large B) wave speed v_b.

REMARK 5.3. It is interesting to determine if $\mu(z;e)$ is in fact a smooth function of z. As mentioned in the Introduction, this is known to be true for one-dimensional random media. For the case of higher-dimensional random media, this is still unknown. If $\mu(z;e)$ is a smooth function, then the proof of the lower bound estimate is greatly simplified. There can be many approaches to showing such a result, but we would like to mention one. For instance, we can show the smoothness of $\mu(z;e)$ by showing the variance bound

$$V^{Q_{te,z}}[\tau_0] = O(t).$$

The smoothness of μ would then follow from the Arzéla–Ascoli theorem. Sznitman (1995) shows that $0 < \limsup_{t\to\infty}(1/t)E^{Q_{te,z}}[\tau_0] < \infty$. It would be interesting to see a central limit theorem proved or disproved $\hat{P}$-a.s. for the (standardized) τ_0 with respect to the tilted measure $Q_{te,z}$, $z < -A$.

REMARK 5.4. We would like to say something in the case $\xi(0)$ is an unbounded nonnegative random variable. It turns out that the speed $v = v_e$ will be infinite for all $e \in R^d \backslash \{0\}$. To see this, let $W_{(k)}$ be the truncation of $\xi(0)$ at k, $k \geq 0$: $W_{(k)} = \xi(0)$ if $\xi(0) \leq k$ and $W_{(k)} = k$ if $\xi(0) > k$. Let $v_{(k)}$ and $\mu_{(k)}(z)$ be the wave front speed and μ-function, respectively, associated with i.i.d. $W_{(k)}$-distributed random media. Here, since we fix an $e \in R^d\backslash\{0\}$, we omit writing the e in our notation. Clearly $v \geq v_{(k)}$ and both $v_{(k)}$ and $\mu_{(k)}$ are nondecreasing in k. The main theorem gives

$$v \geq v_{(k)} = \inf_{z<-k} \frac{z}{\mu_{(k)}(z)} \geq \inf_{z<-k} \frac{z}{\mu_{(0)}(z)}. \tag{5.1}$$

Now $\mu_{(0)}$ can be estimated in all dimensions [see the proof of Lemma 2(a)]. We will demonstrate $v = \infty$ only in the one-dimensional case. In this case, we have an explicit formula for $\mu_{(0)}(z)$, which can be easily shown to be

$$\mu_{(0)}(z) = -2\sinh^{-1}\sqrt{-\frac{z}{2}}.$$

The right-hand side of (5.1) is then readily seen to tend to infinity as k tends to infinity.

REMARK 5.5. An interesting consequence of our result for $d = 1$ is that, $\hat{P}$-a.s.,

$$H(b) \equiv \lim_{t\to\infty} \frac{1}{t} \log E_0\Big[\exp\Big(b\eta_t + \int_0^t \xi(\eta_s)\,ds\Big)\Big] = -z(|b|), \qquad b \in R^1, \tag{5.2}$$

where z stands for the inverse of the function $-\mu = -\mu(z)$. To see (5.2), apply (4.7), (3.1) and the symmetry of the random walk to get

$$\lim_{t\to\infty} \frac{1}{t} \log E_0\Big[\exp\Big(\int_0^t \xi(\eta_s)\,ds\Big)\chi_{\{\eta_t=[at]\}}\Big] = -|a| I\bigg(\frac{1}{|a|}\bigg), \qquad \hat{P}\text{-a.s.} \tag{5.3}$$

Here we take e $= 1$ and omit writing it. Also recall that from (1.3),

$$|a| I\bigg(\frac{1}{|a|}\bigg) = \sup_z \big[z - |a|\mu(z)\big] = \sup_{-\mu} \big[|a|(-\mu) - (-z))\big]; \tag{5.4}$$

thus, this is the Legendre transform of $-z$, evaluated at $|a|$. Now (5.3), together with the asymptotic Laplace method, suggests that, $\hat{P}$-a.s.,

$$H(b) = \sup_{a\in R^1} \bigg[ba - |a| I\bigg(\frac{1}{|a|}\bigg)\bigg] = -z(|b|), \tag{5.5}$$

where the last equality follows from (5.4) and the involution property of the Legendre transform. Set $H(b; B, t) = (1/t) \log E_0[\exp(b\eta_t + \int_0^t \xi(\eta_s)\,ds) \cdot \chi_{\{|\eta_t|/t > B\}}]$. Then, as usual, (5.5) is rigorized by showing the exponential tightness [see, e.g., Dembo and Zeitouni (1993), page 8]

$$\lim_{B\to\infty} \lim_{t\to\infty} H(b; B, t) = -\infty, \qquad \hat{P}\text{-a.s.} \tag{5.6}$$

Now (5.6) is readily verified if we bound ξ by its supremum A and do elementary computations employing heat kernels.

We thank Jack Xin for bringing (5.2) to our attention. In the case of constant potential $\xi = c$, the counterpart of (5.2) is an exact equality and can be seen from the martingale $M_t = \exp(bW_t - (b^2/2)t)$ with mean 1 by using hitting times τ_x. Here, for simplicity, we use Brownian motion W_t. In the case of a periodic potential ξ, we again get an exact equality [as opposed to the asymptotic equality (5.2) for random media] if we take τ_{np}, $n \in Z$, and p is the period, as our hitting times. Xin mentioned the idea of approximating the random potential by periodically extending the potential in $[-L, L]$ and letting L tend to infinity. From such an approach, we should be able to understand the phenomena from the standpoint of spectral analysis, since Freidlin (1985) links the counterpart of (5.2) to a Frobenius eigenvalue of a second order elliptic operator. It would be interesting to have these connections worked out.

In general, for $d > 1$, the analysis above follows except that (5.5) is readily modified to give

$$H(b) = \sup_{\mathrm{e}:\,|\mathrm{e}|=1} -z\big(|b\cdot \mathrm{e}|; \mathrm{e}\big),$$

where now we include the dependence on e. In the case of isotropic media, that is, rotationally invariant, a multidimensional analog of (5.2) holds since in this case we have $z(\cdot) = z(\cdot; \mathrm{e})$ independent of $|\mathrm{e}| = 1$ and the right-hand side of the above display reduces to $-z(|b|)$.

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