

CLUSTERING AND INVARIANT MEASURES FOR SPATIAL BRANCHING MODELS WITH INFINITE VARIANCE¹

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We consider two spatial branching models on \mathbb{R}^d : branching Brownian motion with a branching law in the domain of normal attraction of a $(1+\beta)$ stable law, $0 < \beta \leq 1$, and the corresponding high density limit measure valued diffusion. The longtime behavior of both models depends highly on β and d . We show that for $d \leq 2/\beta$ the only invariant measure is δ_0 , the unit mass on the empty configuration. Furthermore, we give a precise condition for convergence toward δ_0 . For $d > 2/\beta$ it is known that there exists a family $(\nu_\theta, \theta \in [0, \infty))$ of nontrivial invariant measures. We show that every invariant measure is a convex combination of the ν_θ . Both results have been known before only under an additional finite mean assumption. For the critical dimension $d = 2/\beta$ we show that both models display the phenomenon of diffusive clustering. This means that clusters grow spatially on a random scale. We give a precise description of the clusters via multiple scale analysis. Our methods rely mainly on studying sub- and supersolutions of the reaction diffusion equation $\partial u / \partial t - \frac{1}{2} \Delta u + u^{1+\beta} = 0$.

1. Introduction and main results.

1.1. *Survey.* Branching Brownian motion (BBM) is an (infinite) particle system in which particles perform independent Brownian motions and split at random times into a random number of offspring particles. We consider the process $(\psi_t)_{t \geq 0}$, which will be either branching Brownian motion on \mathbb{R}^d with offspring probability generating function $f(z) = z + \frac{1}{2}(1-z)^{1+\beta}$, $0 < \beta \leq 1$, or its high density limit measure valued diffusion, the so-called super-Brownian motion (SBM). (Note that the probability distribution generated by f is in the normal domain of attraction of a stable law with index $1+\beta$. In particular, for $\beta < 1$ this law does not have a variance.) In the SBM the transport of mass is governed by the (deterministic) heat flow while the local “intensity of matter” fluctuates randomly.

A pathwise construction of these processes in terms of excursions of certain random walks and Lévy processes, respectively, can be found in a recent paper by Le Gall and Le Jan (1998). See also Gorostiza, Roelly and Wakolbinger (1992) for a corresponding multitype model.

It is well known that δ_0 , the unit mass on the empty configuration, is the only invariant measure with finite intensity if $d \leq 2/\beta$. One aim of this paper is to show that the finite intensity assumption can be dropped, hence δ_0 is

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the *only* invariant measure for (ψ_t) if $d \leq 2/\beta$. In the case of finite variance branching ($\beta = 1$) this has been done before by Bramson, Cox and Greven (1993). Their approach (like ours) is based on the study of sub- and super-solutions $u(t, x; f)$, $x \in \mathbb{R}^d$, $t \geq 0$, to the reaction diffusion equation

$$(1.1) \quad (\partial_t - \frac{1}{2}\Delta)u + u^{1+\beta} = 0,$$

where $\partial_t = \partial/\partial t$ and Δ denotes the Laplacian in \mathbb{R}^d . While most of their techniques work also in our setting, part of the argument of Bramson, Cox and Greven (1993) relies on a second moment estimate and had to be replaced to cope with $\beta < 1$.

In the high-dimensional case $d > 2/\beta$ it is known that there exists a family $(\nu_\theta, \theta \in [0, \infty))$ of extremal invariant (and translation invariant) measures for (ψ_t) . All invariant measures with finite intensity can be represented as a convex combination of these ν_θ . We show in this paper that the finite intensity assumption can be dropped. For the case $\beta = 1$ this has been shown before by Bramson, Cox and Greven (1997).

The other main aim of this paper is to investigate closer the clustering in the critical dimension $d = 2/\beta$. We show that so-called *diffusive clustering* occurs. This phenomenon was first investigated for the voter model by Cox and Griffeath (1986). Roughly speaking, diffusive clustering means that clusters grow spatially at a random order of magnitude. This phenomenon has been observed for a lot of interacting particle systems and related models such as the voter model, linearly interacting diffusions, critical binary branching Brownian motion and so on. A detailed treatment can be found in Klenke (1996, 1997). All these models have in common that the local random fluctuations (given, e.g., by the branching law or the resampling mechanism) have finite variances and that the critical dimension (in which diffusive clustering occurs) is $d = 2$.

This is, however, the first case in which diffusive clustering is observed in the absence of a second moment. In the finite variance models it turned out that the growth of the cluster height is dominated by the Green function

$$G(t) = \int_1^t p_s(0, 0) ds,$$

where $p_t(\cdot, \cdot)$ is the (symmetric) *interaction kernel* of the model. Of course, here p_t is the heat kernel. In the absence of a second moment we show that the cluster growth is now governed by the quantity

$$G_\beta(t) = \left(\int_1^t p_s(0, 0)^\beta ds \right)^{1/\beta}.$$

This object naturally arises in the investigation of the clusters in terms of Kallenberg's backward tree [see Gorostiza and Wakolbinger (1991)]. In this paper we do not make explicit use of the backward *tree*, but rely on pde methods, the connection being that (1.1) is Kolmogorov's backward equation for the Laplace functionals of (ψ_t) .

1.2. *The models.* We give a short description of the models considered in this paper. For more details we refer the reader to Dawson (1993). Unfortunately, we have to introduce a lot of notation first.

Basic definitions for random measures. Let E be a locally compact Polish space. By $\mathcal{B}(E)$ we denote the Borel σ -field on E . By $C_b(E)$ and $C_c(E)$ we denote the spaces of continuous real-valued functions on E that are bounded, respectively, have compact support. Further let $C_c^+ = \{f \in C_c: f \geq 0\}$ and $C_c^{++} = \{f \in C_c^+: f \neq 0\}$ and define C_b^+ and C_b^{++} analogously.

A measure μ on $\mathcal{B}(E)$ is called *locally finite* if $\mu(K) < \infty$ for all compact sets $K \subset E$. Let

$$(1.2) \quad \mathcal{M}(E) = \{\text{locally finite measures on } E\}$$

and $\mathcal{M}_f(E) = \{\mu \in \mathcal{M}(E): \mu(E) < \infty\}$.

For $\mu \in \mathcal{M}(E)$ and $f: E \rightarrow \mathbb{R}$ measurable and μ -integrable we define $\langle \mu, f \rangle := \int f d\mu$. Now $\mathcal{M}(E)$ is a Polish space with the vague topology, defined by $\mu_n \rightarrow \mu$ iff $\langle \mu_n, f \rangle \rightarrow \langle \mu, f \rangle$ for all $f \in C_c(E)$. The space $\mathcal{M}_1(\mathcal{M}(E))$ of probability measures on $\mathcal{M}(E)$, equipped with the weak topology, is also Polish [see, e.g., Kallenberg (1983)]. For weak convergence of probability measures we use the symbol " \implies ".

The space of (nonnegative) integer-valued measures μ on $\mathcal{B}(E)$ will be denoted by

$$(1.3) \quad \mathcal{N}(E) = \{\mu \in \mathcal{M}(E): \mu(A) \in \{0, 1, 2, \dots, \infty\} \forall A \in \mathcal{B}(E)\}.$$

For $m \in \mathcal{M}(\mathbb{R}^d)$ we denote by $\mathcal{H}(m) \in \mathcal{M}_1(\mathcal{N}(\mathbb{R}^d))$ the distribution of the Poisson point process on \mathbb{R}^d with intensity measure m . That is, for $f \in C_c^+(\mathbb{R}^d)$,

$$(1.4) \quad \int \mathcal{H}(m)(d\mu) e^{-\langle \mu, f \rangle} = \exp(-\langle m, 1 - e^{-f} \rangle).$$

We use the notation $\mathcal{L}[X]$ for the distribution of a random variable X . Let $(X_t)_{t \geq 0}$ be a Markov process with values in E and $x \in E$ or $Q \in \mathcal{M}_1(E)$. By $\mathcal{L}^x[(X_t)_{t \geq 0}]$ and $\mathcal{L}^Q[(X_t)_{t \geq 0}]$ we denote the distributions of $(X_t)_{t \geq 0}$ with $\mathcal{L}^x[X_0] = \delta_x$ and $\mathcal{L}^Q[X_0] = Q$. If (X_t) is càdlàg, convergence of paths will be understood in the Skorohod topology. Convergence of finite-dimensional marginals will be indicated by fdd.

(1 + β)-branching Brownian motion. Let $0 < \beta \leq 1$ and let $(p_k)_{k=0,1,\dots}$ be the probability distribution on \mathbb{N}_0 with p.g.f. $f(z) = z + \frac{1}{2}(1 - z)^{1+\beta}$, $z \in [0, 1]$. That is,

$$(1.5) \quad p_k = \begin{cases} \frac{1}{2}, & \text{if } k = 0, \\ (1 - \beta)/2, & \text{if } k = 1, \\ \frac{1}{2}(-1)^k \binom{1 + \beta}{k}, & \text{if } k = 2, 3, \dots \end{cases}$$

Note that (p_k) is critical, that is, $\sum k p_k = 1$, and is in the normal domain of attraction of a stable law on $[0, \infty)$ with index $(1 + \beta)$. In particular, for $\beta < 1$ the law (p_k) has infinite variance.

We will consider a particle performing a Brownian motion on \mathbb{R}^d and with an exponential lifetime with mean $1/2b > 0$. At the time of death, the particle produces an offspring of k particles with probability p_k . The offspring behave as k independent copies of the one-particle system started at the parent particle's final position. If we start the process with more than one particle at time 0, we assume that all particles are independent.

The process

$$(1.6) \quad \eta_t(A) = \#\{\text{particles in } A\}, \quad A \in \mathcal{B}(\mathbb{R}^d), \quad t \geq 0,$$

will be called the *branching Brownian motion on \mathbb{R}^d with parameters $1 + \beta$ and b* , abbreviated $\text{BBM}(d, 1 + \beta, b)$.

(1 + β) super-Brownian motion. Next we consider the short lifetime high density limit of $\text{BBM}(d, 1 + \beta, b)$. Let $\mu \in \mathcal{M}_f(\mathbb{R}^d)$ and $\mu^N \in \mathcal{M}_f(\mathbb{R}^d)$, $N \in \mathbb{N}$, such that $N^{-1}\mu^N \rightarrow \mu$, as $N \rightarrow \infty$. For $N \in \mathbb{N}$ let $(\eta_t^N)_{t \geq 0}$ be $\text{BBM}(d, 1 + \beta, bN^\beta)$ with initial state $\eta_0^N = \mu^N$. It is well known that there exists a càdlàg Markov process $(\zeta_t)_{t \geq 0}$ with values in $\mathcal{M}_f(\mathbb{R}^d)$ such that

$$(1.7) \quad \mathcal{L}^\mu[(\zeta_t)_{t \geq 0}] = \text{w-} \lim_{N \rightarrow \infty} \mathcal{L}^{\mu^N} \left[\left(\frac{1}{N} \eta_t^N \right)_{t \geq 0} \right]$$

[see Dawson (1993), Section 4.4ff].

The process $(\zeta_t)_{t \geq 0}$ will be called *super-Brownian motion on \mathbb{R}^d with parameters $1 + \beta$ and b* , abbreviated $\text{SBM}(d, 1 + \beta, b)$.

For $\mu \in \mathcal{M}(\mathbb{R}^d)$ we can define $(\zeta_t)_{t \geq 0}$ with initial configuration $\zeta_0 = \mu$ as the increasing limit of $(\zeta_t^n)_{t \geq 0}$ with initial configurations $\mu^n \in \mathcal{M}_f(\mathbb{R}^d)$, $n \in \mathbb{N}$, such that $\mu^n \uparrow \mu$. It is known that $\text{SBM}(d, 1 + \beta, b)$ takes values in $\mathcal{M}(\mathbb{R}^d)$ if we impose a regularity condition on the initial state μ . For example, assume $\langle \mu, (1 + \|\cdot\|^2)^{-p} \rangle < \infty$ for some $p > d/2$. The same condition also assures that $\eta_t \in \mathcal{N}(\mathbb{R}^d)$ a.s. for all $t \geq 0$.

Log-Laplace equation. Let $f \in C_b^+(\mathbb{R}^d)$. A prominent role in this paper is played by the solution $u(t, x; f)$, $x \in \mathbb{R}^d$, $t \geq 0$, of the Cauchy problem

$$(1.8) \quad \begin{aligned} L_\beta u(t, x; f) &= 0, & x \in \mathbb{R}^d, \quad t \geq 0, \\ u(0, x; f) &= f(x), & x \in \mathbb{R}^d, \end{aligned}$$

where

$$(1.9) \quad L_\beta u(t, x; f) = (\partial_t - \frac{1}{2}\Delta)u(t, x; f) + bu(t, x; f)^{1+\beta}.$$

Since (1.8) is time-homogeneous, u has the (nonlinear) semigroup property

$$(1.10) \quad u(t + s, x; f) = u(t, x; u(s, \cdot; f)), \quad x \in \mathbb{R}^d, \quad s, t \geq 0.$$

Note that for $\rho > 0$ the following scaling relation holds:

$$(1.11) \quad u(t, x; f) = \rho^{1/\beta} u(\rho t, \rho^{1/2} x; \rho^{-1/\beta} f(\rho^{-1/2} \cdot)), \quad x \in \mathbb{R}^d, \quad t > 0.$$

The reaction–diffusion equation (1.9) is linked to our branching processes by the equations [see, e.g., Dawson (1993)]

$$(1.12) \quad E^{\delta_x}[\exp(-\langle \eta_t, f \rangle)] = 1 - u(t, x; 1 - e^{-f}),$$

$$(1.13) \quad E^{\delta_x}[\exp(-\langle \zeta_t, f \rangle)] = \exp(-u(t, x; f)).$$

1.3. *Invariant laws.* Recall that $p_t(x) = (2\pi t)^{-d/2} \exp(-\|x\|^2/2t)$ is the heat kernel on \mathbb{R}^d . Define $G_\beta(t)$ by

$$(1.14) \quad G_\beta(t) = \left(\int_1^t p_s(0)^\beta ds \right)^{1/\beta}.$$

For $d \in \mathbb{N}$ define $\phi: (1, \infty) \times \mathbb{R}^d \rightarrow [0, \infty)$ by

$$(1.15) \quad \phi(t, x) = p_t(x)/G_\beta(t).$$

It will turn out that (in the particle language) $G_\beta(t)$ measures the concentration of particles around a certain point (say the origin), given that there is a particle. It is the typical concentration of particles or average “cluster height.” On the other hand, $p_t(x)$ is the expected intensity of particles at the origin if we start in δ_x . Consequently, the function $\phi(t, x)$ measures the probability of seeing a particle at the origin at time $t > 1$ if we start with one “particle” at time 0 at site $x \in \mathbb{R}^d$.

Low dimension. Recall that (ψ_t) is either $\text{BBM}(d, 1 + \beta, b)$ or $\text{SBM}(d, 1 + \beta, b)$.

THEOREM 1. *Assume $d \leq 2/\beta$. Then the following hold.*

(i) $\mathcal{L}[\psi_t] \implies \delta_0$ as $t \rightarrow \infty$ if and only if

$$(1.16) \quad \mathcal{L}[\langle \psi_0, \phi(r, \cdot) \rangle] \implies \delta_0, \quad r \rightarrow \infty.$$

(ii) *If condition (1.16) does not hold, then ψ_t is unstable. That is, for any $f \in C_c^{++}(\mathbb{R}^d)$ the sequence $\langle \psi_t, f \rangle$ is stochastically unbounded.*

(iii) *If $\mathcal{L}[\langle \psi_0, \phi(r, \cdot) \rangle] \implies \delta_\infty$, $r \rightarrow \infty$, then ψ_t explodes. That is, for any $f \in C_c^{++}(\mathbb{R}^d)$ almost surely $\langle \psi_t, f \rangle \rightarrow \infty$, as $t \rightarrow \infty$.*

COROLLARY 1.1. *If $d \leq 2/\beta$, then the only invariant measure for ψ_t is δ_0 .*

In order to check the conditions of Theorem 1 it is useful to note that $\phi(t, x)$ can be bounded from above and below by the function $\Phi: (1, \infty) \times \mathbb{R}^d \rightarrow [0, \infty)$, defined by

$$(1.17) \quad \Phi(t, x) = \begin{cases} t^{d/2-1/\beta} p_t(x), & \text{if } \beta < 2/d, \\ (\log t)^{-1/\beta} p_t(x), & \text{if } \beta = 2/d, \\ p_t(x), & \text{if } \beta > 2/d. \end{cases}$$

More precisely, there exist $c, C > 0$ (depending only on d and β) such that for $t \geq 2$ and $x \in \mathbb{R}^d$,

$$(1.18) \quad c\Phi(t, x) \leq \phi(t, x) \leq C\Phi(t, x).$$

[This is immediate from the fact that $p_t(0) = (2\pi t)^{-d/2}$.] Hence it suffices to verify the conditions of Theorem 1 for Φ instead of ϕ .

High dimension. Let \mathcal{S} be the set of invariant measures for (ψ_t) ; by $\mathcal{S}_e \subset \mathcal{S}$ we denote its extremal elements. It is well known [see Gorostiza and Wakolbinger (1992), Theorem 1] that there exists a one-parameter family $\{\nu_\theta, \theta \in [0, \infty)\} \subset \mathcal{S}_e$ with the following properties. Each ν_θ is translation invariant, ergodic and has intensity θ , that is, $\int \nu_\theta(dm) \langle m, f \rangle = \theta \langle \lambda, f \rangle$ for $f \in C_c^+(\mathbb{R}^d)$. Further, for $\mathcal{L}[\psi_0]$ translation invariant and ergodic with $E[\langle \psi_0, f \rangle] = \theta \langle \lambda, f \rangle$,

$$\mathcal{L}[\psi_t] \implies \nu_\theta \text{ as } t \rightarrow \infty.$$

For any $\mu \in \mathcal{S}$ with σ -finite intensity measure there exists a unique probability measure F_μ on $[0, \infty)$ such that

$$(1.19) \quad \mu = \int \nu_\theta F_\mu(d\theta).$$

Our point is to drop the assumption of the σ -finiteness on μ to allow for a representation as in (1.19).

THEOREM 2. *Let $d > 2/\beta$ and let (ψ_t) either $\text{BBM}(d, 1 + \beta, b)$ or $\text{SBM}(d, 1 + \beta, b)$. Then the following holds: $\mathcal{S}_e = \{\nu_\theta, \theta \in [0, \infty)\}$ and for any $\mu \in \mathcal{S}$ there exists a unique probability distribution F_μ on $[0, \infty)$ such that $\mu = \int \nu_\theta F_\mu(d\theta)$.*

The crucial step to prove Theorem 2 is the following proposition.

PROPOSITION 1.2. *Any invariant measure is translation invariant.*

1.4. Critical dimension: diffusive clustering. Our aim is to give a precise description of the clustering in the critical dimension $d = 2/\beta$. Hence we will assume $\beta = 2/d$. For simplicity of notation we will also assume $b = 1$ in the following discussion.

Proceeding as in Klenke (1997), we introduce the following concepts for the description of the heights of the clusters and their expansions in space.

1. High density rescaling. For time $t > 1$ we define

$$(1.20) \quad \tilde{\psi}_t = \tilde{\psi}_t^0 := (\log t)^{-1/\beta} \psi_t$$

with (recall that λ is the d -dimensional Lebesgue measure)

$$(1.21) \quad \mathcal{L}[\psi_0] = M_t := \begin{cases} \mathcal{H}((\log t)^{1/\beta} \lambda), & \text{if } \psi_t \text{ is BBM,} \\ \delta_{(\log t)^{1/\beta} \cdot \lambda}, & \text{if } \psi_t \text{ is SBM.} \end{cases}$$

2. Spatial rescaling. For (ψ_t) $\text{BBM}(d, 1 + \beta, 1)$ or $\text{SBM}(d, 1 + \beta, 1)$, let $I = [0, 1]$, respectively, $I =] - \infty, 1]$. We fix $\alpha \in I$ and define $(\tilde{\psi}_t^\alpha)$ by

$$(1.22) \quad \tilde{\psi}_t^\alpha := \mathcal{S}_{\alpha, t} \tilde{\psi}_t, \quad \alpha \in I,$$

where

$$\mathcal{S}_{\alpha, t}: \mathcal{M}(\mathbb{R}^d) \rightarrow \mathcal{M}(\mathbb{R}^d), \quad \mu(\cdot) \mapsto t^{-\alpha d/2} \mu(t^{\alpha/2} \cdot).$$

That is, for $B \in \mathcal{B}(\mathbb{R}^d)$ we set $\tilde{\psi}_t^\alpha(B) = t^{-\alpha d/2} \tilde{\psi}_t(t^{\alpha/2} B)$. As above we let $\tilde{\psi}_t = \tilde{\psi}_t^0$.

REMARK. Since we intend to take the limit as $t \rightarrow \infty$, it would not make sense to allow $\alpha < 0$ for BBM. Due to the particle structure in this case, we would get $\mathbb{P}^{M_t}[\tilde{\eta}_t^\alpha(B) = 0] \rightarrow 1$ as $t \rightarrow \infty$ for all bounded sets $B \in \mathcal{B}(\mathbb{R}^d)$. This leads us to the different choices of I .

We introduce the *total mass process* $(Z_t)_{t \geq 0}$ of $\text{SBM}(d, 1 + \beta, 1)$, which is the “diffusion limit” of Galton–Watson processes with offspring probabilities defined in (1.5) above. Here (Z_t) is a process with independent increments which can be characterized by its log-Laplace transform

$$(1.23) \quad v(t, K, \theta) = -\log E^K[\exp(-\theta Z_t)],$$

which is the unique solution of

$$(1.24) \quad \begin{aligned} v(0, K, \theta) &= \theta K, \\ \partial_t v(t, K, \theta) &= -v(t, K, \theta)^{1+\beta}. \end{aligned}$$

The solution can be given explicitly:

$$(1.25) \quad v(t, K, \theta) = (\beta t + (K\theta)^{-\beta})^{-1/\beta}.$$

Let $c_\beta = (2\pi(1 + \beta)^{1/\beta})^{-1}$ and recall that λ is the Lebesgue measure.

THEOREM 3. For (ψ_t) , $\text{BBM}(d, 1 + 2/d, 1)$ or $\text{SBM}(d, 1 + 2/d, 1)$ and $\alpha \in I$ the following holds:

$$(1.26) \quad \mathcal{L}^{M_t}[\tilde{\psi}_t^\alpha] \implies \mathcal{L}^1[Z_{c_\beta(1-\alpha)}\lambda] \text{ as } t \rightarrow \infty.$$

Multiple scale analysis. So far we have considered our rescaled process $\tilde{\psi}^\alpha$ at *one scale* α . A natural task is to investigate the limit behavior of $(\tilde{\psi}_t^{\alpha_1}, \dots, \tilde{\psi}_t^{\alpha_n})$ for $\alpha_1, \dots, \alpha_n \in I$. In order to learn more about the spatial structure of the clusters, we might also wish to choose different *points of observation* $x_t^1, \dots, x_t^n \in \mathbb{R}^d$. Theorem 3 indicates that the distances $\|x_t^e - x_t^f\|$, $e \neq f$, of these points should grow in t on an algebraic scale $\alpha_{e,f} \in I$. Note that a consistent choice of the $\alpha_{e,f}$ implies that $2^{\alpha_{e,f}}$ is an ultrametric on $\{1, \dots, n\}$. Hence we may assume w.l.o.g. that the points of observation are indexed by a finite (rooted) tree \mathbb{T} and that $\alpha_{e,f} = A(e \wedge f)$, where

$$A: \mathbb{T} \rightarrow I$$

is a strictly decreasing map.

To explain this a bit, note that \mathbb{T} carries a natural partial ordering \leq , where $e \leq f$ iff e is an ancestor of f , that is, if e is closer to the root (denoted by \emptyset) than f . Hence $e \wedge f$ is the greatest common ancestor of e and f .

The pair $\mathbb{L} = (\mathbb{T}, A)$ will be called a *multiple space scale*. We will assume that $X = (x_t^e, e \in \mathbb{T}, t \geq 0)$ is a family of points $x_t^e \in \mathbb{R}^d$ such that

$$\|x_t^e - x_t^f\| \approx t^{A(e \wedge f)/2} \text{ as } t \rightarrow \infty.$$

As usual, $a_t \approx b_t$ means $(\log a_t)/(\log b_t) \rightarrow 1$ as $t \rightarrow \infty$. We refer to X as \mathbb{L} -spaced. Our aim is to study the asymptotics of the common distribution of [recall \mathcal{S} from (1.22)]

$$(\mathcal{S}_{A(e), t} \mathcal{T}_{x_t^e} \tilde{\psi}_t)_{e \in \mathbb{T}} \text{ as } t \rightarrow \infty,$$

where $\mathcal{T}_z: \mathcal{M}(\mathbb{R}^d) \rightarrow \mathcal{M}(\mathbb{R}^d)$ is the translation by z , $(\mathcal{T}_z \mu)(\cdot) = \mu(z + \cdot)$.

We give a heuristic motivation for the next definition. Consider the simplest case $\mathbb{T} = \{\emptyset, e, f\}$, $e \wedge f = \emptyset$. Since Brownian motion has range $t^{1/2}$, the common history of the space-time points (t, x_t^e) , and (t, x_t^f) ends at time $t - t^{A(e \wedge f)}$. After that time the histories of these points develop independently. In fact, asymptotically the common history of (t, x_t^e) and (t, x_t^f) is contained in $\tilde{\psi}_{t-t^{A(e \wedge f)/2}}([-t^{A(e \wedge f)/2}, t^{A(e \wedge f)/2}]^d)$. This, together with Theorem 3, suggests that the intensity of $\mathcal{S}_{A(e), t} \mathcal{T}_{x_t^e} \tilde{\psi}_t$ and $\mathcal{S}_{A(f), t} \mathcal{T}_{x_t^f} \tilde{\psi}_t$ should consist of $Z_{c_\beta(1-A(e \wedge f))}$ plus two independent increments.

To be precise, let $(Z_t^e, e \in \mathbb{T})_{t \geq 0}$ be the following Markov process on $[0, \infty)^\mathbb{T}$. Each $(Z_t^e)_{t \geq 0}$ is a $(1 + \beta)$ continuous state branching “diffusion” introduced in (1.23). For $e, f \in \mathbb{T}$ with $e \neq f$ we let $Z_t^e = Z_t^f$ for $t \in [0, 1 - A(e \wedge f)]$. For $t > 1 - A(e \wedge f)$, the evolutions of Z_t^e and Z_t^f shall be independent.

THEOREM 4 (Multiple scale). *Let (ψ_t) be $BBM(d, 1 + 2/d, 1)$ or $SBM(d, 1 + 2/d, 1)$ and $I = [0, 1]$, respectively, $I = (-\infty, 1]$. Then the following holds:*

$$\mathcal{L}^{M_t}[(\mathcal{S}_{A(e), t} \mathcal{T}_{x_t^e} \tilde{\psi}_t)_{e \in \mathbb{T}}] \implies \mathcal{L}[(Z_{c_\beta(1-A(e))}^e \lambda)_{e \in \mathbb{T}}] \text{ as } t \rightarrow \infty.$$

By taking a linear tree \mathbb{T} we obtain the following corollary.

COROLLARY 1.3.

$$\mathcal{L}^{M_t}[(\tilde{\psi}_t^\alpha(B))_{\alpha \in I}] \implies_{\text{fdd}} \mathcal{L}^1[\lambda(B)(Z_{c_\beta(1-\alpha)}}_{\alpha \in I}] \text{ as } t \rightarrow \infty, B \in \mathcal{B}(\mathbb{R}^d).$$

1.5. *Outline.* The rest of the paper is organized as follows. In Section 2 we give upper and lower bounds of $u(t, x; f)$ in terms of the function $\phi(t, x)$ in Proposition 2.6. This is the key for the proof of Theorems 1 and 2 in Sections 2 and 3. In Section 4 we give better bounds, for the special situation $\beta = 2/d$, that serve to prove Proposition 4.1. A coupling technique will be employed to infer Theorem 4.

2. Proof for the low dimensions. In this section we give some lemmas dealing with sub- and supersolutions to the equation $L_\beta u = 0$. With the aid of these lemmas we prove Theorem 1. Some of the lemmas will be used in Section 3 to prove the high-dimensional results.

The main tool for the investigation is a maximum principle for the nonlinear parabolic differential operator L_β . We state the following lemma without proof and refer the reader to Protter and Weinberger [(1967), Chapter 3.7]. (In fact, Protter and Weinberger only deal with the case of a bounded domain. Our lemma follows by approximation arguments.)

LEMMA 2.1 (Maximum principle). *Let $L = L_{t,H}$ be the semiparabolic operator on \mathbb{R}^d defined by*

$$Lu(t, x) = \partial_t u(t, x) - \frac{1}{2} \Delta u(t, x) + H(t, u(t, x)),$$

where $H: [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ is continuous and nondecreasing in u . Let $f, g \in C_c^+(\mathbb{R}^d)$, $f \leq g$ and $T > 0$. Let $\underline{u}(t, x; f)$ and $\bar{u}(t, x; g)$ be sub- and supersolutions to $Lu = 0$ with initial conditions f , respectively, g . That is,

$$\underline{u}(0, x; f) = f(x), \quad \bar{u}(0, x; g) = g(x), \quad x \in \mathbb{R}^d,$$

and

$$L\underline{u}(t, x; f) \leq 0, \quad L\bar{u}(t, x; g) \geq 0, \quad x \in \mathbb{R}^d, t \in [0, T].$$

We also assume that \underline{u} and \bar{u} are bounded and vanishing at infinity (as functions of x for any t). Then

$$\underline{u}(t, x; f) \leq \bar{u}(t, x; g), \quad x \in \mathbb{R}^d, t \in [0, T].$$

To warm up we give two simple applications of the maximum principle. Recall that $u(t, x; f)$ denotes the solution of the Cauchy problem (1.8).

LEMMA 2.2. *Let $f, g \in C_c^+(\mathbb{R}^d)$ and let $c > 1$. Then*

$$(2.1) \quad u(t, x; f + g) \leq u(t, x; f) + u(t, x; g), \quad x \in \mathbb{R}^d, t \geq 0,$$

and

$$(2.2) \quad u(t, x; cf) \leq cu(t, x; f), \quad x \in \mathbb{R}^d, t \geq 0.$$

Furthermore, if $f \leq g$, then

$$(2.3) \quad u(t, x; f) \leq u(t, x; g), \quad x \in \mathbb{R}^d, t \geq 0.$$

PROOF. Check that $L_\beta(u(t, x; f) + u(t, x; g)) \geq 0$ and $L_\beta(cu(t, x; f)) \geq 0$. The last inequality is trivial. \square

LEMMA 2.3. *Let $f \in C_c^+(\mathbb{R}^d)$ and let $v(t, x; f) = (p_t * f)(x)$ be the solution of the heat equation $(\partial_t - \frac{1}{2} \Delta)v = 0$, $v(0, \cdot; f) = f$. Then the following inequality holds:*

$$(2.4) \quad (1 + b\beta \|f\|_\infty^\beta t)^{-1/\beta} v(t, x; f) \leq u(t, x; f) \leq v(t, x; f), \quad x \in \mathbb{R}^d, t \geq 0.$$

PROOF. Let $U(t, x; K) = U(t; K)$, $K > 0$, be the solution of $L_\beta U = 0$, $U(0, \cdot; K) \equiv K$. The explicit solution of this equation is

$$U(t; K) = K(1 + bK^\beta \beta t)^{-1/\beta}.$$

By the maximum principle, for $K \geq \|f\|_\infty$,

$$(2.5) \quad \begin{aligned} L_\beta u(t, x; f) &\leq \partial_t u - \frac{1}{2} \Delta u + bU^\beta u \\ &= \partial_t u - \frac{1}{2} \Delta u + (bK^\beta)(1 + bK^\beta \beta t)^{-1} u. \end{aligned}$$

Hence $\tilde{L}_\beta u \geq 0$, where $\tilde{L}_\beta = \tilde{L}_{\beta,t}$ is defined by

$$\tilde{L}_{\beta,t} \tilde{u} = \partial_t \tilde{u} - \frac{1}{2} \Delta \tilde{u} + (bK^\beta)(1 + bK^\beta \beta t)^{-1} \tilde{u}.$$

Let \tilde{u} be the solution of $\tilde{L}_{\beta,t} \tilde{u} = 0$. The maximum principle implies $u(t, x; f) \geq \tilde{u}(t, x; f)$, $x \in \mathbb{R}^d$, $t \geq 0$. Note that \tilde{u} can be represented as

$$\begin{aligned} (2.6) \quad \tilde{u}(t, x; f) &= v(t, x; f) \exp\left(-\int_0^t bK^\beta(1 + bK^\beta \beta s)^{-1} ds\right) \\ &= v(t, x; f)(1 + bK^\beta \beta t)^{-1/\beta}. \end{aligned}$$

Since the maximum principle implies $u(t, x; f) \leq v(t, x; f)$, $x \in \mathbb{R}^d$, $t \geq 0$, the assertion is proved. \square

Let $A > 0$, recall $G_\beta(t)$ from (1.14) and define $\bar{u}(t, x)$ by

$$(2.7) \quad \bar{u}(t, x) = \begin{cases} At^{-d/2} G_\beta(t)^{-1} \exp\left(-\frac{1}{2} \frac{\|x\|^2}{2t}\right), & \beta < \frac{2}{d}, \\ At^{-d/2} G_\beta(t)^{-1} \exp\left(-\left(1 - G_\beta(t)^{-\beta}\right) \frac{\|x\|^2}{2t}\right), & \beta = \frac{2}{d}, \\ At^{-d/2} \exp\left(-\frac{1}{2} \frac{\|x\|^2}{2t}\right), & \beta > \frac{2}{d}. \end{cases}$$

The definition of \bar{u} for the case $\beta = 2/d$ might look a little strange at first glance. Clearly for t large enough, $\bar{u}(t, x) \leq At^{-d/2} G_\beta(t)^{-1} \exp(-\frac{1}{2}(\|x\|^2/2t))$. However, we will be able to show that \bar{u} is a supersolution only for the definition given in (2.7).

LEMMA 2.4 (Supersolution). *Assume that $\beta \leq 2/d$. There exists $t_0 = t_0(\beta, d) \geq 0$ such that for A large enough and $t \geq t_0$,*

$$(2.8) \quad L_\beta \bar{u}(t, x) \geq 0, \quad x \in \mathbb{R}^d.$$

PROOF. We do the proof separately for critical and low dimensions.

Critical dimension ($\beta = 2/d$). Note that

$$(2.9) \quad G_\beta(t) = (2\pi)^{-d/2} (\log t)^{1/\beta}$$

and that $G'_\beta(t) = (d/dt)G_\beta(t) = (\beta t \log t)^{-1} G_\beta(t) = (1/2\pi)(d/2t)G_\beta(t)^{1-\beta}$. Hence a short calculation shows that for $t \geq \exp(2 + 4\pi)$,

$$\begin{aligned} (2.10) \quad L_\beta \bar{u}(t, x) &= \frac{\bar{u}(t, x)}{t G_\beta(t)^\beta} \left[-\frac{1}{\beta} \left(1 + \frac{1}{2\pi}\right) + \left(1 - \left(1 + \frac{1}{2\pi}\right) G_\beta(t)^{-\beta}\right) \frac{\|x\|^2}{2t} \right. \\ &\quad \left. + bA^\beta \exp\left(-\beta \left(1 - G_\beta(t)^{-\beta}\right) \frac{\|x\|^2}{2t}\right) \right] \\ &\geq \frac{\bar{u}(t, x)}{t G_\beta(t)^\beta} \left[-\frac{1}{\beta} \left(1 + \frac{1}{2\pi}\right) + \frac{1}{2} \frac{\|x\|^2}{2t} + bA^\beta \exp\left(-\beta \frac{\|x\|^2}{2t}\right) \right]. \end{aligned}$$

Check separately that this is nonnegative for $\|x\|^2/4t \geq (1 + 1/2\pi)/\beta$ and for $\|x\|^2/4t < (1 + 1/2\pi)/\beta$ and $A \geq \exp(4/\beta)/b^{1/\beta}$.

Low dimension ($\beta < 2/d$). Note that

$$(2.11) \quad G_\beta(t) = (2\pi)^{-d/2} \left(1 - \frac{\beta d}{2}\right)^{-1/\beta} (t^{1-\beta d/2} - 1)^{1/\beta}$$

and that

$$\begin{aligned} G'_\beta(t) &= \frac{d}{dt} G_\beta(t) = (2\pi)^{-d/2} \left(1 - \frac{\beta d}{2}\right)^{-1/\beta} \frac{1}{\beta} \left(1 - \frac{\beta d}{2}\right) (t^{1-\beta d/2} - 1)^{-1+1/\beta} t^{-\beta d/2} \\ &= \left(\frac{1}{\beta} - \frac{d}{2}\right) (1 - t^{1-\beta d/2})^{-1} \frac{G_\beta(t)}{t} \\ &\leq \left(\frac{2}{\beta} - d\right) \frac{G_\beta(t)}{t} \quad \text{for } t \geq 2^{1/(1-\beta d/2)}. \end{aligned}$$

Hence for $t \geq 2^{1/(1-\beta d/2)}$,

$$\begin{aligned} (2.12) \quad L_\beta \bar{u}(t, x) &= u(t, x) \left[-\frac{d}{4t} - \frac{G'_\beta(t)}{G_\beta(t)} + \frac{\|x\|^2}{8t^2} \right. \\ &\quad \left. + bA^\beta t^{-\beta d/2} G_\beta(t)^{-\beta} \exp\left(-\beta \frac{1}{2} \frac{\|x\|^2}{2t}\right) \right] \\ &\geq u(t, x) \frac{1}{t} \left[-\frac{2}{\beta} + \frac{\|x\|^2}{8t} \right. \\ &\quad \left. + bA^\beta (2\pi)^{\beta d/2} \left(1 - \frac{\beta d}{2}\right) \exp\left(-\beta \frac{1}{2} \frac{\|x\|^2}{2t}\right) \right]. \end{aligned}$$

Check separately that this expression is nonnegative for $\|x\|^2/8t \geq 2/\beta$ and for $\|x\|^2/8t < 2/\beta$ and

$$A \geq \frac{2^{1/\beta} \exp(4/\beta)}{(2\pi)^{d/2} (1 - \beta d/2)^{1/\beta} b^{1/\beta} \beta^{1/\beta}}. \quad \square$$

We continue by giving subsolutions \underline{u} . Recall that $p_t(x)$ is the heat kernel. We define for $\alpha \in (0, (\beta b)^{-1/\beta})$,

$$(2.13) \quad \underline{u}(t, x) = \alpha p_t(x) G_\beta(t)^{-1}.$$

LEMMA 2.5 (Subsolution). *In any dimension for $t > 1$,*

$$(2.14) \quad L_\beta \underline{u}(t, x) \leq 0, \quad x \in \mathbb{R}^d.$$

PROOF. We calculate

$$(2.15) \quad L_\beta \underline{u}(t, x) = \underline{u}(t, x) G_\beta(t)^{-\beta} \left(b a^\beta p_t(x)^\beta - \frac{1}{\beta} p_t(0)^\beta \right) \leq 0,$$

where the inequality holds since $\alpha \leq (\beta b)^{-1/\beta}$. \square

Recall ϕ from (1.15).

PROPOSITION 2.6. *Let $d \in \mathbb{N}$ and $0 < \beta \leq 1$. Let $f \in C_c^{++}(\mathbb{R}^d)$ and let $u(t, x; f)$ be the solution of $L_\beta u = 0$ with $u(0, \cdot; f) = f$. Then there exist constants $a, A, t_0 > 0$ such that*

$$(2.16) \quad a\phi(t/2, x) \leq u(t, x; f) \leq A\phi(4t, x), \quad x \in \mathbb{R}^d, t \geq t_0.$$

PROOF. *Upper bound, low and critical dimensions.* Assume $\beta \leq 2/d$. Let \bar{u} and t_0 be as in Lemma 2.4 with $A' [= A$ in the notation of (2.7)] large enough such that $f(x) \leq \bar{u}(t_0, x)$, $x \in \mathbb{R}^d$. By the maximum principle,

$$u(t, x; f) \leq \bar{u}(t + t_0, x), \quad x \in \mathbb{R}^d, t \geq 0.$$

Note that there exist constants $A, C > 0$ such that for $t \geq t_0$ [note that $G_\beta(t)^{-\beta} \leq \frac{1}{2}$ if $\beta = 2/d$],

$$A\phi(4t, x) \geq C\phi(2(t + t_0), x) \geq \bar{u}(t + t_0, x).$$

Upper bound, high dimension. Assume $\beta > 2/d$. Note that by (1.18) there exist constants $c, c' > 0$ such that

$$A\phi(4t, x) \geq Ac'p_{4t}(x) \geq Ac p_{t+1}(x) \quad \text{for } t \geq 2.$$

Choose A large enough such that $Ac p_1(x) \geq f(x)$. Note that

$$L_\beta(Ac p_t(x)) = b(Ac p_t(x))^{1+\beta} \geq 0.$$

Hence by the maximum principle $Ac p_{t+1}(x) \geq u(t, x; f)$, $t \geq 0$.

Lower bound. Note that there exists $a' > 0$ such that for \underline{u} of (2.13) with a' instead of a ,

$$u(4, x; f) \geq \underline{u}(2, x), \quad x \in \mathbb{R}^d.$$

Arguing as above, another application of the maximum principle yields that for some $c > 0$, $a > 0$, \underline{u} of (2.13) now again with a and for $t \geq 4$,

$$(2.17) \quad u(t, x; f) \geq \underline{u}(t - 2, x) = c\phi(t - 2, x) \geq a\phi(t/2, x). \quad \square$$

With Proposition 2.6 in hand we are able to prove the extinction and explosion in Theorem 1. It is, however, more subtle to show instability. Here we need better lower bounds for u that reflect the clustering in low dimensions. Roughly speaking, the picture is as follows. Let $\varepsilon > 0$ and $B \in \mathcal{B}(\mathbb{R}^d)$ be bounded. For t large with high probability there is no particle in B : $\eta_t(B) = 0$. With a small probability, $\eta_t(B) \gg 1/\varepsilon$ and there is no intermediate regime. Hence for large t with overwhelming probability, $\exp(-\varepsilon\eta_t(B))$ is either 0 or close to 1 implying that $\exp(-\varepsilon\eta_t(B)) \approx \exp(-\eta_t(B))$. This suggests that, for given $\varepsilon > 0$ and indicator function $f = \mathbb{1}_B$, for $t > 0$ large enough we should have $u(t, x; \varepsilon f) \approx u(t, x; f)$. [Recall that $u(t, x; f)$ is the solution of $L_\beta u = 0$ with $u(0, x; f) = f(x)$.] This should carry over to $f \in C_c^+(\mathbb{R}^d)$ by approximation.

For our purpose of showing instability in Theorem 1 it will be sufficient to give a lower bound for $u(t, x; \varepsilon f)$.

LEMMA 2.7. Let $\beta \leq 2/d$ and $f \in C_c^{++}(\mathbb{R}^d)$. There exists a constant $c_f > 0$ such that for $\varepsilon > 0$ and t large enough,

$$(2.18) \quad u(t, x; \varepsilon f) \geq c_f \phi(t/2, x).$$

PROOF. Note that there exists a constant $c_f^1 \in (0, 1)$ such that

$$c_f^1 p_1 \leq (p_2 * f).$$

Hence by Lemmas 2.3 and 2.2 [and the semigroup property of u (1.10)],

$$(2.19) \quad \begin{aligned} u(t, x; \varepsilon f) &= u(t-2, x; u(2, \cdot; \varepsilon f)) \\ &\geq u(t-2, x; (1 + 2b\beta\varepsilon \|f\|_\infty^\beta)^{-1/\beta} \varepsilon (p_2 * f)) \\ &\geq c_f^2 u(t-2, x; \varepsilon p_1), \end{aligned}$$

where $c_f^2 := c_f^1 (1 + 2b\beta\varepsilon \|f\|_\infty^\beta)^{-1/\beta} > 0$. Now we use two different arguments for low and critical dimensions.

Critical dimension. Assume $\beta = 2/d$. Here we make use of the self-similarity of SBM($d, 1 + 2/d, 1$). Let $\alpha \in (0, (b\beta)^{-1/\beta})$ and let \underline{u} be the corresponding subsolution according to Lemma 2.5. That is,

$$\underline{u}(x, t) = (2\pi)^{-d/2} \alpha \cdot t^{-d/2} (\log t)^{-1/\beta} \exp(-\|x\|^2/2t).$$

Abbreviating $\rho = \exp((\alpha/\varepsilon)^\beta/2\pi)$, we get by the scaling property (1.11) that

$$(2.20) \quad \begin{aligned} u(t, x; \varepsilon p_1(\cdot)) &= u(t, x; (2\pi)^{1/\beta} \alpha (\log \rho)^{-1/\beta} \rho^{1/\beta} p_\rho(\rho^{1/2} \cdot)) \\ &= u(t, x; \rho^{1/\beta} \underline{u}(\rho, \rho^{1/2} \cdot)) \\ &= \rho^{1/\beta} u(\rho t, \rho^{1/2} x; \underline{u}(\rho, \cdot)). \end{aligned}$$

By time homogeneity, $\underline{u}(s + \cdot, \cdot)$ is also a subsolution for all $s \geq 0$. Hence \underline{u} has the property

$$(2.21) \quad \underline{u}(t + s, x) \leq u(t, x; \underline{u}(s, \cdot)), \quad s, t \geq 0, \quad x \in \mathbb{R}^d.$$

We can thus continue (2.20) by

$$(2.22) \quad \begin{aligned} u(t, x; \varepsilon p_1(\cdot)) &\geq \rho^{1/\beta} \underline{u}(\rho(t+1), \rho^{1/2} x) \\ &= (\log(t+1) + \log \rho)^{-1/\beta} \alpha p_{t+1}(x) \\ &\geq (\log(t+1) + \log \rho)^{-1/\beta} \alpha 2^{-d/2} p_t(x) \quad (\text{for } t \geq 1) \\ &\geq 2^{-d} \underline{u}(t, x) \quad (\text{for } t > \rho + 1). \end{aligned}$$

[Note that for $t > \rho + 1$, $\log(t+1) + \log(\rho) \leq \log((t+1)(t-1)) \leq 2 \log t$.] Combining (2.19) and (2.22) we get that there exist c_f^3, c_f^4 and $c_f > 0$ such that for $\varepsilon > 0$ and t large enough,

$$(2.23) \quad \begin{aligned} u(t, x; \varepsilon f) &\geq c_f^3 \underline{u}(t-2, x) \geq c_f^4 \underline{u}(t/2, x) \\ &= c_f \phi(t/2, x). \end{aligned}$$

Low dimension. Assume $\beta < 2/d$. We make implicit use of the fact that $\text{SBM}(d, 1 + 2/d, 1)$ has a density in the low dimensions. Recall from (1.13) that u is the Laplace transform of SBM. The existence of a density of SBM is known to be equivalent to the existence of a smooth solution u to $L_\beta u = 0$ with a delta distribution δ_0 as initial condition [see Fleischmann (1988)]. More precisely, $u(t, x; \delta_0)$, $t > 0$, $x \in \mathbb{R}^d$, is the solution of $L_\beta u = 0$ such that for $h \in C_c^+(\mathbb{R}^d)$,

$$\lim_{t \rightarrow 0} \int u(t, x; \delta_0) h(x) dx = h(0).$$

Brezis, Peletier and Terman (1986) show in their Appendix that there exists $t_0 > 0$ such that

$$u(t, x; \delta_0) \geq \frac{1}{2} p_t(x), \quad x \in \mathbb{R}^d, \quad 0 < t \leq t_0.$$

Note that the maximum principle implies

$$u(t, x; \delta_0) \leq p_t(x), \quad x \in \mathbb{R}^d, \quad t > 0.$$

Hence by Lemma 2.3 for $t + s > t_0$ [note that $\|\frac{1}{2} p_{t_0}\|_\infty = \frac{1}{2}(2\pi t_0)^{-d/2}$],

$$\begin{aligned} (2.24) \quad u(t, x; p_s(\cdot)) &\geq u(t + s, x; \delta_0) \\ &= u(t + s - t_0, x; u(t_0, \cdot; \delta_0)) \\ &\geq u(t + s - t_0, x; \frac{1}{2} p_{t_0}(\cdot)) \\ &\geq \frac{1}{2} (1 + 2^{-\beta} b \beta (t + s) (2\pi t_0)^{-\beta d/2})^{-1/\beta} p_{t+s}(x). \end{aligned}$$

Let $c_f^3 = \frac{1}{2} (1 + 2^{-\beta} b \beta (2\pi t_0)^{-\beta d/2})^{-1/\beta}$. By the scaling property (1.11) and by (2.19) there exists $c_f > 0$ such that for $t \geq \varepsilon^{-1/(1/\beta - d/2)}$,

$$\begin{aligned} (2.25) \quad u(t, x; \varepsilon f) &\geq c_f^2 u(t - 2, x; \varepsilon p_1(\cdot)) \\ &= c_f^2 t^{-1/\beta} u\left(1 - \frac{2}{t}, t^{-1/2} x; t^{1/\beta} \varepsilon t^{-d/2} p_{1/t}(\cdot)\right) \\ &\geq c_f^2 t^{-1/\beta} u\left(1 - \frac{2}{t}, t^{-1/2} x; p_{1/t}(\cdot)\right) \\ &\geq c_f^2 c_f^3 t^{-1/\beta} p_{1-1/t}(t^{-1/2} x) \\ &\geq c_f \phi\left(\frac{t}{2}, x\right). \end{aligned} \quad \square$$

PROOF OF THEOREM 1. We do the proof only for $(\psi_t) = (\zeta_t)$ $\text{SBM}(d, 1 + \beta, b)$. An easy application of Jensen's inequality in (1.12) and (1.13) then yields the claim for $\text{BBM}(d, 1 + \beta, b)$.

To prove part (i) note that $\mathcal{L}[\zeta_t] \implies \delta_0$ as $t \rightarrow \infty$ is equivalent to

$$(2.26) \quad \liminf_{t \rightarrow \infty} E[\exp(-\langle \zeta_t, f \rangle)] = 1, \quad f \in C_c^{++}(\mathbb{R}^d).$$

However, for such f , by Proposition 2.6, there exist $\alpha, A > 0$ such that

$$\begin{aligned}
 \liminf_{t \rightarrow \infty} E[\exp(-A \langle \zeta_0, \phi(t, \cdot) \rangle)] &\leq \liminf_{t \rightarrow \infty} E[\exp(-\langle \zeta_0, u(t, \cdot; f) \rangle)] \\
 &= \liminf_{t \rightarrow \infty} E[\exp(-\langle \zeta_t, f \rangle)] \\
 &\leq \liminf_{t \rightarrow \infty} E[\exp(-\alpha \langle \zeta_0, \phi(t, \cdot) \rangle)].
 \end{aligned}
 \tag{2.27}$$

These expressions are equal to 1 if (and only if),

$$\langle \zeta_0, \phi(t, \cdot) \rangle \rightarrow 0 \quad \text{as } t \rightarrow \infty \text{ stochastically.}
 \tag{2.28}$$

This proves part (i).

Now assume that (2.28) does not hold. We have to show that $\langle \zeta_t, f \rangle$ is stochastically unbounded. This is the case if and only if

$$\limsup_{\varepsilon \rightarrow 0} \liminf_{t \rightarrow \infty} E[\exp(-\langle \zeta_t, \varepsilon f \rangle)] < 1.
 \tag{2.29}$$

By Lemma 2.7 there exists a constant $c_f > 0$ such that the left-hand side of (2.29) is dominated by

$$\liminf_{t \rightarrow \infty} E[\exp(-c_f \langle \zeta_0, \phi(t, \cdot) \rangle)],
 \tag{2.30}$$

which is strictly smaller than 1 by assumption. Hence ζ_t is unstable.

Assume that the assumption of part (iii) holds, that is, $\mathcal{L}[\langle \zeta_0, \phi(r, \cdot) \rangle] \Rightarrow \delta_\infty$. The assertion that ζ_t explodes is equivalent to

$$\limsup_{t \rightarrow \infty} E[\exp(-\langle \zeta_t, f \rangle)] = 0
 \tag{2.31}$$

for all $f \in C_c^{++}(\mathbb{R}^d)$. This, however, can be shown with the aid of Proposition 2.6 as in the proof of part (i). \square

3. Proof for the high dimensions. Bramson, Cox and Greven (1997) give a proof for Theorem 2 in the case $\beta = 1$. Most of their proof works without *any* changes for all $\beta \in (0, 1]$. We do not repeat their entire proof here but only give an outline of their strategy and proofs of the lemmas that needed (minor) modifications.

Bramson, Cox and Greven (1997) use Proposition 2.6 to show that if ψ_t is stable, then $\langle \psi_0, p_t(\cdot) \rangle$ is stochastically bounded (as $t \rightarrow \infty$). They infer with the aid of Lemma 3.1 below that if $\mathcal{L}[\psi_0]$ is invariant, then for $f \in C_c^+(\mathbb{R}^d)$ and $z \in \mathbb{R}^d$,

$$E[\exp(-\langle \psi_t, f(z + \cdot) \rangle)] - E[\exp(-\langle \psi_t, f \rangle)] \rightarrow 0 \quad \text{as } t \rightarrow \infty.
 \tag{3.1}$$

Since the left-hand side of (3.1) does not depend on t (by the assumption that $\mathcal{L}[\psi_0]$ is an invariant law), we know that $\mathcal{L}[\psi_t]$ is translation invariant, hence Proposition 1.2 holds.

A standard argument now yields the claim of Theorem 2.

We only give the proof of Lemma 3.1 since it is here where small changes have to be made to cover the case $\beta < 1$.

For $\rho > 0$ let $B(\rho) = \{x \in \mathbb{R}^d, \|x\| < \rho\}$ denote the ball with radius ρ centered at the origin.

LEMMA 3.1. *Let $\varepsilon > 0$ and $0 < M < 1/4\varepsilon$. Let $f \in C_c^{++}(\mathbb{R}^d)$. For t large enough and all $x_1, x_2 \in B(\varepsilon t)$ with $\|x_1 - x_2\| \leq M$,*

$$(3.2) \quad u(t, x_1; f) \leq \exp(8\sqrt{M\varepsilon}/\beta) u(t, x_2; f).$$

PROOF.

Step 1. We show that for $x \in \mathbb{R}^d$ and $t \geq 0$,

$$(3.3) \quad e^{-2\delta/\beta} v(\delta t, x; u((1-\delta)t, \cdot; f)) \leq u(t, x; f) \leq v(\delta t, x; u((1-\delta)t, \cdot; f)),$$

where v is defined as in Lemma 2.3. The right-hand inequality follows immediately from the maximum principle. For the other inequality we proceed as follows.

Fix $t > 0$. Define the linear operator K_β for smooth functions $w(s, x)$ by

$$K_\beta w = \partial_s w - \frac{1}{2} \Delta w + \frac{2}{\beta t} w.$$

By (2.5) for $s > t/2$,

$$K_\beta u(s, x; f) \geq L_\beta u(s, x; f).$$

Let $w = w(s, x; f)$ be the solution of the Cauchy problem $K_\beta w = 0, w(0, \cdot; f) = f$. By the maximum principle (applied to K_β), $w(\delta t, x; u((1-\delta)t, \cdot; f)) \leq u(t, x; f)$. Note that [as in (2.6)] w can be represented as

$$(3.4) \quad \begin{aligned} w(\delta t, x; u((1-\delta)t, \cdot; f)) &= v(\delta t, x; u((1-\delta)t, \cdot; f)) \exp\left(-\int_0^{\delta t} \frac{2}{\beta} ds\right) \\ &= v(\delta t, x; u((1-\delta)t, \cdot; f)) \exp\left(\frac{-2\delta}{\beta}\right). \end{aligned}$$

This however yields (3.3).

Step 2. Finally we show the assertion of the lemma. Let $x \in B(\varepsilon t)$. We write

$$(3.5) \quad \begin{aligned} v(\delta t, x; u((1-\delta)t, \cdot; f)) &= I_1(x) + I_2(x) \\ &:= \int_{B(4\varepsilon t)} p_{\delta t}(z, x) u((1-\delta)t, z; f) dz \\ &\quad + \int_{B(4\varepsilon t)^c} p_{\delta t}(z, x) u((1-\delta)t, z; f) dz. \end{aligned}$$

An easy estimate using $\|x - z\| \geq 3\varepsilon t$ for $x \in B(\varepsilon t)$ and $z \in B(4\varepsilon t)^c$ and Lemma 2.3 yields for t large enough,

$$(3.6) \quad \begin{aligned} I_2(x) &\leq \int_{B(4\varepsilon t)^c} dz p_{\delta t}(x, z) \int_{\mathbb{R}^d} dy p_{(1-\delta)t}(z, y) f(y) \\ &\leq (2\pi\delta t)^{-d/2} \exp(-9\varepsilon^2 t/2) \|f\|_1 \leq \exp(-4\varepsilon^2 t) \|f\|_1. \end{aligned}$$

We obtain $u(t, x; f) \geq \exp(-3\varepsilon^2 t)\|f\|_1$ for t large by a similar estimate using the other inequality in Lemma 2.3. Hence

$$(3.7) \quad I_2(x) \leq \exp(-\varepsilon^2 t)u(t, x; f).$$

Let $x_1, x_2 \in B(\varepsilon t)$ with $\|x_1 - x_2\| \leq M$. For $z \in B(4\varepsilon t)$, clearly

$$\begin{aligned} \|z - x_2\|^2 - \|z - x_1\|^2 &= \|x_2 - x_1\|^2 + 2\langle z, x_2 - x_1 \rangle \\ &\leq \|x_2 - x_1\| \|x_2 - x_1\| + 2\|z\| \|x_2 - x_1\| \\ &\leq M \cdot 2\varepsilon t + 2 \cdot 4\varepsilon t \cdot M = 10\varepsilon M. \end{aligned}$$

Hence $p_{\delta t}(z, x_1) \leq \exp(5\varepsilon M/\delta)p_{\delta t}(z, x_2)$. Together with (3.3) we get (recall $\delta = \sqrt{\varepsilon M}$)

$$(3.8) \quad \begin{aligned} I_1(x_1) &\leq \exp(5\varepsilon M/\delta) \int_{B(4\varepsilon t)} p_{\delta t}(z, x_2)u((1 - \delta)t, z; f) dz \\ &\leq \exp(5\varepsilon M/\delta)v(\delta t, x_2; u((1 - \delta)t, \cdot; f)) \end{aligned}$$

$$(3.9) \quad \begin{aligned} &\leq \exp(2\delta/\beta + 5\varepsilon M/\delta)u(t, x_2; f) \\ &= \exp(7\delta/\beta)u(t, x_2; f). \end{aligned}$$

Together with (3.7) this implies for t large

$$u(t, x_1; f) \leq \exp(8\delta/\beta)u(t, x_2; f). \quad \square$$

4. Proof for the diffusive clustering. In order to show the weak convergence statement (1.26) of Theorem 3 we will show convergence of the Laplace transforms in the case of SBM. This is the content of Proposition 4.1 below. The case of BBM will follow by a comparison argument using the embedded particle system and the law of large numbers (here $\alpha \geq 0$ is needed).

The log-Laplace transform of the left-hand side of (1.26) for a test function $f \in C_c^+(\mathbb{R}^d)$ is [recall (1.13) and (1.20)–(1.22)]

$$(4.1) \quad \begin{aligned} &-\log E^{(\log t)^{1/\beta}\lambda}[\exp(-\langle \tilde{\zeta}_t^\alpha, f \rangle)] \\ &= -\log E^{(\log t)^{1/\beta}\lambda}[\exp(-\langle \tilde{\zeta}_t, t^{-\alpha d/2} f(t^{-\alpha/2} \cdot) \rangle)] \\ &= -\log E^{(\log t)^{1/\beta}\lambda}[\exp(-\langle \zeta_t, (\log t)^{-1/\beta} t^{-\alpha d/2} f(t^{-\alpha/2} \cdot) \rangle)] \\ &= \langle (\log t)^{1/\beta}\lambda, u(t, \cdot; (\log t)^{-1/\beta} t^{-\alpha d/2} f(t^{-\alpha/2} \cdot)) \rangle. \end{aligned}$$

Using the scaling relation (1.11) (with $\rho = t^\alpha$), this quantity equals (recall $\beta = 2/d$)

$$(4.2) \quad \begin{aligned} &\langle (\log t)^{1/\beta}\lambda, t^{-\alpha d/2} u(t^{1-\alpha}, t^{-\alpha/2} \cdot; (\log t)^{-1/\beta} f) \rangle \\ &= \langle (\log t)^{1/\beta}\lambda, u(t^{1-\alpha}, \cdot; (\log t)^{-1/\beta} f) \rangle \\ &= (1 - \alpha)^{-1/\beta} \langle (\log t')^{1/\beta}\lambda, u(t', \cdot; (\log t')^{-1/\beta} (1 - \alpha)^{1/\beta} f) \rangle, \end{aligned}$$

where we put $t' = t^{1-\alpha}$. We have to show that this expression converges as $t' \rightarrow \infty$ to [recall (1.23), (1.25) and that $c_\beta = (2\pi(1 + \beta)^{1/\beta})^{-1}$]

$$(4.3) \quad \begin{aligned} -\log E^1[\exp(-Z_{c_\beta(1-\alpha)}\langle \lambda, f \rangle)] &= (\beta c_\beta(1 - \alpha) + \langle \lambda, f \rangle^{-\beta})^{-1/\beta} \\ &= (1 - \alpha)^{-1/\beta}(c_\beta + \langle \lambda, (1 - \alpha)^{1/\beta} f \rangle^{-\beta})^{-1/\beta}. \end{aligned}$$

Comparing this with (4.2), it is clearly enough to handle the case $\alpha = 0$. This is the content of the following proposition.

PROPOSITION 4.1. *Assume $\beta = 2/d$ and let u be the solution of (1.8). For $f \in C_c^{++}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $-\infty < \alpha \leq 1$ and $s \geq 0$ the following hold.*

- (i)
$$\begin{aligned} \lim_{t \rightarrow \infty} (t \log t)^{1/\beta} u(t - s, t^{1/2} x; (\log t)^{-1/\beta} f) \\ = (2\pi(\beta c_\beta + \langle \lambda, f \rangle^{-\beta}))^{-1/\beta} \exp(-\|x\|^2/2); \end{aligned}$$
- (ii)
$$\lim_{t \rightarrow \infty} \langle (\log t)^{1/\beta} \lambda, u(t, \cdot; (\log t)^{-1/\beta} f) \rangle = (\beta c_\beta + \langle \lambda, f \rangle^{-\beta})^{-1/\beta}.$$

4.1. *Proof of Proposition 4.1.* The proof of Proposition 4.1 copies the proofs of Lee [(1991), Theorems 2.1, 2.3 and 2.4]. Lee's results cover only the case $\beta = 1$, but can easily be adapted to $\beta < 1$. For the sake of completeness we give the proof in detail.

Strategy of the proof. Recall that $b = 1$ and that the (nonlinear) operator L_β is defined by

$$L_\beta u = \partial_t u - \frac{1}{2} \Delta u + u^{1+\beta}.$$

As indicated by Proposition 2.6, a solution of $L_\beta u = 0$ should be "close" to a multiple of

$$\phi(t, x) = t^{-d/2} (\log t)^{-1/\beta} \exp(-\|x\|^2/2t).$$

However, in Proposition 2.6 we had no control of the constants a and A of the upper and lower bounds of $u(t, x; f)$ in terms of $\phi(t, x)$. (The point that ϕ in the upper and lower bounds is evaluated not at time t , but at $t/2$ and $4t$, respectively, could be repaired rather easily.)

More precisely, we would like to find a constant θ_β such that for $\varepsilon > 0$ and t large enough,

$$\begin{aligned} \theta_\beta(1 - \varepsilon)(t \log t)^{-1/\beta} \exp\left(-\frac{\|x\|^2}{2t}\right) &\leq u(t, x; f) \\ &\leq \theta_\beta(1 + \varepsilon)(t \log t)^{-1/\beta} \exp\left(-\frac{\|x\|^2}{2t(1 + \varepsilon)}\right), \quad x \in \mathbb{R}^d. \end{aligned}$$

We make the following ansatz to determine θ_β . For $\theta > 0$ define \tilde{u} by

$$(4.4) \quad \tilde{u} = \theta \phi + t^{-1/\beta} (\log t)^{-(1+1/\beta)} g(t^{-1/2} x),$$

with a "smooth" function g .

Let

$$H = -\left(\frac{1}{\beta}I + \frac{1}{2}\Delta + \frac{x}{2}\nabla\right)$$

and

$$f_\theta(t, x) = \theta^{1+\beta} \exp\left(- (1 + \beta) \frac{\|x\|^2}{2t}\right) - \frac{\theta}{\beta} \exp\left(- \frac{\|x\|^2}{2t}\right).$$

Then

$$L_\beta \tilde{u} = (t \log t)^{-(1+1/\beta)} [f_\theta(t, x) - (Hg)(t^{-1/2}x)] + o((t \log t)^{-(1+1/\beta)}).$$

Hence we search for θ such that $Hg(x) = f_\theta(1, x)$ has a solution g . In Lemma 4.2 we construct a right inverse G of H on a certain subspace $V \subset C_b(\mathbb{R}^d)$. It turns out that $f_\theta(1, \cdot) \in V$ if and only if

$$(4.5) \quad \theta = \theta_\beta = \left(\frac{1}{\beta}(1 + \beta)^{1/\beta}\right)^{1/\beta}.$$

The problem with this ansatz is that we cannot control the tail behavior of $g(x)$ at infinity. So now that we have determined θ_β we come back to the main idea of constructing sub- and supersolutions to $L_\beta u = 0$. We try the following ansatz:

$$(4.6) \quad \begin{aligned} \underline{u}(t, x) &= (\theta_\beta - \varepsilon) \phi(t, x) + t^{-1/\beta} (\log t)^{-1-1/\beta} g^-(t^{-1/2}x), \\ \bar{u}(t, x) &= (\theta_\beta + \varepsilon) \phi(t, x) + t^{-1/\beta} (\log t)^{-1-1/\beta} g^+(t^{-1/2}x), \end{aligned}$$

where the functions g^- and g^+ will be determined in terms of G . In Lemma 4.3 we control the tails of g^- and g^+ and in Lemma 4.4 we show that these \underline{u} and \bar{u} are indeed sub- and supersolutions to our problem.

The second step in the proof of Proposition 4.1 is to deal with the fact that we rescale the initial value of our Cauchy problem, that is, that we start u in $(\log t)^{-1/\beta} f$. We use our new estimates on the (nonrescaled) behavior of u to give a refinement of Lemma 2.7 that also contains an upper bound (Lemma 4.7). First we reduce the problem to initial conditions of the form ap_1 , $a > 0$ (Lemma 4.6). Next we use a similar scaling and comparison argument as in the proof of Lemma 2.7 to deal with general initial data.

The conclusion of part (i) of Proposition 4.1 is then easy. Statement (ii) follows by a simple dominated convergence argument.

The details. We start by giving a right inverse G of

$$H = -\left(\frac{1}{\beta}I + \frac{1}{2}\Delta + \frac{x}{2}\nabla\right)$$

on the linear space V generated by functions $\mathbb{R}^d \rightarrow \mathbb{R}$ of the type

$$q_{a,b}(x) = a^{-d/2} \exp\left(-\frac{\|x\|^2}{2a}\right) - b^{-d/2} \exp\left(-\frac{\|x\|^2}{2b}\right), \quad a, b > 0.$$

Define for $s > 0$ and $y, z \in \mathbb{R}^d$,

$$G(s, y, z) = s^{-1} (2\pi(1-s))^{-d/2} \exp\left(-\frac{\|y - s^{1/2}z\|^2}{2(1-s)}\right).$$

LEMMA 4.2. We can define a linear operator $G: V \rightarrow C_b(\mathbb{R}^d)$ by

$$(Gq_{a,b})(y) = \int_0^1 \int_{\mathbb{R}^d} G(s, y, z)q_{a,b}(z) dz ds.$$

The operator G is the right inverse of H on V . That is, $HG = \text{id}_V$.

PROOF. It is easily verified that the integral converges. Hence G is well defined. Now let

$$v(t, x) = \int_0^t \int_{\mathbb{R}^d} (2\pi)^{-d/2}(t-s)^{-d/2} \exp\left(-\frac{\|x-z\|^2}{2(t-s)}\right) s^{-1-d/2} q_{a,b}(s^{-1/2}z) dz ds.$$

Note that the integral converges, that $v(1, \cdot) = Gq_{a,b}$ and that

$$(\partial_t - \frac{1}{2}\Delta)v(t, x) = t^{-1-d/2}q(t^{-1/2}x).$$

By the substitution $s' = ts$ we obtain $v(t, x) = t^{-d/2}v(1, t^{-1/2}x)$. Thus $(\partial_t - \frac{1}{2}\Delta)v(t, x) = t^{-1-d/2}(Hv(1, \cdot))(t^{-1/2}x)$. Hence $q_{a,b}(y) = Hv(1, y) = H(Gq_{a,b})(y)$ as desired. \square

Note that we can find a function g such that $Hg = f_\theta(1, \cdot)$ if $f_\theta(1, \cdot) \in V$. This, however, is equivalent to $\theta = \theta_\beta$ with θ_β defined in (4.5)

The next aim is to construct the functions g^- and g^+ of (4.6) and to give upper and lower bounds for g^- and g^+ .

For $0 < \varepsilon < \min(\beta/(1-\beta), \theta_\beta/2)$ [where $\beta/(1-\beta) = \infty$ if $\beta = 1$] define

$$q^-(x) = \rho^- q_{1, 1/(1+\beta)}$$

and

$$q^+(x) = \rho^+ q_{1+k\varepsilon, 1/(1+\beta)},$$

where

$$\rho^- = \rho^-(\varepsilon) = \frac{(\theta_\beta - \varepsilon)^{1+\beta}}{(1+\beta)^{1/\beta}},$$

$$\rho^+ = \rho^+(\varepsilon) = \frac{(\theta_\beta + \varepsilon)^{1+\beta}}{(1+\beta)^{1/\beta}}$$

and $k = \beta^2/(2\theta_\beta)$. Now let $g^- = Gq^-$ and $g^+ = Gq^+$.

LEMMA 4.3. There exist constants $0 < m, M < \infty$ such that for all $x \in \mathbb{R}^d$,

$$(4.7) \quad -m \exp\left(-\frac{\|x\|^2}{2}\right) \leq g^-(x) \leq M \exp\left(-\frac{\|x\|^2}{2(1+\varepsilon)}\right),$$

$$(4.8) \quad -m \exp\left(-\frac{\|x\|^2}{2}\right) \leq g^+(x) \leq M \exp\left(-\frac{\|x\|^2}{2(1+k\varepsilon)}\right).$$

PROOF. We only give the proof of (4.8). The proof of (4.7) is similar but easier.

By the substitution $y = s^{1/2}z$ and the Chapman–Kolmogorov equation, we get

$$\begin{aligned}
 g^+(x) &= \rho^+ \int_0^1 ds \int_{\mathbb{R}^d} dz s^{-1} (2\pi(1-s))^{-d/2} \exp\left(-\frac{\|x - s^{1/2}z\|^2}{2(1-s)}\right) \\
 &\quad \times \left[(1+k\varepsilon)^{-d/2} \exp\left(-\frac{\|z\|^2}{2(1+k\varepsilon)}\right) \right. \\
 (4.9) \quad &\quad \left. - (1+\beta)^{d/2} \exp\left(-\frac{\|z\|^2}{2}\right) \right] \\
 &= \rho^+ \int_0^1 ds s^{-1} \left[(1+sk\varepsilon)^{-d/2} \exp\left(-\frac{\|x\|^2}{2(1+sk\varepsilon)}\right) \right. \\
 &\quad \left. - \left(1 - \frac{\beta s}{1+\beta}\right)^{-d/2} \exp\left(-\frac{\|x\|^2}{2(1-\beta s/(1+\beta))}\right) \right].
 \end{aligned}$$

Choose $a \in (0, 1)$. Then clearly,

$$\int_a^1 \dots ds \leq \left(\int_a^1 s^{-1} (1+sk\varepsilon)^{-d/2} ds \right) \exp\left(-\frac{\|x\|^2}{2(1+k\varepsilon)}\right).$$

By partial integration we see that the \int_0^a term equals

$$\begin{aligned}
 &\int_0^a ds \log s \left[k\varepsilon \frac{d(1+sk\varepsilon) + \|x\|^2}{2(1+sk\varepsilon)^{2+d/2}} \exp\left(-\frac{\|x\|^2}{2(1+sk\varepsilon)}\right) \right. \\
 &\quad \left. + \frac{\beta}{1+\beta} \frac{d(1-\beta s/(1+\beta)) + \|x\|^2}{2(1-\beta s/(1+\beta))^{2+d/2}} \right. \\
 &\quad \left. \times \exp\left(-\frac{\|x\|^2}{2(1-\beta s/(1+\beta))}\right) \right] \\
 (4.10) \quad &+ (\log a) \left[(1+ak\varepsilon)^{-d/2} \exp\left(-\frac{\|x\|^2}{2(1+ak\varepsilon)}\right) \right. \\
 &\quad \left. - \left(1 - \frac{\beta a}{1+\beta}\right)^{-d/2} \exp\left(-\frac{\|x\|^2}{2(1-\beta a/(1+\beta))}\right) \right] \\
 &= O\left((1+\|x\|^2) \exp\left(-\frac{\|x\|^2}{2(1+ak\varepsilon)}\right)\right) = O\left(\exp\left(-\frac{\|x\|^2}{2(1+k\varepsilon)}\right)\right). \quad \square
 \end{aligned}$$

Now we are able to construct our sub- and supersolutions \underline{u} and \bar{u} .

LEMMA 4.4. *Let $0 < \varepsilon < \min(\beta/(1-\beta), \theta_\beta/2)$ and g^- and g^+ as in Lemma 4.3. Define \underline{u} and \bar{u} by*

$$\underline{u}(t, x) = (\theta_\beta - \varepsilon) \phi(t, x) + t^{-1/\beta} (\log t)^{-1-1/\beta} g^-(t^{-1/2}x),$$

$$\bar{u}(t, x) = (\theta_\beta + \varepsilon) \phi(t, x) + t^{-1/\beta} (\log t)^{-1-1/\beta} g^+(t^{-1/2}x).$$

Then $L_\beta \underline{u} \leq 0$ and $L_\beta \bar{u} \geq 0$. Furthermore, for t large enough and all $x \in \mathbb{R}^d$,

$$(4.11) \quad 0 \leq (\theta_\beta - 2\varepsilon)\phi(t, x) \leq \underline{u}(t, x) \leq \theta_\beta (t \log t)^{-1/\beta} \exp\left(-\frac{\|x\|^2}{2t(1+\varepsilon)}\right),$$

$$(4.12) \quad 0 \leq \theta_\beta \phi(t, x) \leq \bar{u}(t, x) \leq (\theta_\beta + 2\varepsilon) (t \log t)^{-1/\beta} \exp\left(-\frac{\|x\|^2}{2t(1+\varepsilon)}\right).$$

PROOF. From (4.7) and (4.8) it is clear that (4.11) and (4.12) hold. In order to show that \underline{u} and \bar{u} are sub- and supersolutions, we have to give upper and lower bounds for $L_\beta \underline{u}$ and $L_\beta \bar{u}$. Note that for $a, b \in \mathbb{R}$, $a > 0$ and $|b|/a$ small enough, $|(\alpha + b)^{1+\beta} - \alpha^{1+\beta}| < 4\alpha^\beta b$. We use this estimate to show that

$$\begin{aligned} L_\beta \underline{u}(t, x) &\leq L_\beta((\theta_\beta - \varepsilon)\phi(t, x)) + (t \log t)^{-1-1/\beta} (Hg^-)(t^{-1/2}x) \\ &\quad + 4M((\theta_\beta - \varepsilon)\phi(t, x))^\beta t^{-1/\beta} (\log t)^{-1-1/\beta} \exp\left(-\frac{\|x\|^2}{2t(1+\varepsilon)}\right) \\ &\quad + m\left(1 + \frac{1}{\beta}\right) t^{-1-1/\beta} (\log t)^{-2-1/\beta} \exp\left(-\frac{\|x\|^2}{2t}\right) \\ &= (t \log t)^{-1-1/\beta} \left[(\theta_\beta - \varepsilon)^{1+\beta} \exp\left(-\frac{\|x\|^2}{2t}\right) \right. \\ &\quad \left. - \frac{\theta_\beta - \varepsilon}{\beta} \exp\left(-\frac{\|x\|^2}{2t}\right) + \rho^- \exp\left(-\frac{\|x\|^2}{2t}\right) \right. \\ &\quad \left. - \rho^- (1 + \beta)^{1/\beta} \exp\left(-\frac{\|x\|^2}{2t}\right) \right] \\ &\quad + O\left(t^{-1-1/\beta} (\log t)^{-2-1/\beta} \exp\left(-\frac{\|x\|^2}{2t}\right)\right) \\ &= (t \log t)^{-1-1/\beta} \left(\rho^- - \frac{\theta_\beta - \varepsilon}{\beta} \right) \exp\left(-\frac{\|x\|^2}{2t}\right) \\ &\quad + O\left(t^{-1-1/\beta} (\log t)^{-2-1/\beta} \exp\left(-\frac{\|x\|^2}{2t}\right)\right) \\ &\leq 0 \end{aligned} \tag{4.13}$$

for t large enough, since $\rho^- - (\theta_\beta - \varepsilon)/\beta < 0$ for $0 < \varepsilon < \theta_\beta$.

Similarly, for t large,

$$\begin{aligned} L_\beta \bar{u} &\geq L_\beta((\theta_\beta + \varepsilon)\phi(t, x)) + (t \log t)^{-1-1/\beta} (Hg^+)(t^{-1/2}x) \\ &\quad - 4m((\theta_\beta + \varepsilon)\phi(t, x))^\beta t^{-1/\beta} (\log t)^{-1-1/\beta} \exp\left(-\frac{\|x\|^2}{2t}\right) \\ (4.14) \quad &\quad - M(1 + 1/\beta) \left(t^{-1-1/\beta} (\log t)^{-2-1/\beta} \exp\left(-\frac{\|x\|^2}{2t(1+k\varepsilon)}\right) \right) \end{aligned}$$

$$\begin{aligned}
 &= (t \log t)^{-1-1/\beta} \left[\rho^+(1+k\varepsilon)^{-d/2} \exp\left(-\frac{\|x\|^2}{2t(1+k\varepsilon)}\right) \right. \\
 &\quad \left. - \frac{\theta_\beta + \varepsilon}{\beta} \exp\left(-\frac{\|x\|^2}{2t}\right) \right] \\
 &\quad + O\left(\left(t^{-1-1/\beta}(\log t)^{-2-1/\beta} \exp\left(-\frac{\|x\|^2}{2t(1+k\varepsilon)}\right)\right)\right) \\
 &\geq 0,
 \end{aligned}$$

since $\rho^+(1+k\varepsilon)^{-d/2} - (\theta_\beta + \varepsilon)/\beta > 0$. To see the latter inequality, note that

$$1 + k\varepsilon = 1 + \frac{\beta^2 \varepsilon}{2\theta_\beta} = \frac{\beta^\beta}{1 + \beta} \left[\theta_\beta^{\beta^2} + \frac{1}{2} \beta^2 \theta_\beta^{\beta^2-1} \varepsilon \right] < \frac{\beta^\beta}{1 + \beta} [\theta_\beta + \varepsilon]^{\beta^2}.$$

This inequality holds since $\varepsilon < \theta_\beta$. Thus

$$(\theta_\beta + \varepsilon) \left[\frac{(\theta_\beta + \varepsilon)^\beta}{(1 + \beta)^{1/\beta}} (1 + k\varepsilon)^{-1/\beta} - \frac{1}{\beta} \right] > 0,$$

as claimed. \square

A simple consequence of Lemma 4.4 is the following result on the asymptotic behavior of u .

LEMMA 4.5. *For $f \in C_c^{++}(\mathbb{R}^d)$ the following holds:*

$$\lim_{t \rightarrow \infty} (t \log t)^{1/\beta} u(t, t^{1/2}x, f) = \theta_\beta \exp(-\|x\|^2/2), \quad x \in \mathbb{R}^d.$$

The proof of this statement is simple with Lemma 4.4 at hand. Since we will not make use of this lemma, we omit the details and refer the reader to Lee [(1993), pages 304 and 305].

So far we have considered only a fixed initial condition $u(0, \cdot; f) = f$ of the Cauchy problem $L_\beta u = 0$. The next aim is to change the initial condition to $(\log t)^{-1/\beta} f$.

Define

$$b(x; f) = \inf_{s \geq 0} \liminf_{t \rightarrow \infty} (t \log t)^{1/\beta} u(t - s, t^{1/2}x; (\log t)^{-1/\beta} f)$$

and

$$B(x; f) = \sup_{s \geq 0} \limsup_{t \rightarrow \infty} (t \log t)^{1/\beta} u(t - s, t^{1/2}x; (\log t)^{-1/\beta} f).$$

In order to prove Proposition 4.1 we will give upper and lower bounds for B and b , respectively.

The first step is to reduce the situation to $f(x)$ replaced by $\langle \lambda, f \rangle (2\pi)^{-d/2} \exp(-\|x\|^2/2)$. Recall that p_t is the heat kernel.

LEMMA 4.6. *Let $f \in C_c^{++}(\mathbb{R}^d)$. For $x \in \mathbb{R}^d$ the following inequalities hold:*

$$\begin{aligned} b(x; f) &\geq b(x; \langle \lambda, f \rangle p_1), \\ B(x; f) &\leq B(x; \langle \lambda, f \rangle p_1). \end{aligned}$$

PROOF. Note that for $x, y \in \mathbb{R}^d$ and $t, \delta > 0$,

$$\frac{\|x\|^2}{2t(1+\delta)} - \frac{\|x-y\|^2}{2t} \leq \frac{1}{t} \frac{1+\delta}{2\delta} \|y\|^2.$$

Hence for $K \subset \mathbb{R}^d$ compact and $\varepsilon > 0$ we can find $\delta > 0$ and $t_0 > 0$ such that for $f \in C_c^{++}(\mathbb{R}^d)$ with $\text{supp}(f) \subset K$,

$$p_t * f \leq (1 + \varepsilon) \langle \lambda, f \rangle p_{(1+\delta)t}, \quad t \geq t_0.$$

Similarly we get $\delta > 0$ and $t_0 > 0$ such that

$$p_t * f \geq (1 - \varepsilon) \langle \lambda, f \rangle p_{(1-\delta)t}, \quad t \geq t_0.$$

Hence if we let $d_t = \varepsilon \|f\|_\infty^{-\beta} \log t$, we get with Lemmas 2.3 and 2.2 that for $d_t > t_0$,

$$\begin{aligned} &u(t-s, t^{1/2}x; (\log t)^{-1/\beta} f) \\ &= u(t-s-d_t, t^{1/2}x; u(d_t, \cdot; (\log t)^{-1/\beta} f)) \\ &\geq u(t-s-d_t, t^{1/2}x; (1-\varepsilon) (\log t)^{-1/\beta} (p_{d_t} * f)) \\ &\geq u(t-s-d_t, t^{1/2}x; (1-\varepsilon)^2 (\log t)^{-1/\beta} \langle \lambda, f \rangle p_{(1-\delta)d_t}) \\ &\geq (1-\varepsilon)^2 u(t-s-d_t, t^{1/2}x; (\log t)^{-1/\beta} \langle \lambda, f \rangle p_{(1-\delta)d_t}) \\ &\geq (1-\varepsilon)^2 u(t-s-\delta d_t-1, t^{1/2}x; (\log t)^{-1/\beta} \langle \lambda, f \rangle p_1). \end{aligned}$$

Since $d_t \ll t$, we obtain

$$b(x; f) \geq (1 - \varepsilon)^2 b(x; \langle \lambda, f \rangle p_1).$$

Now let $\varepsilon \rightarrow 0$ to obtain the claim. The claim for B follows analogously using the opposite inequality in Lemma 2.3. \square

Now we give the bounds for $b(x; ap_1)$ and $B(x; ap_1)$. The result is a refinement of Lemma 2.7 and will be obtained by a similar scaling argument.

LEMMA 4.7. *For $a > 0$ and $x \in \mathbb{R}^d$ the following inequalities hold:*

$$\begin{aligned} b\left(x; a \exp\left(-\frac{\|\cdot\|^2}{2}\right)\right) &\geq \left(\frac{(\theta_\beta a)^\beta}{a^\beta + (\theta_\beta)^\beta}\right)^{1/\beta} \exp\left(-\frac{\|x\|^2}{2}\right), \\ B\left(x; a \exp\left(-\frac{\|\cdot\|^2}{2}\right)\right) &\leq \left(\frac{(\theta_\beta a)^\beta}{a^\beta + (\theta_\beta)^\beta}\right)^{1/\beta} \exp\left(-\frac{\|x\|^2}{2}\right). \end{aligned}$$

PROOF. We do the proof only for the second inequality. The proof of the other inequality is quite similar.

Recall from (1.11) that

$$(4.15) \quad u(t, x; \rho^{1/\beta} f(\rho^{1/2} \cdot)) = \rho^{1/\beta} u(\rho t, \rho^{1/2} x; f), \quad \rho > 0.$$

Thus for $s > 0$, $t - s > 0$ large enough and $\rho = \rho(t) = t^{(\theta_\beta)^\beta / \alpha^\beta}$ we have by (4.12),

$$(4.16) \quad \begin{aligned} & u(t - s, x; (\log t)^{-1/\beta} \alpha \exp(-\|\cdot\|^2/2)) \\ &= u(t - s, x; \theta_\beta (\log \rho)^{-1/\beta} \exp(-\|\cdot\|^2/2)) \\ &\leq u(t - s, x; \rho^{1/\beta} \bar{u}(\rho, \rho^{1/2} \cdot)) \\ &= \rho^{1/\beta} u(\rho(t - s), \rho^{1/2} x; \bar{u}(\rho, \cdot)) \\ &\leq \rho^{1/\beta} \bar{u}(\rho(t - s + 1), \rho^{1/2} x). \end{aligned}$$

We infer that for $\varepsilon > 0$ uniformly in $x \in \mathbb{R}^d$,

$$(4.17) \quad \begin{aligned} & \sup_{s \geq 0} \limsup_{t \rightarrow \infty} (t \log t)^{1/\beta} u\left(t - s, t^{1/2} x; (\log t)^{-1/\beta} \alpha \exp\left(-\frac{\|\cdot\|^2}{2}\right)\right) \\ &\leq \sup_{s \geq 0} \limsup_{t \rightarrow \infty} (t \log t)^{1/\beta} \rho^{1/\beta} \bar{u}(\rho(t - s + 1), (\rho t)^{1/2} x) \\ &\leq \sup_{s \geq 0} \limsup_{t \rightarrow \infty} (t \log t)^{1/\beta} (t - s + 1)^{-1/\beta} (\log(\rho(t - s + 1)))^{-1/\beta} \theta_\beta \\ &\quad \times (1 + \varepsilon) \exp\left(-\frac{\|x\|^2 t}{2(1 + \varepsilon)(t - s + 1)}\right) \\ &= \sup_{s \geq 0} \limsup_{t \rightarrow \infty} \left(\frac{\log t}{\log \rho + \log t}\right)^{1/\beta} \theta_\beta (1 + \varepsilon) \exp\left(-\frac{\|x\|^2}{2(1 + \varepsilon)}\right) \\ &= \theta_\beta (1 + \varepsilon) \exp\left(-\frac{\|x\|^2}{2(1 + \varepsilon)}\right) \left(\frac{(\theta_\beta \alpha)^\beta}{\alpha^\beta + (\theta_\beta)^\beta}\right)^{1/\beta}. \end{aligned}$$

Now let $\varepsilon \rightarrow 0$. \square

PROOF OF PROPOSITION 4.1. The proof of Proposition 4.1 is now easy. Combine Lemmas 4.6 and 4.7 to obtain part (i).

In order to prove part (ii), note that by Lemma 2.3,

$$\begin{aligned} (t \log t)^{1/\beta} u(t, t^{1/2} x; (\log t)^{-1/\beta} f) &\leq t^{1/\beta} (p_t * f)(t^{1/2} x) \\ &= (p_1 * (t^{1/\beta} f(t^{1/2} \cdot)))(x) \\ &\leq c_f p_2(x) \quad \text{for } t \geq 1 \end{aligned}$$

for some $c_f > 0$. Hence dominated convergence yields the claim. \square

4.2. *Proof of Theorem 3.* First consider the case where (ψ_t) is SBM($d, 1 + 2/d, 1$). Note that here the assertion is immediate from Proposition 4.1(ii), by (1.13) and the scaling relation [see (4.2)]

$$(4.18) \quad \mathcal{L}^{\rho, \lambda}[t^{-\alpha/\beta} \zeta_t(t^{\alpha/2} \cdot)] = \mathcal{L}^{\rho, \lambda}[\zeta_{t^{1-\alpha}}(\cdot)].$$

A more detailed discussion precedes Proposition 4.1.

Now assume that $(\psi_t) = (\eta_t)$ is BBM($d, 1 + 2/d, 1$). The link to the SBM is the embedded particle system [an idea that goes back to Gorostiza, Roelly-Coppoletta and Wakolbinger (1990), Lemme 1]:

(4.19) "For fixed time horizon t , poissonizing the initial state m first and then running a BBM (η_s) is the same as running SBM (ζ_s) with initial state m and then Poissonizing the random population ζ_t ."

To make this precise we define a new random measure X_t such that $\mathcal{L}[X_t | \zeta_t] = \mathcal{H}(\zeta_t)$ [recall that $\mathcal{H}(m)$ is the law of a Poisson point process with intensity measure m]. Then (4.19) says that, for $m \in \mathcal{M}(\mathbb{R}^d)$,

$$(4.20) \quad \mathcal{L}^{\mathcal{H}(m)}[\eta_t] = \mathcal{L}^m[X_t].$$

To check this let $f \in C_c^+(\mathbb{R}^d)$. Then [recall (1.12) and (1.13)]

$$(4.21) \quad \begin{aligned} \mathbb{E}^{\mathcal{H}(m)}[\exp(-\langle \eta_t, f \rangle)] &= \exp\left(-\int m(dx) (1 - \mathbb{E}^{\delta_x}[\exp(-\langle \eta_t, f \rangle)])\right) \\ &= \exp(-\langle m, u(t, \cdot, 1 - e^{-f}) \rangle) \\ &= \mathbb{E}^m[\exp(-\langle \zeta_t, 1 - e^{-f} \rangle)] \\ &= \mathbb{E}^m[\exp(-\langle X_t, f \rangle)]. \end{aligned}$$

Now for $A \in \mathcal{B}(\mathbb{R}^d)$ bounded and $\alpha \geq 0$ by the law of large numbers,

$$(4.22) \quad \mathbb{E}^{(\log t)^{1/\beta} \lambda}[t^{-\alpha d/2} (\log t)^{-1/\beta} |X_t(t^{\alpha/2} A) - \zeta_t(t^{\alpha/2} A)|] \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad \square$$

4.3. *Proof of Theorem 4.* Note that as above, the case of (ψ_t) BBM can be derived from the case (ψ_t) SBM. Hence we will now assume that $(\psi_t) = (\zeta_t)$ is SBM($d, 1 + 2/d, 1$).

The idea of the proof is an induction over the length of the tree \mathbb{T} . Recall the heuristics given in the discussion preceding Theorem 4. The key point in the induction is to show that the "important" information about $\tilde{\zeta}_{t-t^\alpha} |_{[-t^{\alpha/2}, t^{\alpha/2}]^d}$ is already contained in $\tilde{\zeta}_{t-t^\alpha}([-t^{\alpha/2}, t^{\alpha/2}]^d)$. We do so by constructing a coupling of $(\zeta_{t-t^\alpha+s})_{s \geq 0}$ with a SBM($d, 1 + 2/d, 1$) $(\zeta_s^2)_{s \geq 0}$ started in $t^{-\alpha d/2} \zeta_{t-t^\alpha}([-t^{\alpha/2}, t^{\alpha/2}]^d) \lambda$. We show that the coupling is successful in a certain sense within time $s = t^\alpha$.

We prepare for the proof of Theorem 4 by stating a coupling lemma and a comparison lemma both taken from Klenke (1997) (stated there for the case $\beta = 1$ only). Note that we give a new proof of the comparison lemma since the second moment used in Klenke (1997) is not available here.

LEMMA 4.8 (Coupling). *Let $S > R > 0$. Consider $(\zeta_s^1)_{s \geq 0}$ SBM($d, 1 + 2/d, 1$). Assume that $\mathcal{L}[\zeta_0^1]$ is translation invariant and that $\varepsilon > 0$ and $0 < \rho < \infty$ are chosen such that*

$$\begin{aligned} E[\zeta_0^1([0, 1]^d)] &= \rho, \\ E[|R^{-d}\zeta_0^1([0, R]^d) - S^{-d}\zeta_0^1([0, S]^d)|] &< \varepsilon, \\ E[|\zeta_0^1([0, S]^d) - \zeta_0^1(S(z + [0, 1]^d))|] &< \varepsilon S^d \quad \forall z \in [-1, 1]^d. \end{aligned}$$

Then there exists a coupling $(\zeta_s^1, \zeta_s^2)_{s \geq 0}$ [i.e., (ζ_s^2) is also SBM($d, 1 + 2/d, 1$) and both processes are defined on the same probability space] such that

$$(4.23) \quad \mathcal{L}[\zeta_0^2 | \zeta_0^1] = S^{-d}\zeta_0^1([0, S]^d)\lambda$$

and

$$(4.24) \quad E[\|(\zeta_s^1 - \zeta_s^2)|_B\|] \leq \lambda(B) \left[4\varepsilon + d \exp\left(\frac{-D^2}{2s}\right) + 2\sqrt{\frac{d}{\pi}} \rho R s^{-1/2} \right],$$

where $B \in \mathcal{B}(\mathbb{R}^d)$, $B \subset [0, S]^d$ and $D = \text{dist}(B, \mathbb{R}^d \setminus [0, S]^d)$.

The proof is Corollary 3.7 in Klenke (1997). In fact the proof given there does not rely on the finite variance available there.

With the tool of the coupling lemma we are able to give a proof of the following lemma that does not rely on second moments.

LEMMA 4.9 (Comparison). *Let $\alpha \in (-\infty, 1]$ and let $a(t), b(t) \approx t^\alpha$, that is,*

$$\lim_{t \rightarrow \infty} \frac{\log a(t)}{\log t} = \lim_{t \rightarrow \infty} \frac{\log b(t)}{\log t} = \alpha.$$

Then

$$(4.25) \quad E^{(\log t)^{1/\beta} \lambda} [(\log t)^{-1/\beta} |a(t)^{-d} \zeta_t([0, a(t)]^d) - b(t)^{-d} \zeta_t([0, b(t)]^d)|] \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

PROOF. The main tool for the proof is the coupling lemma, Lemma 4.8. We prepare for the use of it.

By the basic scaling relation (4.18) we may w.l.o.g. assume $\alpha = 0$. Also we can restrict ourselves to the case $b(t) \equiv 1$ and $a(t) \downarrow 0$, $\log a(t) \ll \log t$. Since

$$(4.26) \quad E^{(\log t)^{1/\beta} \lambda} [(\log t)^{-1/\beta} |S^{-d} \zeta_t([0, S]^d) - R^{-d} \zeta_t([0, R]^d)|] \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

for any $R, S > 0$, we may find $R = R(t) \downarrow 0$, $S = S(t) \uparrow \infty$ such that

$$(4.27) \quad \begin{aligned} E^{(\log t)^{1/\beta} \lambda} [(\log t)^{-1/\beta} |S(t)^{-d} \zeta_{t-1}([0, S(t)]^d) - R(t)^{-d} \zeta_{t-1}([0, R(t)]^d)|] \\ =: \varepsilon_t \rightarrow 0 \quad \text{as } t \rightarrow \infty. \end{aligned}$$

By a similar argument we obtain [maybe by enlarging ε_t and changing $R(t)$ and $S(t)$ a little]

$$(4.28) \quad \sup_{z \in [0,1]^d} E^{(\log t)^{1/\beta} \lambda} [|(\log t)^{-1/\beta} S^{-d} \zeta_{t-1}([0, S]^d) - \zeta_{t-1}(S(z + [0, 1]^d))|] \leq \varepsilon_t \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

[For example, for fixed S choose $N \in \mathbb{N}$ large and take the maximum over $z \in \{0, 1/N, \dots, 1\}^d$. This term clearly vanishes as $t \rightarrow \infty$. The error term of the two maxima results from the (less than) $N^{d-1} 3^d$ blocks of size $[0, 1/N]^d$ at the surface of $(z + [0, 1]^d)$ and is thus bounded by $3^d(1/N)$. Now let $S = S(t)$ and $N = N(t)$ increase slowly to ∞ .]

We apply the coupling lemma, Lemma 4.8, to obtain a SBM($d, 1 + \beta, 1$) $(\zeta_s^2)_{s \geq t-1}$ with initial state $\mathcal{L}[\zeta_{t-1}^2 | \zeta_{t-1}] = S(t)^{-d} \zeta_{t-1}([0, S(t)]^d) \lambda$ and

$$(4.29) \quad E^{(\log t)^{1/\beta} \lambda} [(\log t)^{-1/\beta} \|(\zeta_t^1 - \zeta_t^2)|_B\|] \leq \lambda(B) [4\varepsilon_t + d \exp(-D^2) + 2\sqrt{d/\pi} R(t)],$$

where $B \in \mathcal{B}(\mathbb{R}^d)$ is bounded and $D = \text{dist}(B, \mathbb{R}^d \setminus [0, S(t)]^d)$. In particular, for $M < \infty$ there exists $\delta_t^M \rightarrow 0$ as $t \rightarrow \infty$ such that for any Borel set $B \subset [-M, M]^d$,

$$(4.30) \quad E^{(\log t)^{1/\beta} \lambda} [(\log t)^{-1/\beta} \|(\zeta_t^1 - \zeta_t^2)|_B\|] \leq \delta_t^M \lambda(B).$$

Now fix a value $\rho = (\log t)^{-1/\beta} S(t)^{-d} \zeta_{t-1}([0, S(t)]^d)$. According to Theorem 3 (and the basic scaling),

$$(4.31) \quad (\log t)^{-1/\beta} a(t)^{-d} \zeta_t^2([0, a(t)]^d) \rightarrow \rho \quad \text{as } t \rightarrow \infty$$

and

$$(\log t)^{-1/\beta} \zeta_t^2([0, 1]^d) \rightarrow \rho \quad \text{as } t \rightarrow \infty.$$

A simple uniform integrability argument yields

$$(4.32) \quad E^{(\log t)^{1/\beta} \lambda} [|(\log t)^{-1/\beta} a(t)^{-d} \zeta_t'([0, a(t)]^d) - \zeta_t'([0, 1]^d)|] \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Combined with (4.30) the proof is complete. \square

PROOF OF THEOREM 4. Recall that we do the proof for the case in which $(\psi_t) = (\zeta_t)$ is SBM($d, 1 + 2/d, 1$). The proof is almost identical to that given in Klenke (1997). The only "real" difference to the case $\beta = 1$ is the modified proof of Lemma 4.9 given above and some changes in the constants. However, for the sake of completeness we give the proof here in detail.

We do the proof by induction over the length of the tree \mathbb{T} . For $\mathbb{T} = \{\emptyset\}$, this is the assertion of Theorem 3. Now assume that the claim has been shown for all trees shorter than \mathbb{T} .

The idea of the proof is the following. We introduce a time scale $L(t) \approx t^{A(\emptyset)}$ and couple (ζ_s) for $s \geq t - L(t)$ with another process (ζ_s^2) . This process shall have initial configuration $M(\rho)$, where ρ is the empirical population density

of $\zeta_{t-L(t)}^1$ in a box of length $\approx t^{A(\emptyset)/2}$. Then $L(t)$ will be chosen small enough that the evolutions of the subtrees (resulting from eliminating \emptyset from \mathbb{T}) are approximately independent. On the other hand, $L(t)$ has to be chosen large enough so that the coupling of Lemma 4.8 with $R(t) \approx t^{A(\emptyset)/2}$ is successful. Here are the details.

Let $b = \max\{\text{diam}(B^e), e \in \mathbb{T}\}$. Let $d_t \downarrow 0, t \rightarrow \infty$ such that

$$(4.33) \quad \begin{aligned} t^{(A(e \wedge f) - d_t)/2} &\leq \|x_t^e - x_t^f\| - b(t^{A(e)/2} + t^{A(f)/2}) \\ &\leq \|x_t^e - x_t^f\| + b(t^{A(e)/2} + t^{A(f)/2}) \leq \frac{1}{2}t^{(A(e \wedge f) + d_t)/2} \end{aligned}$$

for all $e, f \in \mathbb{T}$. We may assume that $t^{d_t} \rightarrow \infty$ as $t \rightarrow \infty$. Let $\alpha := A(\emptyset)$ and define

$$\begin{aligned} S &= S(t) = t^{(\alpha + d_t)/2}, \\ R &= R(t) = t^{(\alpha - 3d_t)/2}, \\ L &= L(t) = t^{\alpha - 2d_t}. \end{aligned}$$

Let

$$(4.34) \quad B_t^e = x_t^e + t^{A(e)/2} B^e$$

and

$$(4.35) \quad B_t = \bigcup_{e \in \mathbb{T}} B_t^e.$$

By shifting $X = (x_t^e, e \in \mathbb{T})$, if necessary, we can assume that $B_t \subset [0, S]^d$ for all $t > 0$ and that

$$(4.36) \quad L^{-1/2} \text{dist}(B_t, \mathbb{R}^d \setminus [0, S]^d) \rightarrow \infty \quad \text{as } t \rightarrow \infty.$$

Apply Corollary 4.8 with $\zeta_0^1 = \zeta_{t-L(t)}^1, s = L(t), \rho = (\log t)^{1/\beta}$ and with $\varepsilon = (\log t)^{1/\beta} \varepsilon_t$, where $\varepsilon_t \rightarrow 0$ as $t \rightarrow \infty$. This last choice is possible due to Lemma 4.9. Thus we obtain a coupling $(\zeta_s^1, \zeta_s^2)_{s \geq 0}$ with $\mathcal{L}[\zeta_0^1 | \zeta_s^1] = M(S^{-d} \zeta_0^1([0, S]^d))$ such that there exists a sequence $\delta_t \downarrow 0$ with

$$(4.37) \quad \mathbb{E}^{(\log t)^{1/\beta} \lambda} [|(\tilde{\zeta}_{L(t)}^1 - \tilde{\zeta}_{L(t)}^2)(C)|] \leq \delta_t \lambda(C) \quad \forall C \in \mathcal{B}(\mathbb{R}^d) \text{ bounded.}$$

So all we have to show is

$$(4.38) \quad \begin{aligned} &\mathcal{L}^{(\log t)^{1/\beta} \lambda} [(\log t)^{-1/\beta} (t^{-A(e)d/2} \zeta_{L(t)}^2(B_t^e))_{e \in \mathbb{T}}] \\ &\implies \mathcal{L}^1 [(\lambda(B^e) Z_{(1-A(e))c_\beta}^e)_{e \in \mathbb{T}}] \quad \text{as } t \rightarrow \infty. \end{aligned}$$

By Theorem 3 we know that

$$(4.39) \quad \mathcal{L}^{(\log t)^{1/\beta} \lambda} [(\log t)^{-1/\beta} S^{-d} \zeta_0^1([0, S]^d)] \implies \mathcal{L}^1 [Z_{(1-\alpha)c_\beta}] \quad \text{as } t \rightarrow \infty.$$

Hence it suffices to show that for $\rho \geq 0$,

$$\begin{aligned}
 & \mathcal{L}^{\rho(\log t)^{1/\beta}\lambda} [(\log t)^{-1/\beta} (t^{-A(e)d/2} \zeta_{L(t)}(B_t^e))_{e \in \mathbb{T}}] \\
 (4.40) \quad & \implies \mathcal{L}^\rho [(Z_{(\alpha-A(e))c_\beta}^e)_{e \in \mathbb{T}}] \quad (\text{as } t \rightarrow \infty) \\
 & = \mathcal{L}^{\rho\alpha^{-1/\beta}} [(\alpha^{1/\beta} Z_{(1-A(e)/\alpha)c_\beta}^e)_{e \in \mathbb{T}}].
 \end{aligned}$$

In the third line we have used the scaling property of $(1 + \beta)$ branching “diffusion.”

Let $\mathbb{T}_j = \{(j, l_2, \dots, l_n) \in \mathbb{T}, n \in \mathbb{N}\}$, $j = 1, \dots, J$, be the partition of \mathbb{T} into subtrees \mathbb{T}_j according to the offspring of the root ($\mathbb{T} = \{\emptyset\} \cup \mathbb{T}_1 \cup \dots \cup \mathbb{T}_J$). In order to prove (4.40) by the induction hypothesis it suffices to show that

$$\begin{aligned}
 (4.41) \quad & ((\log t)^{-1/\beta} t^{-A(e)d/2} \zeta_{L(t)}(B_t^e))_{e \in \mathbb{T}_j}, \quad j = 1, \dots, J, \\
 & \text{are asymptotically independent random variables.}
 \end{aligned}$$

For each $j = 1, \dots, J$, fix one $e_j \in \mathbb{T}_j$ and let $C_j = C_j(t) = x_{t^{e_j}} + [-R(t), R(t)]^d$ and $C_0 = \mathbb{R}^d \setminus (C_1 \cup \dots \cup C_J)$. Then for t large enough we have $C_i \cap C_j = \emptyset$ for $i \neq j$. Let

$$\Delta_j = \Delta_j(t) = \inf_{e \in \mathbb{T}_j} \text{dist}(B_t^e, \mathbb{R}^d \setminus C_j).$$

Since $A: \mathbb{T} \rightarrow I$ is strictly decreasing, we have $\Delta_j(t)/\sqrt{L(t)} \rightarrow \infty$ as $t \rightarrow \infty$.

Let $(\chi_s^j)_{s \geq 0}$, $j = 0, 1, \dots, J$, be independent SBM($d, 1 + 2/d, 1$) with initial states

$$\chi_0^j = \mathbb{1}_{C_j} \rho(\log t)^{1/\beta}\lambda, \quad j = 0, 1, \dots, J.$$

We can assume $\zeta_s = \chi_s^0 + \dots + \chi_s^J$. Now for $j = 1, \dots, J$ and $e \in \mathbb{T}_j$,

$$\begin{aligned}
 (4.42) \quad & \mathbb{E} \left[(\log t)^{-1/\beta} t^{-A(e)d/2} \sum_{\substack{i=0 \\ i \neq j}}^J \chi_{L(t)}^i(B_t^e) \right] \\
 & \leq \rho\lambda(B^e) t^{-A(e)d/2} \int_{\mathbb{R}^d \setminus C_j} dx \int_{B_t^e} dy p_{L(t)}(x, y) \\
 & \leq \rho\lambda(B^e) \exp(-\Delta_j^2/L(t)) \rightarrow 0 \quad \text{as } t \rightarrow \infty.
 \end{aligned}$$

Thus (4.41) holds and the proof is complete. \square

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