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WHITE NOISE INDEXED BY LOOPS

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Given a Riemannian manifold M and loop $\phi: S^1 \mapsto M$, we construct a Gaussian random process $S^1 \ni \theta \rightsquigarrow X_\theta \in T_{\phi(\theta)}M$, which is an analog of the Brownian motion process in the sense that the formal covariant derivative $\theta \rightsquigarrow \nabla_\theta X_\theta$ appears as a stationary process whose spectral measure is uniformly distributed over some discrete set. We show that X satisfies the two-point Markov property (reciprocal process) if the holonomy along the loop ϕ is nontrivial. The covariance function of X is calculated and the associated abstract Wiener space is described. We also characterize X as a solution of a special (nondiffusion type) stochastic differential equation. Somewhat surprisingly, the nature of X turns out to be very different if the holonomy along ϕ is the identity map $I: T_{\phi(0)}M \mapsto T_{\phi(0)}M$. In this case, we show that the usual periodic Ornstein–Uhlenbeck process, associated with a harmonic oscillator at nonzero temperature, may be viewed as a standard velocity process in which the driving Brownian motion is replaced by the process X.

1. Overview and motivation. The present paper grew out of the desire to construct a stochastic process ϕ_t, $t \in \mathbf{R}_+$, with values in

$$\mathbf{L}(M) := \left\{ \phi \in C([0, 2\pi[\to M) \,|\, \lim_{\theta \nearrow 2\pi} \phi(\theta) = \phi(0) \right\},$$

the space of loops over a Riemannian manifold M, which incorporates *as much symmetry as possible*. More specifically, we want the probability distributions in the cross-section of TM above ϕ of the formal differential $\theta \rightsquigarrow d\phi_t(\theta) \equiv d_t\phi_t(\theta)$ and its (formal) covariant derivative $\theta \rightsquigarrow \nabla_\theta\, d\phi_t(\theta)$ *both to be Gaussian and invariant under rotation along the loop*, with $\theta \rightsquigarrow \nabla_\theta\, d\phi_t(\theta)$ having *uniform spectral density* (white noise along the loop ϕ). If parallel translation along ϕ is possible, by choosing an orthonormal frame in $T_{\phi(0)}M$ and moving it parallel along ϕ in positive direction, $\theta \rightsquigarrow d\phi_t(\theta)$ could be treated as a *Gaussian process* in $\mathbf{R}^d$ indexed by $\theta \in \mathbf{R}$, which is *stationary and pseudoperiodic*, in that its value at $\theta + 2\pi$ is the one at θ twisted by the holonomy along ϕ. *In terms of Dirichlet forms*, processes on $\mathbf{L}(M)$ were constructed in [1] by considering probability distributions in the cross-section of TM above $\phi \in \mathbf{L}(M)$ derived from the abstract Wiener space construction associated with the Hilbert space H of absolutely continuous mappings

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$S^1 \ni \theta \rightsquigarrow h(\theta) \in T_{\phi(\theta)} M$ endowed with the scalar product

$$(h|k) := \frac{1}{2\pi}\int_0^{2\pi} \langle h(\theta), k(\theta)\rangle \, d\theta + \frac{1}{2\pi}\int_0^{2\pi} \langle \nabla_\theta h(\theta), \nabla_\theta k(\theta)\rangle \, d\theta, \; h, k \in H,$$

where $\langle \cdot, \cdot \rangle :=$ the Riemannian metric on M. It is easy to check that although this construction yields a rotation-invariant process the spectrum of its (formal) covariant derivative is not flat. In fact, this is precisely the periodic Ornstein–Uhlenbeck process described in Section 4 with $m = 1$.

Alternatively, one may incorporate symmetry in the process $\theta \rightsquigarrow d\phi_t(\theta)$ by requiring it to be a *reciprocal process*, that is, to share the two-point Markov property, in addition to being rotation-along-the-loop invariant (obviously, the standard Markov property does not make sense for processes indexed by the circle). This approach was taken in the recent work [3], which extends some of the results of [2] on periodic Osterwalder–Schrader positive processes. Unfortunately, if the holonomy along the loop ϕ is trivial, the requirement that $S^1 \ni \theta \rightsquigarrow X_\theta$ be two-point Markov contradicts the requirement that $S^1 \ni \theta \rightsquigarrow \nabla_\theta X_\theta$ has a flat spectrum. On the other hand, in the case of nontrivial holonomy, the flatness of the spectrum of $S^1 \ni \theta \rightsquigarrow \nabla_\theta X_\theta$ entails that $S^1 \ni \theta \rightsquigarrow X_\theta$ is two-point Markov. These statements, which we establish in Section 3, have two interesting consequences. First, while one would expect "white noise" to be a much more restrictive property than "Markov," this is not always the case; in fact, somewhat surprisingly, it fails to be the case in the simplest possible scenario: loops with trivial holonomy. The second observation is that replacing the index space $\mathbf{R}_+$ by some loop $\phi \in \mathbf{L}(M)$ makes the Markovian nature of the white noise process more intriguing. So, we treat the case Holonomy$(\phi) =$ Id separately from the case Holonomy$(\phi) \neq$ Id and in each case give a complete description of the covariance nature and of the Markovian nature (or the lack thereof) [see (4.4) and (4.10)] of the white noise process in TM which is above the loop ϕ. The velocity process associated with the circular white noise from (4.10) is proved to be nothing but the usual periodic Ornstein–Uhlenbeck process—see Section 4. General description in terms of the covariance function of the class of all real two-point Markov processes indexed by the circle is given in [4]. It should be noted that, in the present context, it is essential to work with complex-valued processes and that the respective theory is very different from its real-valued counterpart if the holonomy is nontrivial.

2. Objectives and notation. Throughout, M will denote a fixed Riemannian manifold of dimension $d \geq 2$ endowed with the Riemannian connection and ϕ: $S^1 \mapsto M$ will be some fixed loop on M. Here, S^1 will be identified with the interval $[0, 2\pi[$ in the obvious way, which, in turn, endows ϕ with positive and negative direction. We suppose that parallel translation along ϕ is possible; this is always the case if ϕ is smooth or is a sample path of a semimartingale. Next, let $V(\theta)$, $\theta \in S^1$, $1 \leq d' := \dim V(\theta) \leq d$, be a continuous distribution of tangent spaces along ϕ, which is invariant under parallel translation. That is, $\forall\, \theta \in S^1$ $V(\theta)$ is a vector subspace of $T_{\phi(\theta)}M$ and as such is obtained by parallel translation of $V(0)$ along ϕ in the positive direction from $\phi(0)$ to $\phi(\theta)$. The continuity assumption implies that the parallel translation of $V(0)$ along the entire loop results in a rotation, possibly the trivial one, inside $V(0)$. This rotation splits $V(0)$ into orthogonal subspaces of dimension less than or equal to 2, every one of which is invariant under its action. Thus $\theta \rightsquigarrow V(\theta)$ is just an orthogonal sum of continuous translation invariant distributions of dimension less than or equal to 2.

Our goal is to study continuous random processes X_θ, $\theta \in S^1$, which take values in TM above ϕ and which are supported by the distribution V, in that $\forall\ \theta \in S^1$ X_θ is randomly distributed in the vector space $V(\theta) \subseteq T_{\phi(\theta)}M$. Because of the remark just made, the interesting case is $d' \le 2$. If $d' = 2$, after choosing some orthonormal frame φ_0 in $V(0)$, X_θ, $\theta \in S^1$, could be treated as a complex-valued process $x_\theta \in \mathbf{C}$, $\theta \in [0, 2\pi[$, chosen so that $\Re\, x_\theta$ and $\Im\, x_\theta$ are nothing but the coördinates of X_θ in the frame $\varphi_\theta :=$ the translate of φ_0 along ϕ in positive direction from $\phi(0)$ to $\phi(\theta)$. Furthermore, x_θ could be extended to a continuous process $x_\theta \in \mathbf{C}$, $\theta \in \mathbf{R}$, with the property $x(\theta + 2\pi) = e^{i\alpha} x_\theta$ $\forall\ \theta \in \mathbf{R}$, where $\alpha \in [0, 2\pi[$ describes the rotation of φ_0 inside $V(0)$ caused by parallel translation in positive direction along the entire loop ϕ. It is to be noted that since the rotation group $O(2)$ is commutative, in general, α will depend on the loop ϕ and on the tangent subspace $V(0)$, but not on the frame φ_0. Of course, the process x depends on φ_0 and saying that its distribution is independent of φ_0 is equivalent to the claim that the distribution of X is invariant under the action of $O(2)$. The case $d' = 1$ is easy to reduce to the one just described by taking either $\alpha = \pi$ in case the parallel translation along ϕ reverses the direction in $V(0)$, or $\alpha = 0$ if it preserves it, after which X_θ, $\theta \in \mathbf{R}$, could be identified with the real part of x_θ, $\theta \in \mathbf{R}$.

From now on, our only concern will be the case where $d' = 2$ and where X_θ, $\theta \in S^1$, is continuous, stationary, symmetric and Gaussian in the sense that x_θ, $\theta \in \mathbf{R}$, is a L^2-continuous, stationary (wide sense), complex Gaussian process with $\mathbf{E}\{x_\theta\} = \mathbf{E}\{x_\theta x_{\theta'}\} = 0$, $\forall\ \theta, \theta' \in \mathbf{R}$. Clearly, if $\dim V > 2$, a stationary symmetric Gaussian process supported by V is simply the sum of (appropriately normalized) such processes supported by the orthogonal distributions (of dimension less than or equal to 2); V is being split into by the holonomy along ϕ. So, we will be working only with the case

$$x_\theta = \sum_{k \in \mathbf{Z}} \exp\left(i\left(\frac{\alpha}{2\pi} + k\right)\theta\right)\xi_k, \tag{2.1}$$

where ξ_k, $k \in \mathbf{Z}$, are independent complex Gaussian r.v.'s with

$$\sum_{k \in \mathbf{Z}} \mathbf{E}\left\{\xi_k \overline{\xi_k}\right\} < \infty \quad \text{and} \quad \mathbf{E}\{\xi_k\} = \mathbf{E}\left\{\xi_k^2\right\} = 0 \qquad \forall\ k \in \mathbf{Z}.$$

If, in addition,

$$\sum_{k \in \mathbf{Z}} \left| \frac{\alpha}{2\pi} + k \right|^2 \mathbf{E}\{\xi_k \overline{\xi_k}\} < \infty,$$

then the L^2-derivative of $\theta \rightsquigarrow x_\theta$ (note that this is nothing but the covariant derivative of $\theta \rightsquigarrow X_\theta$) is well defined and given by

$$\sum_{k \in \mathbf{Z}} i\left(\frac{\alpha}{2\pi} + k\right)\exp\left(i\left(\frac{\alpha}{2\pi} + k\right)\theta\right)\xi_k. \tag{2.2}$$

Thus, the spectrum of $\theta \rightsquigarrow x_\theta$ could only be inside the set $\alpha/(2\pi) + k$, $k \in \mathbf{Z}$. Consequently, if ξ_k, $k \in \mathbf{Z}$, are chosen so that

$$\mathbf{E}\{\xi_k \overline{\xi_k}\} = \text{constant} \times \left| \frac{\alpha}{2\pi} + k \right|^{-2}, \qquad k \in \mathbf{Z},\ k \neq -\frac{\alpha}{2\pi}, \tag{2.3}$$

then $\theta \rightsquigarrow \nabla_\theta X_\theta$ ($\nabla :=$ covariant derivative along ϕ) will have the meaning of *white noise process*, for in that case the spectral measure of $\theta \rightsquigarrow x_\theta$ will be uniformly distributed on the biggest possible set the spectrum could live on. Of course, in this later case the expression in (2.2) cannot give meaning to $\theta \rightsquigarrow \nabla_\theta X_\theta$, or, equivalently, to $\theta \rightsquigarrow x(\theta)$.

To summarize, we will only consider the case where $d' = \dim(V) = 2$ and where x_θ, $\theta \in \mathbf{R}$, is given by (2.1), assuming that (2.3) holds with constant $= 1$.

The covariance of x_θ, $\theta \in \mathbf{R}$, is then given by

$$R(\theta' - \theta) = \exp\left(i\frac{\alpha}{2\pi}(\theta' - \theta)\right)\sum_{k \in \mathbf{Z}} \frac{\exp(ik(\theta' - \theta))}{|\alpha/2\pi + k|^2},$$

where $\alpha \in [0, 2\pi[$ is uniquely determined by the loop ϕ and the distribution $\theta \rightsquigarrow V(\theta)$; if $V(0)$ is identified with $\mathbf{C}$ via some orthonormal frame, then after parallel translation along the entire loop ϕ in positive direction any $z \in V(0) \equiv \mathbf{C}$ arrives at $e^{-i\alpha}z \in V(0) \equiv \mathbf{C}$. Our main objective is to find a more tractable expression for $R(\cdot)$ and derive the Markovian nature of the process X, or, equivalently, of x.

3. The Hilbert space of X. The case $\alpha = 0$ differs in an essential way from $\alpha \neq 0$ and requires different treatment: if $\alpha \neq 0$, the path-space of x contains no constants and if $\alpha = 0$, it does. First we study the simpler case $\alpha \neq 0$.

DEFINITION 3.1. A mapping $\mathbf{R} \ni \theta \rightsquigarrow z(\theta) \in \mathbf{C}$ with the property $z(\theta + 2\pi) = e^{i\alpha}z(\theta)\ \forall\ \theta \in \mathbf{R}$ will be called α-periodic. The space of all absolutely continuous, that is, continuous and a.e. differentiable, α-periodic complex-valued functions on $\mathbf{R}$ will be denoted by H_α and will be endowed with the usual scalar product

$$\langle h | k \rangle := \frac{1}{2\pi}\int_{[0, 2\pi[} h'(\theta)\overline{k'(\theta)}\, d\theta, \qquad h, k \in H_\alpha.$$

In fact, H_α consists of absolutely continuous mappings from S^1 into V as explained in the following remark.

REMARK 3.2. Let $\varphi_0 \equiv (e_0^R, e_0^I)$, be some orthonormal frame in $V(0)$ and let $\varphi_\theta \equiv (e_\theta^R, e_\theta^I) :=$ the parallel transport of φ_0 along ϕ from $T_{\phi(0)}M$ to $T_{\phi(\theta)}M$. Given $z \in \mathbf{C}$, set $\varphi_\theta[z] := (\Re z)e_\theta^R + (\Im z)e_\theta^I \in V(\theta)$ and notice that if $\mathbf{R} \ni \theta \rightsquigarrow z(\theta) \in \mathbf{C}$ is α-periodic and continuous then

$$[0, 2\pi[\ \ni \theta \rightsquigarrow (\varphi z)(\theta) := \varphi_\theta[z(\theta)] \in V(\theta)$$

is continuous too. Thus, the correspondence $H_\alpha \ni \mathrm{z}(\varphi \mathrm{z}) \in \mathrm{V}$ allows identifying H_α with the space of all continuous vector fields above ϕ which are supported by V and admit covariant derivatives at almost every point on the loop ϕ. This association will be often blurred in the sequel, and, with a slight abuse of terminology and notation, continuous complex α-periodic functions on $\mathbf{R}$ will be treated also as functions from S^1 into TM supported by the distribution $\theta \rightsquigarrow \mathrm{V}(\theta)$. The space of all such functions will be denoted by $C_\alpha(S^1 \to V)$ and we will no longer distinguish between the $\mathbf{C}$-valued process x and the process $X = \varphi x \in V$.

For $t \in [0, 2\pi[$ let $T_t\colon H_\alpha \mapsto H_\alpha$ be the usual shift operator $T_t h(\theta) := h(\theta - t)$, $\theta \in \mathbf{R}$, and given $h \in H_\alpha$, consider the vector field

$$\rho_t(\varphi h)(\theta) := \varphi_\theta[T_t h(\theta)] \in V(\theta), \qquad \theta \in [0, 2\pi[.$$

It is easy to see that $\rho_t(\varphi h)(\cdot)$ is obtained by rotating (by way of parallel transport) the vector field $(\varphi h)(\cdot)$ along ϕ in positive direction; for example $\rho_t(\varphi h)(t)$ is obtained by parallel translation of $(\varphi h)(0)$ from $T_{\phi(0)}M$ to $T_{\phi(t)}M$.

Next, given $t \in [0, 2\pi[$, let

$$\Lambda_t(\theta) := \left[e^{i\alpha} 1_{[0,t[}(\theta) + 1_{[t,2\pi[}(\theta)\right] \frac{t - \theta}{1 - e^{i\alpha}} - \frac{2\pi e^{i\alpha}}{(1 - e^{i\alpha})^2}, \qquad \theta \in [0, 2\pi[$$

[we assume that for $t = 0$ $[0, t[= \varnothing$ and $1_{[0,t[}(\theta) \equiv 0]$, extend Λ_t to an α-periodic complex function on $\mathbf{R}$ and notice that, so defined, Λ_t is in fact an element of H_α and that $\Lambda_t = T_t \Lambda_0$ $\forall$ $t \in [0, 2\pi[$; equivalently, $\varphi\Lambda_t = \rho_t(\varphi\Lambda_0)$, for any choice of frame φ in $V(0) \subset T_{\phi(0)}M$. More importantly, notice that

$$[0, 2\pi[\setminus \{t\} \ni \theta \rightsquigarrow -\Lambda'_t(\theta) = \frac{e^{i\alpha}}{1 - e^{i\alpha}} 1_{[0,t[}(\theta) + \frac{1}{1 - e^{i\alpha}} 1_{[t,2\pi[}(\theta) \in \mathbf{C}$$

is just the *circular Heaviside function* with jump at $t \in [0, 2\pi[$, in that $-\Lambda'_t(\theta)$ is constant on $S^1 \setminus t$ and

$$\lim_{\theta \searrow t} -\Lambda'_t(\theta) + \lim_{\theta \nearrow t} \Lambda'_t(\theta) = 1;$$

consequently, $\Lambda_t(\cdot)$ is an integral kernel for the evaluation map $H_\alpha \ni h \rightsquigarrow h(t) \in \mathbf{C}$ in the sense of the following proposition.

PROPOSITION 3.3. *For any choice of $h \in H_\alpha$ and $t \in [0, 2\pi[$, one has*

$$\langle h | 2\pi\Lambda_t \rangle \equiv \int_{[0,2\pi[} h'(\theta)\overline{\Lambda'_t(\theta)}\, d\theta = h(t) \in \mathbf{C}.$$

Next, notice that

$$\lambda_k(\theta) := \frac{\exp(i(\alpha/2\pi + k)\theta)}{i(\alpha/2\pi + k)}, \qquad k \in \mathbf{Z},$$

form an orthonormal basis in H_α, consisting, in fact, of eigenfunctions of the Laplacian $\Delta = d^2/d\theta^2$ and that, by choosing ξ_k, $k \in \mathbf{Z}$, as in (2.3) with constant $= 1$, the process x_t, $t \in [0, 2\pi[$, given by (2.1) could be written as

$$X_t = \sum_{k \in \mathbf{Z}} \frac{\exp(i(\alpha/2\pi + k)t)}{i(\alpha/2\pi + k)} \gamma_k \equiv \sum_{k \in \mathbf{Z}} \langle \lambda_k | 2\pi\Lambda_t \rangle \gamma_k, \tag{3.1}$$

where γ_k, $k \in \mathbf{Z}$, are i.i.d. complex Gaussian r.v.'s with $\mathbf{E}\{\gamma_k\} = \mathbf{E}\{\gamma_k^2\} = 0$ and $\mathbf{E}\{\gamma_k \overline{\gamma_k}\} = 1$. Plainly, if $C_\alpha(S^1 \to V)$ is being treated as a probability space equipped with the Borel σ-field and the Gaussian probability law $\wp$ derived from the canonical inclusion $H_\alpha \hookrightarrow C_\alpha(S^1 \to V)$ and the associated abstract Wiener space construction, then the process X_θ, $\theta \in \mathbf{R}$, and the complex-valued random process

$$\mathbf{R} \ni \theta \rightsquigarrow 2\pi z(\theta) \in \mathbf{C}, \qquad z \in C_\alpha(S^1 \to V),$$

are indistinguishable in the sense that their respective finite-dimensional distributions are identical. In particular, this shows that

$$\begin{aligned} R(t) &:= \frac{1}{2\pi}\mathbf{E}\{X_0 \overline{X_t}\} = \langle \Lambda_0 | 2\pi\Lambda_t \rangle = \Lambda_0(t) \\ &\equiv \frac{t}{e^{i\alpha} - 1} + \frac{\pi}{1 - \cos\alpha}, \qquad t \in [0, 2\pi[. \end{aligned} \tag{3.2}$$

Now we develop the counterpart of the above for the case $\alpha = 0$. In view of Definition 3.1, $H := H_{\alpha=0}$ is simply the space of absolutely continuous and periodic (with period 2π) complex functions on $\mathbf{R}$; that is, the space of absolutely continuous functions on the circle. $\langle \cdot | \cdot \rangle^{1/2}$ is no longer a norm and we turn H into a Hilbert space by decomposing it into the sum

$$H = \mathbf{C} + K, \qquad K := \left\{ h \in H \mid \int_{[0,2\pi[} h = 0 \right\},$$

and by setting

$$(c_1 + h_1 | c_2 + h_2) := c_1\overline{c_2} + \langle h_1 | h_2 \rangle \equiv c_1\overline{c_2} + \frac{1}{2\pi}\int_{[0,2\pi[} h_1'(\theta)\overline{h_2'(\theta)}\, d\theta,$$

$c_1, c_2 \in \mathbf{C}$, $h_1, h_2 \in K$. Note that $\langle \cdot | \cdot \rangle^{1/2}$ is a norm on K and therefore H turns into a Hilbert space with norm $(\cdot | \cdot)^{1/2}$. Similarly, by splitting each $z \in C_{\alpha=0}(S^1 \to V)$ into the sum

$$z(\theta) = \frac{1}{2\pi}\int_{[0,2\pi[} z(\theta)\, d\theta + \left(z(\theta) - \frac{1}{2\pi}\int_{[0,2\pi[} z(\theta)\, d\theta \right), \qquad \theta \in [0, 2\pi[,$$

$C_0(S^1 \to V)$ could be written as $\mathbf{C} + \tilde{C}_0(S^1 \to V)$, the second component of which is endowed with the standard sup-norm. We suppose that $C_0(S^1 \to V)$ has the topology of the product $\mathbf{C} \otimes \tilde{C}_0(S^1 \to V)$ and endow $C_0(S^1 \to V)$ with probability law $\wp$ which is the product of two independent probability laws Γ and $\tilde{\wp}$, respectively, on $\mathbf{C}$ and on $\tilde{C}_0(S^1 \to V)$, $\Gamma :=$ the law of a complex Gaussian r.v. ζ with $\mathbf{E}\{\zeta\} = \mathbf{E}\{\zeta^2\} = 0$ and $\tilde{\wp} :=$ the Gaussian measure on $\tilde{C}_0(S^1 \to V)$ derived from the abstract Wiener space associated with the inclusion $K \hookrightarrow \tilde{C}_0(S^1 \to V)$. To this end, we consider the space K as a Hilbert space with scalar product $\langle \cdot | \cdot \rangle$ and choose an orthonormal basis in K formed by the eigenfunctions of the Laplacian (Laplacian on K, that is),

$$\lambda_k(\theta) := \frac{1}{ik} e^{ik\theta}, \qquad k \in \mathbf{Z} \setminus \{0\}.$$

In the present setting the *circular Heaviside function* with jump at $t \in [0, 2\pi[$ is (-1) times the derivative of the following quadratic (rather than linear, as was the case where $\alpha \neq 0$) function:

$$\Lambda_t(\theta) := \frac{1}{4\pi}(\theta - t)^2 - \frac{1}{2}(\theta - t) + (\theta - t) 1_{[0, t[}(\theta) + \frac{\pi}{6}, \qquad \theta \in [0, 2\pi[,$$

which, just as before, we extend to a periodic function on $\mathbf{R}$ and remark that so defined it is an element of K and that $\Lambda_t(\theta) = T_t \Lambda_0(\theta) = \Lambda_0(\theta - t)$. Obviously, Λ'_t is no longer a constant on $[0, 2\pi[\setminus \{t\}$ and the term "Heaviside function" is justified by

PROPOSITION 3.4. *For any choice of $h \in K$ and $t \in [0, 2\pi[$ one has*

$$\langle h | 2\pi \Lambda_t \rangle \equiv \int_{[0, 2\pi[} h'(\theta) \overline{\Lambda'_t(\theta)}\, d\theta = h(t) \in \mathbf{C},$$

where

$$\Lambda'_t(\theta) = \frac{1}{2\pi}(\theta - t) - \frac{1}{2} + 1_{[0, t[}(\theta), \qquad \theta \in [0, 2\pi[\setminus \{t\}.$$

PROOF. It is enough to notice that

$$\Lambda''_t(\theta) = -\delta_t(\theta) + (1/2\pi) 1_{[0, 2\pi[\setminus \{t\}}(\theta), \qquad \theta \in [0, 2\pi[. \qquad \square$$

Consequently, (3.1) now becomes

$$X_t = \xi_0 + \sum_{k \in \mathbf{Z} \setminus \{0\}} \frac{1}{ik} e^{ikt} \gamma_k \equiv \xi_0 + \sum_{k \in \mathbf{Z} \setminus \{0\}} \langle \lambda_k | 2\pi \Lambda_t \rangle \lambda_k =: \xi_0 + Y_t \tag{3.3}$$

and (3.2) turns into

$$R(t) := \frac{1}{2\pi} \mathbf{E}\{Y_0 \overline{Y}_t\} = \langle \Lambda_0 | 2\pi \Lambda_t \rangle = \Lambda_0(t) \equiv \frac{t^2}{4\pi} - \frac{t}{2} + \frac{\pi}{6}, \qquad t \in [0, 2\pi[. \tag{3.4}$$

3.5 REMARK. 1. The Hilbert space associated with the process $Y_t := X_t - \xi_0$, $t \in S^1$, is K, not H.

2. Since the covariance function in (3.2) is real for $\alpha = \pi$, and so is the one in (3.4), it is easy to conclude from the above that if X'_t, $t \in S^1$, is a real-valued (Gaussian) white noise process then, up to a factor, the covariance of X_t, $t \in S^1$, is either

$$R(t) = \mathbf{E}\{X_0 X_t\} = -\frac{t}{2} + \frac{\pi}{2}, \qquad t \in [0, 2\pi[,$$

[this is (3.2) with $\alpha = \pi$] if X is antiperiodic ($X_{t+2\pi} = -X_t$), or is given by

$$\frac{t^2}{4\pi} - \frac{t}{2} + \frac{\pi}{6} + \text{constant}, \qquad t \in [0, 2\pi[,$$

with some arbitrary real positive constant, if X is periodic ($X_{t+2\pi} = X_t$). Note that in this later case the covariance is not unique (unique up to a factor, that is).

4. The Markovian nature of X. Here again we treat the case $\alpha \neq 0$ and $\alpha = 0$ separately and consider the case $\alpha \neq 0$ first.

As scaling makes no difference in our study, we assume that X_t, $t \in [0, 2\pi[$, has covariance $R(t)$, $t \in [0, 2\pi[$, given by (3.2). For $t < 0$ we set $R(t) := \overline{R(-t)}$ and notice that

$$\frac{\partial^2}{\partial t\, \partial s} R(t - s) = \delta_0(t - s), \qquad s, t \in [0, 2\pi[, \tag{4.1}$$

and therefore that stochastic integrals of the form

$$\int_{[0, 2\pi[} f(t)\, dX_t, \qquad f \in L^2(S^1 \to \mathbf{C}),$$

are well defined as is the quadratic variation $dX_t \overline{dX_t} = dt$. Given $0 \leq s < t < 2\pi$, we have

$$\mathbf{E}\{X_0 \mid X_s\} = \frac{R(s)}{R(0)} X_s \quad \text{and} \quad \mathbf{E}\{X_t \mid X_s\} = \frac{\overline{R(t-s)}}{R(0)} X_s$$

and so

$$\mathbf{E}\{X_t \mid X_0, X_s\} = \varkappa(s, t)\left(X_0 - \frac{R(s)}{R(0)} X_s\right) + \frac{\overline{R(t-s)}}{R(0)} X_s,$$

where $\varkappa(s, t)$ is to be determined by the condition

$$\mathbf{E}\{(X_t - \mathbf{E}\{X_t | X_0, X_s\}) \overline{X}_0\} = 0.$$

Somewhat tedious but otherwise trivial calculation yields

$$\varkappa(s, t) = \frac{\overline{R(t)} R(0) - \overline{R(t-s) R(s)}}{R(0)^2 - |R(s)|^2} = e^{i\alpha} \frac{t - s}{2\pi - s},$$

from which one finds that

$$\mathbf{E}\{X_t - X_s \mid X_0, X_s\} = \frac{e^{i\alpha}X_0 - X_s}{2\pi - s}(t-s). \tag{4.2}$$

Now it is easy to see that X is a reciprocal process in the sense of the following proposition.

PROPOSITION 4.1. *Given* $0 \le s < t < 2\pi$, *one has*

$$\mathbf{E}\{X_t \mid X_u, u \in [0,s]\} = \mathbf{E}\{X_t \mid X_0, X_s\}.$$

PROOF. The claim is that

$$\mathbf{E}\{(X_t - \mathbf{E}\{X_t | X_0, X_s\})\overline{X_u}\} = 0 \qquad \forall\, u \in [0,s],$$

or, what amounts to the same thing, that

$$\begin{aligned} &\overline{R(t-u)}R(0) - \varkappa(s,t)\big[R(u)R(0) - R(s)\overline{R(s-u)}\big] \\ &\quad -\overline{R(t-s)R(s-u)} = 0 \qquad \forall\, u \in [0,s], \end{aligned} \tag{4.3}$$

which is certainly true, because the left-hand side is linear in u and $\varkappa(s,t)$ was chosen so that the identity holds for $u = 0$ and $u = s$. □

REMARK 4.2. For a wide sense-stationary, L^2-continuous, periodic Gaussian process with period 2π and covariance R, s.t., $R(t) \ne R(0)$, $\forall\, t \in [0, 2\pi[$, (4.3) is just a rephrasing of the two-point Markov property. It is obvious, then, that *the two-point Markov property always holds if R is linear*. It is easy to see that $R''(t) = \text{constant} \times R(t)$, $t \ne 0$, *implies two-point Markov* too, for, in that case, as a function of $u \in [0,s]$, the left side of (4.3) could only be of the form $Ae^{-Cu} + Be^{+Cu}$ and could vanish for $u = 0$ and for $u = s \ne 0$ only if $A = B = 0$.

In fact, Proposition 4.1, combined with (4.2), comes down to the following statement.

PROPOSITION 4.3. *A continuous, stationary, complex-Gaussian stochastic process X_t, $t \in [0, 2\pi[$, with $\mathbf{E}\{X_t\} = \mathbf{E}\{X_t X_{t'}\} = 0$ and with covariance function $R(t) := \mathbf{E}\{X_0 \overline{X_t}\}$ given by the right side of (2.2) solves the equation*

$$\begin{cases} dX_t = \dfrac{e^{i\alpha}X_0 - X_t}{2\pi - t}\, dt + dW_t, \qquad 0 \le t < 2\pi, \\ X_0 = \textit{complex Gaussian r.v. with } \mathbf{E}\{X_0\} = \mathbf{E}\{X_0^2\} = 0 \\ \qquad \textit{and } \mathbf{E}\{X_0 \overline{X_0}\} = \dfrac{\pi}{1 - \cos\alpha}, \end{cases} \tag{4.4}$$

driven by complex Brownian motion W of intensity $dW_t\, \overline{dW_t} = dt$, which is independent of the initial value X_0.

This, of course, is pinned Brownian motion with appropriately randomized initial value. This leads to the following conclusion.

COROLLARY 4.4. *The solution of* (4.4) *could be written in the* (*W-nonanticipative*) *form*

$$X_t \equiv (2\pi - t)\int_0^t \frac{dW_s}{2\pi - s} + \frac{2\pi + (e^{i\alpha} - 1)t}{2\pi} X_0, \qquad t \in [0, 2\pi[, \tag{4.5a}$$

or, equivalently, in the (*W-anticipative*) *form*

$$X_t = \frac{1}{e^{i\alpha} - 1} W_{2\pi} + W_t, \qquad t \in [0, 2\pi[. \tag{4.5b}$$

It is not hard to compute $R(t) := \mathbf{E}\{X_0 \overline{X_t}\}$ directly from this last representation and, of course, arrive at the expression in (3.2).

Now we work out the case $\alpha = 0$. Instead of studying the process X_t, $t \in \mathbf{R}$, we study $Y_t := X_t - \xi_0$ and assume that its covariance is given by the right side of (3.4). The above calculation with $R(t) = t^2/4\pi - t/2 + \pi/6$, $t \in [0, 2\pi[$, implies that (4.3) holds with

$$\varkappa(s, t) = \frac{(\pi/6)R(t) - R(t - s)R(s)}{\pi^2/36 - R^2(s)}, \qquad 0 \le s < t < 2\pi. \tag{4.6}$$

The left side of (4.3) is a quadratic function of u which, due to the choice of $\varkappa$, vanishes for $u = 0$ and $u = s$ and therefore could be $\equiv 0 \; \forall \; u \in [0, s]$, only if its second derivative w.r.t. u,

$$\frac{1}{12} - \varkappa(s, t)\left[\frac{1}{12} - R(s)\frac{1}{2\pi}\right] - R(t - s)\frac{1}{2\pi},$$

vanishes in $[0, s]$. This however is easily seen to fail *with the implication that* Y_t, $t \in S^1$, *is not a reciprocal process* and that $\mathbf{E}\{Y_t \mid Y_u,\, u \in [0, s]\}$ could be found only in the form

$$aY_0 + bY_s + \int_{[0,s]} f(u) Y_u \, du,$$

with some $a \equiv a(s, t)$, $b \equiv b(s, t) \in \mathbf{C}$ and $f \equiv f_{s,t} \in C([0, 2\pi[\to \mathbf{C})$. Since

$$\mathbf{E}\left\{\int_{[0,s]} f(u) Y_u \, du \;\Big|\; Y_s\right\} = \left(\frac{6}{\pi}\int_{[0,s]} f(u) R(s - u)\, du\right) Y_s$$

we have

$$\begin{aligned}\mathbf{E}\left\{\int_{[0,s]} f(y) Y_u \, du \;\Big|\; Y_0, Y_s\right\} &= \kappa\left(Y_0 - \frac{6}{\pi} R(s) Y_s\right) \\ &\quad + \left(\frac{6}{\pi}\int_{[0,s]} f(u) R(s - u)\, du\right) Y_s,\end{aligned}$$

where

$$\kappa \equiv \kappa(s, t) := \frac{(\pi/6)\int_0^s f(u) R(u)\, du - R(s)\int_0^s f(u) R(s - u)\, du}{(\pi^2/36) - R^2(s)}$$

is found from the condition

$$\mathbf{E}\left\{\left[\int_{[0,s]} f(u) Y_u \, du - \mathbf{E}\left\{\int_{[0,s]} f(u) Y_u \, du \middle| Y_0, Y_s\right\}\right] Y_0\right\} = 0.$$

Consequently,

$$\begin{aligned} Y_t - \mathbf{E}\{Y_t \mid Y_u, u \in [0,s]\} = {} & Y_t - \varkappa\left(Y_0 - \frac{6}{\pi} R(s) Y_s\right) - \frac{6}{\pi} R(t-s) Y_s \\ & - \int_{[0,s]} f(u) Y_u \, du + \kappa\left(Y_0 - \frac{6}{\pi} R(s) Y_s\right) \\ & + \left(\frac{6}{\pi} \int_{[0,s]} f(u) R(s-u) \, du\right) Y_s, \end{aligned}$$

where $\varkappa \equiv \varkappa(s,t)$ is given by (4.6), and this identity determines completely the function $[0,s] \ni u \rightsquigarrow f(u) \equiv f_{s,t}(u)$. Indeed,

$$\mathbf{E}\{[Y_t - \mathbf{E}\{Y_t \mid Y_u, u \in [0,s]\}] Y_\sigma\} = 0 \quad \forall \sigma \in [0,s]$$

yields

$$\begin{aligned} 0 = {} & \frac{\pi}{6} R(t-\sigma) - \varkappa\left(\frac{\pi}{6} R(\sigma) - R(s) R(s-\sigma)\right) \\ & - R(t-s) R(s-\sigma) \\ & - \frac{\pi}{6} \int_{[0,s]} f(u) R(\sigma-u) \, du + \kappa\left(\frac{\pi}{6} R(\sigma) - R(s) R(s-\sigma)\right) \\ & + \left(\int_{[0,s]} f(u) R(s-u) \, du\right) R(s-\sigma). \end{aligned} \tag{4.7}$$

Now, treated as functions of σ for fixed s and t, all terms in the above expression are quadratic with the exception of $\int_0^s f(u) R(\sigma - u) \, du$ and since

$$R''(u) = -\delta_0(u) + \frac{1}{2\pi} \tag{4.8}$$

it follows that

$$\frac{d^2}{d\sigma^2} \int_{[0,s]} f(u) R(\sigma - u) \, du = -f(\sigma) + \frac{1}{2\pi} \int_{[0,s]} f(u) \, du.$$

Thus, differentiating (4.7) w.r.t. σ twice, one sees that actually $[0,s] \ni u \rightsquigarrow f(u) \equiv f_{s,t}(u)$ is a constant. On the other hand, thanks to the choice of $\varkappa$ and κ, (4.7) is automatically satisfied for $\sigma = 0$ and $\sigma = s$, so that (4.7) holds for $\forall \sigma \in [0,s]$ if and only if the second derivative w.r.t. σ of the expression in the right side is identically null. This allows calculating the constant $f_{s,t}$:

$$f_{s,t} = \frac{(1/12) - (1/2\pi)\varkappa[\pi/6 - R(s)] - (1/2\pi) R(t-s)}{(\pi/6)(s/2\pi - 1) - \kappa'(1/2\pi)[\pi/6 - R(s)] - (1/2\pi)\int_{[0,s]} R(u) \, du},$$

where

$$\kappa' = \frac{\int_{[0,s]} R(u)\,du}{\pi/6 + R(s)}$$

and

$$\int_{[0,s]} R(u)\,du \equiv \int_{[0,s]} R(s-u)\,du = \frac{s(s-\pi)(s-2\pi)}{12\pi}.$$

Cumbersome as this last expression may appear, it simplifies to something surprisingly simple:

$$f_{s,t} = \frac{6(t-s)(t-2\pi)}{(2\pi-s)^3}$$

and one finds that

$$\begin{aligned}
&\mathbf{E}\{Y_t - Y_s | Y_u,\, u \in [0,s]\}\\
&\quad = (\varkappa - \kappa)\left(Y_0 - \frac{6}{\pi}R(s)Y_s\right) + \frac{6}{\pi}\left(R(t-s) - \frac{\pi}{6}\right)Y_s\\
&\qquad - \frac{6}{\pi}\,\frac{6(t-s)(t-2\pi)}{(2\pi-s)^3}\,\frac{s(s-\pi)(s-2\pi)}{12\pi}\,Y_s\\
&\qquad + \frac{6(t-s)(t-2\pi)}{(2\pi-s)^3}\int_0^s Y_u\,du.
\end{aligned} \tag{4.9}$$

It is not hard to check that

$$\varkappa(s,t) - \kappa(s,t) = -\frac{2(t-s)}{2\pi - s} + O\big((t-s)^2\big)$$

and, knowing that Y_t has quadratic variation $dY_t\,\overline{dY_t} = dt$ [see (4.8)], conclude from (4.9) that

PROPOSITION 4.5. *A continuous, stationary, complex-Gaussian stochastic process Y_t, $t \in [0, 2\pi[$, with $\mathbf{E}\{Y_t\} = \mathbf{E}\{Y_t Y_{t'}\} = 0$ and with covariance function $R(t) := \mathbf{E}\{Y_0 \overline{Y_t}\}$ given by the right side of* (3.4) *solves*

$$\begin{cases} dY_t = dW_t - \dfrac{2Y_0 + 4Y_t}{2\pi - t}\,dt - \dfrac{6\,dt}{(2\pi - t)^2}\displaystyle\int_0^t Y_\tau\,d\tau, \qquad 0 \le t < 2\pi,\\ Y_0 = \text{complex Gaussian r.v. with } \mathbf{E}\{Y_0\} = \mathbf{E}\{Y_0^2\} = 0\\ \qquad \text{and } \mathbf{E}\{Y_0\overline{Y_0}\} = \dfrac{\pi}{6}, \end{cases} \tag{4.10}$$

driven by complex Brownian motion W of intensity $dW_t\,\overline{dW_t} = dt$, which is independent of the initial value Y_0.

REMARK 4.6. It is possible to write Y as an *explicit* linear functional of W by observing that $[U_t, Y_t, V_t]^\top \equiv [Y_0, Y_t, \int_0^t Y_s\, ds]^\top$ is Markov and solves

$$d\begin{bmatrix} U_t \\ Y_t \\ V_t \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \dfrac{-2}{2\pi - t} & \dfrac{-4}{2\pi - t} & \dfrac{-6}{(2\pi - t)^2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} U_t \\ Y_t \\ V_t \end{bmatrix} dt + \begin{bmatrix} 0 \\ dW_t \\ 0 \end{bmatrix},$$

with initial data $[U_0, Y_0, V_0]^\top = [Y_0, Y_0, 0]^\top$. This leads to a somewhat cumbersome (W-nonanticipative) representation of Y. It is not hard to check that the following (W-anticipative) representation also holds

$$Y_t = W_t - \frac{t + \pi}{2\pi} W_{2\pi} + \frac{1}{2\pi}\int_0^{2\pi} \tau\, dW_\tau, \qquad t \in [0, 2\pi[.$$

5. Periodic Ornstein–Uhlenbeck processes viewed as velocity processes driven by circular white noise. By the *circular white-noise process* we simply mean the process Y_t', $t \in \mathbf{R}$, the formal derivative of the process Y constructed in the second part of Section 3 in the case $\alpha = 0$. Now consider the periodic velocity process Z_t, $t \in \mathbf{R}$, which solves

$$dZ_t + mZ_t\, dt = dY_t, \qquad t \in \mathbf{R},\ m > 0, \tag{5.1}$$

with boundary condition $Z_0 = Z_{2\pi}$. This means that

$$Z_t = e^{-mt}\left[\int_0^t e^{ms}\, dY_s + Z_0\right], \qquad t \in [0, 2\pi[,$$

with

$$Z_0 = \frac{\int_0^{2\pi} e^{(s-2\pi)m}\, dY_s}{1 - e^{-2\pi m}}.$$

Obviously, Z_t, $t \in \mathbf{R}$, is stationary with (discrete) spectral measure

$$\sigma(d\lambda) = \sum_{k \in \mathbf{Z}\setminus\{0\}} \frac{1}{2\pi}\frac{1}{\lambda^2 + m^2}\delta_k(\lambda)\, d\lambda, \qquad \lambda \in \mathbf{R},$$

obtained by multiplying

$$\sum_{k \in \mathbf{Z}\setminus\{0\}} \frac{1}{2\pi}\delta_k(\lambda)\, d\lambda \equiv \text{the spectral measure of } Y_t',$$

by the transfer function

$$\lambda \rightsquigarrow |H(\lambda)|^2 = H(\lambda)H(-\lambda) = \frac{1}{i\lambda + m}\frac{1}{-i\lambda + m} = \frac{1}{\lambda^2 + m^2},$$

corresponding to (5.1). We calculate the covariance $r(t) = \mathbf{E}\{Z_0\overline{Z_t}\}$, $t \in [0, 2\pi[$, next. On account of (4.8) we have

$$\mathbf{E}\left\{dY_s\,\overline{dY_t}\right\} = \delta_0(t - s)\, dt - \frac{1}{2\pi}ds\, dt.$$

Thus

$$
\begin{aligned}
r(0) &\equiv \mathbf{E}\left\{Z_0\overline{Z_0}\right\}\\
&= \frac{1}{(1-e^{-2\pi m})^2}\left[\int_0^{2\pi} e^{2(s-2\pi)m}\,ds - \frac{1}{2\pi}\left(\int_0^{2\pi} e^{(s-2\pi)m}\,ds\right)^2\right]\\
&= \frac{1+e^{-2\pi m}}{2m(1-e^{-2\pi m})} - \frac{1}{2\pi m^2}.
\end{aligned}
$$

On the other hand, (5.1) implies that

$$
r'(t) + mr(t) = \frac{e^{(t-2\pi m)}}{1-e^{-2\pi m}} - \frac{1}{2\pi m},
$$

from which we find that

$$
r(t) + \frac{1}{2\pi m^2} = \frac{e^{-tm} + e^{(t-2\pi)m}}{2m(1-e^{-2\pi m})}. \tag{5.2}
$$

Incidentally, the expression in the right side is precisely the covariance of a periodic Ornstein–Uhlenbeck process with period 2π (see [2]). This means that if γ_0 is some complex Gaussian r.v. with $\mathbf{E}\{\gamma_0\} = \mathbf{E}\{\gamma_0^2\} = 0$, $\mathbf{E}\{\gamma_0\overline{\gamma_0}\} = 1$, which is independent of Y_t, $t \in \mathbf{R}$, then the process

$$
\varpi_t := Z_t + \frac{1}{m\sqrt{2\pi}}\gamma_0, \qquad t \in \mathbf{R},
$$

is the one associated with a harmonic oscillator with frequency m at a nonzero temperature $T = k/2\pi$; k is the Boltzmann constant. This process obviously solves

$$
d\varpi_t + m\varpi_t\,dt = dY_t + \frac{\gamma_0}{\sqrt{2\pi}}\,dt, \qquad t \in \mathbf{R},
$$

with periodic condition $\varpi_t = \varpi_{t+2\pi}$.

REMARK 5.1. The comment in Remark 4.2 implies that ϖ_t, $t \in [0, 2\pi[$, having covariance given by (5.2), *does have the two-point Markov property* (which is well known). Notice also that

$$
\frac{1}{2\pi m^2} = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-tm} + e^{(t-2\pi)m}}{2m(1-e^{-2\pi m})}\,dt,
$$

which shows that $\int_0^{2\pi} r(t)\,dt = 0$. It is trivial to check that the covariance function of Y_t, $t \in [0, 2\pi[$ integrates to 0, too.

Finally, we consider (5.1) with dY_t replaced by dX_t, the circular white noise process with nontrivial holonomy described in the first part of Section 3—the process from (4.5). Now there is precisely one α-periodic solution (i.e., solution for which $Z_{2\pi} = e^{i\alpha}Z_0$):

$$
Z_t = e^{-mt}\left[\int_0^t e^{ms}\,dX_s + Z_0\right], \qquad t \in [0, 2\pi[,
$$

with

$$Z_0 = \frac{\int_0^{2\pi} e^{(s-2\pi)m}\, dX_s}{e^{i\alpha} - e^{-2\pi m}}.$$

On the other hand (4.1) implies that $\mathbf{E}\{dX_t\, \overline{dX_t}\} = dt$ and we find that

$$r(0) \equiv \mathbf{E}\{Z_0 \overline{Z_0}\} = \frac{\int_0^{2\pi} e^{2(s-2\pi)m}\, ds}{(e^{i\alpha} - e^{-2\pi m})(e^{-i\alpha} - e^{-2\pi m})}$$
$$= \frac{1}{2m} \frac{\sinh(2\pi m)}{\cosh(2\pi m) - \cos(\alpha)}.$$

At the same time,

$$\mathbf{E}\left\{Z_0\, \overline{dX_t}\right\} = \frac{e^{(t-2\pi)m}}{e^{i\alpha} - e^{-2\pi m}}\, dt,$$

which implies

$$r'(t) + mr(t) = \frac{e^{(t-2\pi)m}}{e^{i\alpha} - e^{-2\pi m}}$$

and we thus find that

$$r(t) = \frac{1}{2m} \frac{\sinh(mt)(e^{-i\alpha} - e^{-2\pi m}) + e^{-mt}\sinh(2\pi m)}{\cosh(2\pi m) - \cos(\alpha)}.$$

Just as before, since $r'' = \text{const} \times r$, we conclude that this covariance function corresponds to a reciprocal process—the α-periodic (i.e., with nontrivial holonomy) version of a harmonic oscillator at nonzero temperature.

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