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STRONG LAW OF LARGE NUMBERS FOR MULTILINEAR FORMS

By Anda Gadidov

Gannon University

Let $m \geq 2$ be a nonnegative integer and let $\{X^{(l)}, X_i^{(l)}\}_{i\in\mathbb{N}}$, $l = 1, \dots, m$, be m independent sequences of independent and identically distributed symmetric random variables. Define $S_n = \sum_{1\leq i_1,\dots,i_m \leq n} X_{i_1}^{(1)} \cdots X_{i_m}^{(m)}$, and let $\{\gamma_n\}_{n\in\mathbb{N}}$ be a nondecreasing sequence of positive numbers, tending to infinity and satisfying some regularity conditions. For $m = 2$ necessary and sufficient conditions are obtained for the strong law of large numbers $\gamma_n{}^{-1} S_n \to 0$ a.s. to hold, and for $m > 2$ the strong law of large numbers is obtained under a condition on the growth of the truncated variance of the $X^{(l)}$.

1. Introduction. Let $m \geq 2$ be a nonnegative integer and let $\{X^{(l)}, X_i^{(l)}\}_{i\in\mathbb{N}}$, $l = 1, \dots, m$, be m independent sequences of independent and identically distributed (i.i.d.) random variables. Let $\mathbf{\Phi}$ be a Borel measurable function of m variables. The classical U-statistic of order m with kernel $\mathbf{\Phi}$ is defined as

$$U(n) = \binom{n}{m}^{-1} \sum \mathbf{\Phi}\big(X_{j_1}^{(1)}, \dots, X_{j_m}^{(1)}\big),$$

where the sum extends over all $1 \leq j_1 < \cdots < j_m \leq n$. Relaxing the requirement that all arguments of $\mathbf{\Phi}$ be drawn from the same sequence of random variables leads to the notion of the *generalized U-statistic of order m*:

$$U(n) = \binom{n}{m}^{-1} \sum \mathbf{\Phi}\big(X_{j_1}^{(1)}, \dots, X_{j_m}^{(m)}\big).$$

Further generalizations combining both types are possible [see, e.g., Sen (1977).]

This paper is concerned with the study of the strong law of large numbers

$$\lim_{n\to\infty} \frac{1}{\gamma_n} \sum_{1\leq i_1,\dots,i_m\leq n} X_{i_1}^{(1)} \cdots X_{i_m}^{(m)} = 0 \quad \text{a.s.,} \tag{1.1}$$

where $\{X^{(l)}, X_i^{(l)}\}_{i\in\mathbb{N}}$ are i.i.d. symmetric and γ_n satisfies some regularity conditions.

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Sums as in (1.1) are a particular case of generalized U-statistics and their study is of interest in the attempt to tackle the more general problem. An instance in which a problem like (1.1) occurs in the context of U-statistics is the following: consider the kernel $h(x, y) = x^2y + xy^2$ and the U-statistic defined by it $U_n = \binom{n}{2}^{-1} \sum_{1\le i<j\le n} h(X_i, X_j)$. By means of decoupling tail probabilities as in de la Peña and Montgomery-Smith (1995), the study of convergence of U_n can be reduced to the one of

$$\binom{n}{2}^{-1} \sum_{1\le i<j\le n} h(X_i, Y_j),$$

where $\{Y_j\}$ is a copy of $\{X_j\}$, independent of $\{X_j\}$.

For $\{X_i\}$ i.i.d. and γ_n satisfying some regularity conditions, the necessary and sufficient conditions for $\lim_{n\to\infty} \gamma_n^{-1} \sum_{1\le i_1<\cdots<i_m\le n} X_{i_1}\cdots X_{i_m} = 0$ a.s. were obtained by Cuzick, Giné and Zinn (1995) for $m = 2$ and $\{X_i\}$ symmetric or having regular tails, and by Zhang (1996) for general m and X satisfying a condition on the growth of its truncated mean.

Since in both Cuzick, Giné and Zinn's paper and Zhang's paper the decoupled version of the strong law of large numbers was shown to be equivalent to the nondecoupled one, it is not surprising that the analytical necessary and sufficient conditions for (1.1) to hold are similar. Moreover, the diagonal terms in the decoupled version are irrelevant, while in the nondecoupled one they are not. Let, for example, $\gamma_n = n^{2/\alpha}$, $\alpha < 2$. The a.s. convergence of $n^{-2/\alpha}\sum_{1\le i<j\le n} X_iY_j$ is equivalent to the a.s. convergence of $n^{-2/\alpha}\sum_{1\le i,\, j\le n} X_iY_j$, while, as it was pointed out by Giné and Zinn (1992b), there exist $\{X, X_i\}$ i.i.d. symmetric with $E|X|^\alpha = \infty$ and such that $n^{-2/\alpha}\sum_{1\le i<j\le n} X_iX_j \to 0$ a.s. but $\limsup_{n\to\infty} n^{-2/\alpha}\sum_{1\le i,\, j\le n} X_iX_j = \limsup_{n\to\infty} (n^{1/\alpha}\sum_{i\le n} X_i)^2 = \infty$ by the Marcinkiewicz law of large numbers.

We need some definitions and notation in order to state the main results. For x_1, x_2, $x_1 \wedge x_2 = \min\{x_1, x_2\}$; for A, B nonnegative variable quantities, $A \lesssim B$ will mean that there exists a constant $c > 0$, independent of A and B, such that $A \le cB$, and $A \sim B$ will mean that $A \lesssim B$ and $B \lesssim A$. For a nonincreasing left-continuous function with right limits, $G(x)$, define $G^{-1}(x) = \sup\{y\colon G(y) \ge x\}$. Then, if $u = G^{-1}(v)$, we have $G(u+) \le v \le G(u)$. Also, for a nonnegative continuous function increasing to ∞, $b(t)$, $t \ge 0$, denote $b^{-1}(x) = \inf\{y\colon b(y) = x\}$. The following are obvious: $b(b^{-1}(x)) = x$ and $b^{-1}(b(t)) \le t$. For $J \subseteq \{1, \dots, m\}$ denote $J^c = \{1, \dots, m\} \setminus J$, and $|J|$ the cardinality of J. If $a^{(l)}$ are elements indexed by J, let us adopt as a convention $\prod_{l\in\varnothing} a^{(l)} = 1$.

Let $\gamma(t)$, $t \ge 0$, be a nonnegative, continuous function, increasing to ∞ and satisfying the following two conditions:

(i) there exists a constant $c < \infty$ such that $\gamma(2t) \le c\gamma(t)$ for all t large enough, and

(ii) there exist $\beta > 1/2$ and $b_l(t)$, $l = 1, \dots, m$ nonnegative, continuous, increasing to ∞, such that for all $l = 1, \dots, m$, $t^{-\beta}b_l(t)$ are increasing, and moreover $\gamma(t) = \prod_{l=1}^m b_l(t)$ for all $t \ge 0$.

The above requirements are, in particular, satisfied by normalizing sequences such as n^m or $n^{m/\alpha}, \alpha < 2$, that occur in the Kolmogorov strong law or Marcinkiewicz strong law, respectively.

Let $G_l(x) = P\{|X^{(l)}| \geq x\}$, $\gamma_n := \gamma(n)$, $\gamma_k^* := \gamma(2^k)$. Define $u_k^{(l)} = G_l^{-1}(2^{-k})$, $k \in \mathbb{N}$, $l = 1, \dots, m$, and for $J \subsetneq \{1, \dots, m\}$, $J \neq \varnothing$, define $\omega_k^{(J)} = \gamma_k^* / \sqrt{\prod_{l \in J^c} 2^k E(|X^{(l)}| \wedge u_k^{(l)})^2}$.

Under the above assumptions we have the following results.

THEOREM 1.1. *Let $m = 2$, and assume that* $\sup_{t \geq 0} tG_l(b_l(t)) \leq 1$, $l = 1, 2$. *Then the strong law of large numbers*

$$\lim_{n \to \infty} \frac{1}{\gamma_n} \sum_{1 \leq i,\, j \leq n} X_i^{(1)} X_j^{(2)} = 0 \quad a.s. \tag{1.2}$$

holds if and only if for all $\varepsilon > 0$ the following are satisfied:

$$\sum_{k \geq 1} 2^{2k} P\{|X^{(1)} X^{(2)}| > \varepsilon \gamma_k^*, |X^{(l)}| > u_k^{(l)}, l = 1, 2\} < \infty, \tag{1.3}$$

$$\sum_{k \geq 1} 2^k P\{|X^{(l)}| > \varepsilon \omega_k^{(l)}\} < \infty \quad \text{for } l = 1, 2. \tag{1.4}$$

THEOREM 1.2. *Let $m > 2$. Suppose that* $\sup_{t \geq 0} tG_l(b_l(t)) \leq 1$, *and, for all $J \subset \{1, \dots, m\}$,*

$$\sum_{k \geq n} \frac{2^{(m-|J|)k}}{(w_k^{(J)})^2} \lesssim \frac{2^{(m-|J|)n}}{(w_n^{(J)})^2}.$$

Then the strong law of large numbers (1.1) holds if and only if, for any $J \subseteq \{1, \dots, m\}$, $J \neq \varnothing$ and any $\varepsilon > 0$,

$$\sum_{k \geq 1} 2^{|J|k} P\left\{\prod_{j \in J} |X^{(j)}| > \varepsilon \omega_k^{(J)},\ |X^{(l)}| > u_k^{(l)}, l \in J\right\} < \infty. \tag{1.5}$$

The assumption $\sup_{t \geq 0} tG_l(b_l(t)) \leq 1$ was introduced as a control on the tail of the individual factors. We make use of it in proving the sufficiency part of the results. In the case of identical distribution of the factors, $b_l(t) = \gamma^{1/m}(t)$ for all l, and the boundedness of $\sup_{t \geq 0} tG_l(b_l(t))$ follows either from (1.5) for $J = \{1, \dots, m\}$ or directly from (1.1) which implies, in particular, convergence to 0, in probability, of $1/\gamma_n^{1/m} \sum_{i \leq n} X_i^{(l)}$.

The necessity of the conditions is proved in Section 2. The proof is based on a modified version of Hoffmann-Jørgensen's inequality for U-processes as it appears in Giné and Zinn (1992a). In Section 3 we prove the sufficiency parts of the results. An important step is contained in Proposition 3.6 which makes use of a Rosenthal-type inequality for sums of products of independent and symmetric random variables. The section ends with an example which shows that for $s > 2$ there exist $\{X_i\}, \{Y_j\}$ two independent sequences of i.i.d. symmetric random variables for which $n^{-s}\sum_{1\le i,\, j\le n} X_i Y_j \to 0$ a.s. and $n^{-s}\sum_{1\le i<j\le n} X_i X_j \to 0$ a.s., while $n^{-s}\sum_{1\le i<j\le n} Y_i Y_j$ does not.

2. Necessity. Following the same type of approach as in Cuzick, Giné and Zinn (1995), we derive our results by focusing on the maxima of products. The necessary and sufficient conditions for the strong law of large numbers to hold for maxima of products are given in the following.

THEOREM 2.1. *Assume that $\gamma(t)$ is a nonnegative continuous function increasing to ∞, and $X^{(l)}$, $l = 1, \dots, m$, are nonnegative. Then*

$$\lim_{n\to\infty} \frac{1}{\gamma_n} \max_{1\le i_1,\dots,i_m\le n} X^{(1)}_{i_1}\cdots X^{(m)}_{i_m} = 0 \quad \text{a.s.} \tag{2.1}$$

if and only if the following hold:

$$\sum_{k\ge 1} 2^{|J|k} P\left\{\prod_{l\in J} X^{(l)} > \varepsilon \frac{\gamma_k^*}{\prod_{j\in J^c} u_k^{(j)}},\ \ X^{(h)} > u_k^{(h)}, h\in J\right\} < \infty \tag{2.2}$$

for all $J \subseteq \{1, \dots, m\}, |J| \ge 2$ and all $\varepsilon > 0$, and

$$\sum_{k\ge 1} 2^k P\left\{X^{(l)} > \varepsilon \frac{\gamma_k^*}{\prod_{j\ne l} u_k^{(j)}}\right\} < \infty \tag{2.3}$$

for all $l = 1, \dots, m$ and all $\varepsilon > 0$.

Theorem 2.1 generalizes Theorem 2.1′ in Cuzick, Giné and Zinn (1995); its proof is based on similar techniques and we shall omit it. But let us point out two facts that will be used in the sequel.

REMARK 2.2. If $\gamma^{-1}(t)$ denotes the left-continuous inverse of $\gamma(t)$, then, for $J = \{1, \dots, m\}$, (2.2) can be written in integral form

$$E\left[\gamma^{-1}\left(\frac{X^{(1)}\cdots X^{(m)}}{\varepsilon}\right) \wedge \frac{1}{G_1(X^{(1)})} \wedge \cdots \wedge \frac{1}{G_m(X^{(m)})}\right]^m < \infty \tag{2.2′}$$

for all $\varepsilon > 0$. For $m = 2$, $\gamma(t) = t^{2/\alpha}$, (2.2′) gives a condition which is weaker than the classical Marcinkiewicz necessary and sufficient condition (if $m = 1$) $E|X|^\alpha < \infty$, $EX = 0$ if $\alpha \ge 1$.

REMARK 2.3. One idea in the proof of Theorem 2.1 is to break the index set into blocks of exponential size and to use the Borel–Cantelli lemma. One can then obtain that, for *any* normalizing sequence $\{\gamma_n\}_{n\in\mathbb{N}}$,

$$\frac{1}{\gamma_k^*}\max_{2^{k-1}<i_1,\dots,i_m\le 2^k} X_{i_1}^{(1)}\cdots X_{i_m}^{(m)}\to 0 \quad \text{a.s.} \tag{2.4}$$

is equivalent to the analytic conditions (2.2) and (2.3). Moreover, if (2.4) holds, then

$$\frac{u_k^{(m)}}{\gamma_k^*}\max_{2^{k-1}<i_1,\dots,i_{m-1}\le 2^k} X_{i_1}^{(1)}\cdots X_{i_{m-1}}^{(m-1)}\to 0 \quad \text{a.s.} \tag{2.5}$$

LEMMA 2.4. *If $\{X_i^{(l)}\}_{i\in\mathbb{N}}$, $l=1,\dots,m$, satisfy the strong law of large numbers (1.1), and if $\{\varepsilon_i^{(l)}\}_{i\in\mathbb{N}}$, $l=1,\dots,m$, are independent sequences of i.i.d. Rademacher random variables, independent of $\{X_i^{(l)}\}_{i\in\mathbb{N}}$, then*

$$\lim_{n\to\infty}\frac{1}{\gamma_n}\sum_{1\le i_1,\dots,i_m\le n}\varepsilon_{i_1}^{(1)}X_{i_1}^{(1)}\cdots\varepsilon_{i_m}^{(m)}X_{i_m}^{(m)}=0 \quad \text{a.s.} \tag{2.6}$$

and

$$\lim_{n\to\infty}\frac{1}{\gamma_n^2}\sum_{1\le i_1,\dots,i_m\le n}\left(X_{i_1}^{(1)}\cdots X_{i_m}^{(m)}\right)^2=0 \quad \text{a.s.} \tag{2.7}$$

PROOF. Since, for every $l=1,\dots,m$, $\{X_i^{(l)}\}_{i\in\mathbb{N}}$ is a sequence of i.i.d. symmetric random variables, it follows that if $\{\varepsilon_i^{(l)}\}_{i\in\mathbb{N}}$ are i.i.d. Rademacher random variables independent of $\{X_i^{(l)}\}_{i\in\mathbb{N}}$, then $\{\varepsilon_i^{(l)}X_i^{(l)}\}_{i\in\mathbb{N}}$ and $\{X_i^{(l)}\}_{i\in\mathbb{N}}$ have the same joint distribution; moreover, in view of the independence of the sequences, (1.1) holds if and only if (2.6) holds.

The proof of (2.7) follows the same ideas as the proof of Proposition (4.7) in Cuzick, Giné and Zinn (1995), and therefore we omit it. □

PROOF OF THE NECESSITY IN THEOREMS 1.1 AND 1.2. Let us assume (1.1) holds. By Lemma 2.4, (2.7) holds, and therefore the strong law of large numbers for maxima of the products holds. By Theorem 2.1, (2.2) holds, and therefore we obtain (1.5) for $J=\{1,\dots,m\}$. Let now $J\subsetneq\{1,\dots,m\}$. To ease notation, let $H_{k,J}=\{2^{k-1}<i_l\le 2^k, l\in J\}$; in particular, if $J=\{1,\dots,r\}$ we shall use $H_{k,r}$ instead of $H_{k,J}$. We shall prove that, for all $1\le j\le m-1$ and $J\subsetneq\{1,\dots,m\}$, such that $|J|=m-j$, and all $h\in J$, the following hold:

$$\frac{1}{w_k^{(J)}}\max_{H_{k,J}}\prod_{l\in J}\left|X_{i_l}^{(l)}\right|\to 0 \quad \text{a.s.}, \tag{2.8}$$

$$\begin{aligned}&\frac{1}{w_k^{(J)2}}\max_{H_{k,J\setminus\{h\}}}\prod_{l\in J\setminus\{h\}}\left|X_{i_l}^{(l)}\right|^2 I_{\prod_{r\in J,r\ne h}|X_{i_r}^{(r)}|\le w_k^{(J)}/u_k^{(h)}}\\&\qquad\times E_h\max_{2^{k-1}<i\le 2^k}\left[\left(\left|X_i^{(h)}\right|\wedge u_k^{(h)}\right]^2\to 0 \quad \text{a.s.},\end{aligned} \tag{2.9}$$

where E_h denotes expectation with respect to the variables $X_i^{(h)}$ only. Notice that if (2.8) holds, then, in view of Remark 2.3, (1.5) will follow for all $J \subsetneq \{1, \dots, m\}$. We shall prove (2.8) and (2.9) by induction on j.

Let $j = 1$, and for notational convenience let $I = \{1, \dots, m-1\}$, $h = m-1$. Notice that (2.7) yields, in particular,

$$\frac{1}{\gamma_k^{*2}} \max_{H_{k,m-1}} \prod_{l=1}^{m-1} \big|X_{i_l}^{(l)}\big|^2 I_{\prod_{r=1}^{m-1} |X_{i_r}^{(r)}| \le \gamma_k^*/u_k^{(m)}} \sum_{2^{k-1} < i \le 2^k} \big[|X_i^{(m)}| \wedge u_k^{(m)}\big]^2 \to 0 \quad \text{a.s.} \tag{2.10}$$

Conditionally on $X_{i_l}^{(l)}$, $2^{k-1} < i_l \le 2^k$, $l = 1, \dots, m-1$, the above is a normalized sum of independent, nonnegative random variables whose normalized summands are bounded by 1. Therefore, by the Lebesgue dominated convergence theorem,

$$\begin{aligned} &\frac{1}{\gamma_k^{*2}} E_m \left\{ \max_{H_{k,m-1}} \prod_{l=1}^{m-1} \big|X_{i_l}^{(l)}\big|^2 I_{\prod_{r=1}^{m-1} |X_{i_r}^{(r)}| \le \gamma_k^*/u_k^{(m)}} \max_{2^{k-1} < i \le 2^k} \big[\big|X_i^{(m)}\big| \wedge u_k^{(m)}\big]^2 \right\} \\ &\qquad \to 0 \quad \text{a.s.} \end{aligned} \tag{2.10$'$}$$

By Fubini's theorem, conditionally on $\{X_i^{(l)}\}_{i \in \mathbb{N}}$, $l = 1, \dots, m-1$, the expression in (2.10′) converges to 0 P_m-a.s., thus also in probability. Therefore, Hoffmann–Jørgensen's inequality [Hoffmann-Jørgensen (1974)], which also holds for sums of nonnegative i.i.d. random variables, applied to the expression in (2.10), conditionally on $X_{i_l}^{(l)}$, $l = 1, \dots, m-1$, gives

$$\frac{1}{\gamma_k^{*2}} E_m \left\{ \max_{H_{k,m-1}} \prod_{l=1}^{m-1} \big|X_{i_l}^{(l)}\big|^2 I_{\prod_{r=1}^{m-1} |X_{i_r}^{(r)}| \le \gamma_k^*/u_k^{(m)}} \sum_{2^{k-1} < i \le 2^k} \big[\big|X_i^{(m)}\big| \wedge u_k^{(m)}\big]^2 \right\} \to 0 \quad \text{a.s.}$$

or, equivalently,

$$\frac{2^k E_m[|X^{(m)}| \wedge u_k^{(m)}]^2}{\gamma_k^{*2}} \max_{H_{k,m-1}} \prod_{l=1}^{m-1} \big|X_{i_l}^{(l)}\big|^2 I_{\prod_{r=1}^{m-1} |X_{i_r}^{(r)}| \le \gamma_k^*/u_k^{(m)}} \to 0 \quad \text{a.s.} \tag{2.11}$$

By Remark 2.3 the strong law of large numbers for maxima of products also implies that

$$\frac{u_k^{(m)}}{\gamma_k^*} \max_{H_{k,m-1}} \prod_{l=1}^{m-1} \big|X_{i_l}^{(l)}\big| \to 0 \quad \text{a.s.},$$

which together with (2.11) yields (2.8) for $J = \{1, \dots, m-1\}$. In particular, we have

$$\begin{aligned} \frac{2^k E_m[|X^{(m)}| \wedge u_k^{(m)}]^2}{\gamma_k^{*2}} \max_{H_{k,m-2}} \prod_{l=1}^{m-2} \big|X_{i_l}^{(l)}\big|^2 I_{\prod_{r=1}^{m-2} |X_{i_r}^{(r)}| \le \gamma_k^* / \left(\sqrt{2^k E_m[|X^{(m)}| \wedge u_k^{(m)}]^2}\, u_k^{(m-1)}\right)} \\ \times \max_{2^{k-1} < i \le 2^k} \big[\big|X_i^{(m-1)}\big| \wedge u_k^{(m-1)}\big]^2 \to 0 \quad \text{a.s.} \end{aligned}$$

Since the above is a bounded sequence, expectation with respect to the $X_i^{(m-1)}$ and application of the dominated convergence theorem yield

$$\frac{2^k E_m\big[|X^{(m)}|\wedge u_k^{(m)}\big]^2}{\gamma_k^{*2}}\max_{H_{k,m-2}}\prod_{l=1}^{m-2}\big|X_{i_l}^{(l)}\big|^2 I_{\prod_{r=1}^{m-2}|X_{i_r}^{(r)}|\le\gamma_k^*/\left(\sqrt{2^k E_m[|X^{(m)}|\wedge u_k^{(m)}]^2}\,u_k^{(m-1)}\right)}$$
$$\times E_{m-1}\max_{2^{k-1}<i\le 2^k}\big[|X_i^{(m-1)}|\wedge u_k^{(m-1)}\big]^2\to 0\quad\text{a.s.}$$

and therefore (2.9) is proved.

Suppose now that, for some $1\le j<m-1$, (2.8) and (2.9) hold for all subsets $J\subsetneq\{1,\dots,m\}$, $|J|=m-j$, and all $h\in J$. Let $|J|=m-j-1$. Without loss of generality, we may suppose $J=\{1,\dots,m-j-1\}$. From (2.7) we have

$$\frac{1}{\gamma_k^{*2}}\max_{H_{k,m-j-1}}\prod_{l=1}^{m-j-1}\big|X_{i_l}^{(l)}\big|^2 I_{\prod_{r=1}^{m-j-1}|X_{i_r}^{(r)}|\le w_k^{(\{1,\dots,m-j\})}/u_k^{(m-j)}}$$
$$\times\sum_{2^{k-1}<i_{m-j},\dots,i_m\le 2^k}\prod_{l=m-j}^{m}\big(|X_{i_l}^{(l)}|\wedge u_k^{(l)}\big)^2\to 0\quad\text{a.s.}$$

Notice that, conditionally on $\{X_i^{(l)}\}_{i\in\mathbb{N}}$, $1\le l\le m-j-1$, the above expression converges to 0 a.s. Thus it converges to 0 in probability. Therefore, Hoffmann-Jørgensen's inequality for U-processes, which also holds when the random variables are nonnegative, applied conditionally on $X_{i_l}^{(l)}$, $1\le l\le m-j-1$, together with the induction hypothesis (2.9) yields

$$\frac{1}{w_k^{(I)2}}\max_{H_{k,m-j-1}}\prod_{l=1}^{m-j-1}\big|X_{i_l}^{(l)}\big|^2 I_{\prod_{r=1}^{m-j-1}|X_{i_r}^{(r)}|\le w_k^{(\{1,\dots,m-j\})}/u_k^{(m-j)}}\to 0\quad\text{a.s.}$$

But from the induction hypothesis (2.8) and Remark 2.3 we also have

$$\frac{u_k^{(m-j)}}{w_k^{(\{1,\dots,m-j\})}}\max_{H_{k,m-j-1}}\prod_{l=1}^{m-j-1}\big|X_{i_l}^{(l)}\big|\to 0\quad\text{a.s.}$$

and therefore (2.8) follows. The proof of (2.9) is similar to that for the first step of the induction. The proof is complete. □

REMARK 2.5. Notice that no regularity of the normalizing sequence γ_n was needed in the proof of the necessity.

3. Sufficiency. We shall now provide the main ingredients to be used in the proof of the sufficiency in Theorems 1.1 and 1.2. An outline of the proof is as follows: the analytical conditions (1.3) and (1.4) or (1.5) and the use of the Borel–Cantelli lemma reduce the proof of (1.1) to proving the convergence of sums of truncated random variables. The domain of truncation is then split into a disjoint union of events which are handled separately. Since the variables are symmetric, the partial sums of the truncated variables are

martingales, and the tool that we use is Kolmogorov's maximal inequality for martingales. However, we mention that Kolmogorov's maximal inequality does not provide good estimates when the variables are "small." This case is treated separately in Proposition 3.6.

Let $X^{(l)}$, $\{X_i^{(l)}\}_{i\in\mathbb{N}}$, $l = 1, \dots, m$, be as before, and denote

$$S_n = \sum_{1<i_1<\cdots<i_m\le n} X_{i_1}^{(1)} \cdots X_{i_m}^{(m)} \quad \text{for } n \ge m, \qquad S_n^* = \max_{m\le k\le n} |S_k|,$$

and let $\mathscr{F}_n = \sigma\{X_i^{(l)}\colon i \le n, l = 1, \dots, m\}$. The first result is a Rosenthal-type inequality for sums of products of independent and symmetric random variables.

PROPOSITION 3.1. *For each $2 \le p < \infty$ the following holds:*

$$E|S_n^*|^p \lesssim \prod_{l=1}^m \bigl[n^{p/2}\bigl(E|X^{(l)}|^2\bigr)^{p/2} + nE|X^{(l)}|^p\bigr]. \tag{3.1}$$

PROOF. We shall prove (3.1) by induction on m. Denote $\Delta S_k = S_k - S_{k-1}$, $\langle S_n\rangle = \sum_{k=m}^n E((\Delta S_k)^2 \mid \mathscr{F}_{k-1})$ and $\delta S_n^* = \max_{m\le k\le n} |\Delta S_k|$.

Let $m = 2$. Since $\{S_n\}_{n\ge 2}$ is an $\{\mathscr{F}_n\}_n$ martingale, we may apply Theorem 21.1 in Burkholder (1973) to conclude that

$$E|S_n^*|^p \lesssim E\langle S_n\rangle^{p/2} + E(\Delta S_n^*)^p. \tag{3.2}$$

Define the following $n-1$-dimensional vectors:

$$\begin{aligned} Z_1 &= \bigl(X_1^{(1)}, X_1^{(1)}, \dots, X_1^{(1)}\bigr),\\ Z_2 &= \bigl(0, X_2^{(1)}, \dots, X_2^{(1)}\bigr)\\ &\;\;\vdots\\ Z_{n-1} &= \bigl(0, 0, \dots, X_{n-1}^{(1)}\bigr) \end{aligned}$$

and let $\|\cdot\|$ denote the Euclidean norm. Then

$$\langle S_n\rangle = \left\|\sum_{k=1}^{n-1} Z_k\right\|^2 E|X^{(2)}|^2$$

and

$$E\langle S_n\rangle^{p/2} = \bigl(E|X^{(2)}|^2\bigr)^{p/2} E\left\|\sum_{k=1}^{n-1} Z_k\right\|^p.$$

Let $t_0 = \inf\{t > 0\colon P\{\|\sum_{k=1}^{n-1} Z_k\| > t\} \le (3\cdot 4^p)^{-1}\}$. By Hoffmann-Jørgensen's inequality,

$$E\left\|\sum_{k=1}^{n-1} Z_k\right\|^p \le 2\cdot 4^p\Bigl(t_0^p + E\max_{k\le n-1}\|Z_k\|^p\Bigr),$$

and, furthermore, by Chebyshev's inequality,

$$E\left\|\sum_{k=1}^{n-1} Z_k\right\|^p \lesssim \big(n^2 E\big|X^{(1)}\big|^2\big)^{p/2} + E \max_{1\le k\le n-1} \|Z_k\|^p.$$

Hence

$$E\langle S_n\rangle^{p/2} \lesssim n^p \big(E\big|X^{(1)}\big|^2\big)^{p/2} \big(E\big|X^{(2)}\big|^2\big)^{p/2} + n^{p/2+1} E\big|X^{(1)}\big|^p \big(E\big|X^{(2)}\big|^2\big)^{p/2}. \tag{3.3}$$

Now

$$E(\Delta S_n^*)^p \le \sum_{k=2}^{n} E|\Delta S_k|^p = E|X^{(2)}|^p \sum_{k=2}^{n} E\left|\sum_{i=1}^{k-1} X_i^{(1)}\right|^p.$$

By Rosenthal's inequality [e.g., Rosenthal (1970a, b) or Kwapién and Woyczynski (1992)],

$$\begin{aligned} E\left|\sum_{i=1}^{k-1} X_i^{(1)}\right|^p &\lesssim \left(\sum_{i=1}^{k-1} E\big|X_i^{(1)}\big|^2\right)^{p/2} + \sum_{i=1}^{k-1} E\big|X_i^{(1)}\big|^p \\ &\sim \big((k-1)E|X^{(1)}|^2\big)^{p/2} + (k-1)E|X^{(1)}|^p. \end{aligned}$$

Therefore,

$$E(\Delta S_n^*)^p \lesssim E|X^{(2)}|^p \big[n^{(p/2)+1}\big(E|X^{(1)}|^2\big)^{p/2} + n^2 E|X^{(1)}|^p\big]. \tag{3.4}$$

Putting (3.3) and (3.4) together, we get

$$E|S_n^*|^p \lesssim \big(n^{p/2}\big(E|X^{(1)}|^2\big)^{p/2} + nE|X^{(1)}|^p\big)\big(n^{p/2}\big(E|X^{(2)}|^2\big)^{p/2} + nE|X^{(2)}|^p\big).$$

Assume now (3.1) is true for some $m = l$. We shall prove it also holds for $m = l + 1$. We have

$$\langle S_n\rangle = \sum_{k=l+1}^{n} \left(\sum_{1<i_1<\cdots<i_l<k} X_{i_1}^{(1)} \cdots X_{i_l}^{(l)}\right)^2 E\big|X^{(l+1)}\big|^2,$$

and by Jensen's inequality and the induction hypothesis,

$$E\langle S_n\rangle^{p/2} \lesssim n^{p/2}\big(E\big|X^{(l+1)}\big|^2\big)^{p/2} \prod_{j=1}^{l} \big[n^{p/2}\big(E|X^{(j)}|^2\big)^{p/2} + nE|X^{(j)}|^p\big] \tag{3.5}$$

and

$$\begin{aligned} E(\Delta S_n^*)^p &\le E\big|X^{(l+1)}\big|^p \sum_{k=l+1}^{n} E\left|\sum_{1<i_1<\cdots<i_l<k} X_{i_1}^{(1)} \cdots X_{i_l}^{(l)}\right|^p \\ &\lesssim nE\big|X^{(l+1)}\big|^p \prod_{j=1}^{l} \big[n^{p/2}\big(E|X^{(j)}|^2\big)^{p/2} + nE|X^{(j)}|^p\big]. \end{aligned} \tag{3.6}$$

Then the result follows from (3.5), (3.6) and (3.2). □

REMARK 3.2. Notice that a straightforward application of Rosenthal's inequality for sums of i.i.d. and symmetric real random variables yields

$$E\left|\sum_{1\le i_1,\dots,i_m\le n} X_{i_1}^{(1)}\cdots X_{i_m}^{(m)}\right|^p \lesssim \prod_{l=1}^m \left[n^{p/2}\left(E|X^{(l)}|^2\right)^{p/2} + nE|X^{(l)}|^p\right]. \tag{3.1$'$}$$

Although we will use the result in the form (3.1′), the proof of the proposition shows that the inequality also holds for $T_n = \sum_{1<i_1<\cdots<i_m\le n} X_{i_1}\cdots X_{i_m}$, and X, X_i, $i \in \mathbb{N}$ i.i.d. symmetric, that is,

$$E|T_n^*|^p \lesssim \left(n^{p/2}(E|X|^2)^{p/2} + nE|X|^p\right)^m.$$

LEMMA 3.3. *Let b be a continuous, nonnegative, increasing function, such that $t^{-\beta}b(t)$ is increasing for some $\beta > 1/2$, and let X be a real-valued random variable satisfying $\sup tP\{|X| > b(t)\} \le 1$. Then there exists a constant c, depending on β and j only, and such that, for any integer $j \ge 1$ and any $t > 0$,*

$$E\left[\frac{X^{2j}}{(b^{-1}(|X|))^{j-1}} I_{|X|\le t}\right] \le c\frac{t^{2j}}{\left(b^{-1}(t)\right)^j}.$$

PROOF. The result follows immediately by Fubini and using the fact that $s(b^{-1}(s))^{-\beta}$ is increasing. □

COROLLARY 3.4. *Let $b(t)$ and X be as in Lemma 3.3, and let $X, X_1, X_2, \dots, X_r$ be i.i.d. random variables. Then*

$$E\left[X_1^2X_2^2\cdots X_r^2 I_{|X_1|\le|X_2|\le\cdots\le|X_r|}\right] \lesssim E\frac{X_r^{2r}}{\left(b^{-1}(|X_r|)\right)^{r-1}}.$$

PROOF. The result follows by successive conditioning and application of Lemma 3.3. □

LEMMA 3.5. *Let $\gamma(t)$ and $b_l(t)$, $l = 1, \dots, m$, be continuous nondecreasing functions, and such that $\gamma(t) = \prod_{l=1}^m b_l(t)$. Let $X_1, \dots, X_m$ be independent nonnegative random variables such that, for all $l = 1, \dots, m$, $G_l(x) = P\{X_l \ge x\}$ satisfies $\sup tG_l(b_l(t)) \le 1$, and define $\Omega_1 = \{b_m^{-1}(X_m) = \min_{1\le l\le m} b_l^{-1}(X_l)\}$. Then*

$$b_m^{-1}(X_m)I_{\Omega_1} \le \left[\gamma^{-1}(X_1\cdots X_m) \wedge \frac{1}{G_1(X_1)} \wedge\cdots\wedge \frac{1}{G_m(X_m)}\right] I_{\Omega_1}.$$

PROOF. Since $\sup tG_l(b_l(t)) \le 1$, it follows that $b_l^{-1}(X_l)G_l(X_l) \le 1$. Then, for each l,

$$b_m^{-1}(X_m)I_{\Omega_1} \le b_l^{-1}(X_l)I_{\Omega_1} \le \frac{1}{G_l(X_l)} I_{\Omega_1}.$$

Also $\gamma(b_m^{-1}(X_m))I_{\Omega_1} \le \gamma(\gamma^{-1}(X_1\cdots X_m))I_{\Omega_1}$ yields $b_m^{-1}(X_m)I_{\Omega_1} \le \gamma^{-1}(X_1\cdots X_m)I_{\Omega_1}$ by the monotonicity of $\gamma(t)$. The result follows. □

PROPOSITION 3.6. *Let* $\gamma(t)$ *satisfy assumptions* (i) *and* (ii). *Suppose that, for all* $l = 1, \dots, m$, $\sup tG_l(b_l(t)) \le 1$ *and*

$$E\left[\gamma^{-1}\left(|X^{(1)} \cdots X^{(m)}|\right) \wedge \frac{1}{G_1\left(|X^{(1)}|\right)} \wedge \cdots \wedge \frac{1}{G_m|X^{(m)}|}\right]^m < \infty. \tag{3.7}$$

Then

$$\sum_{k\ge 1} P\left\{\max_{1\le n\le 2^k}\left|\sum_{1\le i_1,\dots,i_m\le n}\prod_{l=1}^m X_{i_l}^{(l)} I_{|X_{i_l}^{(l)}|\le b_l(2^k)}\right| > \gamma_k^*\right\} < \infty. \tag{3.8}$$

PROOF. Let $r \ge m$ be an integer such that $\beta \ge (r+m-1)/(2r)$, and denote $S_n = \sum_{1\le i_1,\dots,i_m\le n}\prod_{l=1}^m X_{i_l}^{(l)} I_{|X_{i_l}^{(l)}|\le b_l(2^k)}$. Since the variables are symmetric, $\{S_n\}_{n\ge 1}$ is a martingale with respect to the σ-fields $\mathscr{F}_n = \sigma\{X_i^{(l)}, i \le n, 1 \le l \le m\}$. By Chebyshev's inequality and (3.1′) we have

$$\begin{aligned}
\sum_{k\ge 1} P&\left\{\max_{1\le n\le 2^k}|S_n| > \gamma_k^*\right\} \\
&\le \sum_{k\ge 1}\frac{E\left(\max_{1\le n\le 2^k}|S_n|\right)^{2r}}{(\gamma_k^*)^{2r}} \\
&\le \sum_{k\ge 1}\sum_{l=0}^m\sum_{J:\,|J|=l}\frac{2^{k(lr+m-l)}}{(\gamma_k^*)^{2r}}\prod_{j\in J}\left[E\left(|X^{(j)}|^2 I_{|X^{(j)}|\le b_j(2^k)}\right)\right]^r \\
&\qquad\qquad \times \prod_{j\in J^c} E\left[|X^{(j)}|^{2r} I_{|X^{(j)}|\le b_j(2^k)}\right] \\
&:= \sum_{l=0}^m\sum_{J:\,|J|=l} T_J.
\end{aligned}$$

Then (3.8) will follow if, for each $l = 0, \dots, m$ and each $J \subseteq \{1, \dots, m\}$, $|J| = l$, $T_J < \infty$. Notice first that it will be enough to look at the sets $J = \varnothing$ and, for $1 \le l \le m$, $J = \{1, \dots, l\}$. Fix $0 \le l \le m$, and denote $T_0 := T_\varnothing$, $T_l := T_J$ for $J = \{1, \dots, l\}$, $1 \le l \le m$. We have

$$\begin{aligned}
T_l &= \sum_{k\ge 1}\frac{2^{k(lr+m-l)}}{(\gamma_k^*)^{2r}}\prod_{j=1}^l\left[E\left(|X^{(j)}|^2 I_{|X^{(j)}|\le b_j(2^k)}\right)\right]^r\prod_{i=l+1}^m E\left[|X^{(i)}|^{2r} I_{|X^{(i)}|\le b_i(2^k)}\right] \\
&\sim \sum_{k\ge 1}\frac{2^{k(lr+m-l)}}{(\gamma_k^*)^{2r}} \\
&\qquad \times E\left\{\prod_{j=1}^l\left(X_1^{(j)}\cdots X_r^{(j)}\right)^2 I_{|X_1^{(j)}|\le\cdots\le|X_r^{(j)}|\le b_j(2^k)}\prod_{i=l+1}^m |X^{(i)}|^{2r} I_{|X^{(i)}|\le b_i(2^k)}\right\}.
\end{aligned}$$

By applying Corollary 3.4 we obtain

$$
\begin{aligned}
T_l &\lesssim E\left\{\prod_{j=1}^{l}\frac{|X^{(j)}|^{2r}}{(b_j^{-1}(|X^{(j)}|))^{r-1}}\prod_{i=l+1}^{m}|X^{(i)}|^{2r}\sum_{k:\ 2^k\geq \max_{j\leq l} b_j^{-1}(|X^{(j)}|)}\frac{2^{k(lr+m-l)}}{(\gamma_k^*)^{2r}}\right\}\\
&\lesssim E\left\{\prod_{j=1}^{l}\frac{|X^{(j)}|^{2r}}{(b_j^{-1}(|X^{(j)}|))^{r-1}}\prod_{i=l+1}^{m}|X^{(i)}|^{2r}\frac{(\max_{j\leq l} b_j^{-1}(|X^{(j)}|))^{lr+m-l}}{\gamma^{2r}(\max_{j\leq l} b_j^{-1}(|X^{(j)}|))}\right\}.
\end{aligned}
\tag{3.9}
$$

Case $l=0$. What we shall do is compare the integrand in (3.9) with the one in (3.7), by considering all possible orderings of the $b_j^{-1}(|X^{(j)}|)$'s. Obviously, it will be enough to do the computations for one only. Therefore, let $\Omega_1 = \{b_1^{-1}(|X^{(1)}|) \geq b_2^{-1}(|X^{(2)}|) \geq \cdots \geq b_m^{-1}(|X^{(m)}|)\}$. Since $r \geq m$, it follows, in particular, that $t^{-m}b_m^{2r}(t)$ is nondecreasing, and since

$$
\gamma(b_1^{-1}(|X^{(1)}|))I_{\Omega_1} \geq \left\{\prod_{j=1}^{m-1} b_j(b_j^{-1}(|X^{(j)}|))\right\} b_m(b_m^{-1}(|X^{(m)}|))I_{\Omega_1},
\tag{3.10}
$$

we get

$$
\left\{\prod_{j=1}^{m}|X^{(j)}|^{2r}\right\}\frac{(b_1^{-1}(|X^{(1)}|))^m}{\gamma^{2r}(b_1^{-1}(|X^{(1)}|))}I_{\Omega_1} \leq (b_m^{-1}(|X^{(m)}|))^m I_{\Omega_1}.
\tag{3.11}
$$

In view of Lemma 3.5, (3.11) and (3.7) yield $T_0 < \infty$.

Case $l=m$. Define Ω_1 as before. The assumption $\beta \geq (r+m-1)/(2r)$ ensures that $t^{r-1}b_j(t)$, $j=1,\ldots,m-1$, and $t^{m+r-1}b_m(t)$ are nondecreasing, which together with (3.10) give

$$
\left\{\prod_{j=1}^{m}\frac{|X^{(j)}|^{2r}}{(b_j^{-1}(|X^{(j)}|))^{r-1}}\right\}\frac{(b_1^{-1}(|X^{(1)}|))^{mr}}{\gamma^{2r}(b_1^{-1}(|X^{(1)}|))}I_{\Omega_1} \leq (b_m^{-1}(|X^{(m)}|))^m I_{\Omega_1}.
\tag{3.12}
$$

Lemma 3.5, (3.12) and (3.7) yield $T_m < \infty$.

Case $1 \leq l \leq m-1$. We shall look at

$$
\max_{1\leq j\leq l} b_j^{-1}(|X^{(j)}|)
$$

and

$$
\max_{l+1\leq j\leq m} b_j^{-1}(|X^{(j)}|)
$$

separately. Define the following sets:

$$
\begin{aligned}
\Omega_1 &= \{b_1^{-1}(|X^{(1)}|) \geq \cdots \geq b_l^{-1}(|X^{(l)}|), b_{l+1}^{-1}(|X^{(l+1)}|) \geq \cdots \geq b_m^{-1}(|X^{(m)}|)\},\\
\Omega_2 &= \{b_l^{-1}(|X^{(l)}|) \geq b_m^{-1}(|X^{(m)}|)\},\\
\Omega_3 &= \{b_1^{-1}(|X^{(1)}|) \geq b_{l+1}^{-1}(|X^{(l+1)}|)\}.
\end{aligned}
$$

The computations on the sets $\Omega_1\cap\Omega_2\cap\Omega_3$ and $\Omega_1\cap\Omega_2^c\cap\Omega_3$ are similar to the ones on $\Omega_1\cap\Omega_2\cap\Omega_3^c$ and $\Omega_1\cap\Omega_2^c\cap\Omega_3^c$, respectively, so we will consider the first two only.

Using that $t^{-(r-1)}b_j^{2r}(t)$ are nondecreasing for all $1\le j\le l$, $t^{-m}b_m^{2r}(t)$ is nondecreasing and $b_j(t)$ are nondecreasing for $l+1\le j\le m-1$, we obtain

$$\begin{aligned}&\left\{\prod_{j=1}^{l}\frac{|X^{(j)}|^{2r}}{(b_j^{-1}(|X^{(j)}|))^{r-1}}\right\}\left\{\prod_{i=l+1}^{m}|X^{(i)}|^{2r}\right\}\frac{\left(b_1^{-1}(|X^{(1)}|)\right)^{lr+m-l}}{\gamma^{2r}\left(b_1^{-1}(|X^{(1)}|)\right)}I_{\Omega_1\cap\Omega_2\cap\Omega_3}\\&\qquad\le\left(b_m^{-1}(|X^{(m)}|)\right)^m I_{\Omega_1\cap\Omega_2\cap\Omega_3}.\end{aligned}\tag{3.13}$$

Now, in order to estimate the integrand on $\Omega_1\cap\Omega_2^c\cap\Omega_3$, we use that $t^{-(r-1)}b_j^{2r}(t)$ are nondecreasing for all $1\le j\le l-1$, $t^{-m+r-1}b_l^{2r}(t)$ is nondecreasing and $b_j(t)$ are nondecreasing for $l+1\le j\le m$. We have

$$\begin{aligned}&\left\{\prod_{j=1}^{l}\frac{|X^{(j)}|^{2r}}{\left(b_j^{-1}(|X^{(j)}|)\right)^{r-1}}\right\}\left\{\prod_{i=l+1}^{m}|X^{(i)}|^{2r}\right\}\frac{\left(b_1^{-1}(|X^{(1)}|)\right)^{lr+m-l}}{\gamma^{2r}\left(b_1^{-1}(|X^{(1)}|)\right)}I_{\Omega_1\cap\Omega_2^c\cap\Omega_3}\\&\qquad\le\left(b_l^{-1}(|X^{(l)}|)\right)^m I_{\Omega_1\cap\Omega_2^c\cap\Omega_3}.\end{aligned}\tag{3.14}$$

Equations (3.13) and (3.14) and Lemma 3.5 yield $T_l<\infty$, completing the proof. $\square$

Proof of sufficiency in Theorem 1.1. Notice that $2^kE(|X^{(l)}|\wedge u_k^{(l)})^2\ge(u_k^{(l)})^2$, hence $\omega_k^{(1)}\le\gamma_k^*/u_k^{(2)}$, and similarly $\omega_k^{(2)}\le\gamma_k^*/u_k^{(1)}$. Then (1.3) and (1.4) imply that the law of large numbers for maxima holds. In particular,

$$P\left\{\max_{1\le i,\ j\le 2^k}\left|X_i^{(1)}X_j^{(2)}\right|>\varepsilon\gamma_k^*\ \text{i.o.}\right\}=0,$$

and since $\gamma_{2n}\le c\gamma_n$ it follows that $\max_{1\le i,\ j\le 2^k}|X_i^{(1)}X_j^{(2)}|<\gamma_{k-1}^*$ eventually a.s. Also, (1.4) and the Borel–Cantelli lemma imply $\max_{i\le 2^k}|X_i^{(1)}|<w_{k-1}^{(1)}$ and $\max_{j\le 2^k}|X_j^{(2)}|<w_{k-1}^{(2)}$ eventually a.s. Therefore, it will be sufficient to prove that

$$\frac{1}{\gamma_n}\sum_{1\le i,\ j\le n}X_i^{(1)}X_j^{(2)}I_{|X_i^{(1)}X_j^{(2)}|<\gamma_{k(n)}^*,\ |X_i^{(1)}|<w_{k(n)}^{(1)},\ |X_j^{(2)}|<w_{k(n)}^{(2)}}\to 0\quad\text{a.s.},$$

where $k(n)=\max\{k\colon 2^k<n\}$. By the Borel–Cantelli lemma, it will suffice to prove

$$\begin{aligned}&\sum_{k\ge1}P\left\{\max_{2^{k-1}<n\le 2^k}\frac{1}{\gamma_{k-1}^*}\right.\\&\qquad\left.\times\left|\sum_{1\le i,\ j\le n}X_i^{(1)}X_j^{(2)}I_{|X_i^{(1)}X_j^{(2)}|<\gamma_{k-1}^*,\ |X_i^{(1)}|<w_{k-1}^{(1)},\ |X_j^{(2)}|<w_{k-1}^{(2)}}\right|>\varepsilon\right\}<\infty.\end{aligned}\tag{3.15}$$

Split the event $\{|X_i^{(1)}X_j^{(2)}| < \gamma_k^*, |X_i^{(1)}| < w_k^{(1)}, |X_j^{(2)}| < w_k^{(2)}\}$ into a disjoint union of five events:

$$\big\{|X_i^{(1)}| \le b_1(2^k), |X_j^{(2)}| \le b_2(2^k)\big\},$$

$$\big\{|X_i^{(1)}| \le u_k^{(1)}, b_2(2^k) < |X_j^{(2)}| < w_k^{(2)}\big\},$$

$$\big\{|X_j^{(2)}| \le u_k^{(2)}, b_1(2^k) < |X_i^{(1)}| < w_k^{(1)}\big\},$$

$$\Big\{u_k^{(1)} < |X_i^{(1)}| \le b_1(2^k), b_2(2^k) < |X_j^{(2)}| < w_k^{(2)}, |X_i^{(1)}X_j^{(2)}| < \gamma_k^*\Big\},$$

$$\big\{u_k^{(2)} < |X_j^{(2)}| \le b_2(2^k), b_1(2^k) < |X_i^{(1)}| < w_k^{(1)}, |X_i^{(1)}X_j^{(2)}| < \gamma_k^*\big\}.$$

Then the proof of (3.15) reduces to proving the following:

$$\sum_{k\ge1} P\Bigg\{\max_{2^{k-1}\le n\le 2^k}\Bigg|\sum_{1\le i,\, j\le n} X_i^{(1)} I_{|X_i^{(1)}|\le b_1(2^{k-1})} X_j^{(2)} I_{|X_j^{(2)}|\le b_2(2^{k-1})}\Bigg| > \varepsilon\gamma_{k-1}^*\Bigg\} < \infty, \tag{3.16}$$

$$\sum_{k\ge1} P\Bigg\{\max_{2^{k-1}\le n\le 2^k}\Bigg|\sum_{1\le i,\, j\le n} X_i^{(1)} I_{|X_i^{(1)}|\le u_{k-1}^{(1)}} X_j^{(2)} I_{b_2(2^{k-1})<|X_j^{(2)}|<w_{k-1}^{(2)}}\Bigg| > \varepsilon\gamma_{k-1}^*\Bigg\} < \infty, \tag{3.17}$$

$$\sum_{k\ge1} P\Bigg\{\max_{2^{k-1}\le n\le 2^k}\Bigg|\sum_{1\le i,\, j\le n} X_i^{(1)} I_{b_1(2^{k-1})<|X_i^{(1)}|<w_{k-1}^{(1)}} X_j^{(2)} I_{|X_j^{(2)}|\le u_{k-1}^{(2)}}\Bigg| > \varepsilon\gamma_{k-1}^*\Bigg\} < \infty, \tag{3.17$'$}$$

$$\sum_{k\ge1} P\Bigg\{\max_{2^{k-1}\le n\le 2^k}\Bigg|\sum_{1\le i,\, j\le n} X_i^{(1)} X_j^{(2)} I_{\substack{u_{k-1}^{(1)}<|X_i^{(1)}|\le b_1(2^{k-1})\\ b_2(2^{k-1})<|X_j^{(2)}|<w_{k-1}^{(2)}}} I_{|X_i^{(1)}X_j^{(2)}|<\gamma_{k-1}^*}\Bigg| > \varepsilon\gamma_{k-1}^*\Bigg\} < \infty, \tag{3.18}$$

$$\sum_{k\ge1} P\Bigg\{\max_{2^{k-1}\le n\le 2^k}\Bigg|\sum_{1\le i,\, j\le n} X_i^{(1)} X_j^{(2)} I_{\substack{b_1(2^{k-1})<|X_i^{(1)}|<w_{k-1}^{(1)}\\ u_{k-1}^{(2)}<|X_j^{(2)}|\le b_2(2^{k-1})}} I_{|X_i^{(1)}X_j^{(2)}|<\gamma_{k-1}^*}\Bigg| > \varepsilon\gamma_{k-1}^*\Bigg\} < \infty. \tag{3.18$'$}$$

Since the variables are symmetric, the sums inside these expressions are martingales relative to the σ-fields $\mathscr{F}_n = \sigma\{X_1^{(1)}, \dots, X_n^{(1)}, X_1^{(2)}, \dots, X_n^{(2)}\}$.

By Proposition 3.6, the sum in (3.16) is finite. Notice that (3.17) and (3.18) are similar to (3.17′) and (3.18′), respectively, so we will only prove the first two. To evaluate the sum in (3.17), apply Kolmogorov's maximal inequality; therefore, (3.17) will follow if

$$\sum_{k=1}^{\infty} \frac{2^{2k}}{(\gamma_k^*)^2} E\left[\left| X^{(1)} X^{(2)} \right|^2 I_{|X^{(1)}| \le u_k^{(1)},\, b_2(2^k) < |X^{(2)}| \le w_k^{(2)}} \right] < \infty. \tag{3.19}$$

Denote $S_k := E[|X^{(1)}|^2 I_{|X^{(1)}| \le u_k^{(1)}}]$, $T_k := E[|X^{(2)}|^2 I_{b_2(2^k) < |X^{(2)}| \le w_k^{(2)}}]$ and $Q_k := \sum_{j \ge k} (2^{2j}/(\gamma_j^*)^2)$. Since $t^{-\beta}\gamma(t)$ is nondecreasing, $Q_k \simeq 2^{2k}(\gamma_k^*)^{-2}$. In order to estimate the sum in (3.19), we use summation by parts; we have

$$\begin{aligned}\sum_{k=1}^{n} Q_k S_k T_k &\simeq \sum_{k=1}^{n} (Q_k - Q_{k+1}) S_k T_k \\ &= Q_1 S_1 T_1 + \sum_{k=2}^{n} Q_k (S_k - S_{k-1}) T_k \\ &\quad + \sum_{k=2}^{n} Q_k S_{k-1} (T_k - T_{k-1}) - Q_{n+1} S_n T_n.\end{aligned} \tag{3.20}$$

Let us evaluate

$$\begin{aligned}&\sum_{k \ge 2} Q_k (S_k - S_{k-1}) T_k \\ &\quad \lesssim \sum_{k \ge 2} \frac{2^{2k}}{(\gamma_k^*)^2} E\left[|X^{(1)}|^2 I_{u_{k-1}^{(1)} < |X^{(1)}| \le u_k^{(1)}} |X^{(2)}|^2 I_{b_2(2^k) < |X^{(2)}|} I_{|X^{(1)} X^{(2)}| < \gamma_k^*} \right].\end{aligned}$$

Notice that $2^k P\{|X^{(l)}| \ge b_l(2^k)\} \le 1$ implies that $u_k^{(l)} \le b_l(2^k)$, and therefore

$$\left\{ u_{k-1}^{(1)} < |X^{(1)}| \le u_k^{(1)}, b_2(2^k) < |X^{(2)}|, |X^{(1)} X^{(2)}| < \gamma_k^* \right\} \subseteq \Omega_{1,k},$$

where $\Omega_{1,k} := \{2^{-k+1} \ge G_1(|X^{(1)}|) \ge 2^{-k} \ge G_2(|X^{(2)}|), |X^{(1)} X^{(2)}| < \gamma_k^*\}$. On $\Omega_{1,k}$ the following relations hold:

$$\frac{1}{G_1(|X^{(1)}|)} \wedge \frac{1}{G_2(|X^{(2)}|)} = \frac{1}{G_1(|X^{(1)}|)}, \qquad \gamma^{-1}(|X^{(1)} X^{(2)}|) \le 2^k \le \frac{2}{G_1(|X^{(1)}|)}.$$

Let us denote

$$\Omega_2 := \{1/G_1(|X^{(1)}|) \le 1/G_2(|X^{(2)}|), \qquad \gamma^{-1}(|X^{(1)} X^{(2)}|) \le 2/G_1(|X^{(1)}|)\}.$$

Then $\Omega_{1,k} \subseteq \Omega_2$ for all k, and for n large enough we have

$$
\begin{aligned}
&\sum_{k\ge n} Q_k(S_k - S_{k-1})T_k \\
&\quad\lesssim E\bigg[|X^{(1)}X^{(2)}|^2 \sum_{k\ge n} \frac{2^{2k}}{(\gamma_k^*)^2} I_{\Omega_{1,k}}\bigg] \\
&\quad\lesssim E\bigg[|X^{(1)}X^{(2)}|^2 \sum_{k:\ 2^k \ge \gamma^{-1}(|X^{(1)}X^{(2)}|)\vee 1/G_1(|X^{(1)}|)} \frac{2^{2k}}{(\gamma_k^*)^2} I_{\Omega_2}\bigg] \qquad (3.21)\\
&\quad\lesssim E\left[|X^{(1)}X^{(2)}|^2 \frac{\left(\gamma^{-1}(|X^{(1)}X^{(2)}|) \vee \dfrac{1}{G_1(|X^{(1)}|)}\right)^2}{\gamma^2\left(\gamma^{-1}(|X^{(1)}X^{(2)}|) \vee \dfrac{1}{G_1(|X^{(1)}|)}\right)} I_{\Omega_2}\right].
\end{aligned}
$$

Let

$$\Omega_3 = \{\gamma^{-1}(|X^{(1)}X^{(2)}|) \ge 1/G_1(|X^{(1)}|)\}.$$

In (3.21) we write $I_{\Omega_2} = I_{\Omega_2\cap\Omega_3} + I_{\Omega_2\cap\Omega_3^c}$ and we shall compare each of the resulting integrands with $[\gamma^{-1}(|X^{(1)}X^{(2)}|) \wedge (1/G_1(|X^{(1)}|)) \wedge (1/G_2(|X^{(2)}|))]^2$. We have

$$
\begin{aligned}
|X^{(1)}X^{(2)}| \frac{\gamma^{-1}(|X^{(1)}X^{(2)}|)}{\gamma(\gamma^{-1}(|X^{(1)}X^{(2)}|))} I_{\Omega_2\cap\Omega_3} &= \gamma^{-1}(|X^{(1)}X^{(2)}|) I_{\Omega_2\cap\Omega_3} \\
&\le \frac{2}{G_1(|X^{(1)}|)} I_{\Omega_2\cap\Omega_3},
\end{aligned}
$$

and therefore

$$E\left\{\left[|X^{(1)}X^{(2)}| \frac{\gamma^{-1}(|X^{(1)}X^{(2)}|)}{\gamma(\gamma^{-1}(|X^{(1)}X^{(2)}|))}\right]^2 I_{\Omega_2\cap\Omega_3}\right\} < \infty. \tag{3.22}$$

Using that $t^{-1}\gamma(t)$ is nondecreasing and $\gamma(2t) \le c\gamma(t)$, we obtain the following estimates:

$$|X^{(1)}X^{(2)}| \frac{\dfrac{1}{G_1(|X^{(1)}|)}}{\gamma\left(\dfrac{1}{G_1(|X^{(1)}|)}\right)} I_{\Omega_2\cap\Omega_3^c} \le \gamma^{-1}(|X^{(1)}X^{(2)}|) I_{\Omega_2\cap\Omega_3^c},$$

$$
\begin{aligned}
|X^{(1)}X^{(2)}| \frac{\dfrac{1}{G_1(|X^{(1)}|)}}{\gamma\left(\dfrac{1}{G_1(|X^{(1)}|)}\right)} I_{\Omega_2\cap\Omega_3^c} &\le c\gamma(|X^{(1)}X^{(2)}|) \frac{\dfrac{1}{G_1(|X^{(1)}|)}}{\gamma\left(\dfrac{2}{G_1(|X^{(1)}|)}\right)} I_{\Omega_2\cap\Omega_3^c} \\
&\le c \frac{1}{G_1(|X^{(1)}|)} I_{\Omega_2\cap\Omega_3^c},
\end{aligned}
$$

and therefore

$$E\left\{\left[|X^{(1)}X^{(2)}|\frac{\dfrac{1}{G_1(|X^{(1)}|)}}{\gamma\left(\dfrac{1}{G_1(|X^{(1)}|)}\right)}\right]^2 I_{\Omega_2\cap\Omega_3^c}\right\}<\infty. \tag{3.23}$$

By (3.22) and (3.23), the expression in (3.21) is finite.

To evaluate the series $\sum_{k\ge 2} Q_k S_{k-1}(T_k-T_{k-1})$, notice that the series of positive terms is dominated by

$$\begin{aligned}
&\sum_{k\ge 2}\frac{2^{2k}}{(\gamma_k^*)^2}E\big[|X^{(1)}|^2 I_{|X^{(1)}|\le u_{k-1}^{(1)}}\big]E\big[|X^{(2)}|^2 I_{w_{k-1}^{(2)}<|X^{(2)}|\le w_k^{(2)}}\big]\\
&\qquad\le \sum_{k:\ w_{k-1}^{(2)}<w_k^{(2)}}\frac{2^{2k}}{(\gamma_k^*)^2}E\big(|X^{(1)}|\wedge u_{k-1}^{(1)}\big)^2 {w_k^{(2)}}^2 P\big\{|X^{(2)}|>w_{k-1}^{(2)}\big\}\\
&\qquad\le 2\sum_{k\ge 2}2^k P\big\{|X^{(2)}|>w_k^{(2)}\big\}<\infty.
\end{aligned}$$

Therefore, (3.17) is proved.

Let us now prove (3.18). By Kolmogorov's maximal inequality, the series in (3.18) is bounded from above by

$$\sum_k \frac{2^{2k}}{(\gamma_k^*)^2}E\big[|X^{(1)}|^2 I_{u_k^{(1)}<|X^{(1)}|\le b_1(2^k)}|X^{(2)}|^2 I_{b_2(2^k)<|X^{(2)}|<w_k^{(2)}}I_{|X^{(1)}X^{(2)}|<\gamma_k^*}\big].$$

Now, if $u_k^{(1)}<|X^{(1)}|$, $b_2(2^k)<|X^{(2)}|$ and $|X^{(1)}X^{(2)}|<\gamma_k^*$, then $G_1(|X^{(1)}|)\le 2^{-k}$, $G_2(|X^{(2)}|)\le 2^{-k}$ and $\gamma^{-1}(|X^{(1)}X^{(2)}|)\le 2^k$. Hence

$$\gamma^{-1}(|X^{(1)}X^{(2)}|)\le 2^k\le \frac{1}{G_1(|X^{(1)}|)}\wedge\frac{1}{G_2(|X^{(2)}|)},$$

and we have

$$\begin{aligned}
&\sum_k \frac{2^{2k}}{(\gamma_k^*)^2}E\left[|X^{(1)}|^2 I_{u_k^{(1)}<|X^{(1)}|\le b_1(2^k)}|X^{(2)}|^2 I_{b_2(2^k)<|X^{(2)}|<w_k^{(2)}}I_{|X^{(1)}X^{(2)}|<\gamma_k^*}\right]\\
&\qquad\le E\left[|X^{(1)}X^{(2)}|^2 I_{\gamma^{-1}(|X^{(1)}X^{(2)}|)\le 1/G_1(|X^{(1)}|)\wedge 1/G_2(|X^{(2)}|)}\sum_{k:\ 2^k\ge\gamma^{-1}(|X^{(1)}X^{(2)}|)}\frac{2^{2k}}{(\gamma_k^*)^2}\right]\\
&\qquad\lesssim E\left[|X^{(1)}X^{(2)}|^2 I_{\gamma^{-1}(|X^{(1)}X^{(2)}|)\le 1/G_1(|X^{(1)}|)\wedge 1/G_2(|X^{(2)}|)}\frac{(\gamma^{-1}(|X^{(1)}X^{(2)}|))^2}{|X^{(1)}X^{(2)}|^2}\right]\\
&\qquad= E\left[(\gamma^{-1}(|X^{(1)}X^{(2)}|))^2 I_{\gamma^{-1}(|X^{(1)}X^{(2)}|)\le 1/G_1(|X^{(1)}|)\wedge 1/G_2(|X^{(2)}|)}\right]\\
&\qquad< E\left[\gamma^{-1}(|X^{(1)}X^{(2)}|)\wedge\frac{1}{G_1(|X^{(1)}|)}\wedge\frac{1}{G_2(|X^{(2)}|)}\right]^2<\infty,
\end{aligned}$$

and (3.18) is proved. □

PROOF OF SUFFICIENCY IN THEOREM 1.2. Since $2^k E(|X^{(l)}| \wedge u_k^{(l)})^2 \geq (u_k^{(l)})^2$, it follows that, for $J \subsetneq \{1, \dots, m\}$, $J \neq \varnothing$, $\omega_k^{(J)} \leq \gamma_k^* / \prod_{l \in J^c} u_k^{(l)}$. Then the law of large numbers for maxima holds, and by arguments similar to the ones in the proof of sufficiency for $m = 2$, it will be enough to prove that, for all $\varepsilon > 0$ and all $J \subsetneq \{1, 2, \dots, m\}$,

$$\sum_{k \geq 1} P\Bigg\{\max_{2^{k-1} < n \leq 2^k} \frac{1}{\gamma_{k-1}^*}\Bigg|\sum_{1 \leq i_1, \dots, i_m \leq n} \prod_{l=1}^{m} X_{i_l}^{(l)} I_{\prod_{h=1}^m |X_{i_h}^{(h)}| < \gamma_{k-1}^*} \times I_{\prod_{j \in J} |X_{i_j}^{(j)}| \leq w_{k-1}^{(J)}}\Bigg| > \varepsilon\Bigg\} < \infty. \tag{3.24}$$

For $k \geq 1$, $\mathrm{i} = (i_1, \dots, i_m)$, $i_l \leq n$ and $J \subsetneq \{1, 2, \dots, m\}$, define the following sets:

$$\Omega_{k,\mathrm{i}} = \{|X_{i_l}^{(l)}| \leq b_l(2^k), l = 1, \dots, m\},$$

$$\Omega_{k,\mathrm{i}}^{(J)} = \Bigg\{|X_{i_h}^{(h)}| \leq u_k^{(h)}, h \in J, |X_{i_l}^{(l)}| > u_k^{(l)}, l \in J^c, |X_{i_j}^{(j)}| > b_j(2^k) \text{ for at least one } j \in J^c, \prod_{l \in J^c} |X_{i_l}^{(l)}| \leq w_k^{(J)}\Bigg\},$$

$$\Omega_{k,1}^{(J)} = \{|X^{(l)}| \leq u_k^{(l)}, l \in J\},$$

$$\Omega_{k,2}^{(J)} = \Bigg\{|X^{(h)}| > u_k^{(h)}, h \in J^c, |X^{(j)}| > b_j(2^k) \text{ for at least one } j \in J^c, \prod_{l \in J^c} |X^{(l)}| \leq w_k^{(J)}\Bigg\}.$$

Then (3.24) will follow if for all $J \subsetneq \{1, 2, \dots, m\}$ the following hold:

$$\sum_{k \geq 1} P\Bigg\{\max_{2^{k-1} < n \leq 2^k}\Bigg|\sum_{1 \leq i_1, \dots, i_m \leq n} \prod_{l=1}^{m} X_{i_l}^{(l)} I_{\Omega_{k,\mathrm{i}}}\Bigg| > \varepsilon \gamma_k^*\Bigg\} < \infty, \tag{3.25}$$

$$\sum_{k \geq 1} P\Bigg\{\max_{2^{k-1} < n \leq 2^k}\Bigg|\sum_{1 \leq i_1, \dots, i_m \leq n} \prod_{l=1}^{m} X_{i_l}^{(l)} I_{\Omega_{k,\mathrm{i}}^{(J)}}| > \varepsilon \gamma_k^*\} < \infty. \tag{3.26}$$

Now, (3.25) follows by Proposition 3.6. By Kolmogorov's maximal inequality for martingales, (3.26) will follow if

$$\sum_{k \geq 1} \frac{2^{mk}}{(\gamma_k^*)^2} E\Bigg[\prod_{l=1}^{m} |X^{(l)}|^2 I_{\Omega_{k,1}^{(J)} \cap \Omega_{k,2}^{(J)}}\Bigg] < \infty.$$

We have

$$\begin{aligned}
&\sum_{k\ge 1}\frac{2^{mk}}{(\gamma_k^*)^2}E\bigg[\prod_{l=1}^{m}|X^{(l)}|^2 I_{\Omega_{k,1}^{(J)}\cap\Omega_{k,2}^{(J)}}\bigg]\\
&\qquad\le \sum_{k\ge 1}\frac{2^{mk}\prod_{l\in J}E(|X^{(l)}|\wedge u_k^{(l)})^2}{(\gamma_k^*)^2}E\bigg[\prod_{l\in J^c}|X^{(l)}|^2 I_{\Omega_{k,2}^{(J)}}\bigg]\\
&\qquad\le \sum_{k\ge 1}\frac{2^{(m-|J|)k}}{(w_k^{(J)})^2}E\bigg[\prod_{l\in J^c}|X^{(l)}|^2 I_{\Omega_{k,2}^{(J)}}\bigg].
\end{aligned}\tag{3.27}$$

Since $\sum_{k\ge n}2^{(m-|J|)k}(w_k^{(J)})^{-2}\lesssim 2^{(m-|J|)n}(w_n^{(J)})^{-2}$, we may apply an argument of summation by parts. Let $Q_k := 2^{(m-|J|)k}(w_k^{(J)})^{-2}$, $S_k := E[\prod_{l\in J^c}|X^{(l)}|^2 I_{\Omega_{k,2}^{(J)}}]$. We have

$$\begin{aligned}
\sum_{k=1}^{n}Q_kS_k &\sim \sum_{k=1}^{n}(Q_k-Q_{k+1})S_k\\
&= Q_1S_1 - Q_{n+1}S_n + \sum_{k=2}^{n}Q_k(S_k-S_{k-1})\\
&\le Q_1S_1 + \sum_{k=2}^{n}\frac{2^{(m-|J|)k}}{(w_k^{(J)})^2}E\bigg[\prod_{l\in J^c}|X^{(l)}|^2 I_{\Omega_{k,2}^{(J)}\setminus\Omega_{k-1,2}^{(J)}}\bigg].
\end{aligned}\tag{3.28}$$

Note that the series in (3.27) has only nonnegative terms; therefore, it will be sufficient to prove that the series whose partial n-sum appears in (3.28) is finite. We have

$$\begin{aligned}
&\sum_{k\ge 2}\frac{2^{(m-|J|)k}}{(w_k^{(J)})^2}E\bigg[\prod_{l\in J^c}|X^{(l)}|^2 I_{\Omega_{k,2}^{(J)}\setminus\Omega_{k-1,2}^{(J)}}\bigg]\\
&\qquad\le \sum_{k:\ w_{k-1}^{(J)}<w_k^{(J)}}E\bigg[\prod_{l\in J^c}|X^{(l)}|^2 I_{|X^{(h)}|>u_k^{(h)},\,h\in J^c,\,w_{k-1}^{(J)}<\prod_{h\in J^c}|X^{(h)}|\le w_k^{(J)}}\bigg]\\
&\qquad\le \sum_{k\ge 1}2^{(m-|J|)k}P\bigg\{\prod_{l\in J^c}|X^{(l)}|>w_k^{(J)},\,|X^{(h)}|>u_k^{(h)},\,h\in J^c\bigg\}<\infty,
\end{aligned}$$

completing the proof. $\square$

As mentioned in the Introduction, for $m=2$ the diagonal terms are irrelevant in the decoupled case. The same holds in our setting: (1.2) implies, in particular, a.s. convergence to 0 of $\gamma_n^{-1}\max_{i\le n}|X_i^{(1)}X_i^{(2)}|$, so $\gamma_n^{-1}\sum_{i\le n}|X_i^{(1)}X_i^{(2)}|\to 0$ a.s. For $m>2$ we obtain that all normalized sums that have terms with two or more identical indices converge to 0 a.s. Let us state this as the following.

COROLLARY 3.7. *If (1.1) holds then, any $l = 2, \ldots, m$,*

$$\frac{1}{\gamma_n} \sum_{i, i_{l+1}, \ldots, i_m = 1}^{n} X_i^{(1)} \cdots X_i^{(l)} X_{i_{l+1}}^{(l+1)} \cdots X_{i_m}^{(m)} \to 0 \quad \textit{a.s.}$$

PROOF. The preceding sum has n^{m-l+1} terms, and by the argument used in the proof of sufficiency in Theorem 1.2, the conclusion follows. □

EXAMPLE 3.8. Let $s > 1$, $\delta > 0$ such that $s - \delta > 1$, and define $\alpha = 2(s-\delta)^{-1}$, $\beta = 2(s+\delta)^{-1}$. Let $\{X, X_i\}, \{Y, Y_j\}$ be independent sequences of i.i.d. and symmetric random variables with tail probabilities given by

$$P\{|X| > t\} \sim \frac{1}{t^\alpha \log t}, \qquad t \geq 2$$

and

$$P\{|Y| > t\} \sim \frac{1}{t^\beta \log t}, \qquad t \geq 2.$$

We prove that

$$n^{-s} \sum_{i, j=1}^{n} X_i Y_j \to 0 \quad \text{a.s.} \tag{3.29}$$

Let

$$a = \frac{s-\delta}{4(s-\delta-1/2)}, \qquad b = \frac{s+\delta}{4(s+\delta-1/2)}.$$

In order to prove that (3.29) holds, it will be sufficient, by the Borel–Cantelli, lemma to show that, for all $\varepsilon > 0$,

$$\sum_k P\left\{2^{-ks} \max_{2^{k-1} < n \leq 2^k} \left| \sum_{i, j \leq n} X_i Y_j \right| > \varepsilon \right\} < \infty. \tag{3.30}$$

Let $u_k = 2^{k/\alpha} k^a$, $v_k = 2^{k/\beta} k^b$ and for $i, j \leq 2^k$, $\bar{X}_i = X_i 1_{|X_i| \leq u_k}$, $\bar{Y}_j = Y_j 1_{|Y_j| \leq v_k}$. We have

$$\begin{aligned}
&P\left\{2^{-ks} \max_{2^{k-1} < n \leq 2^k} \left| \sum_{i, j \leq n} X_i Y_j \right| > \varepsilon \right\} \\
&\quad \leq P\left\{2^{-ks} \max_{2^{k-1} < n \leq 2^k} \left| \sum_{i, j \leq n} \bar{X}_i \bar{Y}_j \right| > \varepsilon \right\} \\
&\qquad + P\left\{\max_{i \leq 2^k} |X_i| > u_k \text{ or } \max_{j \leq 2^k} |Y_j| > v_k \right\} \\
&\quad \leq \frac{2^{2k} E\bar{X}^2 E\bar{Y}^2}{\varepsilon^2 2^{2ks}} + 2^k P\{|X| > u_k\} + 2^k P\{|Y| > v_k\}.
\end{aligned}$$

But notice that for k large

$$P\{|X| > u_k\} \lesssim \frac{1}{2^k k^{a\alpha} k}, \qquad P\{|Y| > v_k\} \lesssim \frac{1}{2^k k^{b\beta} k}$$

and therefore

$$\sum_k 2^k \big(P\{|X| > u_k\} + P\{|Y| > v_k\} \big) < \infty.$$

In order to evaluate $E\bar{X}^2$ and $E\bar{Y}^2$, notice that X and Y have regularly varying tails with exponents $(-\alpha)$ and $(-\beta)$, respectively; therefore, using the properties of regularly varying functions, as in Feller (1971), we have

$$E\bar{X}^2 \lesssim \int_2^{u_k} \frac{1}{t^{\alpha-1}\log t}\, dt \lesssim \frac{u_k^{2-\alpha}}{\log u_k}, \qquad E\bar{Y}^2 \lesssim \frac{v_k^{2-\beta}}{\log v_k}.$$

Therefore,

$$\begin{aligned}\sum_k P\bigg\{ 2^{-ks} \max_{2^{k-1} < n \le 2^k} \bigg| \sum_{i,\, j \le n} \bar{X}_i \bar{Y}_j \bigg| > \varepsilon \bigg\} &\lesssim \sum_k \frac{2^{2k} u_k^{2-\alpha} v_k^{2-\beta}}{2^{2ks} \log u_k \log v_k} \\ &\sim \sum_k k^{a(2-\alpha)+b(2-\beta)-2} < \infty,\end{aligned}$$

since $a(2-\alpha)+b(2-\beta)-2 < -1$, and (3.30) follows.

Notice that $EX^{2/s} < \infty$ and therefore, by the Marcinkiewicz strong law of large numbers, the U-statistic of order 2 in X_i satisfies

$$n^{-s} \sum_{\substack{i,\, j=1 \\ i \ne j}}^{n} X_i X_j = n^{-s} \bigg[\sum_{i \le n} X_i \bigg]^2 - n^{-s} \sum_{i \le n} X_i^2 \to 0 \quad \text{a.s.},$$

but the U-statistic of order 2 in Y_j does not. If it did, it would have implied, as in Giné and Zinn (1992b), that

$$nP\big\{ |Y| > n^{s/2} \big\} \to 0$$

and hence $E|Y|^{2/s-\eta} < \infty$ for all $0 < \eta < 2/s$. But $E|Y|^{2/(s+\delta)} = \infty$, and $E|Y|^{2/s-\eta} = \infty$ for all $\eta \le 2\delta/(s-\delta)$.

REMARK 3.9. The truncation levels u_k and v_k in the above are not the u_k's used in Theorems 1.1 and 1.2. The asymptotic order of magnitude of $u_k^{(l)}$, $w_k^{(l)}$ is $u_k^{(1)} \sim 2^{k/\alpha} k^{-1/\alpha}$, $w_k^{(1)} \sim k^{1/\beta} 2^{k/\alpha}$; $u_k^{(2)} \sim 2^{k/\beta} k^{-1/\beta}$, $w_k^{(2)} \sim k^{1/\alpha} 2^{k/\beta}$ for X and Y, respectively. Condition (1.4) can be easily checked, but (1.3) leads to tedious computations, so we prefered a more direct proof of (3.29).

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DEPARTMENT OF MATHEMATICS AND PHYSICS
GANNON UNIVERSITY
ERIE, PENNSYLVANIA 16541-0001
E-MAIL: gadidov001@mail1.gannon.edu