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CONTINUOUS DEPENDENCE OF A CLASS OF SUPERPROCESSES ON BRANCHING PARAMETERS AND APPLICATIONS

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A general class of finite variance critical (ξ, Φ, k)-superprocesses X in a Luzin space E with cadlag right Markov motion process ξ, regular local branching mechanism Φ and branching functional k of bounded characteristic are shown to continuously depend on (Φ, k). As an application we show that the processes with a classical branching functional $k(ds) = \varrho_s(\xi_s)ds$ [that is, a branching functional k generated by a classical branching rate $\varrho_s(y)$] are dense in the above class of (ξ, Φ, k)-superprocesses X. Moreover, we show that, if the phase space E is a compact metric space and ξ is a Feller process, then always a Hunt version of the (ξ, Φ, k)-superprocess X exists. Moreover, under this assumption, we even get continuity in (Φ, k) in terms of weak convergence of laws on Skorohod path spaces.

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1. Introduction.

1.1. *Motivation, purpose and main results.* While the characterization of the class of (ξ, Φ, k)-superprocesses X is obviously a fundamental part of the theory of measure-valued branching processes, it cannot alone fully describe the reach structure of this class. In particular, it would be natural to define a meaningful metric in terms of only the parameters (ξ, Φ, k). Topological properties of this metric, such as, for instance, the description of dense or compact subsets, or such as the completeness property, would give further insight into the nature of superprocesses. As a long-term goal, it seems to be desirable to express properties of (ξ, Φ, k)-superprocesses (as their path properties, for example) in terms of the properties of the parameters (ξ, Φ, k), and this paper should be seen as a step in this direction.

Indeed, we focus here on the question of jointly continuous dependence on the branching mechanism Φ and the branching functional k. Once one has such a continuous dependence, one can, for instance, use it to derive certain properties of a class of superprocesses by starting from more elementary processes, rather than by a direct analysis. We will in fact include such applications below.

The problem of continuous dependence of superprocesses on their branching rate is not entirely new. For instance in [4], Lemma 2.3.5 and its application in Sections 2.4 and 2.5, it was used to construct a class of one-dimensional superprocesses with catalytic branching rate $\varrho_s(dy)$ by starting from superprocesses with classical branching rate $\varrho_s(y)\,dy$. In [5], Proposition 1 and Subsection 3.1, continuity in k was exploited to construct super-Brownian motions in R^d with (only) locally admissible branching functional k by approximating them by (globally) admissible ones. In this way, a class of super-Brownian motions constructed in [9] could be extended. Finally, in [12], a truncation procedure of branching rate was applied to construct a one-dimensional super-Brownian motion with the locally infinite catalytic mass $|y|^{-2}dy$. (In contrast to the present paper, this superprocess does not have a finite variance even though the branching mechanism is "binary critical.")

The question of continuous dependence of superprocesses on their branching mechanism Φ and branching functional k is studied here on its own and in a general (ξ, Φ, k)-superprocess setting. Then we use this continuity to prove that for each (ξ, Φ, k)-superprocess considered in this paper, a Hunt version exists, provided that the phase space is a compact metric space and the motion process ξ is Feller (Theorem 40). In this case we even get continuity in (Φ, k) in terms of weak convergence of the laws on the Skorohod space of cadlag paths (Theorem 42).

The construction of superprocesses with regularity properties of the paths has a long history. Concerning recent general results, in the first place we refer to [11], which proved the existence of a right or even Hunt version of a superprocess if the motion process is right or Hunt, respectively, provided that the branching mechanism is time-homogeneous and the branching functional is given by $k(ds) = ds$. Fitzsimmons' right version result is generalized in [8] and [14]. Then [15] generalized Fitzsimmons' Hunt result to a general class of (ξ, Φ, k)-superprocesses with finite variance and admissible (in the sense of Dynkin) functional k. One of our motivations was to obtain such result for nonadmissible k of bounded characteristic. Finally, we mention the recent paper [17], which deals with the construction and path regularity of (ξ, Φ, k)-superprocesses with metrizable co-Souslin spaces as phase space.

We also note that the results of the present paper play a crucial role in [16] where a martingale problem is established for a class of (ξ, Φ, k)-superprocesses under mild conditions.

1.2. *Setup.* Before going further, recall that the main steps of the method of construction of superprocesses via the analysis of the related evolution equation (see, for instance, [2, 4, 7, 9, 15, 12, 5]) more or less resemble the following procedure. First, find for fixed n a measure-valued process X^n whose log-Laplace functional $v^n = v^n(f) = v^n_{\bullet,t}(f)$ solves an evolution equation

$$v^n = \Psi^n(v^n). \tag{1}$$

Second, show that, for a certain norm $\|\cdot\|$ (typically a supremum norm $\|\cdot\|_\infty$, or a closely related one),

$$\|v^m - v^n\| \le \tfrac{1}{2}\|v^m - v^n\| + q_{m,n},$$

where $q_{m,n}$ is a nonnegative quantity converging to zero as $m, n \to \infty$. By completeness, this shows, that v^n converges. It is usually possible to conclude the following.

1. The limit v again satisfies an evolution equation

$$v = \Psi(v). \tag{2}$$

2. v is the unique solution to that equation.
3. Each $v_{r,t}(x)$ is the log-Laplace functional of a random measure.
4. v determines a semigroup.

This semigroup then uniquely characterizes a superprocess X (log-Laplace functional characterization).

Suppose now that (1) is the (ξ, Φ^n, k^n)-evolution equation of the so-called (ξ, Φ^n, k^n)-superprocess X^n. Here Ψ^n is a functional of ξ, Φ^n, k^n, where we have the following.

1. The particles' motion process $\xi = (\xi_t, \Im, \pi_{r,x})$ is cadlag right Markov.
2. Φ^n is a critical local branching mechanism with finite variance [see Assumption 13(f)].
3. The branching functional k^n is a continuous additive functional of ξ of bounded characteristic.

Our key result can briefly be described as follows. Suppose that k^n converges to a continuous additive functional k of ξ in an appropriate sense, and the Φ^n converge uniformly to a regular branching mechanism Φ, then the log-Laplace functionals v^n converge to some v solving the (ξ, Φ, k)-evolution equation (2). As in [15], this equation is then used to construct a (ξ, Φ, k)-superprocess X with v as its log-Laplace functional. Since the convergence $v^n \to_n v$ of log-Laplace functionals implies the convergence $X^n \Rightarrow_n X$ in the sense of (weak) convergence of all finite-dimensional distributions (fdd), the (ξ, Φ, k)-superprocess continuously depends on (Φ, k) (Theorem 20).

This fdd continuity theorem can be extended to weak convergence on some Skorohod path spaces, and several applications are supplied. In particular, if the phase space is a compact metric space and ξ is Feller, we show that a Hunt version of X exists and "classical" $(\xi, \Phi, \varrho_s(\xi_s)ds)$-superprocesses are weakly dense in the set of all (ξ, Φ, k)-superprocesses.

1.3. *Outline.* To prove the continuity theorem, we follow essentially the method described in the previous subsection. We use the norm $\|\cdot\|_C$ defined to be the supremum over the set C of all those points (r, x) such that

$$(\alpha) \qquad \pi_{r,x} \bigvee_{n=1}^{\infty} k^n(r, t] < \infty,$$

$$(\beta) \qquad \pi_{r,x}\Big\{k^n \underset{n}{\Longrightarrow} k\Big\} = 1$$

(recall $\pi_{r,x}$ refers to the law of the motion process ξ with initial data r, x). Starting from a point $(r, x) \in C$, it is crucial to know that $\pi_{r,x}$-a.s. all points (s, ξ_s), $s > r$, also belong to C. This is essentially what we will cover in Section 2.

After introducing more carefully in the beginning of Section 3 the model we deal with in detail, we formulate our key result, the fdd continuity Theorem 20. Then we discuss the assumptions on the branching functional in that theorem and review the log-Laplace functional characterization of (ξ, Φ, k)-superprocesses. However, the central part of our argument is Proposition 34. It states that in the case $\Phi^n \equiv \Phi$, for "small" test functions f (the parameter entering into the linear term of the evolution equation (2) coming from the log-Laplace functional), and for starting points (r, x) in C, the log-Laplace functionals v^n converge to some v.

The derived fdd continuity theorem has strong implications. First, as an application we establish in Theorem 23 that each (ξ, Φ, k)-superprocess can be approximated by ones with "classical" branching functional k. "Classical" here means that the branching functional k can be represented as $k(ds) = \varrho_s(\xi_s)\,ds$ with ϱ a bounded (classical) function. In this case, a particle at time s at site y splits with branching rate $\varrho_s(y)$. In other words, the approximating processes are "classical" superprocesses.

We mention that by fdd convergence of X^n to X we actually mean

$$E \exp\left[\sum_{i=1}^{m} \langle X^n_{t_i}, -f_i\rangle\right] \to E \exp\left[\sum_{i=1}^{m} \langle X_{t_i}, -f_i\rangle\right] \quad \text{as } n \to \infty$$

for any choice of bounded measurable nonnegative functions $f_1, \ldots, f_m$ on E. [$\langle \mu, f\rangle$ abbreviates the integral $\int f(x)\,\mu(dx)$.] In other words, we have fdd convergence in every topology on E compatible with the measurability structure $(E, \mathscr{E})$ of our Luzin space E.

A more subtle question is the convergence of laws on path spaces. Here one needs some further restrictive assumptions on the data (ξ, Φ, k). In order to avoid expensive technicalities, in Section 4 we restrict our attention to the special case of a Feller motion process ξ in a compact metric space (E, d). Then the continuity and approximation theorems can be used to construct a Hunt version of the (ξ, Φ, k)-superprocesses (Theorem 40). These Hunt (ξ, Φ, k)-superprocesses depend continuously on (Φ, k) in terms of weak convergence of the laws on the Skorohod path spaces, rather than only fdd (Theorem 42).

In the Appendix, we collect some results, which are purely technical.

As a standard reference for weak convergence we refer to [10] and for (ξ, Φ, k)-superprocesses to [9].

1.4. *Basic assumptions: motion process ξ and branching functional k.* In this paper, "nonnegative" always means R_+-valued, $\mathsf{R}_+ := [0, \infty)$. But in some cases we also need to consider variables with values in the one-point compactification $\overline{\mathsf{R}}_+ := [0, \infty]$ of R_+. In this case, we will explicitly refer to this.

Throughout this paper, the following assumptions are in force.

ASSUMPTION 1 (Motion process and branching functional).

(a) (Phase space) The phase space E is a Luzin space. That is a topological space E which is homeomorphic to a Borel subset of a compact metrizable space. [Note that, for example, every complete separable metric space is Luzin (see, e.g., [18], page 370).] Let $\mathscr{E}$ denote the Borel σ-algebra of E and $\mathscr{E}_+ = \mathscr{E}_+(E)$ the set of all $\overline{\mathsf{R}}_+$-valued measurable functions f on E. Moreover, write

$b\mathscr{E}_+ = b\mathscr{E}_+(E)$ for the subset of all bounded $f \in \mathscr{E}_+$, equipped with the topology of bounded pointwise convergence.

(b) (Measure space) Let $\mathscr{M}_\mathrm{f} = \mathscr{M}_\mathrm{f}(E) = \mathscr{M}_\mathrm{f}(\mathscr{E})$ denote the set of all finite measures on $\mathscr{E}$. Endowed with the topology of weak convergence, $\mathscr{M}_\mathrm{f}$ is a Luzin space.

(c) (Time interval) We consider first of all stochastic processes on a fixed finite interval $I := [0, T]$, $T > 0$, or on subintervals of I; later, in Section 4, we extend to R_+.

(d) (Underlying particle's motion process ξ) Once and for all, fix an E-valued process ξ on I satisfying the following conditions.

(d1) (Markov process) ξ is a (time-inhomogeneous) Markov process $(\xi_t, \Im, \pi_{r,x})$ in the setting of [9], Section 2.2.1.

(d2) (Right process) This Markov process ξ is assumed to be a right process which means the following:

(i) $t \mapsto \xi_t(\omega)$ is right continuous (in the Luzin E), for each ω.

(ii) For $0 \le r \le t \le T$, $\mu \in \mathscr{M}_\mathrm{f}$, and $f \in \mathscr{E}_+$ fixed, the function $s \mapsto \pi_{s,\xi_s} f(\xi_t)$, $s \in [r, t)$, is right continuous $\pi_{r,\mu}$-almost everywhere. [Note that our terminology differs slightly from [9] which includes the cadlag property (d3) in the notion of a right process. In this situation we call it a cadlag right process.]

(d3) (Cadlag) The process ξ is required to be cadlag [additionally to (i)]; that is, for each ω, the limits $\lim_{s\uparrow t} \xi_s =: \xi_{t-}$ exist in E for all $t \in (0, T]$.

(d4) (Hunt) Sometimes we additionally assume that the cadlag right Markov process ξ is Hunt. In this case we work with $I = \mathsf{R}_+$ as the time axis.

(e) (Branching functional) As a rule, the letter k refers to a (nonnegative) continuous additive functional of ξ ([9], Section 2.4.1) of bounded characteristic:

$$\sup_{(r,x)\in I\times E} \pi_{r,x} k(r, T] < \infty. \tag{3}$$

We call such k a branching functional. Intuitively, $k(ds)$ is the rate of branching of a particle with position ξ_s at time s.

REMARK 2 (Admissible functionals). Note that condition (3) is weaker than the admissibility requirement in [9], Section 3.3.3:

$$\sup_{x\in E} \pi_{r,x} k(r, t] \to 0, \qquad s \in I \text{ as } r, t \to s. \tag{4}$$

(In fact, read the proof of Lemma 3 in [5] with ϕ_p replaced by 1.)

REMARK 3 (Natural functionals k). Several partial results in the present paper remain valid if the (limiting) additive functional k is only natural (instead of continuous). But we stress the fact that in our key Theorem 20, the assumption on the continuity of k cannot be dropped.

2. Path and preservation properties. In this section we investigate the following question. Suppose that for a "starting point" (r, x) a certain property

$\wp$ of particles' motion process ξ holds $\pi_{r,x}$-a.s. When can we say that, $\pi_{r,x}$-a.s., the process $s \mapsto (s, \xi_s)$ passes only through those points (s, y) such that the property $\wp$ is valid $\pi_{s,y}$-a.s.?

For example, suppose that $k^1, k^2, \dots$ are (continuous) additive functionals of the (cadlag right Markov) process $\xi = (\xi_t, \Im, \pi_{r,x})$. Fix a starting point $(r, x) \in I \times E$. Assume that $\pi_{r,x}$-almost surely the measures k^n (as finite measures on $[r, T]$) converge weakly to k as $n \to \infty$. Is it then the case that $\pi_{r,x}$-almost surely, for every $s \in [r, T]$, with π_{s,ξ_s}-probability 1, k^n converges weakly to k (as measures on $[s, T]$)?

With Proposition 9, we will give a positive answer to this type of question. At this place it might be helpful to give a heuristic reasoning which indicates the strategy we will use. Suppose that the following expectation vanishes:

$$\pi_{r,x}\Big(\sup_{s\in[r,T]} \limsup_n \big|k^n(s,T] - k(s,T]\big|\Big) = 0.$$

Then, for any point $s \in [r, T]$, the Markov property gives that

$$\pi_{s,\xi_s}\Big(\sup_{t\in[s,T]} \limsup_n \big|k^n(t,T] - k(t,T]\big|\Big) = 0, \qquad \pi_{r,x}\text{-a.s.}$$

Obviously, this remains true for a countable dense set of times $s \in [r, T]$. Hence, if the process

$$s \mapsto \pi_{s,\xi_s}\Big(\sup_{t\in[s,T]} \limsup_n \big|k^n(t,T] - k(t,T]\big|\Big)$$

could be verified to be right continuous, we get that

$$\sup_{s\in[r,T]} \pi_{s,\xi_s}\Big(\sup_{t\in[s,T]} \limsup_n \big|k^n(t,T] - k(t,T]\big|\Big) = 0, \qquad \pi_{r,x}\text{-a.s.},$$

as wanted.

This reasoning motivates in particular the following subsection.

2.1. *Path properties of a class of processes.* For convenience, we impose the following assumption (which will be in force throughout this subsection).

ASSUMPTION 4 (A pair of processes). Fix a starting point $(r, x) \in I \times E$. For $s \in [r, T]$, let Y_s and Z_s be $\overline{\mathsf{R}}_+$-valued $\Im[s, T]$-measurable variables. [Note that $s \mapsto \Im[s, T]$ is not a filtration since $\Im[s, T] \supseteq \Im[s', T]$, $s \le s' \le T$. Here $\Im[s, T]$ is the sub-σ-field of $\Im$ of "events observable during" the interval $[s, T]$.] Define $y_s := \pi_{s,\xi_s} Y_s$ and $z_s := \pi_{s,\xi_s} Z_s$ (which could be infinite at this stage). Suppose $\pi_{r,x} Y_r < \infty$.

The main result of this subsection is the following proposition.

PROPOSITION 5 [Nonnegative cadlag processes of class (D)]. *Let (Y, Z) be a pair of processes satisfying Assumption 4. In addition, suppose $s \mapsto Y_s$ is right continuous and nonincreasing (for each ω, as $\overline{\mathsf{R}}_+$-valued function). Then the following statements hold.*

(i) *The process* $y = \{y_s\colon r \le s \le T\}$ *is* $\pi_{r,x}$*-indistinguishable from a nonnegative cadlag process of class* (D).

(ii) *If additionally* $Z \le Y$ *and* $s \mapsto Z_s$ *is cadlag (as* $\overline{\mathsf{R}}_+$*-valued function), then* $z = \{z_s\colon r \le s \le T\}$ *is also* $\pi_{r,x}$*-indistinguishable from a nonnegative cadlag process of class* (D).

Before providing the proof, we need some preparation. Consider Y, y as in Assumption 4. For every $c \in [0,\infty]$, define

$$y_s^c := \pi_{s,\xi_s} Y_s^c, \qquad Y_s^c := c \wedge Y_s.$$

Note that $Y_s^\infty = Y_s$ and $y_s^\infty = y_s$.

LEMMA 6 (Preparations). *Let* $c \in [0,\infty]$.

(a) *Suppose that with respect to* $\pi_{r,x}$ *the process* y^c *is indistinguishable from a nonnegative process and belongs to class* (D). *Then it is* $\pi_{r,x}$*-almost surely right continuous.*

(b) *For every* $c \in [0,\infty)$, *the nonnegative process* y^c *is* $\pi_{r,x}$*-a.s. right continuous and belongs to class* (D).

(c) *The* $\overline{\mathsf{R}}_+$*-valued process* y *is* $\pi_{r,x}$*-indistinguishable from a nonnegative process (that is,* R_+*-valued process).*

(d) *With* $\pi_{r,x}$*-probability 1,* y *is nonnegative and belongs to class* (D).

PROOF. (a) We first establish that y^c is optional. For $n \ge 1$, introduce the step function

$$y_s^{n,c} := \sum_{n=0}^{n-1} 1_{[s_i^n,\, s_{i+1}^n)}(s)\, \pi_{s,\xi_s} Y_{s_{i+1}^n}^c, \qquad r \le s \le T, \tag{5}$$

where $s_i^n := r + (i/n)(T-r)$, for $i = 0,\dots,n$. Obviously, the $\pi_{r,x}$-almost surely nonnegative process $y^{n,c}$ is $\pi_{r,x}$-a.s. right continuous and thus optional. [If Y_s has the form $Y_s := f(s,\xi_s)$ for a measurable bounded f then the $\pi_{r,x}$-a.s. right continuity of $y^{n,c}$ is immediate from the definition of a right process (see [9], page 27). The more general case reduces to the just mentioned one by taking the conditional expectation.] Clearly, pointwise $y_s^c = \lim_n y_s^{n,c}$ holds. Therefore y^c is also optional.

Let $\sigma_n \le T$ be r-stopping times nonincreasing to (the r-stopping time) σ as $n \to \infty$. Then by the definition of y^c, the strong Markov property, right continuity of Y^c and the monotone convergence theorem, we have

$$\lim_n \pi_{r,x} y_{\sigma_n}^c = \lim_n \pi_{r,x}\pi_{\sigma_n,\xi_{\sigma_n}} Y_{\sigma_n}^c = \lim_n \pi_{r,x} Y_{\sigma_n}^c = \pi_{r,x} Y_\sigma^c = \pi_{r,x} y_\sigma^c.$$

Hence, according to [9], A.1.1.D, page 116, the $\pi_{r,x}$-a.s. nonnegative process y^c is $\pi_{r,x}$-a.s. right continuous.

(b) This is immediate from (a) and the fact that these processes are bounded (by the constant c).

(c) According to (b), for c finite, the nonnegative process y^c is $\pi_{r,x}$-a.s. right continuous. Therefore, $\sup_{r\le s\le T} y_s^c$ is measurable and monotonously converges to $\sup_{r\le s\le T} y_s$ as $c\uparrow\infty$. Hence, for $\eta>0$,

$$\pi_{r,x}\Big\{\sup_{r\le s\le T} y_s>\eta\Big\}=\lim_{c\to\infty}\pi_{r,x}\Big\{\sup_{r\le s\le T} y_s^c>\eta\Big\}.$$

We can thus invoke Proposition A2 in the Appendix, and continue with

$$\begin{aligned}\pi_{r,x}\Big\{\sup_{r\le s\le T} y_s>\eta\Big\}&\le\eta^{-1}\lim_{c\to\infty}\sup_{r\le\sigma\le T}\pi_{r,x}y_\sigma^c\\&=\eta^{-1}\lim_{c\to\infty}\sup_{r\le\sigma\le T}\pi_{r,x}Y_\sigma^c\\&\le\eta^{-1}\lim_{c\to\infty}\pi_{r,x}Y_r^c\\&\le\eta^{-1}\pi_{r,x}Y_r<\infty.\end{aligned}$$

Letting $\eta\to\infty$ gives the claim.

(d) First, for r-stopping times $\sigma\le T$,

$$\sup_{r\le\sigma\le T}\pi_{r,x}y_\sigma=\sup_{r\le\sigma\le T}\pi_{r,x}Y_\sigma\le\pi_{r,x}Y_r<\infty$$

by the Markov property and monotonicity of Y.

Consider a collection of measurable sets Γ_n with the property $\pi_{r,x}(\Gamma_n)\searrow 0$ as $n\to\infty$. Let us denote by $\pi_{r,x}^{\Im[r,\sigma]}$ the conditional expectation with respect to $\Im[r,\sigma]$. We have that

$$\pi_{r,x}1_{\Gamma_n}y_\sigma=\pi_{r,x}\pi_{r,x}^{\Im[r,\sigma]}1_{\Gamma_n}y_\sigma=\pi_{r,x}\big(\pi_{r,x}^{\Im[r,\sigma]}1_{\Gamma_n}\big)y_\sigma$$

since y_σ is measurable with respect to $\Im[r,\sigma]$.

By the strong Markov property, we can continue with

$$\begin{aligned}\pi_{r,x}\big(\pi_{r,x}^{\Im[r,\sigma]}1_{\Gamma_n}\big)y_\sigma&=\pi_{r,x}\big(\pi_{r,x}^{\Im[r,\sigma]}1_{\Gamma_n}\big)\pi_{\sigma,\xi_\sigma}Y_\sigma=\pi_{r,x}\big(\pi_{r,x}^{\Im[r,\sigma]}1_{\Gamma_n}\big)Y_\sigma\\&\le\pi_{r,x}\big(\pi_{r,x}^{\Im[r,\sigma]}1_{\Gamma_n}\big)Y_r.\end{aligned}$$

Because Y_r is measurable with respect to $\Im[r,\sigma]$, the chain of inequalities can be continued with

$$=\pi_{r,x}\big(\pi_{r,x}^{\Im[r,\sigma]}1_{\Gamma_n}Y_r\big)=\pi_{r,x}1_{\Gamma_n}Y_r.$$

Appealing to the dominated convergence theorem (in the version of [10], Theorem A.1.2), the latter expression tends to zero as $n\to\infty$. Hence

$$\limsup_n\sup_{r\le\sigma\le T}\pi_{r,x}1_{\Gamma_n}y_\sigma=0.$$

That is, y belongs to class (D). $\square$

PROOF OF PROPOSITION 5. We start with part (ii). Besides Y, consider Z as in the theorem. Immediately from Lemma 6(c), (d) and (a) it follows that y is $\pi_{r,x}$-a.s. a nonnegative right continuous process of class (D). Since $0 \le Z_s \le Y_s$ we get that $0 \le z_s \le y_s$, and therefore z belongs to class (D). We have to show that z is $\pi_{r,x}$-a.s. cadlag.

Consider

$$z_s^n := \sum_{n=0}^{n-1} 1_{[s_i^n, s_{i+1}^n)}(s)\, \pi_{s,\xi_s} Z_{s_{i+1}^n}, \qquad r \le s \le T,$$

where again $s_i^n := r + (i/n)(T-r)$, for $i = 0, \dots, n$. The process z^n is cadlag $\pi_{r,x}$-a.s. and thus optional. We have that

$$1_{[s_i^n, s_{i+1}^n)}(s) Z_{s_{i+1}^n} \le 1_{[s_i^n, s_{i+1}^n)}(s) Y_{s_{i+1}^n} \le 1_{[s_i^n, s_{i+1}^n)}(s) Y_s.$$

Since $y_s = \pi_{s,\xi_s} Y_s < \infty$, $\pi_{r,x}$-a.s., the above inequalities allow invoking the dominated convergence theorem and we obtain

$$\sum_{n=0}^{n-1} 1_{[s_i^n, s_{i+1}^n)}(s)\, \pi_{s,\xi_s} Z_{s_{i+1}^n} \underset{n}{\longrightarrow} \pi_{s,\xi_s} Z_s.$$

That is $z_s^n \to_n z_s$. Therefore the process z is optional.

Let $\sigma_1, \sigma_2, \ldots \le T$ be a nonincreasing sequence of r-stopping times converging to σ. Recall that by assumption Z is $\overline{\mathsf{R}}_+$-valued cadlag, and that

$$0 \le \sup_{r \le s \le T} Z_s \le Y_r \in L^1(\pi_{r,x}).$$

Hence, Z is $\pi_{r,x}$-a.s. nonnegative and by definition,

$$\pi_{r,x} z_{\sigma_n} = \pi_{r,x} \pi_{\sigma_n, \xi_{\sigma_n}} Z_{\sigma_n} = \pi_{r,x} Z_{\sigma_n}.$$

Invoking the dominated convergence theorem, we get

$$\lim_n \pi_{r,x} z_{\sigma_n} = \lim_n \pi_{r,x} Z_{\sigma_n} = \pi_{r,x} Z_\sigma = \pi_{r,x} z_\sigma.$$

Hence, z is $\pi_{r,x}$-a.s. right continuous (recall [9], A.1.1.D, page 116). An analogous reasoning, invoking Lemma A1 from the Appendix, shows that z has also left limits $\pi_{r,x}$-a.s. Consequently, z is $\pi_{r,x}$-a.s. nonnegative cadlag, proving (ii).

It remains to prove part (i). Now Y itself satisfies the assumptions on Z in (ii), since it is in particular cadlag. Hence, by the already proved statement (ii), together with z, also y is $\pi_{r,x}$-a.s. nonnegative cadlag, completing the proof. □

2.2. *The case of indistinguishability from zero.* Recall that in this section we investigate conditions under which the following holds. If a certain property $\wp$ is true $\pi_{r,x}$-a.s., then $\pi_{r,x}$-a.s., the property $\wp$ is true π_{s,ξ_s}-a.s. for all s in $[r,T]$. In this subsection, $\wp$ is the property of being indistinguishable from zero. The following result is an immediate consequence of a standard result; see, for instance, [9], A.1.1.E, page 116.

LEMMA 7 (Preservation of indistinguishability from zero). *Fix a starting point $(r,x) \in I \times E$. Let Y_s, $s \in [r,T]$, again be $\overline{\mathsf{R}}_+$-valued $\Im[s,T]$-measurable* *variables. Suppose that $Y = \{Y_s\}_{s\in[r,T]}$ is nonincreasing and right continuous and that $\pi_{r,x} Y_r < \infty$. If $\{Y_s\}_{s\in[r,T]}$ is $\pi_{r,x}$-indistinguishable from zero, then*

$$\pi_{r,x}\big\{\{Y_t\}_{t\in[s,T]} \text{ is } \pi_{s,\xi_s}\text{-indistinguishable from zero, } \forall s \in [r,T]\big\} = 1, \tag{6}$$

or equivalently

$$\pi_{r,x}\Big(\sup_{s\in[r,T]} \pi_{s,\xi_s}\Big(\sup_{t\in[s,T]} Y_t\Big)\Big) = 0. \tag{7}$$

2.3. *Preservation of initial properties for additive functionals.*

ASSUMPTION 8 (Initial properties of additive functionals). Denote by $k^1, \ldots, k^\infty$ (nonnegative) continuous additive functionals of our cadlag right process $\xi = (\xi_t, \Im, \pi_{r,x})$. In the sequel we also write k instead of k^∞. We assume that, for the starting point $(r,x) \in I \times E$ we have the following:

(α) $\pi_{r,x} \bigvee_{n=1}^\infty k^n(r,T] < \infty$;
(β) with $\pi_{r,x}$-probability 1, $k^n(s,T] \to_n k(s,T]$ for every $s \in [r,T]$.

Note that we included k^∞ in the definition of k^n, so that k^∞ is also involved in a supremum expression such as in (α).

Also note that the requirement "for every $s \in [r,T]$" in part (β) can be replaced by "for every rational $s \in (r,T]$ and $s = r$," hence it is a measurable assertion. In fact, $k^n(s,T]$ and $k(s,T]$ are monotone and continuous in s. Note finally that (β) implies that

$$\pi_{r,x}\text{-almost surely,} \quad k^n(s,t] \to_n k(s,t] \text{ whenever } r \le s \le t \le T \tag{8}$$

(indeed, consider differences).

The main result of this section is the following proposition.

PROPOSITION 9 (Preservation of initial properties). *Under Assumption 8, with $\pi_{r,x}$-probability 1 the process $s \mapsto (s,\xi_s)$, $s \in [r,T]$, will pass only through those points (s,y) such that the following hold:*

(α) *$\pi_{s,y} \bigvee_{n=1}^\infty k^n(s,T] < \infty$;*
(β) *with $\pi_{s,y}$-probability 1, $k^n(t,T] \to_n k(t,T]$ for every $t \in [s,T]$.*

Before providing the proof of Proposition 9, we need to establish some preliminary results. For this purpose, for $s \in [r,T]$ introduce the following notation:

$$Y_s^1 := \bigvee_{n=1}^\infty k^n(s,T], \qquad Y_s^2 := \sup_{t\in(s,T]} \limsup_n |k^n(t,T] - k(t,T]|, \tag{9}$$

$$Y_s^3 := \limsup_n |k^n(s,T] - k(s,T]| \tag{10}$$

and set $y_s^i := \pi_{s,\xi_s} Y_s^i$ for $i = 1,2,3$. Note that the variables Y_s^i, $i = 1,2,3$, $s \in [r,T]$, are measurable.

LEMMA 10. *Under Assumption 8, the nonincreasing $\overline{\mathrm{R}}_+$-valued processes Y^1 and Y^2 are right continuous.*

PROOF. First, Y^2 is right continuous, since for any function g, the nonincreasing process $s \mapsto \sup_{t>s} g(s)$ is right continuous.

Next, suppose that Y^1 is not right continuous. That is, for some s (and a fixed ω),

$$\bigvee_{n=1}^{\infty} k^n(s,T] := \alpha > \beta := \lim_{t\searrow s} \bigvee_{n=1}^{\infty} k^n(t,T].$$

Then, for every n,

$$\beta \geq \lim_{t\searrow s} k^n(t,T] = k^n(s,T],$$

since k^n is a measure. Thus $\beta \geq \bigvee_{n=1}^{\infty} k^n(s,T] = \alpha$ which is a contradiction. Therefore Y^1 is right continuous. □

REMARK 11. Note that under Assumption 8, by Lemma 10 and according to Proposition 5(i), the processes y^1 and y^2 are $\pi_{r,x}$-a.s. nonnegative cadlag and of class (D).

LEMMA 12. *Under Assumption 8, for $l = 1,\dots,\infty$ and $s \in [r,T]$, let ψ_s^l be $\Im[s,T]$-measurable nonnegative variables. Suppose that with respect to $\pi_{r,x}$ the random functions $\psi^1, \psi^2, \dots, \psi^\infty$ are measurable processes uniformly bounded by a (nonrandom) constant. For $r \leq s \leq T$ and $M \in \{1,\dots,\infty\}$ put*

$$Z_s(M) := \bigvee_{n=1}^{M} \left| \int_{(s,T]} \psi_t^n \, k^n(dt) - \int_{(s,T]} \psi_t^\infty \, k^\infty(dt) \right|$$

and

$$z_s(M) := \pi_{s,\xi_s} Z_s(M).$$

Then the process $z(\infty)$ is $\pi_{r,x}$-indistinguishable from a nonnegative cadlag process of class (D)*.*

PROOF. Set $B := \sup_{n,s} |\psi_s^n|$, and let M be finite. Note that

$$Z_s(M) \leq 2B \bigvee_{n=1}^{\infty} k^n(s,T] = 2B\, Y_s^1 \in L^1(\pi_{r,x}) \tag{11}$$

and that $Z(M)$ is nonnegative cadlag. Hence, by Lemma 10 and Proposition 5(ii), the process $z(M)$ is $\pi_{r,x}$-a.s. a nonnegative cadlag process of class (D). By monotone convergence, $z_s(\infty) = \lim_M z_s(M)$, and therefore $z(\infty)$ is optional. For all M, from (11) we get $z_s(M) \leq 2B\, y_s^1$, and recalling Remark 11, we conclude that $z(\infty)$ is $\pi_{r,x}$-a.s. nonnegative and belongs to class (D). All that remains to be proved is that $z(\infty)$ is cadlag, $\pi_{r,x}$-a.s. Clearly, because of (11) and the monotonicity of Y^1, we have that $Z(\infty)$ is $\pi_{r,x}$-a.s. nonnegative.

From the elementary identity

$$\bigvee_n |a_n - a| = \Big(\bigvee_n a_n - a\Big) \vee \Big(a - \bigwedge_n a_n\Big)$$

we conclude with Lemma 10 and Corollary A4 that $Z(\infty)$ is $\pi_{r,x}$-a.s. a right continuous nonnegative process. Now, if $\sigma_n \le T$ are r-stopping times nonincreasing to σ, by the strong Markov property, $\pi_{r,x}\, z_{\sigma_n}(\infty) = \pi_{r,x}\, Z_{\sigma_n}(\infty)$. By right continuity, $Z_{\sigma_n}(\infty)$ converges to $Z_\sigma(\infty)$ as $n \to \infty$. Because of (11) we can invoke the dominated convergence theorem to derive that $\lim_n \pi_{r,x}\, z_{\sigma_n}(\infty) = \pi_{r,x}\, Z_\sigma(\infty)$. But again $\pi_{r,x}\, Z_\sigma(\infty) = \pi_{r,x}\, z_\sigma(\infty)$, and hence $\lim_n \pi_{r,x}\, z_{\sigma_n}(\infty) = \pi_{r,x}\, z_\sigma(\infty)$. This proves that z is $\pi_{r,x}$-a.s. nonnegative right continuous. A similar reasoning, invoking Lemma A1 shows that z also has left limits $\pi_{r,x}$-a.s. $\square$

PROOF OF PROPOSITION 9.

Step 1. According to Remark 11, the processes y^1 and y^2 are $\pi_{r,x}$-a.s. nonnegative cadlag processes of class (D). By Lemma 12, if we put for $N \ge 1$,

$$Y_s^3(N) := \bigvee_{n=N}^{\infty} \big|k^n(s,T] - k(s,T]\big|, \qquad y_s^3(N) := \pi_{s,\xi_s}\, Y_s^3(N),$$

then $y^3(N)$ is also $\pi_{r,x}$-a.s. a nonnegative cadlag process of class (D). Since $Y_s^3(N) \le Y_s^1 < \infty$, $\pi_{r,x}$-a.s., and $Y_s^3(N) \searrow Y_s^3$ [defined in (10)] as $N \to \infty$, we get by dominated convergence that $y_s^3(N) \searrow y_s^3$ as $N \to \infty$. This establishes that y^3 is a nonnegative optional process of class (D).

Step 2. Recall that y^1 is in particular $\pi_{r,x}$-indistinguishable from a nonnegative process by Remark 11. In other words, $\pi_{r,x}$-a.s. the process $s \mapsto (s, \xi_s)$ passes only through points (s, y) such that $\pi_{s,y} \bigvee_{n=1}^{\infty} k^n(s,T] < \infty$.

Step 3. Recall that Y^2 defined in (9) is $\overline{\mathsf{R}}_+$-valued nonincreasing and right continuous, and by Assumption 8, $\pi_{r,x}$-indistinguishable from 0. Hence, by Lemma 7, the statement (7) holds (with Y^2 instead of Y). In other words, with $\pi_{r,x}$-probability 1, the process (s, ξ_s) passes only through points (s, y) such that $\pi_{s,y}$-almost surely, $k^n(t,T] \to_n k(t,T]$ for every $t \in (s,T]$. (Note that $t = s$ is not yet included in the statement.)

Step 4. From Step 1 we know that y^3 is a nonnegative optional process of class (D). Moreover, by the strong Markov property, we have for every r-stopping time $\sigma \le T$ that

$$\pi_{r,x}\, y_\sigma^3 = \pi_{r,x} \limsup_n \big|k^n(\sigma,T] - k(\sigma,T]\big| = 0.$$

And therefore, according to [9], A.1.1.E, page 116, the process y^3 is $\pi_{r,x}$-a.s. indistinguishable from zero. In other words, with $\pi_{r,x}$-probability 1, the process (s, ξ_s) passes only through points (s, y) such that $\pi_{s,y}$-almost surely, $k^n(t,T] \to_n k(t,T]$, for $t \in [s,T]$. $\square$

3. Key result: fdd continuity in $(\mathbf{\Phi}, k)$. After the preparations in the previous section, we turn to the continuous dependence of finite-dimensional

distributions of (ξ, Φ, k)-superprocesses on their regular branching mechanism Φ and branching functional k (Theorem 20). A key step in deriving this will be Proposition 34 describing the convergence of log-Laplace functionals for those starting points (r, x) such that $s \mapsto (s, \xi_s)$ will pass $\pi_{r,x}$-a.s. only through those points which preserve some moment and convergence properties of the branching functionals in the sense of Proposition 9. As an application we prove that (ξ, Φ, k)-superprocesses can fdd be approximated by "classical" superprocesses (Theorem 23).

3.1. *Basic assumptions: branching mechanism* Φ. Now we complement the basic Assumption 1 concerning the motion process ξ and branching functional k.

ASSUMPTION 13 (Branching mechanism Φ).

(f) (Branching mechanism) Φ is always a (local) branching mechanism of the form

$$\Phi(r, x, \lambda) = b^r(x)\lambda^2 + \int_0^\infty e(u\lambda)\, n(r, x, du), \qquad (r, x, \lambda) \in I \times E \times \mathsf{R}_+,$$

where $e(z) := \mathrm{e}^{-z} + z - 1$, where $0 \le b^r(x) \le 1$ is measurable in (r, x) and where n is a kernel satisfying the condition

$$0 \le \int_0^\infty u^2\, n(r, x, du) \le 1, \qquad (r, x) \in I \times E.$$

Here "kernel" means that n: $\mathsf{R}_+ \times E \to \mathscr{M}$ is measurable, where $\mathscr{M} = \mathscr{M}(0, \infty)$ is the set of all measures on the locally compact space $(0, \infty)$, finite on compact subsets, endowed with the topology of vague convergence (Polish space).

(g) (Regular Φ) Additionally, the branching mechanism Φ is often assumed to be regular in the following sense: if for each starting point (r, x) in $I \times E$ the process $s \mapsto z_s$ is nonnegative cadlag with $\pi_{r,x}$-probability 1, then so is $s \mapsto \Phi(s, \xi_s, z_s)$.

The following result is taken from Leduc [15], Theorem 1.2, which generalized Theorem 5.2.1 of [9] where the admissibility (4) on k was imposed rather than only the boundedness (3) of characteristic.

LEMMA 14 ("Unique" existence of the (ξ, Φ, k)-superprocess X). *The* (ξ, Φ, k)-*superprocess* X *exists, for each cadlag right Markov process* ξ, *branching mechanism* Φ *and branching functional* k. *More precisely, an* $\mathscr{M}_{\mathsf{f}}$-*valued* (*time-inhomogeneous*) *Markov process* $(X_t, \mathscr{F}, P_{r,\mu})$ *exists* [*in the sense of Assumption* 1(d1)] *with log-Laplace transition functional*

$$-\log P_{r,\mu} \exp\langle X_t, -f\rangle = \int v_{r,t}(f)(x)\, \mu(\mathrm{d}x), \tag{12}$$

$0 \le r \le t \le T$, $x \in E$, $f \in b\mathscr{E}_+$, *where* $v = v(f) = v_{\bullet,t}(f) \ge 0$ *solves the* (ξ, Φ, k)-*evolution equation*

$$v_{r,t}(f)(x) = \pi_{r,x} f(\xi_t) - \pi_{r,x} \int_{(r,t]} \Phi\big(s, \xi_s, v_{s,t}(\xi_s)\big)\, k(ds). \tag{13}$$

Uniqueness of the solution $v(\lambda f)$ to (13) can be either formulated for small nonnegative λ, or in terms of the analyticity of the map $\lambda \mapsto v(\lambda f)$, $\lambda \geq 0$. See, for instance, Proposition 32(i) or Section 3.10 below.

TERMINOLOGY 15. From now on, when we refer to (ξ, Φ, k)-superprocesses, we in particular assume that ξ is a cadlag right process, k a branching functional and Φ a branching mechanism, all according to our basic Assumptions 1 and 13. Moreover, since the log-Laplace transition functional (12) of the (ξ, Φ, k)-superprocess X is uniquely determined by v, for simplicity we call v the log-Laplace functional related to X (as we already did in Section 1).

REMARK 16 (Projection, criticality, total mass process). The motion process ξ of the (ξ, Φ, k)-superprocess X (which we consider in this paper) can be recovered by projection (expectation formula):

$$P_{r,\mu}\langle X_t, f\rangle = \pi_{r,\mu} f(\xi_t), \qquad 0 \leq r \leq t \leq T,\ \mu \in \mathscr{M}_{\mathrm{f}},\ f \in b\mathscr{E}_+.$$

This in particular implies that X is critical; that is, the total mass process $t \mapsto \langle X_t, 1\rangle$ is a martingale (with respect to the natural filtration of X).

REMARK 17 (Finite variances). The (present) (ξ, Φ, k)-superprocesses have (uniformly) finite second moments:

$$\sup_{r \leq t} P_{r,\mu}\langle X_t, 1\rangle^2 < \infty, \qquad t \in I,\ \mu \in \mathscr{M}_{\mathrm{f}}.$$

3.2. *The fdd joint continuity theorem.* The formulation of our main result will be based on the following definition.

DEFINITION 18 (Uniformly of bounded characteristic). If the branching functionals $k^1, \ldots, k^\infty = k$ satisfy

$$\bigvee_{n=1}^{\infty} \sup_{(r,x) \in I \times E} \pi_{r,x} k^n(r, T] < \infty, \tag{14}$$

they are said to be uniformly of bounded characteristic.

For convenience, we introduce the following assumption.

ASSUMPTION 19. Consider branching mechanisms $\Phi^1, \Phi^2, \ldots$ converging uniformly to a regular branching mechanism Φ. Moreover, consider branching functionals $k^1, \ldots, k^\infty = k$ being uniformly of bounded characteristic. Suppose that for every starting point $(r, x) \in I \times E$ and every r-stopping time $\sigma \leq T$ we know that $k^n(r, \sigma]$ converges to $k(r, \sigma]$ in $L^1(\pi_{r,x})$ as $n \to \infty$.

THEOREM 20 (Joint continuity in fdd). *If Assumption 19 is satisfied, the related log-Laplace functionals converge:*

$$v^n_{r,t}(f)(x) \to_n v_{r,t}(f)(x), \qquad 0 \leq r \leq t \leq T,\ x \in E,\ f \in b\mathscr{E}_+. \tag{15}$$

Consequently, the related superprocesses converge fdd.

For fixed branching functional k, the fdd continuity in the branching mechanism Φ can be sharpened by using a weaker convergence concept for Φ and by allowing nonregular limiting Φ.

PROPOSITION 21 (Fdd continuity in Φ only). *Fix a branching functional k. If the branching mechanisms Φ^n converge boundedly pointwise to the branching mechanism Φ as $n \to \infty$, then the related log-Laplace functionals v_n and v converge as expressed in (15).*

The proof of Theorem 20 requires some preparation, provided in the following subsections. We first consider the case $\Phi^n \equiv \Phi$. After some preliminaries, we prove the result in this case in Section 3.10 following the arguments given in [15], Proposition 4.20. Then in Section 3.11 we remove the $\Phi^n \equiv \Phi$ restriction by an approximation procedure.

The proof of Proposition 21 is postponed to Section 3.12.

The following example demonstrates that the requirement in Theorem 20 that the limiting Φ is regular cannot be dropped.

EXAMPLE 22 (Fdd discontinuity for a nonregular Φ). Let $I = [0, 1]$ and C the Cantor subset of I. Consider the following nonregular branching mechanism $\Phi(s, x, \lambda) :\equiv \lambda^2 1_{I\setminus \mathrm{C}}(s)$. That is, consider the "binary splitting," but only at time points s outside the Cantor set C. Let k denote a singularly continuous (with respect to Lebesgue measure) law on I with support C. Assume that k^n be (deterministic) absolutely continuous probability laws on I converging weakly to k as $n \to \infty$. Note that $\Phi(s, \xi_s, \lambda)\, k^n(ds) \equiv \lambda^2 k^n(ds)$, for any motion process ξ. Hence, the (ξ, Φ, k^n)-superprocess is precisely the (ξ, λ^2, k^n)-superprocess. Therefore, by Theorem 20, the (ξ, Φ, k^n)-superprocesses converge fdd to the (ξ, λ^2, k)-superprocess as $n \to \infty$, which is different from the (ξ, Φ, k)-superprocess. In fact, the (ξ, λ^2, k)-superprocess is nondegenerate, since it has nonzero variance: $\mathrm{Var}_{0,\, \delta_x} \langle X_1, 1\rangle \equiv 2k(I) = 2$. On the other hand, $\Phi(s, \xi_s, \lambda)\, k(ds) \equiv 0$. Thus, the (ξ, Φ, k)-superprocess is degenerate. In fact it is the deterministic mass flow according to the semigroup of the motion process. Summarizing, for this nonregular Φ, fdd continuity in k is violated.

3.3. *Application: fdd approximation by classical processes.* Before we come to the proofs of Theorem 20 and Proposition 21, we want to give an application of our continuity result. Indeed, we can use our fdd continuity Theorem 20 to show that all the (ξ, Φ, k)-superprocesses (of the present paper) with regular branching mechanism Φ can be approximated by superprocesses with a "classical" branching rate. Note that the approximating branching functionals k^n are in particular absolutely continuous with respect to the Lebesgue measure.

THEOREM 23 (Fdd approximation by classical processes). *Let Φ be a regular branching mechanism and k be a branching functional. Then there exist bounded measurable functions $\varrho^n\colon I \times E \to R_+$, $n \geq 1$, such that the (ξ, Φ, k^n)-*

superprocesses X^n with "classical" branching functional

$$k^n(ds) := \varrho_s^n(\xi_s)\,ds \tag{16}$$

converge fdd to the (ξ, Φ, k)-superprocess X as $n \to \infty$.

The proof of this theorem will be provided in Section 3.13.

3.4. *Convergence of branching functionals.* Next we want to reformulate the convergence of additive functionals occurring in Assumption 19.

PROPOSITION 24 (Convergence criterion for additive functionals). *Let $k^1, \dots, k^\infty = k$ be continuous additive functionals of ξ. Fix a time point $r \in I$, and a measure $\mu \in \mathscr{M}_{\mathrm{f}}$. The following two conditions are equivalent.*

(i) *$k^n(r, \sigma]$ converges to $k(r, \sigma]$ in $L^1(\pi_{r,\mu})$ as $n \to \infty$, for each r-stopping time $\sigma \le T$.*

(ii) *For every subsequence $\{k^{n_m}\}$ of $\{k^n\}$ there exists a subsequence $\{k^{n_{m_i}}\}$ of $\{k^{n_m}\}$ such that*

$$\pi_{r,\mu} \bigvee_{i=1}^{\infty} k^{n_{m_i}}(r, T] < \infty; \tag{α}$$

$$\sup_{s,t:\, r \le s \le t \le T} \bigl|k^{n_{m_i}}(s, t] - k(s, t]\bigr| \to 0, \qquad \pi_{r,\mu}\text{-a.e. as } i \to \infty \tag{β}$$

PROOF (i) $\Rightarrow$ (ii)(α). Let $\{k^{n_m}\}$ be a subsequence of $\{k^n\}$. Since $k^{n_m}(r, T]$ converges to $k(r, T]$ in $L^1(\pi_{r,\mu})$ as $m \to \infty$, it is uniformly integrable. Hence,

$$\pi_{r,\mu}\bigl(1\bigl\{k^{n_m}(r, T] > k(r, T] + 1\bigr\}k^{n_m}(r, T]\bigr) \to 0 \quad \text{as } m \to \infty.$$

By choosing a subsequence such that the above terms not only converge to zero but also form a convergent series, we get (ii)(α).

(i) $\Rightarrow$ (ii)(β). Let $\{k^{n_m}\}$ be a subsequence of $\{k^n\}$. With the use of Cantor's diagonalization method, one finds a subsequence $\{k^{n_{m_i}}\}$ such that

$$\bigl|k^{n_{m_i}}(r, q] - k(r, q]\bigr| \underset{i}{\to} 0 \text{ for every rational } q \in (r, T] \text{ and } q = T, \tag{17}$$

$\pi_{r,\mu}$-a.e. However, then, because the mappings $t \mapsto k^{n_{m_i}}(r, t]$ are nondecreasing, that implies that $\pi_{r,\mu}$-almost everywhere, $k^{n_{m_i}}(r, t] \to_i k(r, t]$ for all t in $(r, T]$.

(ii) $\Rightarrow$ (i). To show this implication, suppose that (i) is not verified. Then, for some r-stopping time $\sigma \le T$, it is possible to find an $\varepsilon > 0$ and a subsequence $\{k^{n_m}\}$ of $\{k^n\}$ such that for every m,

$$\pi_{r,\mu}\bigl|k^{n_m}(r, \sigma] - k(r, \sigma]\bigr| > \varepsilon. \tag{18}$$

On the other hand, according to (ii), it is possible to choose a subsequence $\{k^{n_{m_i}}\}$ of $\{k^{n_m}\}$ such that (ii)(α) and (ii)(β) are satisfied. Passing to differences, with Lebesgue's theorem this implies that $k^{n_{m_i}}(r, \sigma]$ converges to $k(r, \sigma]$ in

$L^1(\pi_{r,\mu})$. This obviously contradicts (18), and the proof of the proposition is finished. $\square$

For applications of our main Theorem 20 the following sufficient criterion for the convergence of additive functionals might be helpful. (The proof is left to the reader.)

LEMMA 25 (Sufficient criterion). *Let $k^1, \ldots, k^\infty = k$ be branching functionals which are uniformly of bounded characteristic. Fix $r \in I = [0, T]$ and $\mu \in \mathscr{M}_{\mathrm{f}}$. Let $\pi_{r,\mu}$-almost everywhere k^n weakly converge to k as $n \to \infty$. Then the assertions* (i) *and* (ii) *in Proposition 24 hold.*

3.5. *Review: the log-Laplace characterization of (ξ, Φ, k)-superprocesses.* For convenience, here we review the log-Laplace functional characterization of (ξ, Φ, k)-superprocesses and some related facts on log-Laplace functionals; the latter are versions of Proposition 4.20 and Lemmas 4.23, 4.25 and 4.26 in [15].

LEMMA 26 (Log-Laplace characterization). *Suppose that $f \mapsto v_{r,t}(f)(x)$, $f \in b\mathscr{E}_+$, is the log-Laplace functional of an $\mathscr{M}_{\mathrm{f}}$-valued random measure, for every choice of $0 \le r \le t \le T$ and $x \in E$. Moreover, let $x \mapsto v_{r,t}(f)(x)$ be measurable. Finally, let $\{v_{r,t}: 0 \le r \le t \le T\}$ form a semigroup on $b\mathscr{E}_+$:*

$$v_{r,s}\big(v_{s,t}(f)\big)(x) = v_{r,t}(f)(x), \qquad 0 \le r \le s \le t \le T,\ x \in E,\ f \in b\mathscr{E}_+. \tag{19}$$

Then there exists a unique (in the sense of finite-dimensional distributions) $\mathscr{M}_{\mathrm{f}}$-valued Markov process X with log-Laplace functional v [recall (12)].

For $c > 0$, let us introduce the following set:

$$b\mathscr{E}_+^c := \big\{f \in b\mathscr{E}_+ : f \le c\big\}. \tag{20}$$

LEMMA 27 (Continuity in f). *Let Φ be any branching mechanism. Fix $t \in I$ and $\delta > 0$. Let $(r, x) \to v_{r,t}f(x)$ be a nonnegative solution of the (ξ, Φ, k)-evolution equation (13), for each $f \in b\mathscr{E}_+^{2\delta}$. Moreover, let $f \mapsto v_{\bullet,t}(f)$ be increasing. Then, for each $(r, x) \in [0, t] \times E$ fixed, the functional $f \mapsto v_{r,t}(f)(x)$ is continuous on $b\mathscr{E}_+^\delta$ (in the topology of bounded pointwise convergence induced by $b\mathscr{E}_+$).*

LEMMA 28 (Convergence of Laplace functionals). *Assume that L_{P_n} is the Laplace functional of some $\mathscr{M}_{\mathrm{f}}$-valued random variable, for each $n \ge 1$. Suppose there exists $\delta > 0$ such that $L_{P_n}(f) \to L(f)$ as $n \to \infty$, for every $f \in b\mathscr{E}_+^\delta$ and that L is continuous on that set. Then there exists an extension of L to all of $b\mathscr{E}_+$ and a probability measure P_∞ on $\mathscr{M}_{\mathrm{f}}$ such that L is the Laplace functional of P_∞ and $L_{P_n}(f) \to L(f)$ as $n \to \infty$, for every f in $b\mathscr{E}_+$.*

LEMMA 29 (Semigroup property of solutions). *Suppose*

$$f \mapsto \langle \mu, v_{r,t}(f) \rangle, \qquad f \in b\mathscr{E}_+,$$

is the log-Laplace functional of an $\mathscr{M}_{\mathrm{f}}$-valued random measure, for every choice of $0 \le r \le t \le T$ and $\mu \in \mathscr{M}_{\mathrm{f}}$. Moreover, let Φ be a branching mechanism, k be a branching functional, and let $(r, x) \to v_{r,t} f(x)$ solve the (ξ, Φ, k)-evolution equation (13), for each $t \in I$ and $f \in b\mathscr{E}_+$ fixed. Then the semigroup property (19) holds.

3.6. *Solutions to the evolution equation in the case of small f.* By a slight abuse of notation, we adopt the following convention.

CONVENTION 30. For convenience, we will often write $\|g(r,x)\|_\infty$ instead of $\|g(\cdot,\cdot)\|_\infty = \sup_{r,x} |g(r,x)|$. That is, even though the time space variable (r,x) in $I \times E$ appears under the norm sign, the supremum is always taken over them, even if extra parameters are involved.

The following lemma is taken from [15], Lemma 4.21.

LEMMA 31 (Local Lipschitz continuity). *Let Φ be a branching mechanism. Then, $\Phi(r, x, 0) \equiv 0$. Moreover, for every $c > 0$ and $\lambda_1, \lambda_2 \in [0, c]$,*

$$\big\|\Phi(r, x, \lambda_1) - \Phi(r, x, \lambda_2)\big\|_\infty \le 3\,c|\lambda_1 - \lambda_2|. \tag{21}$$

Finally, if $0 \le \lambda_1 \le \lambda_2$ then $0 \le \Phi(r, x, \lambda_1) \le \Phi(r, x, \lambda_2)$, $(r, x) \in I \times E$.

As a first step toward the proof of our main theorem, here we want to give an independent construction of a solution to the (ξ, Φ, k)-evolution equation (13) in the case of small f.

PROPOSITION 32 (Solution for small f). *Fix $t \in I$, a regular branching mechanism Φ and a branching functional k. Let $\delta > 0$ satisfy*

$$3\,\delta \sup_{(r,x)\in[0,t]\times E} \pi_{r,x} k(r, t] \le \tfrac{1}{2}. \tag{22}$$

Then, for $f \in b\mathscr{E}_+^\delta$, we have the following.

(i) (Unique existence) *A unique measurable function $v_{\bullet,t}(f) \ge 0$ exists which solves the (ξ, Φ, k)-evolution equation (13).*

(ii) (Cadlag regularity) *The process $s \mapsto v_{s,t}(f)(\xi_s)$, $s \in [r, t]$, is cadlag $\pi_{r,x}$-a.s., for every starting point $(r, x) \in [0, t] \times E$.*

PROOF. Fix t, Φ, k, f as in the proposition. Let $\mathscr{B}^{t,\delta}$ be the set of all measurable mappings u from $[0, t] \times E$ to $[0, \delta]$ such that $s \mapsto u_s(\xi_s)$ is cadlag. Equipped with the metric generated by the supremum norm $\|\cdot\|_\infty$, this is a complete metric space. Define an operator G on $\mathscr{B}^{t,\delta}$ by

$$G(u)(r, x) := \pi_{r,x} f(\xi_t) - \pi_{r,x} f(\xi_t) \wedge \pi_{r,x} \int_{(r,t]} \Phi(s, \xi_s, u_s(\xi_s))\, k(ds).$$

We want to show that G maps $\mathscr{B}^{t,\delta}$ into $\mathscr{B}^{t,\delta}$. Let $\sigma_n \le t$ be nondecreasing r-stopping times converging to σ as $n \to \infty$. Only by the Markov property,

$$\pi_{r,x}\pi_{\sigma_n,\xi_{\sigma_n}} f(\xi_t) \equiv \pi_{r,x}\pi_{\sigma,\xi_\sigma} f(\xi_t).$$

Similarly, together with monotone convergence, we get

$$\begin{aligned}\lim_{n\to\infty} \pi_{r,x}\pi_{\sigma_n,\xi_{\sigma_n}} \int_{(\sigma_n,t]} \Phi(s,\xi_s,u_s(\xi_s))\,k(ds)\\ = \pi_{r,x}\pi_{\sigma,\xi_\sigma} \int_{(\sigma,t]} \Phi(s,\xi_s,u_s(\xi_s))\,k(ds).\end{aligned}$$

By [9], A.1.1.D, page 116 and our Lemma A1, this establishes that the processes

$$s \mapsto \pi_{s,\xi_s} f(\xi_t) \quad \text{and} \quad s \mapsto \pi_{s,\xi_s} \int_{(s,t]} \Phi\big(s',\xi_{s'},u_{s'}(\xi_{s'})\big)\,k(ds')$$

are cadlag $\pi_{r,x}$-a.s. for every starting point $(r,x) \in I \times E$. Thus

$$\lim_{n\to\infty} \pi_{r,x} G(u)(\sigma_n,\xi_{\sigma_n}) = \pi_{r,x} G(u)(\sigma,\xi_\sigma),$$

showing that $s \mapsto G(u)(s,\xi_s)$ is cadlag. Hence, G maps $\mathscr{B}^{t,\delta}$ into itself.

Let z^1 and z^2 be two mappings in $\mathscr{B}^{t,\delta}$. From (21), we get

$$\big|\Phi\big(s,\xi_s,z_s^1(x)\big) - \Phi\big(s,\xi_s,z_s^2(x)\big)\big| \le 3\,\delta \|z^1 - z^2\|_\infty. \tag{23}$$

Thus,

$$\begin{aligned}\big|G(z^1)(r,x) - G(z^2)(r,x)\big| &\le 3\,\delta\,\pi_{r,x} \int_{(r,t]} \|z^1 - z^2\|_\infty\,k(ds)\\ &\le 3\,\delta \|z^1 - z^2\|_\infty \sup_{r,x} \pi_{r,x} k(r,t]\\ &\le \tfrac{1}{2}\,\|z^1 - z^2\|_\infty,\end{aligned} \tag{24}$$

where we used (22). Hence, G is a *contraction* on $\mathscr{B}^{t,\delta}$. By the Banach fixed point theorem, there exists a (unique) element u in $\mathscr{B}^{t,\delta}$ which solves

$$u_r(x) = G(u)(r,x) = \pi_{r,x} f(\xi_t) - \pi_{r,x} f(\xi_t) \wedge \pi_{r,x} \int_{(r,t]} \Phi(s,\xi_s,u_s(\xi_s))\,k(ds)$$

on $I \times E$. Let us now show that, indeed, u solves (13). To do this, let

$$\sigma^r := \inf\left\{ s \in (r,t] \colon \pi_{s,\xi_s} \int_{(s,t]} \Phi(s',\xi_{s'},u_{s'}(\xi_{s'}))\,k(ds') \le \pi_{s,\xi_s} f(\xi_t) \right\}.$$

Note that $u_s(\xi_s) = G(u)(s,\xi_s) = 0$ for $s \in (r,\sigma^r]$, hence $\Phi(s,\xi_s,u_s(\xi_s))$ vanishes for those s. Thus, using the strong Markov property, we are allowed to write

$$u_r(x) = \pi_{r,x} f(\xi_t) - \pi_{r,x} f(\xi_t) \wedge \pi_{r,x}\pi_{\sigma^r,\xi_{\sigma^r}} \int_{(\sigma^r,t]} \Phi\big(s,\xi_s,u_s(\xi_s)\big)\,k(ds),$$

for all r, x. But, by definition of σ^r,

$$\pi_{r,x}\pi_{\sigma^r,\xi_{\sigma^r}} \int_{(\sigma^r,t]} \Phi(s,\xi_s,u_s(\xi_s))\,k(ds) \le \pi_{r,x}\pi_{\sigma^r,\xi_{\sigma^r}} f(\xi_t) = \pi_{r,x} f(\xi_t).$$

Consequently,

$$\pi_{r,x} f(\xi_t) \wedge \pi_{r,x} \int_{(r,t]} \Phi(s, \xi_s, u_s(\xi_s))\, k(ds) = \pi_{r,x} \int_{(r,t]} \Phi(s, \xi_s, u_s(\xi_s))\, k(ds).$$

Therefore, u solves (13), proving the existence part of the proposition.

Assume now we have two nonnegative solutions u^1 and u^2.

Estimate the differences of related right-hand sides of (13) as in (23) and (24), and uniqueness follows. This completes the proof. □

3.7. *Special notation.* For convenience, we introduce the following special notation.

NOTATION 33. Consider a regular branching mechanism Φ and branching functionals $k^1, \dots, k^\infty = k$ of uniformly bounded characteristic. For $n \geq 1$, let v^n denote the log-Laplace functional related to the (ξ, Φ, k^n)-superprocess.

(i) (Nice starting points) Denote by $C = C(k^1, \dots, k^\infty)$ the set of all points $(r, x) \in [0, T] \times E$ such that

$$(\alpha) \qquad \pi_{r,x} \bigvee_{n=1}^{\infty} k^n(r, T] < \infty;$$

$$(\beta) \qquad \pi_{r,x}\text{-a.s., } k^n(s, t] \to_n k(s, t] \text{ whenever } r \leq s \leq t \leq T.$$

(ii) (Special norm) For any mapping h: $[0, T] \times E \to \mathsf{R}$, we set

$$\|h(r, x)\|_C := \sup_{(r,x) \in C} |h(r, x)|$$

(applying the Convention 30 introduced for $\|\cdot\|_\infty$ analogously to $\|\cdot\|_C$).

(iii) For $t \in I$ and $f \in b\mathscr{E}_+$ fixed, for $n \geq 1$ and $r \in I$ we pose

$$v_r^n := v_{r,t}^n(\xi_r); \qquad v_r := v_{r,t}(\xi_r);$$
$$\Phi_r^n := \Phi(r, \xi_r, v_{r,t}^n(\xi_r)); \qquad \Phi_r := \Phi(r, \xi_r, v_{r,t}(\xi_r));$$
$$S_r^n := \sup_{l \geq n} \left| \int_{(r,t]} \Phi_s^l\, k^l(ds) - \int_{(r,t]} \Phi_s\, k(ds) \right|,$$

reading such quantities as 0 if $r > t$.

(iv) B will denote the following supremum expression:

$$\sup_{t \in I} \left\{ \left\| \pi_{r,x} \lim_n S_r^n \right\|_C \vee \sup_l \left\| \pi_{r,x} \left| \int_{(r,t]} \Phi_s^l\, k^l(ds) - \int_{(r,t]} \Phi_s\, k(ds) \right| \right\|_\infty \right\}.$$

3.8. *Key step: convergence of log-Laplace functionals for nice starting points.* The central part in deriving our key result is the following proposition concerning the convergence of log-Laplace functionals for small test functions f, and for starting points in C (guaranteeing some convergence of the functionals k^n).

PROPOSITION 34 (Convergence starting in C). *Consider a regular branching mechanism Φ and branching functionals $k^1, \dots, k^\infty = k$ which are uniformly of bounded characteristic. Let $f \in b\mathscr{E}_+$ be such that*

$$3\,\|f\|_\infty \|\pi_{r,x} k(r,T]\|_\infty \le \tfrac{1}{2}. \tag{25}$$

Then for the log-Laplace functionals $v^n(f) = v^n$, $n \ge 1$, of (12) related to $k^1, k^2, \dots,$ respectively, we have

$$\lim_n v^n_{r,t}(x) = v_{r,t}(x), \qquad (r,x) \in C,\ t \in [r,T],$$

with $v = v(f)$ the (unique) "small solution" of the (ξ, Φ, k)-evolution equation (13) constructed in Proposition 32.

PROOF. For r, x, t as in the proposition, we clearly have

$$\left|v^n_{r,t}(x) - v_{r,t}(x)\right| \le \pi_{r,x}\left|\int_{(r,t]} \Phi^n_s\, k^n(ds) - \int_{(r,t]} \Phi_s\, k(ds)\right|$$

and thus

$$\left|v^n_{r,t}(x) - v_{r,t}(x)\right| \le \pi_{r,x} S^n_r. \tag{26}$$

Assume for the moment that we have already showed the following statement:

$$\lim_n \pi_{r,x} S^n_r = 0 \quad \text{for all } (r,x) \in C \text{ and } t \in [r,T]. \tag{27}$$

Then (26) will establish the claim in Proposition 34. □

It remains to verify (27). Start with the following fact.

LEMMA 35. *We have $B < \infty$.*

PROOF. From the definition of Φ in Assumption 13(f) we obtain

$$\|\Phi(r,x,\lambda)\|_\infty \le \tfrac{3}{2}\,\lambda^2, \tag{28}$$

since $0 \le e(z) \le z^2/2$, $z \ge 0$. Recall that the log-Laplace functionals v^n solve the (ξ, Φ, k^n)-evolution equation (13) (with k replaced by k^n). Hence,

$$0 \le v^n_{r,t}(f)(x) \le \|f\|_\infty. \tag{29}$$

Using this domination, altogether we get the estimate

$$\left|\int_{(r,t]} \Phi^l_s\, k^l(ds) - \int_{(r,t]} \Phi_s\, k(ds)\right| \le \tfrac{3}{2}\,\|f\|^2_\infty (k^l(r,T] + k(r,T]). \tag{30}$$

Taking the $\pi_{r,x}$-expectation, the finiteness of the second part in the definition of B immediately follows from (14). On the other hand, for the first part, take the supremum on $l \geq n$ and the limit as $n \to \infty$ of the r.h.s. of (30) to get $3\,\|f\|_\infty^2\, k(r,T]$ with $\pi_{r,x}$-probability 1, for each $(r,x) \in C$. Hence,

$$\Big\|\pi_{r,x} \lim_n S_r^n\Big\|_C \leq \text{const}\|\pi_{r,x} k(r,T]\|_\infty$$

which is finite, again by (14). $\square$

We also need the following simple fact.

LEMMA 36 (Convergence of functionals). *Fix a starting point* $(r,x) \in C$ *[with* C *defined in Notation* 33(i)*] and* $t \in [r,T]$. *For* $s \in [r,t]$, *let* ψ_s *denote* $\Im[s,t]$*-measurable nonnegative variables, and let* $s \mapsto \psi_s$ *be* $\pi_{r,x}$*-indistinguishable from a cadlag process, bounded by a (nonrandom) constant. Then,*

$$\int_{(r,t]} \psi_s\, k^l(ds) \underset{l}{\to} \int_{(r,t]} \psi_s\, k(ds)$$

with $\pi_{r,x}$*-probability 1.*

The proof immediately follows from [1], Theorem 5.1.

3.9. *Proof of* (27).

Step 0. For the moment, fix $t \in I$. For $n \geq 0$, $r \in [0,t]$ and $x \in E$, set

$$o_{r,x}^n := \pi_{r,x} \sup_{l \geq n} \left| \int_{(r,t]} \Phi_s\, k^l(ds) - \int_{(r,t]} \Phi_s\, k(ds) \right|. \tag{31}$$

Just as we derived (30),

$$\sup_{l \geq n} \left| \int_{(r,t]} \Phi_s\, k^l(ds) - \int_{(r,t]} \Phi_s\, k(ds) \right| \leq 3\|f\|_\infty^2 \bigvee_{l=1}^{\infty} k^l(r,t] \in L^1(\pi_{r,x}).$$

Therefore, we can invoke Lebesgue's theorem, Proposition 32(ii), the regularity of Φ and Lemma 36 to obtain that, for every $(r,x) \in C$, $r \leq t$,

$$\lim_n o_{r,x}^n = 0.$$

Step 1. We next establish that, for $(r,x) \in C$, $r \leq t$ and $n \geq m$,

$$\pi_{r,x} S_r^n \leq 3\|f\|_\infty \pi_{r,x} \left(\sup_{l \geq n} \int_{(r,t]} B \wedge (\pi_{s,\xi_s} S_s^m) k^l(ds) \right) + o_{r,x}^n. \tag{32}$$

In fact, we have

$$S_r^n \leq \sup_{l \geq n} \left| \int_{(r,t]} (\Phi_s^l - \Phi_s)\, k^l(ds) \right| + \sup_{l \geq n} \left| \int_{(r,t]} \Phi_s\, k^l(ds) - \int_{(r,t]} \Phi_s\, k(ds) \right|,$$

and therefore [by (31)],

$$\pi_{r,x} S_r^n \leq \pi_{r,x} \left(\sup_{l \geq n} \left| \int_{(r,t]} (\Phi_s^l - \Phi_s)\, k^l(ds) \right| \right) + o_{r,x}^n.$$

Using the Lipschitz inequality (21) and domination (29) we can continue with

$$\pi_{r,x} S_r^n \le 3\|f\|_\infty \pi_{r,x}\left(\sup_{l\ge n} \int_{(r,t]} \left|v_s^l - v_s\right| k^l(ds)\right) + o_{r,x}^n$$

and thus, from (26),

$$\pi_{r,x} S_r^n \le 3\|f\|_\infty \pi_{r,x}\left(\sup_{l\ge n} \int_{(r,t]} B \wedge \left(\pi_{s,\xi_s} S_s^l\right) k^l(ds)\right) + o_{r,x}^n. \tag{33}$$

However, for $l \ge n \ge m$, we have $S^l \le S^n \le S^m$ and (33) yields (32).

Step 2. We will now derive from (32) that for $(r,x) \in C$ and $t \in [r,T]$ fixed,

$$\pi_{r,x} \lim_n S_r^n \le 3\|f\|_\infty \pi_{r,x}\left(\int_{(r,t]} B \wedge \left(\pi_{s,\xi_s} \lim_n S_s^n\right) k(ds)\right). \tag{34}$$

Indeed, $s \mapsto B \wedge \pi_{s,\xi_s} S_s^m$ is cadlag $\pi_{r,x}$-a.s., according to Lemma 12. Therefore, in view of Lemma 36,

$$\int_{(r,t]} B \wedge \left(\pi_{s,\xi_s} S_s^m\right) k^l(ds) \underset{l}{\to} \int_{(r,t]} B \wedge \left(\pi_{s,\xi_s} S_s^m\right) k(ds) \tag{35}$$

with $\pi_{r,x}$-probability 1. Note that

$$0 \le \sup_{l\ge n} \int_{(r,t]} B \wedge \left(\pi_{s,\xi_s} S_s^m\right) k^l(ds) \le B \bigvee_{l=1}^{\infty} k^l(r,T] \in L^1(\pi_{r,x}).$$

Hence, from monotone convergence, inequality (32), Lebesgue's theorem and (35), we get

$$\pi_{r,x} \lim_n S_r^n = \lim_n \pi_{r,x} S_r^n \le 3\|f\|_\infty \pi_{r,x}\left(\int_{(r,t]} B \wedge \left(\pi_{s,\xi_s} S_s^m\right) k(ds)\right).$$

Passing to the monotone limit as $m \to \infty$, this yields (34).

Step 3. We will show that (34) implies

$$\left\|\pi_{r,x} \lim_n S_r^n\right\|_C \le 3\|f\|_\infty \left\|\pi_{r,x}\, k(r,T]\right\|_\infty \left\|\pi_{r,x} \lim_n S_r^n\right\|_C. \tag{36}$$

In fact, according to Proposition 9, for every point $(r,x) \in C$,

$$\pi_{r,x}\left\{(s,\xi_s) \in C \text{ for every } s \in [r,T]\right\} = 1.$$

Moreover, for any point $(r,x) \in C$, we have, by definition of B, that

$$B \wedge \pi_{r,x} \lim_n S_r^n = \pi_{r,x} \lim_n S_r^n.$$

Hence, for any point $(r,x) \in C$, inequality (34) implies that

$$\pi_{r,x} \lim_n S_r^n \le 3\|f\|_\infty \left\|\pi_{r,x} \lim_n S_r^n\right\|_C \pi_{r,x}\, k(r,T].$$

Taking the supremum over $(r,x) \in C$, we obtain (36).

Step 4. Recall that according to Lemma 35, $\|\pi_{r,x} \lim_n S_r^n\|_C \le B < \infty$. Using assumption (25), therefore (36) implies that $\|\pi_{r,x} \lim_n S_r^n\|_C = 0$, and, in particular, $\pi_{r,x} \lim_n S_r^n = 0$ for r, x, t as considered in the lemma. By monotone convergence, this completes the proof of (27). □

3.10. *Final steps of proof of fdd continuity if* $\Phi^n \equiv \Phi$. Here we complete the proof of Theorem 20 in the case $\Phi^n \equiv \Phi$. Consider branching functionals $k^1, \dots, k^\infty = k$ which are uniformly of bounded characteristic. Let $f \in b\mathscr{E}_+$ satisfy the smallness property (25). Fix a starting point $(r, x) \in I \times E$. Consider a subsequence $\{k^{n_m}\}$ of $\{k^n\}$. By Assumption 19, and by the convergence criterion Proposition 24, there exists a subsequence $\{k^{n_{m_i}}\}$ of $\{k^{n_m}\}$ such that (α) and (β) in (ii) of this proposition hold. We conclude that (r, x) belongs to the set C introduced in Notation 33(i), related to this sequence $\{k^{n_{m_i}}\}$. By Proposition 34, we then get that $v_{r,t}^{n_{m_i}}(f)(x)$ converges to $v_{r,t}(f)(x)$ as $i \to \infty$ for each $t \in [r, T]$, with $v_{\bullet,t}(f)$ the (unique) small solution to (13). Hence, the limit is independent of the choice of the subsequences, and we get the latter convergence statement along the whole sequence $\{k^n\}$.

But each $v_{r,t}^n(f)(x)$ is monotone as a functional of f satisfying assumption (25) (since it is a log-Laplace functional), and therefore this property is shared by $v_{r,t}(f)(x)$. According to Lemma 27, the mapping $f \mapsto v_{r,t}(f)(x)$ must then be continuous, for all sufficiently small f. As a consequence, Lemma 28 implies that $v_{r,t}^n(f)(x)$ converges to some $v_{r,t}(f)(x)$ as $n \to \infty$, for any f in $b\mathscr{E}_+$, where $v_{r,t}(\cdot)(x)$ is the log-Laplace functional of some random measure. In order to finish the proof, it suffices to show according to Lemma 26 that the family $\{v_{r,t}: 0 \le r \le t \le T\}$ determines a semigroup on $b\mathscr{E}_+$, and that in fact $v_{\bullet,t}(f)$ solves the (ξ, Φ, k)-evolution equation (13).

Recall that $v_{\bullet,t}(f)$ solves (13) for f small in the sense of (25). On the other hand, for any $f \in b\mathscr{E}_+$, the mapping $\theta \mapsto v_{r,t}(\theta f)(x)$ is analytic on the half line $(0, \infty)$, since $\exp(-v_{r,t}(\cdot)(x))$ is a Laplace functional. By replacing f by θf, we get that both sides of the (ξ, Φ, k)-evolution equation (13) are analytic mappings of θ (since Φ is analytic in its third variable, and by the imposed moment assumptions). Since both sides of (13) coincide for small values of θ, by the uniqueness of analytic continuation, they are hence equal for every θ. Specializing to $\theta = 1$, this shows that $v_{\bullet,t}(f)$ solves (13) not only for small f but in fact for every $f \in b\mathscr{E}_+$. Since (r, x) is arbitrary, by Lemma 29, the semigroup property (19) holds, and the proof is complete. $\square$

3.11. *Extension to fdd joint continuity.* To complete the proof of Theorem 20, we have to remove the $\Phi^n \equiv \Phi$ restriction. Consider $\Phi^1, \dots, \Phi^\infty = \Phi$ and $k^1, \dots, k^\infty = k$ as in Assumption 19. Fix $f \in b\mathscr{E}_+$. Write $v^{n,m} = v^{n,m}(f)$ for the log-Laplace functional related to Φ^n, k^m, where $n, m = 1, \dots, \infty$. For $0 \le r \le t \le T$ and $x \in E$, consider

$$\left|v_{r,t}^{n,n}(x) - v_{r,t}^{\infty,n}(x)\right|. \tag{37}$$

We use the abbreviation $\Phi^i(v^{n,m})$ for $\Phi^i(r, \xi_r, v_{r,t}^{n,m}(\xi_r))$, where $i, n, m = 1, \dots, \infty$. In view of the evolution equation (13), we obtain the following upper bound of (37):

$$\pi_{r,x} \int_{(r,t]} \left|\Phi^n(v^{n,n}) - \Phi^\infty(v^{\infty,n})\right| k^n(ds).$$

Compare now both terms in the latter formula line with $\Phi^n(v^{\infty,n})$. In the first case, by the Lipschitz property (21) and the domination (29), we get the bound

$$3\|f\|_\infty \big\|v^{n,n}_{\bullet,t} - v^{\infty,n}_{\bullet,t}\big\|_\infty \pi_{r,x} k^n(r,t].$$

The other part is bounded by $\|\Phi^n - \Phi^\infty\|_\infty \pi_{r,x} k^n(r,t]$. Since all the branching functionals are uniformly of bounded characteristic and $\Phi^n \to \Phi$ in uniform convergence, putting both together, for $\|f\|_\infty$ small enough we get

$$\lim_{n\to\infty} \big\|v^{n,n}_{\bullet,t} - v^{\infty,n}_{\bullet,t}\big\|_\infty = 0.$$

However, $v^{\infty,n}_{s,t}(x)$ converges pointwise to $v^{\infty,\infty}_{s,t}(x)$ as $n \to \infty$, hence $v^{n,n}_{s,t}(x)$ approaches $v^{\infty,\infty}_{s,t}(x)$ as $n \to \infty$, too, for all sufficiently small f. By Lemma 28, this extends to all $f \in b\mathscr{E}_+$, completing the proof of Theorem 20. □

REMARK 37 (Indexed sequences of branching functionals). In the beginning of Section 3.10, we fixed a starting point (r,x), constructed $v_{r,t}(f)(x)$, for any t and f, and verified the properties we needed. Note that all the arguments would work, if the sequence of branching functionals $k^1, k^2, \dots$ we started from depended on (r,x), provided that only the "limiting" $k^\infty = k$ is independent of (r,x). Hence, the fact that in Theorem 20 the sequence $\{k^n\}$ of branching functionals is assumed to be independent of the choice of the starting point r, x is not essential. One could consider a family $\{k^n_{r,x}\}$ of sequences indexed by (r,x), with the "limiting" $k^\infty = k$ independent of (r,x).

3.12. *Fdd continuity in only the branching mechanism.* The purpose of this subsection is to provide the proof of Proposition 21. First, note that the log-Laplace functionals v_n and v exist by Lemma 14. Set

$$\overline{v}_{r,t}(f)(x) := \limsup_n v^n_{r,t}(f)(x), \qquad \underline{v}_{r,t}(f)(x) := \liminf_n v^n_{r,t}(f)(x).$$

By the evolution equation (13), we have

$$\overline{v}_{r,t}(f)(x) = \pi_{r,x} f(x) - \liminf_n \pi_{r,x} \int_r^t \Phi^n\big(s, \xi_s, v^n_{s,t}(f)(\xi_s)\big)\, k(ds).$$

Since Φ is nondecreasing in its third variable, for each $M \ge 1$, we may continue with

$$\le \pi_{r,x} f(x) - \liminf_n \pi_{r,x} \int_r^t \Phi^n\Big(s, \xi_s, \inf_{m\ge M} v^m_{s,t}(f)(\xi_s)\Big)\, k(ds),$$

which equals

$$\pi_{r,x} f(x) - \pi_{r,x} \int_r^t \Phi\Big(s, \xi_s, \inf_{m\ge M} v^m_{s,t}(f)(\xi_s)\Big) k(ds).$$

Letting $M \to \infty$, we conclude that

$$\overline{v}_{r,t}(f)(x) \le \pi_{r,x} f(x) - \pi_{r,x} \int_r^t \Phi\big(s, \xi_s, \underline{v}_{s,t}(f)(\xi_s)\big)\, k(ds). \tag{38}$$

Analogously,

$$\underline{v}_{r,t}(f)(x) \geq \pi_{r,x} f(x) - \pi_{r,x} \int_r^t \Phi\big(s, \xi_s, \overline{v}_{s,t}(f)(\xi_s)\big)\, k(ds). \tag{39}$$

By the local Lipschitz Lemma 31, from (38) and (39) we get

$$\overline{v}_{r,t}(f)(x) - \underline{v}_{r,t}(f)(x) \leq 3\|f\|_\infty \pi_{r,x} \int_r^t \big(\overline{v}_{s,t}(f)(\xi_s) - \underline{v}_{s,t}(f)(\xi_s)\big) k(ds).$$

Hence,

$$\big\|\overline{v}_{r,t}(f)(x) - \underline{v}_{r,t}(f)(x)\big\|_\infty \leq 3\|f\|_\infty \big\|\overline{v}_{r,t}(f)(x) - \underline{v}_{r,t}(f)(x)\big\|_\infty \big\|\pi_{r,x} k(r,t]\big\|_\infty$$

(recall Convention 30). Thus, for functions f small enough, the limit of the l.h.s. in (15) exists. Repeating the argument with v instead of $\overline{v}$ and $\underline{v}$ we conclude that the inequalities (38) and (39) hold for v. That is, v solves the log-Laplace equation (12). By uniqueness [Proposition 32(ii)], we arrive at the desired limit $v(f)$ in (15), for these small f.

Now $v(f)$ is the limit of functionals which are monotone in f and is therefore monotone in f. The rest of the proof is identical to the arguments to our main Theorem in the end of Section 3.10. □

3.13. *Proof of the fdd approximation by classical processes.* For the proof of Theorem 23, by Theorem 20 it obviously suffices to verify the following lemma.

LEMMA 38 (Approximation by classical branching functionals). *Let k be a branching functional. Then there exist bounded measurable functions $\varrho^n: I \times E \to \mathsf{R}_+$, $n \geq 1$, such that the classical branching functionals $k^n(ds) = \varrho^n_s(\xi_s)\,ds$ of (16) are uniformly of bounded characteristic and have the following property: for every starting point $(r,x) \in I \times E$ and every r-stopping time $\sigma \leq T$ fixed, $k^n(r,\sigma]$ converges to $k(r,\sigma]$ in $L^1(\pi_{r,x})$ as $n \to \infty$.*

PROOF. Fix k, r, x as in the lemma. Consider $\pi_{r,x}$. To the branching functional k there corresponds the supermartingale

$$t \mapsto h_T^t(\xi_t) := \pi_{t,\xi_t} k(t,T], \qquad t \in [r,T],$$

with compensator $t \mapsto k(r,t]$. Following [6], Remark VII.22(b), we also consider the approximating sequence of supermartingales

$$\begin{aligned} t \mapsto {}^n h_T^t(\xi_t) &:= \pi_{t,\xi_t} n \int_0^{(1/n)\wedge(T-t)} h_T^{t+u}(\xi_{t+u})\,du \\ &= \pi_{t,\xi_t} n \int_0^{1/n} k(t+u, T]\,du \end{aligned}$$

with compensator

$$\begin{aligned} t \mapsto k^n(r,t] &:= n \int_r^t \bigg(h_T^s(\xi_s) - \pi_{s,\xi_s} k\bigg(s + \frac{1}{n}, T\bigg]\bigg)\,ds \\ &= n \int_r^t h^s_{(s+1/n)\wedge T}(\xi_s)\,ds, \qquad n \geq 1. \end{aligned} \tag{40}$$

Note that ${}^n h^t_T(\xi_t)$ increases to $h^t_T(\xi_t)$ as $n \to \infty$. It follows from Proposition 5(i) (with $Y_s = k(s, T]$) that $s \mapsto h^s_T(\xi_s)$ is $\pi_{r,x}$-indistinguishable from a nonnegative cadlag process of class (D). Moreover, for every r-stopping time $\sigma \le T$, by the strong Markov property,

$$\pi_{r,x}\big(h^\sigma_T(\xi_\sigma) - h^{(\sigma+\delta)\wedge T}_T(\xi_{(\sigma+\delta)\wedge T})\big) = \pi_{r,x}\big(k(\sigma, T] - k(\sigma+\delta, T]\big),$$

which converges to 0 as $\delta \downarrow 0$, uniformly in σ. In fact, $s \mapsto k(s, T]$ is uniformly continuous, and the integrand is bounded by $2k(r, T] \in L^1(\pi_{r,x})$. By Proposition A2 their uniform convergence to zero implies that

$$\pi_{r,x}\bigg\{\sup_{t\in[r,T]}\Big|h^t_T(\xi_t) - h^{(t+\delta)\wedge T}_T(\xi_{(t+\delta)\wedge T})\Big| > \varepsilon\bigg\} \to 0 \quad \text{as } \delta \downarrow 0,$$

for all $\varepsilon > 0$. Hence, for any sequence of r-stopping times $\sigma_n \le T$ and $\varepsilon > 0$,

$$\pi_{r,x}\bigg\{\Big|h^{\sigma_n}_T(\xi_{\sigma_n}) - h^{(\sigma_n+\delta)\wedge T}_T(\xi_{(\sigma_n+\delta)\wedge T})\Big| > \varepsilon\bigg\} \to 0 \quad \text{as } n \to \infty.$$

In other words, the process $t \mapsto h^t_T(\xi_t)$ satisfies Aldous's criterion, hence it is quasi-left continuous (see [13], Remark VI.4.7, page 321). We can then invoke Theorem VII.20 of [6] to conclude that $k^n(r, \sigma]$ defined in (40) converges to $k(r, \sigma]$ in $L^1(\pi_{r,x})$ as $n \to \infty$, for every r-stopping time $\sigma \le T$. Finally, it is easy to see that the k^n are uniformly of bounded characteristic (recall Definition 18). Altogether, the function

$$(s, x) \mapsto \varrho^n_s(x) := n\, h^s_{(s+1/n)\wedge T}(x)$$

entering into (40) satisfies all requirements. This completes the proof. $\square$

4. Special case: Feller $\boldsymbol{\xi}$ on a compactum. Since T is arbitrary, the (ξ, Φ, k)-superprocesses on the interval $I = [0, T]$ considered so far can easily be extended to the whole time half axis R_+. This we will actually do from now on. Of course, conditions as (3) and (14) are then required to hold for all $T > 0$.

Recall that a cadlag right Markov process $\xi = (\xi_t, \Im, \pi_{r,x})$ in a Luzin space is called a Hunt process if it is quasi-left continuous. That is, for $0 \le r \le T < \infty$ and $\mu \in \mathscr{M}_\mathsf{f}$ fixed, we have $\xi_{\sigma_n} \to_n \xi_\sigma$, $\pi_{r,\mu}$-a.e. for every sequence of r-stopping times $\sigma_n \le T$ nondecreasing to (the r-stopping time) σ as $n \to \infty$.

From now on we will pay attention to the following special case, although some of our results below—such as the existence of a Hunt version—can be extended to a more general situation by making use of Ray–Knight methods as exploited in [15]. However, this would require considerably more technical proofs, and the Feller case on a compact space perfectly illustrates our method.

Assumption 39 (Feller on a compactum). Suppose that the phase space is a compact metric space (E, d). Moreover, let ξ be time-homogeneous and indeed be a Feller process.

However note that the related (ξ, Φ, k)-superprocess is in general time-inhomogeneous.

Recall that we introduced in $\mathscr{M}_{\mathrm{f}} = \mathscr{M}_{\mathrm{f}}(\mathscr{E})$ the weak topology [Assumption 1(b)]. It can be generated by the Prohorov metric in the sense of [10], Problem 9.5.6, page 408, which we denote by w_d. Recall that $(\mathscr{M}_{\mathrm{f}}, w_d)$ is separable ([10], Theorem 3.1.7).

Moreover, for each $r \geq 0$ we will introduce the Skorohod spaces $\mathscr{D}_r = \mathscr{D}[[r,\infty), \mathscr{M}_{\mathrm{f}}]$, of all $\mathscr{M}_{\mathrm{f}}$-valued cadlag functions on $[r,\infty)$ equipped with the Skorohod metric s_d, based on d (actually on w_d). Recall that $(\mathscr{D}_r, s_d)$ is separable ([10], Theorem 3.5.6), since $\mathscr{M}_{\mathrm{f}}$ is separable.

4.1. *Results under the Feller assumption.* So far we have considered a (ξ, Φ, k)-superprocess only as a Markov process in the sense of Assumption 1(d1). Now we will be concerned with regularity properties of its (measure-valued) paths. In fact, in this section, under Assumption 39, we extend the fdd convergence results of Section 3 to convergence in law on path space. Also, we show that for our (ξ, Φ, k)-superprocesses a Hunt version exists.

THEOREM 40 (Existence of a Hunt version). *Impose Assumption 39. Let Φ be a branching mechanism and k be a branching functional. Then there exists a Hunt version of the (ξ, Φ, k)-superprocess.*

The proof of this theorem is postponed to Section 4.4.1.

As an application of the previous Theorem 40, using an argument from [9], Chapter 6, we show that under the present Feller assumption the (ξ, Φ, k)-superprocess is continuous exactly in the "binary splitting" case, regardless of the choice of the branching functional k.

COROLLARY 41 (Characterization of continuous processes). *Under the assumptions of Theorem* 40, *the* (*Hunt*) (ξ, Φ, k)-*superprocess X has almost surely continuous paths if and only if Φ has the form $\Phi(s,x,\lambda) = b^s(x)\lambda^2$* [*recall Assumption* 13(f)].

PROOF. Note that X is Hunt by the previous theorem. Moreover, X is almost surely continuous if and only if its modified Lévy measure vanishes, which occurs if and only if the projection of the latter ([9], Section 6.8.1) disappears. But this happens if and only if $n = 0$ in the definition of Φ [recall Assumption 13(f)]. □

Based on Theorem 40, our fdd continuity Theorem 20 can be sharpened in terms of convergence in law on Skorohod path spaces.

THEOREM 42 (Continuity in law on path spaces). *Under Assumptions* 39 *and* 19, *for r, μ fixed*, *the laws $P^n_{r,\mu}$ on the Skorohod space $\mathscr{D}_r$ of the Hunt (ξ, Φ^n, k^n)-superprocesses converge weakly towards the law $P_{r,\mu}$ of the Hunt (ξ, Φ, k)-superprocess.*

The proof of this theorem will follow in Section 4.4.2.

For fixed branching functional k, the continuity in the branching mechanism Φ can be sharpened by using a weaker convergence concept for Φ, just as in the fdd case (Proposition 21).

PROPOSITION 43 (Continuity on path spaces concerning Φ only). *Fix a branching functional k. If the branching mechanisms Φ^n converge boundedly pointwise to a (not necessarily regular) branching mechanism Φ as $n \to \infty$, then, under Assumption* 39, *the related superprocesses converge in law on the Skorohod path spaces $\mathscr{D}_r$.*

The proof of this result is postponed to Section 4.4.3.

We can combine Theorem 42 with Lemma 38 to conclude for the following approximation in law by classical superprocesses (detailed arguments will follow in Section 4.4.4).

THEOREM 44 (Approximation by classical processes). *Impose Assumption* 39.

(a) (Regular Φ) *If Φ is a regular branching mechanism, then, on Skorohod spaces $\mathscr{D}_r$, any (ξ, Φ, k)-superprocess X can be approximated in law by classical Hunt superprocesses X^n [based on classical branching functionals k^n as in* (16)].

(b) (Arbitrary Φ) *If Φ is an arbitrary branching mechanism, then, for every $r \geq 0$ and $\mu \in \mathscr{M}_{\mathsf{f}}$, there exists a collection of regular branching mechanisms Φ^n and classical branching functionals k^n such that the laws $P^n_{r,\mu}$ on $\mathscr{D}_r$ of the (ξ, Φ^n, k^n)-superprocesses X^n converge weakly to the law $P_{r,\mu}$ on $\mathscr{D}_r$ of the (ξ, Φ, k)-superprocess X.*

4.2. *A sufficient criterion for tightness on path space.* The proofs of the claims listed in Section 4.1 will be provided in a slightly different order. A basic step will be the verification of the following criterion, which extends a result from [15], Proposition 6.40. Write $\mathscr{C}_d(E)$ for the set of all nonnegative d-uniformly continuous functions defined on E.

PROPOSITION 45 (Tightness on path space). *Let $\Phi^1, \Phi^2, \ldots$ be a collection of branching mechanisms and let $k, k^1, k^2, \ldots$ be branching functionals which are uniformly of bounded characteristic (on bounded intervals). Assume that for each starting point $(r, x) \in \mathsf{R}_+ \times E$, each $T \geq r$ and each r-stopping time $\sigma \leq T$, we know that $k^n(r, \sigma]$ converges to $k(r, \sigma]$ in $L^1(\pi_{r,x})$ as $n \to \infty$. Suppose that each $X^n = (X^n, \mathscr{F}, P^n_{r,\mu})$ is a cadlag right (ξ, Φ^n, k^n)-superprocess, $n \geq 1$. Then, for $r \geq 0$ and $\mu \in \mathscr{M}_{\mathsf{f}}$ fixed, the laws $P^n_{r,\mu}$ of the X^n, as measures on the Skorohod space $\mathscr{D}_r$, are tight. Moreover, for $T \geq r$ and r-stopping times $\mathscr{T}_n$ bounded by T and $\delta_n \searrow 0$, we have*

$$\lim_{n\to\infty} P^n_{r,\mu} \left|\exp\langle X^n_{\mathscr{T}_n}, -f\rangle - \exp\langle X^n_{\mathscr{T}_n+\delta_n}, -f\rangle\right|^2 = 0, \tag{41}$$

for each $f \in \mathscr{C}_d(E)$.

To prepare for the proof, define $\mathbb{F}$ as the linear span of all functions F_f,

$$F_f(\mu) := \exp\langle \mu, -f \rangle, \qquad \mu \in \mathscr{M}_{\mathrm{f}},$$

where f varies in $\mathscr{C}_d(E)$.

LEMMA 46 (Separation of points). *Each $F \in \mathbb{F}$ is a bounded nonnegative continuous function on $\mathscr{M}_{\mathrm{f}}$. Moreover, $\mathbb{F}$ separates the points of $\mathscr{M}_{\mathrm{f}}$.*

PROOF. Note that $\mathbb{F}$ separates points if the collection of all functions $-\log F_f,\ f \in \mathscr{C}_d(E)$, is separating. Therefore, it suffices to show that $\mathscr{C}_d(E)$ separates the points of E ([10], Theorem 3.4.5(a)). But this is obvious (use d). $\square$

PROOF OF PROPOSITION 45. Fix $r \geq 0$ and μ in $\mathscr{M}_{\mathrm{f}}$, and consider the laws $P^n_{r,\mu}$ on $\mathscr{D}_r$ of the X^n, $n \geq 1$. We will use Jakubowski's criterion (see, e.g., [3], Theorem 3.6.4) to verify the tightness of these laws.

To check the first condition in Jakubowski's criterion, we show that the processes X^n "almost live" on a common compact subset of $\mathscr{M}_{\mathrm{f}}$. More precisely, we verify that for $T > r$ and $\varepsilon > 0$ fixed,

$$P^n_{r,\mu}\left(\sup_{s\in[r,T]} \langle X^n_s, 1\rangle > \frac{1}{\varepsilon}\right) \leq \varepsilon\langle \mu, 1\rangle, \qquad n \geq 1. \tag{42}$$

But using the Doob type inequality of Proposition A2, the l.h.s. can be estimated by

$$\leq \varepsilon \sup_{\mathscr{T}} P^n_{r,\mu}\langle X^n_{\mathscr{T}}, 1\rangle \tag{43}$$

with the supremum running over all r-stopping times $\mathscr{T} \leq T$. But the right superprocesses X^n are critical; hence the processes $t \to \langle X^n_t, 1\rangle$ are right-continuous martingales (recall Remark 16). So our estimate (43) equals $\varepsilon P^n_{r,\mu}\langle X^n_r, 1\rangle = \varepsilon\langle \mu, 1\rangle$, proving (42).

Next, for the second condition in Jakubowski's criterion, using the separation Lemma 46 it is sufficient to check the tightness of the laws of the cadlag processes $t \mapsto F_f(X^n_t)$, $n \geq 1$, on the Skorohod space $\mathscr{D}[[r,\infty), \mathsf{R}_+]$, for each fixed f in $\mathscr{C}_d(E)$. For this purpose, we use Aldous's criterion (see, for instance, [3], Theorem 3.6.5), from which we get that it suffices to show that, given $T \geq r$ and r-stopping times $\mathscr{T}_n$ bounded by T and $\delta_n \searrow 0$, claim (41) holds. Expanding the binomial in (41), we get, in particular, a term $\exp\langle X^n_{\mathscr{T}_n+\delta_n}, -2f\rangle$. Its $P^n_{r,\mu}$-expectation can be written as

$$P^n_{r,\mu} \exp\big\langle X^n_{\mathscr{T}_n}, -v^n_{\mathscr{T}_n,\mathscr{T}_n+\delta_n}(2f)\big\rangle,$$

using the strong Markov property at time $\mathscr{T}_n$, and the log-Laplace transition functional representation (12). Here $v^n(2f)$ solves the evolution equation (13) with f, Φ, k replaced by $2f, \Phi^n, k^n$, respectively. We will compare this term with

$$P^n_{r,\mu} \exp\big\langle X^n_{\mathscr{T}_n}, -2\, v^n_{\mathscr{T}_n,\mathscr{T}_n+\delta_n}(f)\big\rangle.$$

Calculating the other term similarly, for the expectation expression in (41) we get

$$P^n_{r,\mu}\big|\exp\langle X^n_{\mathscr{T}_n}, -f\rangle - \exp\langle X^n_{\mathscr{T}_n}, -v^n_{\mathscr{T}_n,\mathscr{T}_n+\delta_n}(f)\rangle\big|^2$$
$$+ P^n_{r,\mu}\big(\exp\langle X^n_{\mathscr{T}_n}, -v^n_{\mathscr{T}_n,\mathscr{T}_n+\delta_n}(2f)\rangle - \exp\langle X^n_{\mathscr{T}_n}, -2\,v^n_{\mathscr{T}_n,\mathscr{T}_n+\delta_n}(f)\rangle\big).$$

To get an upper bound for this, we may drop the exponent 2 and continue with

$$\le P^n_{r,\mu}\langle X^n_{\mathscr{T}_n}, \big|f - v^n_{\mathscr{T}_n,\mathscr{T}_n+\delta_n}(f)\big|\rangle$$
$$+ P^n_{r,\mu}\langle X^n_{\mathscr{T}_n}, \big|v^n_{\mathscr{T}_n,\mathscr{T}_n+\delta_n}(2f) - 2\,v^n_{\mathscr{T}_n,\mathscr{T}_n+\delta_n}(f)\big|\rangle.$$

Using again [9], Theorem 6.2.1, to each $\mathscr{T}_n$ there exists an r-randomized stopping time $\tau_n \le T$ for ξ such that the latter equals

$$\begin{aligned}&\pi_{r,\mu}\big|f(\xi_{\tau_n}) - v^n_{\tau_n,\tau_n+\delta_n}(f)(\xi_{\tau_n})\big|\\ &\quad + \pi_{r,\mu}\big|v^n_{\tau_n,\tau_n+\delta_n}(2f)(\xi_{\tau_n}) - 2\,v^n_{\tau_n,\tau_n+\delta_n}(f)(\xi_{\tau_n})\big|.\end{aligned} \tag{44}$$

Applying the evolution equation (13), and the strong Markov property for ξ, for the first term in (44) we get the bound

$$\begin{aligned}&\pi_{r,\mu}\big|f(\xi_{\tau_n}) - \pi_{\tau_n,\xi_{\tau_n}} f(\xi_{\tau_n+\delta_n})\big|\\ &\quad + \pi_{r,\mu}\int_{\tau_n}^{\tau_n+\delta_n} \Phi^n\big(s,\xi_s, v^n_{s,\tau_n+\delta_n}(f)(\xi_s)\big)k^n(ds).\end{aligned} \tag{45}$$

Since ξ is a time-homogeneous strong Markov process, the first term is bounded by $\langle\mu,1\rangle \sup_x |f(x) - \pi_{0,x} f(\xi_{\delta_n})|$, and by the Feller property this will disappear as $n \to \infty$. If now $\{k^{n_m}\}$ is a subsequence of $\{k^n\}$, by the reformulation Proposition 24, there exists a subsequence $\{k^{n_{m_i}}\}$ of $\{k^{n_m}\}$ such that

$$(\alpha)\qquad \pi_{r,\mu}\bigvee_{i=1}^{\infty} k^{n_{m_i}}(r,T] < \infty,$$

$$(\beta)\qquad \sup_{s\in[r,T]}\big|k^{n_{m_i}}(s,T] - k(s,T]\big| \to 0, \qquad \pi_{r,\mu}\text{-a.e. as } i\to\infty.$$

Combined with the uniform bound (28) of the Φ^n and (29), we get that the second term in (45) will vanish as $n_{m_i} \to \infty$, hence as $n \to \infty$. So (45) will disappear in the limit.

The proof that the second term in (44) goes to zero is similar. Consequently, (44) will vanish in the limit, hence (41) is true, and Jakubowski's criterion is fulfilled. □

COROLLARY 47 (Convergence on path space). *Suppose in addition to the hypotheses of Proposition 45 that the $X^n = (X^n_t, \mathscr{F}, P^n_{r,\mu})$ converge fdd to a (ξ,Φ,k)-superprocess X with a regular branching mechanism Φ. Then for each r,μ, the laws $P^n_{r,\mu}$ on $\mathscr{D}_r$ converge weakly to some distribution $P^\infty_{r,\mu}$.*

PROOF. Since tightness on path space plus fdd convergence implies weak convergence on path space, we immediately get from Proposition 45 and the assumed fdd convergence that $P^n_{r,\mu}$ converges weakly to some $P^\infty_{r,\mu}$ as $n \to \infty$. $\square$

4.3. *Existence of a cadlag right version* X. Recall that (E, d) is a compact metric space. For convenience, we introduce the following notion.

DEFINITION 48 (Almost sure notions). For the moment, consider an $\mathscr{M}_{\mathrm{f}}$-valued Markov process $X = (X_t, \mathscr{F}, P_{r,\mu})$ with phase space (E, d). We say that X is an a.s. cadlag right process if we have the following.

(i) For $r \geq 0$ and $\mu \in \mathscr{M}_{\mathrm{f}}$,

$$P_{r,\mu}\bigl\{t \to X_t \text{ is cadlag, } t \in [r, \infty)\bigr\} = 1$$

(which implicitly contains the measurability requirement);

(ii) For $0 \leq r < t$, for $\mu \in \mathscr{M}_{\mathrm{f}}$ and for measurable $F\colon \mathscr{M}_{\mathrm{f}} \to \mathrm{R}_+$, the function

$$s \mapsto 1_{s<t} P_{s, X_s} F(X_t), \qquad s \in [r, t)$$

is $P_{r,\mu}$-a.s. right continuous.

An a.s. cadlag right process X is said to be an a.s. Hunt process if it is quasi-left continuous.

As shown in [15], Lemma 5.29, the two introduced a.s. notions are not substantially different from the ones without "a.s."

LEMMA 49 (Dropping "a.s."). *Let X be an a.s. Hunt (respectively, a.s. cadlag right) process. Then there exists a Hunt (respectively, cadlag right) version of X.*

Now we are ready to state the following result.

LEMMA 50 (Cadlag right version). *Impose Assumption* 39. *Let Φ be a branching mechanism and k a branching functional. Then there exists a cadlag right version of the (ξ, Φ, k)-superprocess.*

PROOF. Recall that the (ξ, Φ, k)-superprocess X exists by Lemma 14. According to [8], Theorem 2.1, there is a right version $X = (X_t, \mathscr{F}, P_{r,\mu})$ of this process. Let $(A, \mathscr{D}(A))$ be the strong generator of the Feller process ξ. Recall that $\mathscr{D}(A) \subseteq \mathscr{C}_d(E)$. Fix $r \geq 0$ and $\mu \in \mathscr{M}_{\mathrm{f}}$. Note that for $f \in \mathscr{D}(A)$ the processes $t \mapsto \langle X_t, f\rangle - \int_r^t \langle X_s, Af\rangle\, ds$, $t \geq r$, are right continuous $P_{r,\mu}$-martingales and therefore, with $P_{r,\mu}$-probability 1, cadlag martingales. Hence, the process $t \mapsto \langle X_t, f\rangle$, $t \geq r$, is $P_{r,\mu}$-a.s. cadlag. Let $\{f_m\colon m \geq 1\} \subseteq \mathscr{D}(A)$ be a convergence determining set (for the weak topology in $\mathscr{M}_{\mathrm{f}}$). Recall that $\{f_m\colon m \geq 1\}$ is separating. Let

$$\Omega_r := \bigl\{\omega\colon t \mapsto \langle X_t(\omega), f_n\rangle,\ t \geq r,\ \text{ is cadlag, } n \geq 1\bigr\}.$$

Note that $P_{r,\mu}(\Omega_r)=1$. Recall also that on every bounded interval $[r,T]$, the cadlag trajectory $t\mapsto\langle X_t(\omega),1\rangle$ is bounded. Also, the sets $\{\mu\colon \langle\mu,1\rangle\le N\}$ are compact in $\mathscr{M}_{\mathrm{f}}$. Consider $\omega\in\Omega_r$, $t>r$, and let $t_n\uparrow t$, $t_n<t$. It follows that the family $\{X_{t_n}(\omega)\}_{n\ge 1}\subseteq\mathscr{M}_{\mathrm{f}}$ is tight. Hence, it has an accumulation point $X_{t-}(\omega)$. But since $\omega\in\Omega_r$, this accumulation point is unique and independent of the choice of the sequence $\{t_n\colon n\ge 1\}$. Thus $\lim_{s\uparrow t}X_s(\omega)=X_{t-}(\omega)$. Since t was arbitrary, it follows that $t\mapsto X_t(\omega)$ is cadlag, for $\omega\in\Omega_r$. An appeal to Lemma 49 completes the proof. □

4.4. *Remaining proofs.*

4.4.1. *Proof of existence of a Hunt version.* The next result is taken from [15], Lemma 6.39.

LEMMA 51. *Let $\{y_t\colon 0\le t\le T\}$ and $\{z_t\colon 0\le t\le T\}$ be $[0,1]$-valued stochastic processes over a filtered probability space $(\Omega,\Im,P)$. Suppose that y is P-indistinguishable from a right-continuous process. Let $\tau_n\le T$ be stopping times converging to some stopping time τ as $n\to\infty$. Then there exists a sequence $\delta_n\searrow_n 0$ such that*

$$\lim_{n\to\infty}P\left|z_{\tau_n}y_\tau-z_{\tau_n}y_{\tau_n+\delta_n}\right|=0.$$

Recall that a cadlag right process $X=(X_t,\mathscr{F},P_{r,\mu})$ is a Hunt process if and only if $P_{r,\mu}\{X_{\mathscr{T}-}=X_{\mathscr{T}}\}=1$ for $r\ge 0$, $\mu\in\mathscr{M}_{\mathrm{f}}$ and every bounded predictable r-stopping time $\mathscr{T}$.

PROOF OF THEOREM 40. Take ξ,Φ,k as in the theorem. Recalling Lemma 50, let $X=(X_t,\mathscr{F},P_{r,\mu})$ be a cadlag right version of the (ξ,Φ,k)-superprocess. Fix $r\ge 0$, $\mu\in\mathscr{M}_{\mathrm{f}}$ and $f\in\mathscr{C}_d(E)$. Consider a collection of r-stopping times $\mathscr{T}_n<\mathscr{T}$ nondecreasing to the bounded predictable stopping time $\mathscr{T}$. From Lemma 51 we conclude that there exists $\delta_n\downarrow 0$ such that

$$\begin{aligned}&\lim_{n\to\infty}P_{r,\mu}\left|\exp\langle X_{\mathscr{T}_n},-f\rangle-\exp\langle X_{\mathscr{T}},-f\rangle\right|\\ &\qquad=\lim_{n\to\infty}P_{r,\mu}\left|\exp\langle X_{\mathscr{T}_n},-f\rangle-\exp\langle X_{\mathscr{T}_n+\delta_n},-f\rangle\right|.\end{aligned}\tag{46}$$

Applying the tightness Proposition 45 with $X^n\equiv X$, we obtain

$$\lim_{n\to\infty}P_{r,\mu}\left|\exp\langle X_{\mathscr{T}_n},-f\rangle-\exp\langle X_{\mathscr{T}_n+\delta_n},-f\rangle\right|^2=0,$$

which implies that (46) vanishes. Using Fatou's lemma, we conclude

$$P_{r,\mu}\left|\exp\langle X_{\mathscr{T}-},-f\rangle-\exp\langle X_{\mathscr{T}},-f\rangle\right|=0.$$

Hence $\langle X_{\mathscr{T}-},f\rangle=\langle X_{\mathscr{T}},f\rangle$ with $P_{r,\mu}$-probability 1. Arguing with a separating sequence of functions $f\in\mathscr{C}_d(E)$ yields $X_{\mathscr{T}-}=X_{\mathscr{T}}$ with $P_{r,\mu}$-probability 1, completing the proof. □

4.4.2. *Proof of the joint continuity result.* Theorem 42 follows directly from Theorem 40 (the process is Hunt), Theorem 20 (which guaranties fdd convergence) and Corollary 47 (from which we conclude the weak convergence).

4.4.3. *Proof of the continuity in* Φ *only.* Proposition 43 is derived from Theorem 40 (which guaranties the existence of a Hunt version), from Proposition 21 (which yields the fdd continuity in Φ) and from Corollary 47 (from which we conclude the desired weak convergence).

4.4.4. *Proof of approximation by classical superprocesses.* We will need the following lemma.

LEMMA 52 ("Approximation" by regular Φ). *Every branching mechanism* Φ *belongs to the* bp*-closure of the set of all regular branching mechanisms.*

PROOF. If the maps $(s, x) \mapsto b^s(x)$ and $(s, x) \mapsto n(s, x, du)$ in Assumption 13(f) on a branching mechanism Φ are additionally continuous, then the corresponding branching mechanisms Φ are regular. Thus, the bp-closure of all regular branching mechanisms contains all ([0, 1]-valued) measurable $(s, x) \mapsto b^s(x)$ and continuous $(s, x) \mapsto n(s, x, du)$, ([10], Proposition 3.4.2). In particular, this is true for $n(s, x, du)$ of the form $f(s, x)\, n(du)$, where f is continuous. Hence, the bp-closure contains all measurable functions $(s, x) \mapsto b^s(x)$ and $(s, x) \mapsto 1_A(s, x)\, n(du)$ with A denoting a measurable subset of $\mathsf{R}_+ \times E$. Now let $n^1(du), n^2(du), \ldots$ be a dense subset of $\mathscr{M} = \mathscr{M}(0, \infty)$ [introduced in Assumption 13(f)]. Then every $n(s, x, du)$ is the pointwise limit of kernels of the form $n_N(s, x, du) := \sum_{l=1}^{\infty} 1_{A_N^l}(s, x)\, n^l(du)$ where

$$A_N^l := \left\{(s, x)\colon d_v(n^l, n) < \frac{1}{N} \text{ and } d_v(n^i, n) \geq \frac{1}{N},\ i = 1, \ldots, l-1\right\},$$

with d_v denoting a metric on $\mathscr{M}$ which generates the vague topology in $\mathscr{M}$. Using this fact completes the proof. $\square$

PROOF OF THEOREM 44.

Step 1. First we start from a (ξ, Φ, k)-superprocess X where Φ is regular. Note that, from Theorem 23 and Lemma 38, we can fdd approximate X by classical (ξ, Φ, k)-superprocesses X^n in such a way that the k^n satisfy the conditions imposed in Proposition 45. Note that the X^n are Hunt. It suffices to invoke Corollary 47 to conclude that there exist laws $P^{\infty}_{r,\mu}$ on path space such that $P^n_{r,\mu} \Rightarrow_n P^{\infty}_{r,\mu}$.

Step 2. Suppose now that Φ is arbitrary. Fix $r \geq 0$, $\mu \in \mathscr{M}_{\mathsf{f}}$, and denote by $P^{(\xi, \Phi, k)}_{r,\mu}$ the law on $\mathscr{D}_r$ of the (ξ, Φ, k)-superprocess with initial data (r, μ). Let $\mathscr{K}$ refer to the closure of the set of all laws $P^{(\xi, \Phi, k)}_{r,\mu}$ for which the branching functional k is classical [recall (16)] and the branching mechanism Φ is regular. As shown in Step 1, the set $\mathscr{K}$ contains all $P^{(\xi, \Phi, k)}_{r,\mu}$ with arbitrary k and regular Φ. Consider the set $\mathbf{\Phi}_{\xi, k}$ of all Φ such that $P^{(\xi, \Phi, k)}_{r,\mu}$ belongs to $\mathscr{K}$. From Theorem 40 (Hunt) and Proposition 21 (fdd convergence) we can invoke

Corollary 47 (weak convergence) and therefore conclude that the set $\mathbf{\Phi}_{\xi,k}$ is bp-closed. Therefore, since it contains all regular branching mechanisms, $\mathbf{\Phi}_{\xi,k}$ finally contains *all* branching mechanisms, by Lemma 52. In other words, all $P^{(\xi,\Phi,k)}_{r,\mu}$ belong to $\mathscr{K}$. Hence, for every (k,Φ) there exists a sequence (k^n,Φ^n) with classical k^n and regular Φ^n such that

$$P^{(\xi,\Phi^n,k^n)}_{r,\mu} \Rightarrow P^{(\xi,\Phi,k)}_{r,\mu} \quad \text{as } n\to\infty.$$

This completes the proof. □

APPENDIX

Here we collect some technical results. The following lemma is a slight modification of [9], A.1.1.A, page 116.

LEMMA A1 (Characterization of the existence of left limits). *Let $y = \{y_t\colon 0 \le t \le T\}$ denote a nonnegative right continuous process of class* (D) *over a filtered space $(\Omega, \Im, P)$. Then y is P-a.s. cadlag if and only if for every sequence of nondecreasing stopping times $\sigma_n \le T$ we have that $\lim_n P y_{\sigma_n}$ exists.*

PROOF. *Step* 1 ($\Rightarrow$). Suppose that y is cadlag. Let y_{s-} denote the left limit $\lim_{t\uparrow s} y_t$. Hence if $\sigma_n \nearrow \sigma$ as $n\to\infty$ then $\lim_n y_{\sigma_n} = y_{\sigma-}$. But since y belongs to class (D),

$$P y_{\sigma-} = P \lim_n y_{\sigma_n} = \lim_n P y_{\sigma_n}.$$

Therefore $\lim_n P y_{\sigma_n}$ exists.

Step 2 ($\Leftarrow$). Suppose now that y is not P-a.s. cadlag, but assume that for every sequence of nondecreasing stopping times $\sigma_n \le T$, the limit $\lim_n P y_{\sigma_n}$ exists. Hence, there exists a set $\mathscr{N}$ of positive P-probability such that for every $\omega \in \mathscr{N}$, (i) the process $y_\bullet(\omega)$ has a left oscillation, or (ii) the process $y_\bullet(\omega)$ has a left explosion.

We will show that each of these statements yield a contradiction.

(i) Suppose that the trajectory $y(\omega)$ has a left oscillation. Then there exist numbers q,δ in the set Q_+ of all nonnegative rationales such that $y(\omega)$ oscillates around q with oscillations of magnitude larger than δ. In other words, the sequence $\{\sigma^{q,\delta}_n(\omega)\}_{n=0}^\infty$ defined by $\sigma^{q,\delta}_0(\omega) := 0$ and, for $m \ge 0$,

$$\sigma^{q,\delta}_{2m+1}(\omega) := \inf\bigl\{t > \sigma^{q,\delta}_{2m}(\omega)\colon y_t(\omega) - q > \delta\bigr\},$$

$$\sigma^{q,\delta}_{2m+2}(\omega) := \inf\bigl\{t > \sigma^{q,\delta}_{2m+1}(\omega)\colon y_t(\omega) - q < -\delta\bigr\}$$

has the property that $\sigma^{q,\delta}_0(\omega) < \cdots < \sigma^{q,\delta}_n(\omega) < \sigma^{q,\delta}_{n+1}(\omega) < \cdots < T$. Setting again $\inf\varnothing := T$, then clearly, the random times $\sigma^{q,\delta}_n$ are stopping times. Let us define

$$A_{q,\delta} := \{\omega\colon \sigma_0^{q,\delta}(\omega) < \cdots < \sigma_n^{q,\delta}(\omega) < \sigma_{n+1}^{q,\delta}(\omega) < \cdots < T\}.$$

Moreover, let $y_t^*(\omega) := 1_{A_{q,\delta}^c}(\omega) y_t(\omega)$ where $A_{q,\delta}^c := \Omega \setminus A_{q,\delta}$. Note that for $\omega \in A_{q,\delta}^c$, the sequence $\sigma_n^{q,\delta}(\omega)$ eventually reaches T. Thus, since y is assumed to be right continuous, $y^*_{\sigma_n^{q,\delta}}$ converges to $y_T^*(\omega)$. Because, by assumption, y, hence y^*, belongs to class (D), this implies that

$$\lim_{n\to\infty} P\big(y^*_{\sigma_{n+1}^{q,\delta}} - y^*_{\sigma_n^{q,\delta}}\big) = 0. \tag{A1}$$

On the other hand, we have

$$\lim_{n\to\infty} P\big(1_{A_{q,\delta}}\big(y_{\sigma_{2n+1}^{q,\delta}} - y_{\sigma_{2n}^{q,\delta}}\big)\big) \geq 2\delta P(A_{q,\delta}).$$

From (A1) and the assumption that $\lim_{n\to\infty} P(y_{\sigma_{2n+1}^{q,\delta}} - y_{\sigma_{2n}^{q,\delta}}) = 0$, we conclude that $P(A_{q,\delta}) = 0$. Therefore, we obtain

$$P\bigg(\bigcup_{q,\delta\in Q_+} A_{q,\delta}\bigg) = 0.$$

That is, with probability 1, there is no left oscillation, yielding a contradiction.

(ii) The proof is analogous. Write $\sigma_0 := 0$, and for $n \geq 0$, define $\sigma_{n+1} := \inf\{t > \sigma_n\colon y_t > n\}$. (Here again, $\inf \varnothing := T$.) We put

$$A := \{\sigma_n < T \text{ for every } n \geq 0\}.$$

In the same way as in (i) we have that the existing limit of $P(y_{\sigma_n})$ implies that $P(A) = 0$. Thus there is no explosion towards $+\infty$. This completes the proof altogether. □

PROPOSITION A2 (A Doob type inequality). *Let $\{y_t\colon t \in [0,T]\}$ denote a real-valued right-continuous process of class* (D) *on a filtered probability space $(\Omega, \mathscr{F}, P)$. Then, for each $\eta > 0$,*

$$P\Big\{\sup_{s\leq T} |y_s| > \eta\Big\} \leq \bigg(\frac{2}{\eta}\sup_\sigma |Py_\sigma| + P|y_T|\bigg) \wedge \bigg(\frac{1}{\eta}\sup_\sigma P|y_\sigma|\bigg),$$

where σ denotes any stopping time (bounded by T).

PROOF. Let $\sigma_+^\eta := \inf\{s \in I\colon y_s > \eta\}$. Then by right continuity,

$$P\Big\{\sup_s y_s > \eta\Big\} \leq P\{y_{\sigma_+^\eta} \geq \eta\},$$

and by Markov's inequality we can continue with

$$\leq \frac{1}{\eta}\big(Py_{\sigma_+^\eta} + P|y_T|\big).$$

On the other hand, with $\sigma_{-}^{\eta} := \inf\{s \in I\colon y_s < -\eta\}$, using again right continuity and Markov's inequality,

$$P\Big\{ \inf_s y_s < -\eta \Big\} \le P\{y_{\sigma_{-}^{\eta}} \le -\eta\} = P\{-y_{\sigma_{-}^{\eta}} \ge \eta\} \le \frac{1}{\eta}\big(- P y_{\sigma_{-}^{\eta}} + P|y_T|\big).$$

Adding both cases, the first part of the claim follows. To get the other one, start with $\sigma^{\eta} := \inf\{s \in I\colon |y_s| > \eta\}$, and proceed directly in order to complete the proof. □

LEMMA A3. *Let a_n, b_n be real numbers. Then*

$$\left| \bigwedge_{n=1}^{\infty} a_n - \bigwedge_{n=1}^{\infty} b_n \right| \le \bigvee_{n=1}^{\infty} |a_n - b_n|$$

provided that at least one of the infimum expressions is finite.

PROOF. Obviously,

$$\bigwedge_{n=1}^{\infty} a_n \le \bigwedge_{n=1}^{\infty} (b_n + |a_n - b_n|) \le \bigwedge_{n=1}^{\infty} \Big(b_n + \bigvee_{n=1}^{\infty} |a_n - b_n| \Big) = \bigwedge_{n=1}^{\infty} b_n + \bigvee_{n=1}^{\infty} |a_n - b_n|.$$

By symmetry, the claim follows. □

COROLLARY A4. *Suppose $k^n(ds), k(ds)$ are finite (deterministic) measures on $I = [0, T]$ such that $k^n(r, t]$ converges to $k(r, t]$ as $n \to \infty$, for every $r < t \le T$. For each $n \ge 1$, let $s \mapsto \psi_s^n$ be uniformly bounded nonnegative measurable functions on I. Then the function $t \mapsto F(t) := \bigwedge_{n=1}^{\infty} \int_{(t,\,T]} \psi_s^n \, k^n(ds)$ is right continuous.*

PROOF. Fix t. Consider $t < t + \delta \le T$ and set $B := \sup_n \|\psi^n\|_{\infty}$. By Lemma A3 we have

$$|F(t) - F(t+\delta)| \le \bigvee_{n=1}^{\infty} \left| \int_{(t,\,T]} \psi_s^n \, k^n(ds) - \int_{(t+\delta,\,T]} \psi_s^n \, k^n(ds) \right| = \bigvee_{n=1}^{\infty} \int_{(t,\,t+\delta]} \psi_s^n \, k^n(ds).$$

Thus,

$$|F(t) - F(t+\delta)| \le B \bigvee_{n=1}^{\infty} k^n(t, t+\delta]. \tag{A2}$$

Take any $\varepsilon > 0$ and choose $\delta = \delta_\varepsilon$ so small that $k(t, t+\delta] \le \varepsilon$. Then there exists $N = N_{\varepsilon,\delta}$ such that for every $n \ge N$ we have $|k_n(t, t+\delta] - k(t, t+\delta]| \le \varepsilon$. Thus, for all $\delta_0 \in (0, \delta)$,

$$\bigvee_{n=N}^{\infty} k^n(t, t+\delta_0] \le \bigvee_{n=N}^{\infty} k^n(t, t+\delta] \le k(t, t+\delta] + \varepsilon \le 2\varepsilon.$$

But for $\delta_0 \in (0, \delta)$ small enough (keeping the $N = N_{\varepsilon,\delta}$), we have $\bigvee_{n=1}^{N-1} k^n(t, t+\delta_0] \le 2\varepsilon$. Consequently, for $\delta_0 > 0$ sufficiently small,

$$\bigvee_{n=1}^{\infty} k^n(t, t+\delta_0] \le 2\varepsilon.$$

Returning to (A2), we get

$$|F(t) - F(t+\delta_0)| \le 2B\,\varepsilon.$$

This completes the proof. □

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