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# ORNSTEIN–UHLENBECK PROCESSES INDEXED BY THE CIRCLE

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We consider the class of stationary, zero-mean Gaussian processes, indexed by the circle, satisfying a two-point Markov property and taking values in a vector bundle over the circle with given holonomy. We establish, subject to certain additional symmetry properties, a classification of all such processes. We then propose a construction of a Brownian motion of loops, in which these processes provide the infinitesimal increments.

**1. Introduction.** The problem considered in this paper arises from a desire to understand what are the simplest and most natural evolutions of loops in a Riemannian manifold $M$. A loop is a continuous map $x\colon S \to M$, where $S = \mathbb{R}/\mathbb{Z}$ is the circle. For a loop to move, one must specify a direction, which is a field of tangent vectors $v(t) \in T_{x(t)}M$, and if the loop is not to break apart, $(v(t)\colon t \in S)$ should be continuous. In the deterministic case one might specify that each point should follow a geodesic, but how is the initial direction to be chosen? If $v(0) \in T_{x(0)}M$ is given, there is no canonical way to lift to a continuous field of vectors $(v(t)\colon t \in S)$.

Our interest is, however, in the random case. We would like to construct a "Brownian motion of loops," such that, in particular, each point on the loop follows a Brownian motion in $M$. Let us assume that our loops are sufficiently regular that there is a well-defined notion of parallel translation along the loop—smooth or semimartingale loops would do. Then the tangent space over the loop has the structure of a Euclidean vector bundle over $S$ with connection. We choose the look for a vector field along each loop determined by this structure alone. Whilst one might reasonably use more of the local differential structure of the manifold along the loop, for example, curvature, our choice is minimal and motivated therefore by economy. Considerations of continuity and infinite divisibility suggest a Gaussian distribution for the vector field. Further, we wish to impose maximal symmetry, so that, in particular, the distribution is invariant under rotation of loops. There remain a great many choices, so we impose in addition a two-point Markov property: this leads to a simple stochastic structure and my be considered as a type of locality condition. At this point, in contrast to the deterministic case, where there was no canonical vector field, we find, associated to each invariant subspace of the holonomy of the loop, a canonical two-parameter family of

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vector fields. The parameters are associated to scale and correlation. This classification theorem is our main result, Theorem 3.1. It is followed by a brief investigation of the properties of these canonical vector fields, in the final section. We finish with a section, of a more speculative nature, explaining how the Ornstein–Uhlenbeck processes described in this paper might provide the infinitesimal increments for processes of Brownian motion on loops. There appear to be some major technical problems, which have yet to be resolved and which expose the inadequacies of present techniques for multiparameter stochastic differential equations. For recent progress on related problems, see [1], [3], [4], [5] and [7].

**2. Vector bundles over the circle.** In this section we establish some basic notation and terminology. Let $E = (E_t\colon t \in S)$ be a Euclidean vector bundle over the circle with connection, having holonomy $\tau$. By this we mean that each $E_t$ is equipped with an inner product and, for each pair $t, t' \in S$, there is an isometry

$$\tau^+_{t't}\colon E_t \to E_{t'}.$$

These isometries satisfy $\tau^+_{tt} = I$ and, for $t < t' < t'' < t$, the relation

$$\tau^+_{t''t} = \tau^+_{t''t'}\tau^+_{t't}.$$

Here and throughout, we write $t < t' < t'' < t$ or simply $t < t' < t''$ to mean that one passes $t, t', t''$ in that order as one moves in the positive sense, that is, anticlockwise, around the circle. The holonomy $\tau$ is the isometry of $E_0$ given, for all $t \neq 0$, by

$$\tau = \tau^+_{0t}\tau^+_{t0}.$$

The isometries $\tau^+_{t't}$ define parallel translations anticlockwise around $S$. The clockwise parallel translations are denoted

$$\tau^-_{tt'} = (\tau^+_{t't})^{-1}\colon E_{t'} \to E_t.$$

An isomorphism of two such bundles $E, \tilde{E}$ is specified by an isometry $\phi$: $E_0 \to \tilde{E}_0$ such that

$$\tilde{\tau} = \phi\tau\phi^{-1}.$$

In the case $\tilde{E}_t = E_{s+t}$ there is an isomorphism given by $\phi = \tau^+_{0s}$. Given a connected Euclidean vector bundle $E$, its isomorphism class is determined uniquely by integers $k \geq 1$, $n_1, \dots, n_k \geq 1$ and distinct angles of rotation $\theta_1, \dots, \theta_k \in (-\pi, \pi]$, for which there exists an orthogonal decomposition

$$E_0 = E_0^1 \oplus \cdots \oplus E_0^k$$

such that

$$\tau = \tau_1 \otimes \cdots \otimes \tau_k,$$

where $E_0^i$ has dimension $n_i$, with $n_i$ even unless $\theta_i = 0$ or $\theta_i = \pi$, and where $\tau_i = \tau(n_i, \theta_i)$ acts on $E_0^i$ as a rotation through $\theta_i$. Thus

$$\tau(n, 0) = I, \qquad \tau(n, \pi) = -I,$$

$$\tau(2n, \theta) = I\cos\theta + J\sin\theta, \qquad 0 < |\theta| < \pi,$$

where $J$ is an isometry with $J^2 = -I$. The bundle then decomposes as

$$E = E^1 \oplus \cdots \oplus E^k,$$

where

$$E_t^i = \tau_{t0}^+ E_0^i.$$

LEMMA 2.1. *Suppose $\gamma\colon E_0 \to E_0$ is a linear map, which commutes with every isometry of $E_0$ commuting with $\tau$. Then $\gamma$ respects the decomposition*

$$E_0 = E_0^1 \oplus \cdots \oplus E_0^k$$

*and is given on $E_0^i$, for some $\lambda_i, \mu_i \in \mathbb{R}$, by*

$$\gamma = \lambda_i I + \mu_i \tau_i.$$

PROOF. For any orthogonal decomposition $E_0 = E_0^- \oplus E_0^+$ respected by $\tau$, $\tau$ commutes with the isometry which acts as $\pm 1$ on $E_0^{\pm}$, so must $\gamma$, and hence $\gamma$ must also respect the decomposition. Thus we reduce to the following two special cases. When $\tau = I$, we must have $\gamma = \lambda I$ for some $\lambda \in \mathbb{R}$. When $\tau = J$ with $J^2 = -I$, we must have $\gamma = \lambda I + \mu J$ for some $\lambda, \mu \in \mathbb{R}$. □

**3. Ornstein–Uhlenbeck processes.** Let $E$ be a Euclidean vector bundle over $S$ with connection. Given a section $x\colon S \to E$, we denote by $\tilde{x}\colon \mathbb{R} \to E_0$ the lifting given by

$$\tilde{x}_{n+t} = \tau^{-n}\tau_{0t}^- x_t, \qquad n \in \mathbb{Z}, 0 \le t < 1.$$

Note, in particular, that $\tilde{x}_1 = \tau^{-1}\tilde{x}_0$.

We call a *random section* of $E$ any measurable map

$$X\colon \Omega \times S \to E.$$

We now define a number of properties that such a random section might have. Although these will be expressed in terms of $\tilde{X}$, it is easy to check that the point 0 plays no special role. We say $(X_t\colon t \in S)$ is *Gaussian* if $(\tilde{X}_t\colon t \in \mathbb{R})$ is Gaussian. We say $(X_t\colon t \in S)$ is *stationary* if $(\tilde{X}_{s+t}\colon t \in \mathbb{R})$ has the same distribution as $(\tilde{X}_t\colon t \in \mathbb{R})$ for all $s \in \mathbb{R}$. We say $(X_t\colon t \in S)$ is *rotation invariant* if $(\phi\tilde{X}_t\colon t \in \mathbb{R})$ has the same distribution as $(\tilde{X}_t\colon t \in \mathbb{R})$, for all isometries $\phi$ commuting with $\tau$. We say $(X_t\colon t \in S)$ is *Markovian* if, for all $s, t \in S$, conditional on $X_s$ and $X_t$, $(X_r\colon s < r < t)$ and $(X_r\colon t < r < s)$ are independent. We say $(X_t\colon t \in S)$ is *locally reversible* if, for all $s, t \in \mathbb{R}$ with $0 < t - s < 1$, for all $x, y \in E_0$, the distribution of $(\tilde{X}_{s+r}\colon 0 < r < t - s)$ given $\tilde{X}_s = x$, $\tilde{X}_t = y$ is the same as the distribution of $(\tilde{X}_{t-r}\colon 0 < r < t - s)$ given $\tilde{X}_t = x$, $\tilde{X}_s = y$. We say $(X_t\colon t \in S)$ is *reversible* if $(\tilde{X}_{-t}\colon t \in \mathbb{R})$ has the same distribution as $(\tilde{X}_t\colon t \in \mathbb{R})$. Note that, given stationarity, reversibility implies local reversibility, but, since

$$\tilde{X}_1 = \tau^{-1}\tilde{X}_0 = \tau^{-2}\tilde{X}_{-1},$$

reversibility must in general fail, unless $\tau^2 = I$.

We say that a random section $X$ of $E$ is an *Ornstein–Uhlenbeck process* if it is stationary, centered, Gaussian, rotation invariant, Markovian and locally reversible. We write $X \sim \mathrm{OU}(E)$ for short. We emphasize that we do not assume that $X$ is continuous. This will turn out to be a consequence of the other defining properties.

The distribution of any stationary centered Gaussian section $(X_t: t \in S)$ in $E$ is determined by its *covariance*

$$c(t) = \mathbb{E}\, \tilde{X}_0 \otimes \tilde{X}_t^*, \qquad 0 \le t \le 1,$$

which is a measurable map $c: [0,1] \to E_0 \otimes E_0^*$ with $c(1) = c(0)\tau$.

For $\tau$ an isometry of $E_0$ and for $\lambda > 0$, $0 < \mu < \pi$ and $0 \le t \le 1$, define

$$c_\lambda(t,\tau) = \frac{\tau \sinh \lambda t + I \sinh \lambda(1-t)}{\sinh \lambda},$$

$$c_0(t,\tau) = \tau t + I(1-t),$$

$$c_{i\mu}(t,\tau) = \frac{\tau \sin \mu t + I \sin \mu(1-t)}{\sin \mu},$$

$$c_{i\pi}(t,-I) = I \cos \pi t.$$

Observe that these functions may be regarded as a single family, parametrized by $\lambda$, where $-\pi^2 < \lambda^2 < \infty$, with continuous extension to $\lambda = i\pi$ in the case $\tau = -I$.

We come to the main result. In the case $n = 1$ and $\tau = 1$, this has been discovered at least twice before (see [9] and [6]). Our method is a development of that used by Pitt [9]. Klein and Landau [6] prove a general classification theorem for periodic Gaussian processes having Osterwalder–Schrader positivity, based on a representation theorem of Krein [2].

THEOREM 3.1. *For $\tau = \tau(n,\theta)$, $0 \le \sigma^2 < \infty$ and $-\theta^2 \le \lambda^2 < \infty$, there exists a continuous $\mathrm{OU}(E)$ process with covariance $\sigma^2 c_\lambda(\cdot,\tau)$, and we obtain all $\mathrm{OU}(E)$ processes in this way.*

*In general, if $\tau$ has decomposition*

$$\tau = \tau_1 \otimes \cdots \otimes \tau_k,$$

*where $\tau_i = \tau(n_i, \theta_i)$ with all $\theta_i$ distinct, and if $X^i \sim \mathrm{OU}(E^i)$ with $X^1, \dots, X^k$ independent, then*

$$X = X^1 \oplus \cdots \oplus X^k$$

*is an $\mathrm{OU}(E)$ process, and we obtain all $\mathrm{OU}(E)$ processes in this way.*

PROOF. We deal first with existence. Let $\tau = \tau(n,\theta)$ and fix $0 < \mu < |\theta|$. It suffices to treat the case $\sigma = 1$. Set

$$c(t) = c_{i\mu}(t,\tau) = \frac{\tau \sin \mu t + I \sin \mu(1-t)}{\sin \mu}, \qquad 0 \le t \le 1.$$

Note the identity for $x + y \neq m\pi$,

$$\sin(x+z) = \frac{\sin x}{\sin(x+y)}\sin(x+y+z) + \frac{\sin y}{\sin(x+y)}\sin z,$$

from which it follows that, for $0 \leq s < t < u \leq 1$,

$$c(u-s) = \frac{\sin \mu(t-s)}{\sin \mu t}c(u) + \frac{\sin \mu s}{\sin \mu t}c(u-t).$$

Let $\tilde{X}_0$ be a centered Gaussian random variable in $E_0$, of variance $I$. For $k \in \mathbb{Z}$, set $\tilde{X}_k = \tau^{-k}\tilde{X}_0$. Suppose inductively, for $m = 0, 1, 2, \dots,$ that we can construct a centered Gaussian process $(\tilde{X}_t\colon t \in 2^{-m}\mathbb{Z})$ such that, for all $s, t \in 2^{-m}\mathbb{Z}$ with $0 \leq t \leq 1$, we have

$$\mathbb{E}\,\tilde{X}_s \otimes \tilde{X}^*_{s+t} = c(t).$$

Set $h = 2^{-m-1}$ and compute

$$\begin{aligned}\operatorname{var}\left(\frac{\sin \mu h}{\sin 2\mu h}\left(\tilde{X}_0 + \tilde{X}_{2h}\right)\right) &= \left(\frac{\sin \mu h}{\sin 2\mu h}\right)^2 (2I + c(2h) + c(2h)^*) \\ &= I\left(1 - (\cos \mu - \cos \theta)\frac{\tan \mu h}{\sin \mu}\right).\end{aligned}$$

Since $\mu < |\theta|$, we have $\cos \mu > \cos \theta$, so there is an independent centered Gaussian random variable $Y$ such that $\tilde{X}_h$ has variance $I$, where

$$\tilde{X}_h = \frac{\sin \mu h}{\sin 2\mu h}\left(\tilde{X}_0 + \tilde{X}_{2h}\right) + Y.$$

We repeat this construction, independently, $2^m$ times to obtain $(\tilde{X}_t\colon t \in 2^{-m-1}\mathbb{Z} \cap [0,1])$, then set

$$\tilde{X}_{n+t} = \tau^{-n}\tilde{X}_t, \qquad n \in \mathbb{Z},\ t \in 2^{-m-1}\mathbb{Z} \cap [0,1].$$

Now, for $u \in 2^{-m}\mathbb{Z} \cap (0,1]$,

$$\mathbb{E}\,\tilde{X}_h \otimes \tilde{X}^*_u = \frac{\sin \mu h}{\sin 2\mu h}(c(u) + c(u-2h)) = c(u-h)$$

and, for $u \in 2^{-m}\mathbb{Z} \cap (0,1)$,

$$\mathbb{E}\,\tilde{X}_h \otimes \tilde{X}^*_{u+h} = \left(\frac{\sin \mu h}{\sin 2\mu h}\right)^2 (c(u-2h) + 2c(u) + c(u+h)) = c(u).$$

Hence, for $s, t \in 2^{-m-1}\mathbb{Z}$ with $0 \leq t \leq 1$, we have

$$\mathbb{E}\,\tilde{X}_s \otimes \tilde{X}^*_{s+t} = c(t).$$

This completes the inductive step. Since $c$ is Lipschitz continuous, the usual application of Kolmogorov's criterion now gives us a continuous centered

Gaussian process $(\tilde{X}_t: t \in \mathbb{R})$ with

$$\mathbb{E}\, \tilde{X}_s \otimes \tilde{X}^*_{s+t} = c(t)$$

for all $s \in \mathbb{R}$ and $t \in [0, 1]$. Hence we obtain a continuous stationary centered Gaussian section of $E$ on setting

$$X_t = \tau^+_{t0} \tilde{X}_t.$$

We now have to check that $(X_t: t \in S)$ is rotation invariant, Markovian and locally reversible.

The first is easy, for if $\phi$ is an isometry of $E_0$, commuting with $\tau$, then, for all $s \in \mathbb{R}$ and $t \in [0, 1]$,

$$\mathbb{E}\, \phi \tilde{X}_s \otimes \left(\phi \tilde{X}_{s+t}\right)^* = \phi c(t) \phi^{-1} = c(t).$$

So $(X_t: t \in S)$ is rotation invariant. Next, for $0 \le s < t \le u \le 1$, we can write

$$\tilde{X}_s = \frac{\sin \mu(t-s)}{\sin \mu t} \tilde{X}_0 + \frac{\sin \mu s}{\sin \mu t} \tilde{X}_t + Y_s.$$

Then we have

$$\mathbb{E}\, Y_s \otimes \tilde{X}^*_u = c(u-s) - \frac{\sin \mu(t-s)}{\sin \mu t} c(u) - \frac{\sin \mu s}{\sin \mu t} c(u-t) = 0$$

so $Y_s$ and $\tilde{X}_u$ are independent. This shows that $(X_t: t \in S)$ is Markovian. To establish local reversibility, it suffices to show that $(Y_s: 0 \le s \le t)$ and $(Y_{t-s}: 0 \le s \le t)$ have the same distribution. Then, by the Markov property, it suffices to show that the centered Gaussian random variables $Y_s$ and $Y_{t-s}$ have the same distribution for each $s \in [0, t]$. However,

$$\begin{aligned} I &= \operatorname{var} \tilde{X}_s \\ &= \frac{I \sin^2 \mu(t-s) + \left(c(t) + c(t)^*\right) \sin \mu(t-s) \sin \mu s + I \sin^2 \mu s}{\sin^2 \mu t} + \operatorname{var} Y_s, \end{aligned}$$

so $\operatorname{var} Y_s = \operatorname{var} Y_{t-s}$ as required.

We have shown that $(X_t: t \in S) \sim \mathrm{OU}(E)$. The argument we used, with obvious modifications, shows also that there exist continuous stationary centered Gaussian sections with covariances $c_\lambda(\cdot, \tau)$ for $\lambda = i|\theta|$ and $\lambda \ge 0$ and, moreover these are all $\mathrm{OU}(E)$.

In the general case we have

$$E_0 = E^1_0 \oplus \cdots \oplus E^k_0$$

and

$$\tau = \tau_1 \otimes \cdots \otimes \tau_k$$

with $\tau_i = \tau(n_i, \theta_i)$ for all $i$ and all $\theta_i$ distinct. Then the only isometries of $E_0$ commuting with $\tau$ are those which preserve $E^i_0$ and commute with $\tau_i$ for all $i$. Hence, it is easy to check that, if $X^i \sim \mathrm{OU}(E^i)$ for all $i$, and $X^1, \dots, X^k$ are independent, then $X = X^1 \oplus \cdots \oplus X^k \sim \mathrm{OU}(E)$.

We turn to the problem of uniqueness. Suppose that $(X_t: t \in S) \sim \mathrm{OU}(E)$. Since $(X_t: t \in S)$ is rotation invariant, we have

$$c(t) = \phi c(t) \phi^{-1}$$

for all isometries $\phi$ commuting with $\tau$. By Lemma 2.1, $c(t)$ respects the decomposition $E_0^1 \oplus \cdots \oplus E_0^k$ and is given on $E_0^i$ for some $a_i(t), b_i(t) \in \mathbb{R}$, by

$$c(t) = a_i(t) I + b_i(t) \tau_i.$$

The components $X^1, \dots, X^k$ are therefore independent and it is easy to check $X^i \sim \mathrm{OU}(E^i)$ for all $i$. Thus we are reduced to the case where $\tau = \tau(n, \theta)$ for some $\theta \in (-\pi, \pi]$ and

$$c(t) = a(t) I + b(t) \tau$$

for some $a(t), b(t) \in \mathbb{R}$. Since $c(0)$ is nonnegative definite, either $X$ is identically zero or, after a suitable rescaling, we can assume that $c(0) = I$.

Since $(X_t: t \in S)$ is Gaussian, for each $t \in [0, 1/2]$, there are linear maps $\gamma_-(t), \gamma_+(t)$ on $E_0$ such that

$$\text{(A)} \qquad \tilde{X}_0 = \gamma_-(t) \tilde{X}_{-t} + \gamma_+(t) \tilde{X}_t + Y,$$

where $Y$ is independent of $\tilde{X}_{-t}, \tilde{X}_t$. By local reversibility we can take $\gamma_-(t) = \gamma_+(t) = \gamma(t)$ say. Moreover

$$c(t) + c(t)^* = \mathbb{E} \tilde{X}_0 \otimes \left( \tilde{X}_{-t} + \tilde{X}_t \right)^* = \gamma(t) \mathrm{var}\left( \tilde{X}_{-t} + \tilde{X}_t \right).$$

If $\tilde{X}_{-t} + \tilde{X}_t = 0$, we can take $\gamma(t) = 0$. If not, then $\mathrm{var}(\tilde{X}_{-t} + \tilde{X}_t) = 2I + c(2t) + c(2t)^*$ is invertible. So, in any case, we can take $\gamma(t)$ to be a scalar.

We now show that the covariance $c$ must be continuous on $[0, 1]$ and smooth on $(0, 1)$. By the Markov property and stationarity, for $u \in [t, 1 - t]$, on multiplying (A) by $\tilde{X}_u^*$ and taking expectations, we obtain

$$\text{(B)} \qquad c(u) = \gamma(t)(c(u + t) + c(u - t)).$$

If $c = 0$ almost everywhere, then for any bounded measurable function $\phi$,

$$\mathrm{var} \int_0^1 \tilde{X}_s \phi(s)\, ds = 0.$$

So, by Fubini's theorem, $\tilde{X}_s = 0$ almost surely, for almost all $s$, so $c$ is identically zero by stationarity. We have already excluded this trivial case, so there must exist an $\varepsilon > 0$ and a smooth function $\phi$ on $\mathbb{R}$, supported in $[\varepsilon, 1 - \varepsilon]$, such that

$$\int_{\mathbb{R}} c(u) \phi(u)\, du \neq 0.$$

For $t \in [0, \varepsilon]$, we have

$$\int_{\mathbb{R}} c(u) \phi(u)\, du = \gamma(t) \int_{\mathbb{R}} (c(u + t) + c(u - t)) \phi(u)\, du.$$

Since the integral on the right-hand side is a smooth function of $t$, so is $\gamma$ on $[0, \varepsilon]$. Choose now a smooth function $\psi$ on $\mathbb{R}$, supported in $[0, \varepsilon]$ and of integral 1. Then $\psi\phi$ is smooth on $\mathbb{R}$ and, for $u \in [\varepsilon, 1 - \varepsilon]$,

$$c(u) = \int_{\mathbb{R}} (\psi\phi)(t)(c(u + t) + c(u - t))\, dt.$$

Since the integral is a smooth function of $u$ and $\varepsilon$ may be chosen arbitrarily small, this shows that $c$ is smooth on $(0, 1)$. Now, since $\gamma(0) = 1/2$, there exists $t > 0$ with $\gamma(t) \neq 0$. On letting $u \downarrow t$ and $u \uparrow 1 - t$ in (B), we obtain $\gamma(t)c(0+) = \gamma(t)c(0)$ and $\gamma(t)c(1-) = \gamma(t)c(1)$. Hence $c$ is continuous on $[0, 1]$.

On differentiating (B) twice with respect to $t$ at 0, we find $\gamma'(0)c''(u) = 0$ for all $u \in (0, 1)$. So, either $c'' = 0$ or $\gamma'(0) = 0$. In the latter case, on rearranging (B), dividing by $t^2$ and letting $t \downarrow 0$, we obtain $\gamma(0)c''(u) = \gamma''(0)c(u)$. So, in any case, $c$ satisfies an equation of the form

$$c'' + \nu c = 0$$

on $(0, 1)$ for some $\nu \in \mathbb{R}$. If we now impose the boundary conditions $c(0) = I$ and $c(1) = \tau$, together with the necessary constraint that, for $e \in E_0$,

$$e^* c(u) e = \mathbb{E}\big(e^* \tilde{X}_0\big)\big(e^* \tilde{X}_u\big) \le \mathbb{E}\big(e^* \tilde{X}_0\big)^2 = |e|^2,$$

then we obtain precisely the covariances $c_\lambda$, $\lambda^2 \ge -\theta^2$, as required. □

**4. Further properties.** We have classified, in terms of their covariance structure, all the Ornstein–Uhlenbeck processes in a Euclidean vector bundle $E$ over $S$ with connection. In particular, we have shown that, if $X \sim \mathrm{OU}(E)$, then $X$ decomposes into independent components $X^i \sim \mathrm{OU}(E^i)$ corresponding to the canonical decomposition of $E$. So, for convenience and without loss, we shall assume in this section that $\tau = \tau(n, \theta)$ for some $\theta \in (-\pi, \pi]$. Just as Brownian motion and more classical Ornstein–Uhlenbeck processes have many different characterizations, so do the processes considered here. These alternative characterizations form the subject of this section.

First of all, from the proof of Theorem 3.1, we have the following Markovian description. The Ornstein–Uhlenbeck process with covariance $c_\lambda(\cdot, \tau)$, for $-\theta^2 \le \lambda^2 < \infty$, is characterized among stationary continuous Gaussian sections by the properties:

1. $\mathrm{var}(\tilde{X}_0) = I$;
2. for $0 \le s < t \le 1$ we have

$$\tilde{X}_s = \frac{\sinh \lambda(t - s)}{\sinh \lambda t} \tilde{X}_0 + \frac{\sinh \lambda s}{\sinh \lambda t} \tilde{X}_t + Y$$

for some random variable $Y$, independent of $(\tilde{X}_u \colon t \le u \le 1)$.

Moreover, property 2 may be replaced by the weaker condition:

2′. for $0 < t \le 1/2$ we have

$$\tilde{X}_t = \operatorname{sech} \lambda t \Big( \tilde{X}_0 + \tilde{X}_{2t} \Big)/2 + Y$$

for some random variable $Y$, independent of $(\tilde{X}_u \colon 2t \le u \le 1)$.

Second, there is a Hilbert space approach. The space of (covariantly) absolutely continuous sections $h\colon S \to E$ contains a family of Hilbert spaces, parametrized by $\lambda$, where $-\theta^2 < \lambda^2 < \infty$, with norms

$$\|h\|_\lambda^2 = \int_0^1 |\nabla h_t|^2 \, dt + \lambda^2 \int_0^1 |h_t|^2 \, dt,$$

where

$$\nabla h_t = \tau_{t0}^{+} \frac{d}{dt} \left( \tau_{0t}^{-} h_t \right).$$

The generalized Wiener space construction then gives rise to a continuous centered Gaussian section $X$ characterized by

$$\mathbb{E} \langle h, X \rangle_\lambda^2 = \|h\|_\lambda^2.$$

By a standard type of calculation, we can then show that $X$ has covariance given by

$$\mathbb{E}\, \tilde{X}_0 \otimes \tilde{X}_t^* = \frac{\sinh \lambda}{2\lambda(\cosh \lambda - \cos \theta)} c_\lambda(t, \tau).$$

Hence we obtain in this way, up to a scaling, all OU($E$) processes, except the maximally correlated, smooth process corresponding to $\lambda = i\theta$. Notice that the marginal variance of $X$ depends on the angle of rotation $\theta$, so if this construction is applied in the general case $\tau = \tau_1 \otimes \cdots \otimes \tau_k$, then the marginal variance of $X$ will not be a multiple of the identity. Of course, if this is regarded as a defect, it can be remedied by a suitable reweighting of the Hilbert norms across the decomposition $E = E^1 \oplus \cdots \oplus E^k$. We note that the generalized Wiener space machinery deals quickly with the construction problem, which we considered at greater length by elementary means. It would be interesting to have a direct proof of uniqueness by this method: one would set up the canonical reproducing-kernel Hilbert space for the process, and then seek to identify it with one of the Hilbert spaces of sections considered above by using the special properties of an OU($E$) process. We have not attempted to carry this through.

In the case where $\lambda \ge 0$, the differential equation

$$\dot{\tilde{h}}_t = \dot{g}_t - \lambda \tilde{h}_t$$

gives rise to an isomorphism of the Hilbert space of $X$ and the classical Cameron–Martin space of Brownian motion for

$$\|g\|^2 \equiv \int_0^1 |\dot{g}_t|^2\, dt = \int_0^1 |\nabla h_t + \lambda h_t|^2\, dt = \|h\|_\lambda^2.$$

So we can construct $X$ by solving the linear stochastic differential equation with boundary conditions, driven by Brownian motion

$$d\tilde{X}_t = dB_t - \lambda \tilde{X}_t\, dt, \qquad \tilde{X}_1 = \tau^{-1}\tilde{X}_0.$$

We note that no genuine stochastic integrals are needed for this equation, so the nonadapted boundary condition poses no problem. A general study of some related equations was made by Ocone and Pardoux [8].

Explicit solution of the differential equation leads to another representation of $X$, in terms of an integral kernel with respect to Brownian motion. We shall first present the integral kernel construction, and then rederive the differential equation.

Let $(B_t\colon 0 \le t < 1)$ be a Brownian motion in $E_0$ and define a Gaussian process $(\tilde{B}_t\colon t \in \mathbb{R})$ with stationary increments by $\tilde{B}_0 = 0$ and

$$d\tilde{B}_{n+t} = \tau^{-n}\, dB_t, \qquad n \in \mathbb{Z},\ 0 \le t < 1.$$

Fix $\lambda > 0$ and set

$$\phi(t) = e^{-\lambda(t-1/2)} 1_{0 \le t < 1}.$$

Then define

$$\text{(C)}\qquad \begin{aligned} \tilde{X}_t &= \left(\frac{\lambda}{\sinh \lambda}\right)^{1/2} \int_{\mathbb{R}} \phi(t-r)\, d\tilde{B}_r \\ &= \left(\frac{\lambda}{\sinh \lambda}\right)^{1/2} \left\{ \int_0^t e^{-\lambda(t-r-1/2)}\, dB_r + \tau \int_t^1 e^{-\lambda(t-r+1/2)}\, dB_r \right\}. \end{aligned}$$

Then $\tilde{X}_{n+t} = \tau^{-n}\tilde{X}_t$ for $n \in \mathbb{Z}$ and $0 \le t < 1$, so we obtain a continuous, stationary, centered, Gaussian, rotation invariant section of $E$ on setting

$$X_t = \tau_{t0}^{+}\tilde{X}_t, \qquad 0 \le t < 1.$$

Moreover, for $0 \le t \le 1$, we have

$$\begin{aligned} \mathbb{E}\, \tilde{X}_0 \otimes \tilde{X}_t^* &= \left(\frac{\lambda}{\sinh \lambda}\right) \left\{ \tau \int_0^t e^{\lambda(r-1/2)} e^{\lambda(r-t+1/2)}\, dr \right. \\ &\qquad \left. + I \int_t^1 e^{\lambda(r-1/2)} e^{\lambda(r-t-1/2)}\, dr \right\} \\ &= \frac{\tau \sinh \lambda t + I \sinh \lambda(1-t)}{\sinh \lambda} = c_\lambda(\tau, t), \end{aligned}$$

and on differentiating (C) we obtain

$$d\tilde{X}_t = (\lambda/\sinh \lambda)^{1/2}(e^{\lambda/2} - \tau e^{-\lambda/2})\, dB_t - \lambda \tilde{X}_t\, dt = \sigma(\lambda, \theta)\, dW_t - \lambda \tilde{X}_t\, dt,$$

where

$$\sigma^2(\lambda, \theta) = 2\lambda(\cosh\lambda - \cos\theta)/\sinh\lambda$$

and where $(W_t: 0 \le t < 1)$ is a Brownian motion in $E_0$. So we recover, up to a scaling, the differential equation discussed above.

We can also characterize $X$ by means of a stochastic differential equation, adapted to the filtration

$$F_t = \sigma\{\tilde{X}_s: 0 \le s \le t\}.$$

Consider for now the case $\lambda > 0$. For $0 \le t < t + h \le 1$ we know that

$$\tilde{X}_{t+h} = \frac{\sinh\lambda h}{\sinh\lambda(1-t)}\tilde{X}_1 + \frac{\sinh\lambda(1-t-h)}{\sinh\lambda(1-t)}\tilde{X}_t + Y_{t,t+h},$$

where $Y_{t,t+h}$ is a centered Gaussian, independent of $F_t$. So, as $h \downarrow 0$, uniformly in $t \le 1 - \varepsilon$ for each $\varepsilon > 0$, we have

$$\begin{aligned}\tilde{X}_{t+h} - \tilde{X}_t = &\left(\frac{\lambda h}{\sinh\lambda(1-t)} + o(h)\right)\tau^{-1}\tilde{X}_0 \\ &- \left(\frac{\lambda h\cosh\lambda(1-t)}{\sinh\lambda(1-t)} + o(h)\right)\tilde{X}_t + Y_{t,t+h}.\end{aligned}$$

Moreover

$$\begin{aligned}\operatorname{var}(Y_{t,t+h}) &= \operatorname{var}\left(\tilde{X}_{t+h} - \tilde{X}_t\right) + o(h) \\ &= -2c'_\lambda(0+, \tau)h + o(h) \\ &= \sigma^2(\lambda, \theta)hI + o(h).\end{aligned}$$

Define $(B_t: 0 \le t < 1)$ by

$$\begin{aligned}\sigma(\lambda,\theta)B_t &= \tilde{X}_t - \tilde{X}_0 - \int_0^t \lambda\tau^{-1}\tilde{X}_0 \operatorname{csch}\lambda(1-s)\,ds + \int_0^t \lambda\tilde{X}_t \coth\lambda(1-s)\,ds \\ &= \lim_{n\to\infty}\sum_{k=0}^{[nt]-1} Y_{k/n,(k+1)/n}.\end{aligned}$$

Then $B$ is a standard Brownian motion in $E_0$, adapted to $(F_t)_{0\le t<1}$ and $\tilde{X}$ satisfies the stochastic differential equation

$$d\tilde{X}_t = \sigma(\lambda,\theta)\,dB_t + \lambda\tau^{-1}\tilde{X}_0 \operatorname{csch}\lambda(1-t)\,dt - \lambda\tilde{X}_t\coth\lambda(1-t)\,dt.$$

This all works, with obvious modifications, for the cases $\lambda = 0$ and $\lambda = i\mu$, $0 < \mu < |\theta|$. We note that the representation of $X$ using a stochastic differential equation with boundary conditions works, by contrast, only for $\lambda \ge 0$.

**5. Brownian motion of loops.** Let $E$ be a Euclidean vector bundle over $S$ with connection, having holonomy $\tau$. Fix $\lambda > 0$ and recall that we set

$$c_\lambda(s,\tau) = \frac{\tau\sinh\lambda s + I\sinh\lambda(1-s)}{\sinh\lambda}, \qquad 0 \le s \le 1.$$

We can use $c_\lambda$ to construct a Brownian motion $(\bar{x}_t\colon t \geq 0)$ in the space of sections of $E$. Here $\bar{x}_t = (\bar{x}_{st}\colon s \in S)$ is an $\mathrm{OU}(E)$ process with covariance $tc_\lambda(\cdot, \tau)$ for each $t \geq 0$ and $(\bar{x}_t\colon t \geq 0)$ has stationary independent increments. Then $(\bar{x}_{st}\colon s \in S,\ t \geq 0)$ has a jointly continuous version and for each $s \in S$ the ray $(\bar{x}_{st}\colon t \geq 0)$ is a standard Brownian motion in $E_s$. We call $(\bar{x}_{st}\colon s \in S,$ $t \geq 0)$ the standard Ornstein–Uhlenbeck sheet in $E$ of parameter $\lambda$.

In Section 4 we obtained three representations of $\mathrm{OU}(E)$ processes in terms of Brownian motion, either by a linear stochastic differential equation with boundary conditions or by means of an integral kernel or by a nonlinear, but adapted, stochastic differential equation. To each of these there corresponds a representation of the standard Ornstein–Uhlenbeck sheet in terms of a Brownian sheet $(w_{st}\colon 0 \leq s \leq 1,\ t \geq 0)$, taking values in $E_0$. We give details for the first of these: $(\bar{x}_{st}\colon s \in S,\ t \geq 0)$ satisfies a two-parameter linear stochastic differential equation of the form

$$D_s\, d_t \bar{x}_{st} = \tau_{s0}^{+} \sigma(\lambda, \tau)\, \partial_s\, d_t w_{st} - \lambda\, \partial s\, d_t \bar{x}_{st}.$$

Here we have use notation consistent with [7]: $d$ stands for the Itô differential, $\partial$ for the Stratonovich differential and $D$ for the covariant Stratonovich differential. We wrote $\tau_{s0}^{+}$ for the parallel translation $E_0 \to E_s$ in the positive sense, as in Section 2, and

$$\sigma(\lambda, \tau) = \left(\frac{\lambda}{\sinh \lambda}\right)^{1/2} \left(2I \cosh \lambda - \tau - \tau^{-1}\right)^{1/2}.$$

Consider now a Riemannian manifold $M$ and fix some connection $\nabla$ on $M$ which is compatible with the metric. Choose an initial loop $(x_{s0}\colon s \in S)$ having holonomy $\tau_0$. Assume that parallel translation is well defined and continuous along $(x_{s0}\colon s \in S)$. Then we can take $E_s = T_{x_{s0}}M$ in the above discussion and obtain a sheet $(\bar{x}_{st}\colon s \in S,\ t \geq 0)$ of sections of $(T_{x_{s0}}M\colon s \in S)$. For each $s \in S$, denote by $(x_{st}\colon t \geq 0)$ the stochastic development of $(\bar{x}_{st}\colon$ $t \geq 0)$ in $M$, which is a $\nabla$-Brownian motion in $M$, starting from $x_{s0}$. We can show that $(x_{st}\colon s \in S,\ t \geq 0)$ has a jointly continuous version, so

$$t \mapsto x_t = (x_{st}\colon s \in S)$$

is a process of continuous loops in $M$.

We shall write

$$\tau_{s,tt'}\colon T_{x_{st'}}M \to T_{x_{st}}M$$

for the parallel translation along $(x_{su}\colon u \geq 0)$ and

$$\tau_{ss',t}^{\pm} = \tau_{ss'}^{\pm}(x_t)\colon T_{x_{s't}}M \to T_{x_{st}}M$$

for parallel translation along $(x_{rt}\colon r \in S)$ in the positive and negative senses. Also, write $\tau_t = \tau(x_t)$ for the holonomy of $x_t$.

In general, $(x_t\colon t \geq 0)$ will not be Markov. In particular it remembers the initial holonomy $\tau_0$, so there is little to recommend $(x_t\colon t \geq 0)$ as a natural evolution of loops. Let us suppose, however, that the connection is flat. Then

$$\tau_t = \tau_{0,t0} \tau_0 \tau_{0,0t}$$

and

$$\tau_{s,t0} = \tau^{+}_{s0,t}\tau_{0,t0}\tau^{-}_{0s,0}.$$

Set, for $T \geq 0$,

$$\bar{x}^{T}_{st} = \tau_{s,T0}(\bar{x}_{s,T+t} - \bar{x}_{s,T}), \qquad x^{T}_{st} = x_{s,T+t}.$$

Then $(\bar{x}^{T}_{st}: s \in S, t \geq 0)$ is the standard Ornstein–Uhlenbeck sheet in $(T_{x_{sT}}M: s \in S)$, of parameter $\lambda$ and, for each $s \in S$, $(x^{T}_{st}: t \geq 0)$ is the stochastic development of $(\bar{x}^{T}_{st}: t \geq 0)$, starting from $x_{sT}$. Hence, in the flat case, $(x_t: t \geq 0)$ is Markov.

To move beyond the flat case, we propose to adopt a differential equations approach, reflecting the fact that the holonomy of an evolving loop will change. In the flat case we have

$$d_t x_{st} = \tau_{s,t0}\, d_t \bar{x}_{st}$$

and

$$D_s\, d_t \bar{x}_{st} = \tau^{+}_{s0,0}\sigma(\lambda, \tau_0)\partial_s\, d_t w_{st} - \lambda\,\partial s\, d_t \bar{x}_{st}.$$

Hence, at least formally,

$$\begin{aligned} D_s\, d_t x_{st} &= D_s(\tau_{s,t0})\, d_t \bar{x}_{st} + \tau_{s,t0} D_s\, d_t \bar{x}_{st} \\ &= \tau_{s,t0}\tau^{+}_{s0,0}\sigma(\lambda, \tau_0)\partial_s\, d_t w_{st} - \lambda\,\partial s\, d_t x_{st} \\ &= \tau^{+}_{s0,t}\sigma(\lambda, \tau_t)(\tau_{0,t0}\partial_s\, d_t w_{st}) - \lambda\,\partial s\, d_t x_{st}. \end{aligned}$$

Again formally, in conjunction with the periodic boundary condition $d_t x_{0t} = d_t x_{1t}$, this differential equation should determine $(x_t: t \geq 0)$ uniquely, expressing the fact that the Itô differential $(d_t x_{st}: s \in S)$ is an Ornstein–Uhlenbeck process in $(T_{x_{st}}M: s \in S)$ with covariance $c_\lambda(\cdot, \tau_t)\, dt$. We propose this differential equation as a good definition of a Brownian motion of loops, even when the connection is not flat. We would expect the solution $(x_t: t \geq 0)$ to be Markov and to inherit certain symmetry properties from its infinitesimal increments. Indeed, by Theorem 3.1, if one accepts that the increments of a loop space Brownian motion should have the various natural properties characterizing Ornstein–Uhlenbeck processes, then any such Brownian motion would arise in this way.

However, so far we are unable to make sense of the equation in general. The difficulty lies in the fact that the existing theory of two-parameter stochastic differential equations (see, e.g., [7]) relies heavily on adaptedness, in both parameter directions, whereas here the equation, and in particular the periodic boundary condition, allows adaptedness in only one direction. A resolution of this sort of adaptedness problem for stochastic partial differential equations of hyperbolic type remains an open problem, of importance going beyond the geometrical exercise considered in this paper.

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