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BOOK REVIEW

G. Fayolle, V. A. Malyshev and M. V. Menshikov, *Topics in the Constructive Theory of Countable Markov Chains.*

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A fundamental concept of interest in the analysis of a physical or formal system is its stability. If the time evolution of the system is described by an oridinary differential equation, stability can be studied with the aid of a Lyapunov function (see, e.g., [1]).

Lyapunov-type criteria can as well be used in the context of stochastic systems. For systems modelled as Markov chains, stability (ergodicity) criteria were first presented by Foster [2]. Let $X = (X_n;\ n = 0, 1, \dots)$ be an irreducible Markov chain defined on some countable state space S and with transition probabilities p_{ij}, $i, j \in S$. According to the criteria by Foster, the Markov chain (X_n) is ergodic if (and only if) there exists a finite set $S_0 \in S$, nonnegative numbers $f(i)$, $i \in S$ and a constant $\varepsilon > 0$ such that

$$(1) \qquad \sum_{j \in S} p_{ij} f(j) < \infty \quad \text{for all } i \in S_0$$

and

$$(2) \qquad \sum_{j \in S} p_{ij} f(j) \le f(i) - \varepsilon \quad \text{for all } i \notin S_0.$$

The map $f\colon S \to \mathbb{R}_+$ is commonly referred to as a Lyapunov–Foster (LF) function. Writing (1) and (2) in the equivalent forms

$$E\big(f(X_{n+1}) | X_n = i\big) < \infty \quad \text{for } i \in S_0$$

and

$$E\big(f(X_{n+1}) - f(X_n) | X_n = i\big) \le -\varepsilon \quad \text{for } i \notin S_0$$

shows that the LF criteria can be interpreted as a drift condition in analogy with Lyapunov's stability criteria for ordinary differential equations.

It is possible to distinguish also the subclasses of null recurrent and transient chains among nonergodic Markov chains ([2]). Furthermore, the technique applies to Markov chains defined on general measurable state spaces (see e.g., [3]).

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If the state space S is one-dimensional, $S = \mathbb{N}$, then the finding of an LF function is often easy. It can be, for example, linear, as is the case for a reflecting random walk, or it can be geometric, as for $M/G/1$-queues ([2]).

The topic of the monograph by Fayolle, Malyshev and Menshikov is random walks on the lattices $\mathbb{Z}^d$. These comprise an important subclass of Markov chains having connections to diffusion processes and associated partial differential equations as well as to applications such as queueing networks. If the state space is finite, the situation is trivial. For an infinite state space, the problem of finding an LF function becomes a challenging problem. Three phenomena must be taken into account, namely, the geometry of the state space, the reflections at the boundary and the stabilizing drift from "far away."

Chapters 1 and 2 of the book are devoted to general stability results for Markov chains. Martingale techniques play a central role in the approach. In a remark, the authors state: "... some deeper meta-theory for producing such [martingale based] criteria might well exist, too."

The results of Chapter 3 deal with random walks on the plane lattice $\mathbb{Z}_+^2$. For these chains, it is possible to obtain a complete classification into ergodic, null and transient chains in terms of LF criteria. As the main application come Jackson networks.

The theme of Chapter 4 is the exploitation of the vector fields generated by the mean drifts at the boundaries. The vector field gives rise to a (deterministic) dynamical system $\Gamma = (\Gamma(t);\ t \geq 0)$ whose stability properties reflect the stability properties of the random walk under study. Using this device, it is possible to obtain a complete classification in $\mathbb{Z}_+^3$.

In Chapter 5, a new phenomenon is studied, namely, random scattering within the dynamical system Γ when it hits one-dimensional faces. The Martin boundary theory plays an important role here. As an example, a queueing model for a database is presented.

The rest of the book is devoted to Markov chains whose transition probabilities depend smoothly on a parameter. The natural question posed and investigated here is "the stability of the stability criterion."

The monograph by Fayolle, Malyshev and Menshikov presents a collection of interesting recent results in Markov chain theory. It promises, however, somewhat more than it is able to offer. Although the main motive comes from applications to stochastic networks, the book does not contain a systematic accessible account of these, but only two brief subsections placed at the ends of highly technical main sections. Also, a theoretically oriented reader may feel slightly disappointed: after all, there is nothing about the promised connections to other fields such as Toeplitz operators or diffusion processes (cf. the introduction of the book). As regards these potentially interesting relations, the authors refer to their plans to write a new monograph.

REFERENCES

[1] Coddington, E. A. and Levinson N. (1955). *Theory of Orginary Differential Equations.* McGraw-Hill, New York.

[2] FOSTER, F. G. (1953). On stochastic matrices associated with certain queueing processes. *Ann. Math. Stat.* **24** 355–365.

[3] MEYN, S. P. and TWEEDIE, R. L. (1993). *Markov Chains and Stochastic Stability*. Springer, New York.

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