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CENTRAL LIMIT THEOREMS FOR QUADRATIC FORMS WITH TIME-DOMAIN CONDITIONS

By Liudas Giraitis[1] and Murad S. Taqqu[2]

Boston University

We establish the central limit theorem for quadratic forms $\sum_{t,s=1}^N b(t-s)P_{m,n}(X_t, X_s)$ of the bivariate Appell polynomials $P_{m,n}(X_t, X_s)$ under time-domain conditions. These conditions relate the weights $b(t)$ and the covariances of the sequences $(P_{m,n}(X_t, X_s))$ and (X_t). The time-domain approach, together with the spectral domain approach developed earlier, yields a general set of conditions for central limit theorems.

1. Introduction.

The bivariate Appell polynomials provide a helpful tool for establishing both central limit theorems (CLT) and noncentral limit theorems (NCLT) for quadratic forms in random variables $G(X_t)$, when (X_t) is a linear or Gaussian process and G is a polynomial. These limit theorems can be used to derive properties of statistical estimators that involve quadratic forms, for example, the Whittle estimator (see [13]). Hermite polynomials suffice when X_t is Gaussian, but Appell polynomials appear when X_t is a linear process. The Hermite polynomials are Appell polynomials associated with the Gaussian distribution.

In this paper we study central limit theorems for quadratic forms

$$Q_N := \sum_{t,s=1}^{N} b(t-s)P_{m,n}(X_t, X_s) \tag{1}$$

involving the Appell polynomials

$$P_{m,n}(X_t, X_s) =: \underbrace{X_t, \dots, X_t}_{m}, \underbrace{X_s, \dots, X_s}_{n} : .$$

Here $P_{m,n}(X_t, X_s)$ is a bivariate Appell polynomial (Wick power) of the linear variables X_t and X_s, $m, n \geq 0$, $m+n \geq 1$ and

$$X_t = \sum_{u \in \mathrm{Z}} a(t-u)\xi_u, \quad t \in \mathrm{Z} \tag{2}$$

is a linear process; that is, the random variables $\xi_t,\ t \in \mathrm{Z}$ are independent and identically distributed, $E\xi_0 = 0$, $E\xi_0^2 = 1$ and the sequence $a(t), t \in \mathrm{Z}$ of real-valued weights satisfies the condition $\sum_s a^2(s) < \infty$. We assume $E|\xi_u|^{2(m+n)} < \infty$ in order to ensure that Q_N has a finite variance. Our goal

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is to provide sufficient conditions for $N^{-1/2}Q_N$ to converge to a normal distribution. The definition of multivariate Appell polynomials is given in Section 3. These polynomials are a multivariate generalization of the univariate Appell polynomials [or Hermite polynomials if (X_t) is Gaussian] and, like them, they play an important role in the limit theory of quadratic forms of dependent variables.

Central limit theorems involving Hermite or Appell polynomials have been studied by Sun [21, 22], Breuer and Major [6], Giraitis [10], Giraitis and Surgailis [11, 12], Fox and Taqqu [9], Ho and Sun [17, 18], Ho [16], Avram [2, 3], Avram and Fox [4], Giraitis and Taqqu [14] and Arcones [1].

We will show, in particular, that the assumption

$$\sum_{l_1, l_2, t\in \mathrm{Z}} \left| b(l_1)b(l_2)\,\mathrm{Cov}(P_{m,n}(X_t, X_{t+l_1}), P_{m,n}(X_0, X_{l_2}))\right| < \infty, \tag{3}$$

which ensures that the relation Var $Q_N \le$ const N holds, yields the CLT. In fact, the relation

$$\sum_{t\in \mathrm{Z}} \left|\mathrm{Cov}(P_{m,n}(X_t, X_{t+l_1}), P_{m,n}(X_0, X_{l_2}))\right| < \infty, \qquad l_1, l_2 \in \mathrm{Z},$$

turns out to be equivalent to the condition

$$\sum_t |r(t)|^{m+n} < \infty,$$

where $r(t) = EX_tX_0$ is the covariance function of the process (X_t) and $m+n$ is the order of the Appell polynomial $P_{m,n}(X_t, X_{t+l_1})$.

Assumption (3) underlines the similarities between the central limit theorem for quadratic forms in linear or Gaussian variables with long-range dependence and the central limit theorem for univariate sums $S_N = N^{-1/2}\sum_{t=0}^N G(X_t)$. Breuer and Major [6] proved that if the covariance of the Gaussian stationary process (X_t) satisfies the condition $\sum_{t\in \mathrm{Z}} |r(t)|^m < \infty$ where m is the so-called Hermite rank of the function G, $EG(X_t) = 0$ and $EG(X_t)^2 < \infty$, then the CLT for S_N holds. Giraitis and Surgailis [11] showed that the asymptotic normality of $N^{-1/2}\sum_{t=1}^N G(X_t)$ can be established in terms of the correlation function $r_G(t) = EG(X_t)G(X_0)$, without referring specifically to the concept of Hermite or Appell rank (it plays an important role in the proof). They showed that the CLT holds if $\sum_{t\in \mathrm{Z}} |r_G(t)| < \infty$ when X_t is Gaussian. In addition, Giraitis [10] proved that if $\sum_{t\in \mathrm{Z}} |r_G(t)| < \infty$, then the CLT holds also for the linear process X_t (2) if the function G is sufficiently smooth, for example, if G is a polynomial. Condition (3) thus extends the univariate condition $\sum_{t\in \mathrm{Z}} |r_G(t)| < \infty$ to quadratic forms. However, in contrast to the CLT for univariate sums, it does not cover certain additional cases, first discovered by Fox and Taqqu [8], where the CLT for quadratic form also holds. Specifically, it excludes the possible compensation of the long-range dependence of (X_t) by a fast decay of the weights $b(t)$. These cases, which do not have a simple formulation in the time domain, are best characterized in the spectral domain. Conditions for the CLT in these situations were obtained by Giraitis and Taqqu [14], under assumptions involving m and n and the spectral density of (X_t). (For NCLT results, see [15].) The conditions of this paper are stated in the time domain and are, in general, not equivalent to those in [14].

The method of proof for establishing the CLT is also different in the time domain. Whereas in [14] we use approximation methods, here we apply the method of moments. This is because, in the spectral domain, one can approximate the possibly unbounded spectrum by a bounded one. In the time domain, however, one has to deal directly with the covariances, which decrease slowly.

More specifically, Theorem 2.2 uses conditions formulated in terms of the L^p norms of the covariance $r(t)$ and the weights $b(t)$. These conditions are of a global nature and do not require the power (or regular variation) decay of $r(t)$ and $b(t)$ as $t \to \infty$. Therefore, they do not imply the regular variation of the spectral density $f(x)$ [Fourier transform of the covariance $r(t)$], nor of the Fourier transform $\widehat{b}(x)$ of the weights $b(t)$ as the frequency $x \to 0$. They also do not imply the finiteness of the norms $\|f\|_{L_{p'}}$ and $\|\widehat{b}\|_{L_{q'}}$ of the spectral density f and the function $\widehat{b}$ in the spectral domain, with p', q' complementary to $p, q (p' = p/(1-p),\ q' = q/(1-q),\ p, q \geq 1)$, if $p \neq 2$ and $q \neq 2$. Therefore, our time-domain conditions require different methods of proof. The CLT with spectral domain conditions was obtained in [14]. We considered long-memory sequences and showed that their long-memory behavior, expressed in the spectral domain (the spectral density blows up at the origin), can be compensated by the decay of the Fourier transforms $\widehat{b}(x)$ at the origin. We used an approximation technique, essentially replacing the spectral density by a bounded one, which allowed us to approximate the bivariate quadratic forms by univariate sums of m-dependent random variables. Such an approximation technique, however, does not work with time-domain conditions. We thus use here diagrams and the method of moments. It was already well known that certain diagram formulas can be used to bound the moments and cumulants for functionals of a Gaussian process (see, [6], [11] and [9]). It was, in fact, possible to select a special class of diagrams in such a way that the contribution of the rest was negligible. In the case of Appell polynomials, analogous diagram formulas can be written down, but some additional, more complicated diagrams appear. Hence, in this case, similar results can be expected. But to bound the contribution of the additional diagrams is a hard technical problem. This is done in this paper under time-domain L^p-type conditions on both the covariance r and the weights b. These time-domain conditions clarify the underlying dependence structure and are easy to apply.

The paper is structured as follows: Section 2 contains the main results, Section 3, a description of the Appell polynomials and Section 4, the proofs. Multivariate extensions are given in Section 5.

2. Main results.

Condition (3) ensures the finiteness of the limiting variance and implies the CLT. This fact is stated in the following theorem.

THEOREM 2.1. *Suppose*

$$\sum_{l_1, l_2, t \in \mathbb{Z}} \left| b(l_1) b(l_2) \operatorname{Cov}(P_{m,n}(X_t, X_{t+l_1}), P_{m,n}(X_0, X_{l_2})) \right| < \infty. \tag{4}$$

If $b(0) = 0$, *suppose in addition* $\sum_{t \in \mathbb{Z}} |r(t)|^{m+n} < \infty$. *Then the CLT holds:*

$$N^{-1/2} Q_N \Rightarrow \mathcal{N}(0, \sigma^2), \qquad N \to \infty \tag{5}$$

and the limiting variance is

$$\sigma^2 = \sum_{l_1, l_2, t \in \mathbb{Z}} b(l_1) b(l_2) \operatorname{Cov}(P_{m,n}(X_t, X_{t+l_1}), P_{m,n}(X_0, X_{l_2})). \tag{6}$$

The next theorem provides a condition on r and b, under which (4) is satisfied.

THEOREM 2.2. *If*

$$r \in L^p, \qquad b \in L^q, \qquad p, q \geq 1 \tag{7}$$

and

$$\min(mp^{-1}, 1) + \min(np^{-1}, 1) + 2q^{-1} \geq 3 \tag{8}$$

then the CLT holds and the limiting variance is (6).

The next theorem makes use of Theorem 2.4 in [14] and shows that under some rather restrictive conditions on the covariance $r(t)$ and the weights $b(t)$, the long-range dependence of the (X_t) can be compensated by the fast decay of $b(t)$ in such a way that the CLT holds. These conditions ensure, in fact, that the sufficient assumptions in the spectral domain, provided in [14], are satisfied. The theorem involves *quasi-monotone* sequences: a sequence $a(t)$ is quasi-monotonically convergent to 0 if $a(t) \to 0$ and $a(t+1) \leq a(t)(1 + c/t)$ as $t \to \infty$ for some $c > 0$. The sequence $a(t)$ has *bounded variation* if $\sum_{t=1}^{\infty} |a(t+1) - a(t)| < \infty$.

THEOREM 2.3. *Suppose* $r(t) = |t|^{-\gamma_1} L_1(|t|)$, $b(t) = |t|^{-\gamma_2} L_2(|t|)$ $(0 < \gamma_1, \gamma_2 < 3)$ *and*

$$\min(m\gamma_1, 1) + \min(n\gamma_1, 1) + 2\gamma_2 > 3, \tag{9}$$

where $m, n \geq 1$. *Suppose in addition that both sequences* $\{r(t)\}$ *and* $\{b(t)\}$ *have bounded variation and are quasi-monotonically convergent to 0; if* $1 < \gamma_1 < 3$, $r(t)$ *has the same sign for large* t *and satisfies* $\sum_{t \in \mathbb{Z}} r(t) = 0$; *if* $1 < \gamma_2 < 3$, $b(t)$ *has the same sign for large* t *and satisfies* $\sum_{t \in \mathbb{Z}} b(t) = 0$. *Then the CLT holds. The limiting variance is expressed by* (6) *if* $0 < \gamma_1, \gamma_2 < 1$.

The assumptions in Theorem 2.3 were used in [7], Theorem 3.

REMARKS.

(i) Under the assumptions of the previous theorems, the finite-dimensional distributions of $\{Q_{[Nt]}/\sqrt{N},\ t \geq 0\}$ converge to those of $\{\sigma B(t),\ t \geq 0\}$ where $B(t)$ is standard Brownian motion. The proof of this more general statement is similar to the proof of the one-dimensional theorems.

(ii) Condition (4) is not as general as the spectral domain condition (2.5) in [14] but it is formulated entirely in the time domain. Theorem 2.2 can be viewed as the time-domain analogue of Theorem 2.3 of [14]. Relations (7) and (8) correspond to (2.10) and (2.11) of [14] but one cannot, in general, pass from one to the other. Note also that the direction of the inequalities (8) here and (2.13) in [14] is reversed.

Theorems 2.1, 2.2 and 2.3 are proved in Section 4. The value of a constant C that appears in the proofs may change from line to line.

3. Appell polynomials and their cumulants. Let μ be some probability measure on R with mean $\int x\, d\mu(x) = 0$. The (*univariate*) *Appell polynomials* $P_n(x)$ corresponding to the distribution μ generalize the ordinary "powers" because they are defined by the differential equation

$$P'_n(x) = nP_{n-1}(x)$$

with

$$EP_n(X) = \int P_n(x) d\mu(x) = 0, \qquad n = 1, 2, \dots,$$

providing the constants of integration. Thus, $P_n(x)$ is defined whenever μ has moments of order n. We note that

$$P_0(x) = \int d\mu(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = x^2 - \int x^2\, d\mu(x).$$

The Appell polynomials are orthogonal only when μ is a Gaussian measure [20]. In that case, they are identical to the Hermite polynomials. The Appell polynomials $P_n(x)$ can also be defined by the generating function

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} P_n(x) = \frac{\exp(zx)}{\int e^{zx}\, d\mu(x)}.$$

If the moment generating function $M(z) = \int e^{zx}\, d\mu(x)$ does not exist, this relation has to be understood as a formal expansion that allows $P_n(x)$ to be obtained by formal differentiation:

$$P_n(x) = \frac{d^n}{dz^n} \frac{\exp(zx)}{\int e^{zx}\, d\mu(x)}\bigg|_{z=0},$$

where one sets $(d^k/dz^k)M(z)|_{z=0} = EX^k,\ k \geq 0$. Finally, $P_n(x)$ can also be expressed as

$$P_n(x) = \sum_{k=0}^{r} \biggl(\sum_{\{V\}(n-k)} (-1)^r \prod_{i=1}^{r} \chi_{|V_i|} \biggr) x^k$$

(see [23]), where $\chi_k = \chi(k, \mu)$ is the kth cumulant of the measure μ, and the sum is taken over all partitions $(V_1, \dots, V_r)$, $r \geq 1$ of the set $\{1, \dots, n-k\}$ such that $|V_i| \geq 2$. We set $\sum_{\{V\}(0)} \cdots = 1$, so that the coefficient of x^n is 1. See also [5] and [12] for an introduction to Appell polynomials.

To define the multidimensional analog of the Appell polynomials it is useful to first introduce the *Wick products* (also called *Wick powers*) (cf. [12], [23]). These are multivariate polynomials

$$:y_1, \dots, y_n:^{(\nu)} = \frac{\partial^n}{\partial z_1 \cdots \partial z_n} \left[\exp\left(\sum_1^n z_j y_j \right) \Big/ \int_{\mathrm{R}^n} \exp\left(\sum_1^n z_j y_j \right) d\nu(y) \right] \bigg|_{z_1 = \cdots = z_n = 0}$$

corresponding to a probability measure ν on R^n. Interpret this again as a formal expression if ν does not have a moment generating function, the Wick products being then obtained by formal differentiation. A sufficient condition for the Wick products $:y_1, \dots, y_n:^{(\nu)}$ to exist is $E|Y_i|^n < \infty$, $i = 1, \dots, n$. If $Y_1, \dots, Y_n$ are random variables with joint distribution $\nu(dx) = P((Y_1, \dots, Y_n) \in dx)$, then

$$:Y_1, \dots, Y_n: = :y_1, \dots, y_n:^{(\nu)}\big|_{y_1 = Y_1, \dots, y_n = Y_n}$$

is also called the Wick product of the random variables $Y_1, \dots Y_n$. It is convenient to use the notation

$$:\underbrace{Y_{t_1}, \dots, Y_{t_1}}_{n_1}, \dots, \underbrace{Y_{t_k}, \dots, Y_{t_k}}_{n_k}: = P_{n_1, \dots, n_k}(Y_{t_1}, \dots, Y_{t_k})$$

(the indices in P correspond to the number of times that the variables in ": :" are repeated). The polynomials $P_{n_1, \dots, n_k}$ can be defined also by the recurrence relations

$$\frac{\partial}{\partial y_j} P_{n_1, \dots, n_k}(y_1, \dots, y_k) = n_j P_{n_1, \dots, n_j - 1, \dots, n_k}(y_1, \dots, y_k),$$

$$E P_{n_1, \dots, n_k}(Y_{t_1}, \dots, Y_{t_k}) = 0,$$

setting $P_0 \equiv 1$.

We can now relate Wick products to Appell polynomials. If P_n, $n \geq 1$ is the univariate Appell polynomial corresponding to the distribution $\mu(dx) = P(Y \in dx)$, defined earlier, then

$$:\underbrace{Y, \dots, Y}_{n}: = P_n(Y).$$

We provide below some properties of the Wick products (cf. [12], [23]). Let W be a finite set and Y_i, $i \in W$ be a system of random variables. Let $Y^W = \prod_{i \in W} Y_i$ be the ordinary product, $:Y^W:$ the Wick product, and $\chi(Y^W) = \chi(Y_i, i \in W)$

be the cumulant of the variables $Y_i,\ i \in W$, respectively. We now recall the definition of the mixed cumulant

$$\chi(Y_1, \dots, Y_n) = \frac{\partial^n}{\partial z_1 \cdots \partial z_n} \log E \exp\left(\sum_{i=1}^{n} z_j Y_j \right)\Big|_{z_1 = \cdots = z_n = 0}.$$

The following relations hold ([23], Prop. 1):

$$:Y^W: = \sum_{U \subset W} Y^U \sum_{\{V\}} (-1)^r \chi(Y^{V_1}) \cdots \chi(Y^{V_r}),$$

$$Y^W = \sum_{U \subset W} :Y^U: \sum_{\{V\}} \chi(Y^{V_1}) \cdots \chi(Y^{V_r}) = \sum_{U \subset W} :Y^U: E(Y^{W \setminus U}),$$

where the sum $\sum_{U \subset W}$ is taken over all subsets $U \subset W$, including $U = \varnothing$, and the sum $\sum_{\{V\}}$ is over all partitions $\{V\} = (V_1, \dots, V_r)$, $r \geq 1$ of the set $W \setminus U$. We define $Y^{\varnothing} =: Y^{\varnothing} := \chi(Y^{\varnothing}) = 1$.

It follows that Wick products are multilinear in the sense that, if $\xi_u,\ u \geq 1$, are independent random variables, then for any $N \geq 1$,

$$: \sum_{u_1=1}^{N} \lambda_{u_1}^{(1)} \xi_{u_1}, \dots, \sum_{u_n=1}^{N} \lambda_{u_n}^{(n)} \xi_{u_n} := \sum_{u_1=1}^{N} \cdots \sum_{u_n=1}^{N} \lambda_{u_1}^{(1)} \cdots \lambda_{u_n}^{(n)} : \xi_{u_1}, \dots, \xi_{u_n}:$$

(see also [5]). In particular, Q_N defined by (1), (2) can be written

$$Q_N = \sum_{u_1, \dots, u_{m+n} \in \mathbb{Z}} d_N(u_1, \dots, u_{m+n}) : \xi_{u_1}, \dots, \xi_{u_{m+n}} :, \tag{10}$$

where

$$\begin{aligned} &d_N(u_1, \dots, u_{m+n}) \\ &\quad = \sum_{t, s \in \mathbb{Z}} b(t-s) a(t-u_1) \cdots a(t-u_m) a(s-u_{m+1}) \cdots a(s-u_{m+n}). \end{aligned} \tag{11}$$

Relations of this type will be used implicitly in the sequel.

An important property of the Appell polynomials is the existence of simple combinatorial rules for calculation of the (mixed) cumulants, analogous to the familiar diagrammatic formalism for the mixed cumulants of the Hermite polynomials with respect to a Gaussian measure [19]. Let us assume that W is a union of (disjoint) subsets $W_1, \dots, W_k$. If $(i, 1), (i, 2), \dots, (i, n_i)$ represent the elements of the subset W_i, $i = 1, \dots, k$, then we can represent W as a table consisting of rows $W_1, \dots, W_k$, as follows:

$$\begin{pmatrix} (1,1), \dots, (1, n_1) \\ \dots\dots\dots \\ (k,1), \dots, (k, n_k) \end{pmatrix} = W. \tag{12}$$

By a *diagram* γ we mean a partition $\gamma = (V_1, \dots, V_r)$, $r = 1, 2, \dots$ of the table W into nonempty sets V_i (the "edges" of the diagram) such that $|V_i| \geq 1$. We shall call the edge V_i of the diagram γ *flat*, if it is contained in one row of the table W; and *free*, if it consists of one element, that is, $|V_i| = 1$. We shall call the diagram *connected* if it does not split the rows of the table W into two or more disjoint subsets. We shall call the diagram $\gamma = (V_1, \dots, V_r)$ *Gaussian* if $|V_1| = \cdots = |V_r| = 2$. Suppose as given a system of random variables $Y_{i,j}$ indexed by $(i, j) \in W$. Set for $V \subset W$,

$$Y^V = \prod_{(i,j)\in V} Y_{i,j} \quad \text{and} \quad :Y^V:=:(Y_{i,j}, (i,j) \in V):.$$

For each diagram $\gamma = (V_1, \dots, V_r)$ we define the number

$$I_\gamma = \prod_{j=1}^{r} \chi(Y^{V_j}). \tag{13}$$

PROPOSITION 3.1 (cf. [12], [23]). *Each of the numbers*

(i) $EY^W = E(Y^{W_1} \cdots Y^{W_k})$,
(ii) $E(:Y^{W_1}: \cdots :Y^{W_k}:)$,
(iii) $\chi(Y^{W_1}, \dots, Y^{W_k})$,
(iv) $\chi(:Y^{W_1}:, \dots, :Y^{W_k}:)$

is equal to

$$\sum I_\gamma,$$

where the sum is taken, respectively, over (i) *all diagrams,* (ii) *all diagrams without flat edges,* (iii) *all connected diagrams,* (iv) *all connected diagrams without flat edges. If* $EY_{i,j} = 0$ *for all* $(i, j) \in W$*, then the diagrams in* (i)–(iv) *have no free edges.*

It follows, for example, that $E{:}Y^W := 0$ (take $W = W_1$, then W has only 1 row and all diagrams have flat edges).

We shall now apply the proposition to linear random variables X_t, that is, of the form (2). The cumulant of X_t can be expressed in the form

$$\begin{aligned}\chi(X_{t_1}, \dots, X_{t_k}) &= \sum_{s_1,\dots,s_k} \chi(\xi_{s_1}, \dots, \xi_{s_k}) a(t_1 - s_1) \cdots a(t_k - s_k)\\ &= \chi_k(\xi_0) \sum_s a(t_1 - s) \cdots a(t_k - s),\end{aligned}$$

where

$$\chi_k(\xi_0) = \chi_k(\underbrace{\xi_0, \dots, \xi_0}_{k})$$

is the kth cumulant of ξ_0. This formula holds in view of the multilinearity of the cumulant and the fact that for i.i.d. variables (ξ_j), $\chi_k(\xi_{j_1}, \dots, \xi_{j_k}) = 0$ if $j_i \neq j_l$ for some $i \neq l$. This fact, part (iv) of Proposition 3.1 and the definition (13) of $I\gamma$ imply the following formula for the cumulants of the Wick products of linear variables (2):

$$\text{cum}(:X_{t_{1,1}}, \dots, X_{t_{1,n_1}}:, \dots, :X_{t_{k,1}}, \dots, X_{t_{k,n_k}}:) = \sum_{\gamma \in \Gamma(n_1,\dots,n_k)} d_\gamma J_\gamma, \tag{14}$$

where $\Gamma(n_1, \dots, n_k)$ denotes the set of all connected diagrams $\gamma = (V_1, \dots, V_r)$ of the table W (12) without flat edges, $d_\gamma = \chi_{|V_1|}(\xi_0) \cdots \chi_{|V_r|}(\xi_0)$ and

$$J_\gamma = \sum_{s_1, \dots, s_r \in \mathbb{Z}} \prod_{j=1}^{k} \Big[a(t_{j,1} - s_{j,1}) \cdots a(t_{j,n_1} - s_{j,n_1}) \cdots a(t_{k,1} - s_{k,1}) \cdots a(t_{k,n_k} - s_{k,n_k}) \Big], \tag{15}$$

where $s_{i,j} \equiv s_l$ if $(i, j) \in V_l$, $l = 1, \dots, r$. We refer to (14) as the diagram formula.

4. Proof of the time-domain theorems. We start with the following lemma, which relates the covariances of the polynomials $P_{m,n}(X_t, X_{t+l})$ to those of the variables X_t.

LEMMA 4.1.

$$\sum_{t \in \mathbb{Z}} \big| \mathrm{Cov}(P_{m,n}(X_t, X_{t+l_1}), P_{m,n}(X_0, X_{l_2})) \big| < \infty \quad \text{for any } l_1, l_2 \in \mathbb{Z} \tag{16}$$

if and only if

$$\sum_{t \in \mathbb{Z}} |r(t)|^{m+n} < \infty. \tag{17}$$

PROOF. First we show that (16) implies (17). Recall that $\Gamma(m+n, m+n)$ denotes the set of connected diagrams of the table

$$W = \begin{pmatrix} (1,1), \dots, (1, m+n) \\ (2,1), \dots, (2, m+n) \end{pmatrix} = \begin{pmatrix} (1,1), \dots, (1,m), (1, m+1), \dots, (1, m+n) \\ (2,1), \dots, (2,m), (2, m+1), \dots, (2, m+n) \end{pmatrix}$$

without flat edges. Here W consists of two rows and $m+n$ columns. By the diagram formula (14), we have

$$\begin{aligned} R(t; l_1, l_2) &:= \mathrm{Cov}(P_{m,n}(X_t, X_{t+l_1}), P_{m,n}(X_0, X_{l_2})) \\ &= \sum_{\gamma = (V_1, \dots, V_r) \in \Gamma(m+n, m+n)} d(\gamma) r_{V_1}(t; l_1, l_2) \cdots r_{V_r}(t; l_1, l_2), \end{aligned}$$

where

$$r_{V_l}(t; l_1, l_2) = \sum_{u \in \mathbb{Z}} \prod_{(i,j) \in V_l} a(v_{i,j} - u), \tag{18}$$

and $v_{i,j} = t$ if $(i,j) \in ((1,1), \dots, (1,m))$; $v_{i,j} = t + l_1$ if $(i,j) \in ((1, m+1), \dots, (1, m+n))$; $v_{i,j} = 0$ if $(i,j) \in ((2,1), \dots, (2,m))$; $v_{i,j} = l_2$ if $(i,j) \in ((2, m+1), \dots, (2, m+n))$. Rewrite $\Gamma(m+n, m+n)$ as the disjoint union $\Gamma_2 \cup \Gamma_{>2}$, where Γ_2 denotes the Gaussian diagrams $\gamma = (V_1, \dots, V_r)$: $|V_1| = \cdots |V_r| = 2$ and the $\Gamma_{>2}$ the diagrams γ containing $V_l \in \gamma$ such that $|V_l| > 2$. Thus,

$$R(t; l_1, l_2) = \sum_{\gamma \in \Gamma_2} + \sum_{\gamma \in \Gamma_{>2}} =: R_2(t; l_1, l_2) + R_{>2}(t; l_1, l_2). \tag{19}$$

In the case $l_1 = l_2 = 0$, we have

$$R_2(t;0,0) = \sum_{\gamma\in\Gamma_2} d(\gamma) r_{V_1}(t;0,0)\cdots r_{V_{m+n}}(t;0,0) = r(t)^{m+n} \sum_{\gamma\in\Gamma_2} 1$$

since $r_V(t;0,0) = r(t)$ if $|V| = 2$ and $d(\gamma) = (E\xi_0^2)^{m+n} = 1$. Using (16), we will get

$$\left[\sum_{\gamma\in\Gamma_2} 1\right] \sum_{t\in Z} |r(t)|^{m+n} \le \sum_{t\in Z} |R(t,0,0)| + \sum_{t\in Z} |R_{>2}(t,0,0)| < \infty,$$

provided

$$\sum_{t\in Z} |R_{>2}(t;0,0)| < \infty. \tag{20}$$

We shall now prove (20). By definition of $\Gamma_{>2}$, there is V_i such that $|V_i| > 2$. Two cases are possible.

(a) Suppose V_i contains more than one element of both rows of the table W. Using inequalities of the form $a(t-u)a(t-v) \le 2(a^2(t-u) + a^2(t-v))$ and (5), we have

$$\sum_t |r_{V_i}(t,0,0)| \le 4 \sup_t |a(t)|^{|V_i|-4} \sum_{t,u\in Z} a^2(t-u)a^2(-u) < \infty$$

since $\sum_{t\in Z} a^2(t) < \infty$. Therefore,

$$\sum_t |r_{V_1}(t;0,0)|\cdots|r_{V_r}(t;0,0)| \le C \sum_t |r_{V_i}(t;0,0)| < \infty$$

since $|r_{V_j}(t,0,0)| \le C\sum_{u\in Z} |a(t-u)a(-u)| \le C\sum_{u\in Z} |a(u)|^2 < \infty$ uniformly in $t \in Z$ for any $V_j \in \gamma$.

(b) $|V_i| > 2$ and V_i is a " triangle" of γ; that is, it contains only one element from the first or second row of the table W. This time, we consider $\sum r_{V_i}^2$ instead of $\sum |r_{V_i}|$. Estimating as above, we see that

$$\begin{aligned}\sum_t r_{V_i}^2(t;0,0) &\le \text{const} \sum_{t\in Z}\left[\sum_{u\in Z} |a^2(u)a(t-u)|\right]^2 \\ &\le \text{const} \sum_{u,u'\in Z} a^2(u)a^2(u') \sum_{t\in Z} |a(t-u)a(t-u')| \\ &\le \text{const}\left[\sum_{u\in Z} a^2(u)\right]^2 \sum_{t\in Z} a^2(t) < \infty.\end{aligned}$$

In the case (b), γ contains at least two $V_i, V_j \in \gamma$ such that $|V_i| > 2$, $|V_j| > 2$, because W has the same number of elements in each row. Hence the Cauchy inequality yields

$$\begin{aligned}\sum_t |r_{V_1}(t,0,0)\cdots r_{V_r}(t,0,0)| &\le C\sum_t |r_{V_i}(t,0,0)||r_{V_j}(t,0,0)| \\ &\le C\left(\sum_t r_{V_i}^2(t,0,0)\right)^{1/2}\left(\sum_t r_{V_j}^2(t,0,0)\right)^{1/2} < \infty.\end{aligned}$$

Combining cases (a) and (b), we get

$$\sum_t |R_{>2}(t;0,0)| \le \sum_{\gamma\in\Gamma_{>2}} |d(\gamma)| \sum_t |r_{V_1}(t,0,0)| \cdots |r_{V_r}(t,0,0)| < \infty.$$

This proves (20) and hence (17).

We now prove the converse, namely that (17) implies (16). Observe that in the first part of the proof, relation (20) was a consequence only of $\sum_t a^2(t) < \infty$, and that, in fact, it can be strengthened to

$$\sum_{t\in Z} |R_{>2}(t;l_1,l_2)| < \infty \quad \text{for any } l_1, l_2 \in Z.$$

In view of (19), it is then sufficient to show that (17) implies

$$\sum_{t\in Z} |R_2(t;l_1,l_2)| < \infty.$$

But

$$\begin{aligned}\sum_{t\in Z} |R_2(t;l_1,l_2)| &\le \sum_{\gamma\in\Gamma_2} |d(\gamma)| \sum_t |r_{V_1}(t;l_1,l_2)| \cdots |r_{V_{m+n}}(t;l_1,l_2)| \\ &\le 4 \sum_{\gamma\in\Gamma_2} |d(\gamma)| \sum_t |r(t)|^{m+n} < \infty\end{aligned}$$

by (17), since l_1 and l_2 are fixed and

$$|r_{V_i}(t,l_1,l_2)| \le \max\{|r(t)|, |r(t-l_2)|, |r(t+l_1)|, |r(t+l_1-l_2)|\}. \qquad \square$$

PROOF OF THEOREM 2.1. We divide Q_N (1) into two parts:

$$\begin{aligned}Q_N &:= \sum_{t,s=1}^{N} b(t-s) P_{m,n}(X_t, X_s) \\ &= \sum_{t,s=1:|t-s|>K}^{N} + \sum_{t,s=1:|t-s|\le K}^{N} = Q_{N,K}^{(1)} + Q_{N,K}^{(2)},\end{aligned} \tag{21}$$

where $K > 1$ is a fixed constant. The CLT in Theorem 2.1 will be proved if we show that

$$\lim_{N\to\infty} N^{-1} \operatorname{Var} Q_{N,K}^{(1)} \le \delta(K) \to 0, \qquad K \to \infty \tag{22}$$

and

$$N^{-1/2}\big(Q_{N,K}^{(2)} - EQ_{N,K}^{(2)}\big) \Rightarrow N(0, \sigma_K^2), \qquad N \to \infty, \tag{23}$$

where the limit

$$\sigma_K^2 \to \sigma^2, \qquad K \to \infty \tag{24}$$

exists and is defined by (6).

(i) We first prove (22). Note that

$$N^{-1}\,\mathrm{Var}\,Q^{(1)}_{N,K} = \sum_{t,s,t',s'=1}^{N} b(t-s)\check{(}|t-s|>K)b(t'-s')\check{(}|t'-s'|>K)$$
$$\times\,\mathrm{Cov}(P_{m,n}(X_t, X_s), P_{m,n}(X_{t',s'}))$$

Since X is strictly stationary,

$$\mathrm{Cov}(P_{m,n}(X_t, X_s), P_{m,n}(X_{t'}, X_{s'}))$$
$$= \mathrm{Cov}(P_{m,n}(X_{t-t'}, X_{s-t'}), P_{m,n}(X_0, X_{s'-t'})).$$

Setting $-l_1 = t-s$ and $-l_2 = t'-s'$, and denoting $t-t'$ by t, we get

$$N^{-1}\,\mathrm{Var}\,Q^{(1)}_{N,K}$$
$$\le \sum_{l_1,l_2,t\in \mathrm{Z}} |b(-l_1)\check{\{}|l_1| \ge K\}b(-l_2)\,\mathrm{Cov}(P_{m,n}(X_t, X_{t+l_1})), P_{m,n}(X_0, X_{l_2})|.$$

Relation (4) implies that this tends to 0 as $K\to\infty$ and hence yields (22).

(ii) We now establish the CLT (23) for $Q^{(2)}_{N,K}$. Assume first that all the moments of ξ_0 exist. It is then sufficient to show that all the cumulants of $N^{-1/2}Q^{(2)}_{N,K}$ of order higher than two converge to zero. We have

$$Q^{(2)}_{N,K} = \sum_{l=-K}^{K} b(-l)\sum_{t=1}^{N} P_{m,n}(X_t, X_{t+l}) + R_{N,K},$$

where the correction $R_{N,K}$ involves a finite number of terms independent of N. Since $\mathrm{Var}\,R_{N,K} < \infty$ uniformly bounded in $N \ge 1$, we have $N^{-1/2}R_{N,K} \to 0$ in probability as $N\to\infty$ for any fixed $K>0$. Set $U_N(l) = \sum_{t=1}^{N} P_{m,n}(X_t, X_{t+l})$. Since cumulants are multilinear, it is sufficient to show

(25) $$\mathrm{cum}(U_N(l_1),\dots,U_N(l_k)) = o(N^{k/2})$$

as $N\to\infty$ for any $l_1,\dots,l_k \in \mathrm{Z}$ and $k\ge 3$ and to show convergence of the covariances

(26) $$N^{-1}\,\mathrm{Cov}(U_N(l_1), U_N(l_2)) \to \sigma_{l_1,l_2} := \sum_{t\in\mathrm{Z}} \mathrm{Cov}(P_{m,n}(X_t, X_{t+l_1}), P_{m,n}(X_0, X_{l_2}))$$

as $N\to\infty$ for any $l_1, l_2 \in \mathrm{Z}$. Observe that in view of Lemma 4.1, the conditions of our theorem imply both (17) and (16).

The proof of the convergence (25) in the case $l_1 = \cdots = l_k = 0$ is given in [10], Proposition 6 and is based on the diagram formula (14). In that paper it is shown that (17) implies the CLT for univariate Appell polynomials $P_{m+n}(X_t) := P_{m,n}(X_t, X_t)$. In the case $l_i \in \mathrm{Z}$ the cumulants of $P_{m,n}(X_t, X_{t+l_1})$ are also calculated by the diagram formula (14). Since the

l_i's are fixed, the proof of (25) turns out to be the same as in the case $l_1 = \cdots = l_k = 0$.

We now turn to relation (26):

$$
\begin{aligned}
N^{-1}&\operatorname{Cov}(U_N(l_1), U_N(l_2)) \\
&= N^{-1}\sum_{t,s=1}^{N}\operatorname{Cov}(P_{m,n}(X_{t-s}, X_{t-s+l_1}), P_{m,n}(X_0, X_{l_2})) \\
&= \sum_{v=-(N-1)}^{N-1}\left[N^{-1}\sum_{s=1\vee(1-v)}^{N\wedge(N-v)}1\right]\operatorname{Cov}(P_{m,n}(X_v, X_{v+l_1}),\ P_{m,n}(X_0, X_{l_2})).
\end{aligned}
$$

The dominated convergence theorem applies by (16) and hence relation (26) follows by letting $N \to \infty$. This completes the proof of the CLT (23) when ξ_0 has all moments.

If ξ_0 has only $2(m+n)$ moments, replace it by $\xi_0^{<a} + \xi_0^{>a}$, where $\xi_0^{<a} = \xi_0\check{\{}|\xi_0| \le a\} - E\xi_0\check{\{}|\xi_0| \le a\}$ and $\xi_0^{>a} = \xi_0\check{\{}|\xi_0| > a\} - E\xi_0\check{\{}|\xi_0| > a\}$. Use the multilinearity of the Wick powers to decompose $Q_{N,K}^{(2)} = Q_{N,K}^{(2),<a} + Q_{N,K}^{(2),>a}$ where $Q_{N,K}^{(2),<a}$ involves only $\xi_j^{<a}$ and $Q_{N,K}^{(2),>a}$ involves at least one $\xi_j^{>a}$. (Note that the definitions of $Q_{N,K}^{(2),<a}$ and $Q_{N,K}^{(2),>a}$ are not symmetric.) More precisely, using the multilinearity property of the Wick powers,

$$
\begin{aligned}
P_{n_1,n_2}(X_t, X_{t+l}) &= \sum_{u_1,\ldots,u_{m+n}\in\mathbb{Z}} a(t-u_1)\cdots a(t-u_m)a(t+l-u_{m+1}) \\
&\qquad \cdots a(t+l-u_{m+n}) : \xi_{u_1}, \ldots, \xi_{u_{m+n}} \\
&:= \sum[\cdots] : \xi_{u_1}^{<a}, \ldots, \xi_{u_{m+n}}^{<a} : \\
&\qquad + \sum[\cdots] : \xi_{u_1}, \ldots, \xi_{u_{m+n}} : \check{\{}\exists s = 1, \ldots, m+n : |\xi_{u_s}| > a\} \\
&:= P_{n_1,n_2}(X_t^{<a}, X_{t+l}^{<a}) + P_{n_1,n_2}^{>a}(X_t, X_{t+l})
\end{aligned}
$$

where $P_{n_1,n_2}(X_t^{<a}, X_{t+l}^{<a})$ denotes the first sum above and $P_{n_1,n_2}^{>a}(X_t, X_{t+l}) = P_{n_1,n_2}(X_t, X_{t+l}) - P_{n_1,n_2}(X_t^{<a}, X_{t+l}^{<a})$. Here $X_t^{<a} = \sum_u a(t-u)\xi_u^{<a}$ denotes the linear process with truncated i.i.d. sequence $(\xi_u^{<a})$. Note that the Appell polynomials $P_{n_1,n_2}(X_t, X_{t+l})$ and $P_{n_1,n_2}(X_t^{<a}, X_{t+l}^{<a})$ correspond to different random variables (X_t, X_{t+l}) and $(X_t^{<a}, X_{t+l}^{<a})$ and are different and that $P_{n_1,n_2}^{>a}(X_t, X_{t+l})$ involves at least one $\xi_j^{>a}$. We define $Q_{N,K}^{(2),<a}$ and $Q_{N,K}^{(2),>a}$ by replacing in $Q_{N,K}^{(2)}$ the polynomial $P_{n_1,n_2}(X_t, X_{t+l})$ by $P_{n_1,n_2}(X_t^{<a}, X_{t+l}^{<a})$ and $P_{n_1,n_2}^{>a}(X_t, X_{t+l})$ respectively.

It is sufficient to show that

$$
\limsup_{a}\lim_{N} N^{-1}\operatorname{Var} Q_{N,K}^{(2),>a} = 0. \tag{27}
$$

Indeed, we already know that

$$
N^{-1/2}Q_{N,K}^{(2),<a} \Rightarrow N(0, \sigma_K^2(a)) \tag{28}
$$

since all moments of $Q_{N,K}^{(2),<a}$ exist. Relation (27) also implies that $\sigma_K^2(a) \to \sigma_K^2$ $(a \to \infty)$. Then (23) follows from (27) and (28).

We now establish (27). From $\sum_{t\in Z} |r(t)|^{m+n} < \infty$, using the same argument as in the proof of Lemma 4.1, we get

$$\sum_{t\in Z} \left|\mathrm{Cov}(P_{m,n}(X_t^{<a}, X_{t+l_1}^{<a}), P_{m,n}(X_0^{<a}, X_{l_2}^{<a}))\right| < \infty \quad \text{for any } l_1, l_2 \in Z,$$

$$\sum_{t\in Z} \left|\mathrm{Cov}(P_{m,n}(X_t, X_{t+l_1}), P_{m,n}(X_0^{<a}, X_{l_2}^{<a}))\right| < \infty \quad \text{for any } l_1, l_2 \in Z.$$

[Only $d(\gamma)$ in Lemma 4.1 is modified.] It follows, as in the proof of (26), that the following limits exist:

$$\lim_N N^{-1} \mathrm{Var}\, Q_{N,K}^{(2),<a} =: \sigma_K^{(1,1)}(a),$$

$$\lim_N N^{-1} \mathrm{Cov}(Q_{N,K}^{(2)}, Q_{N,K}^{(2),<a}) =: \sigma_K^{(1,2)}(a),$$

$$\lim_N N^{-1} \mathrm{Var}\, Q_{N,K}^{(2)} =: \sigma_K^2.$$

Moreover,

$$\lim_{a\to\infty} \sigma_K^{(1,1)}(a) = \lim_{a\to\infty} \sigma_K^{(1,2)}(a) = \sigma_K^2,$$

because

$$\mathrm{cum}(\xi_{s_1}, \dots, \xi_{s_l}, \xi_{s_{l+1}}^{<a}, \dots, \xi_{s_{l+p}}^{<a}) \to \mathrm{cum}(\xi_{s_1}, \dots \xi_{s_{l+p}}) \quad \text{as } a \to \infty,$$

which equals 0 if $s_i \neq s_j$ for some $i, j = 1, \dots, l+p$ by the independence of the ξ's. Then

$$\begin{aligned} &\lim_N N^{-1} \mathrm{Var}\, Q_{N,K}^{(2),>a} \\ &\quad = \lim_N N^{-1}\left[\mathrm{Var}\, Q_{N,K}^{(2)} - 2\,\mathrm{Cov}(Q_{N,K}^{(2)}, Q_{N,K}^{(2),<a}) + \mathrm{Var}\, Q_{N,K}^{(2),<a}\right] \\ &\quad = \sigma_K^{(1,1)}(a) - 2\sigma_K^{(1,2)}(a) + \sigma_K^2 \to 0, \qquad a \to \infty. \end{aligned}$$

Thus, (27), and hence the CLT (23) holds.

(iii) We now establish relation (24). Since

$$\sigma_K^2 = N^{-1} \mathrm{Var}\, Q_{N,K}^{(2)} = \sum_{l_1, l_2 = -K}^{K} b(-l_1) b(-l_2) N^{-1} \mathrm{Cov}(U_N(l_1), U_N(l_2)),$$

(24) follows from (26) and (4). Theorem 2.1 is now proved. □

LEMMA 4.2. *Let* $p_1 \geq 1, \dots, p_k \geq 1$, $p_1^{-1} + \cdots + p_k^{-1} = k - 1$. *If* $b_j \in L^{p_j}$ *for* $j = 1, \dots, k$ *then*

$$\sum_{v_1,\dots,v_{k-1}\in Z} |b_1(v_1)\cdots b_{k-1}(v_{k-1}) b_k(v_1 + \cdots + v_{k-1})| \leq \prod_{j=1}^{k} \|b_j\|_{L^{p_j}}.$$

PROOF. If

$$B_l(y) = \sum_{v_1,\dots,v_{l-1}\in Z} |b_1(v_1)\cdots b_{l-1}(v_{l-1})b_l(y-(v_1+\cdots+v_{l-1}))|,$$

then, after a change of variables,

$$\sum_{v_1,\dots,v_{k-1}\in Z} |b_1(v_1)\cdots b_{k-1}(v_{k-1})b_k(v_1+\cdots+v_{k-1})| = \sum_v b_k(v)B_{k-1}(v)$$
$$\le \|b_k\|_{L^{p_k}}\|B_{k-1}\|_{L^{r_{k-1}}},$$

where $p_k^{-1}+r_{k-1}^{-1}=1$ by Hölder's inequality. We shall now use Young's inequality

$$\|f*g\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q}, \qquad 1+r^{-1}=p^{-1}+q^{-1},\ p,q,r\ge 1.$$

Applying it $k-1$ times gives

$$\|B_{k-1}\|_{L^{r_{k-1}}} \le \|b_{k-1}\|_{L^{p_{k-1}}}\|B_{k-2}\|_{L^{r_{k-2}}} \le \cdots \le \prod_{j=1}^{k}\|b_j\|_{L^{p_j}},$$

where $1+r_{k-1}^{-1}=p_{k-1}^{-1}+r_{k-2}^{-1},\dots,1+r_2^{-1}=p_2^{-1}+p_1^{-1}$. Summing these relations and relation $1=p_k^{-1}+r_k^{-1}$ gives $k-1=p_1^{-1}+\cdots+p_k^{-1}$. (Observe also that $r_j\ge 1$, $j=1,\dots,k$.) $\square$

PROOF OF THEOREM 2.2. As in the proof of Theorem 2.1, it is sufficient to show that the relations (22)–(24) are satisfied. Note, that $\sum_{t\in Z}|r(t)|^{m+n}<\infty$, since (7) and (8) imply that $m+n\ge p$. Therefore, (23) holds by the same arguments as in the proof of Theorem 2.1. Let us now verify (22). Observe that if

$$S := \sum_{u_1,\dots u_k\in Z} c(u_1,\dots,u_k)\colon \xi_{u_1}\dots\xi_{u_k}\colon$$

with weights $c(u_1,\dots,u_k)$ satisfying the condition $c(\cdot)\in L^2(Z^k)$, then

$$\operatorname{Var} S = ES^2 = \sum_{\gamma\in\Gamma(k,k)}\sum_{u_1,\dots,u_r\in Z} c(u_{11},\dots,u_{1k})c(u_{21},\dots,u_{2k})\,d(\gamma)$$

by the diagram formula (14). Since the first sum involves only finitely many diagrams, we can use the Cauchy formula to get

$$\operatorname{Var} S \le C(k,\xi_0)\sum_{u_1,\dots,u_k\in Z} c^2(u_1,\dots,u_k),$$

where the constant $C(k, \xi_0)$ depends only on k and on the distribution of ξ_0. Therefore, for $Q^{(1)}_{N,K}$ defined as (21), we get

$$
\begin{aligned}
\operatorname{Var} Q^{(1)}_{N,K} &\le C(\xi_0) \sum_{t_1, s_1, t_2, s_2=1}^{N} \big| b(t_1 - s_1)\check{\{}|t_1 - s_1| \ge K\} b(t_2 - s_2) \\
&\qquad\qquad \times r^m(t_1 - t_2) r^n(s_1 - s_2)\big| \\
&\le C(\xi_0) N \sum_{v_1, v_2, v_3 \in \mathrm{Z}} \big| b(v_1) 1\{|v_1| \ge K\} b(v_2) \\
&\qquad\qquad \times r^m(v_3) r^n(v_1 + v_2 + v_3)\big| =: N V_K.
\end{aligned}
\tag{29}
$$

Let $m, n \ge 1$. By assumption, $p_1^{-1} + p_2^{-1} + 2q^{-1} \ge 3$ where $p_1^{-1} = \min(mp^{-1}, 1)$ and $p_2^{-1} = \min(np^{-1}, 1)$. (Since for sequences $r \in L^p \Rightarrow r \in L^{p'}$ with $p' \ge p$, we can assume $p_1^{-1} + p_2^{-1} + 2q^{-1} = 3$, by increasing for example p_1). We can now apply Lemma 4.2 to get

$$V_K \le C(\xi_0) \| b\check{(}|v| \ge K) \|_{L^q} \|b\|_{L^q} \|r^m\|_{L^{p_1}} \|r^n\|_{L^{p_2}} \to 0, \qquad K \to \infty$$

since

$$
\begin{aligned}
\|r^m\|_{L^{p_1}} &\le \Big(\sum |r(t)|^{m(\max(m^{-1}p, 1))} \Big)^{1/p_1} \le \Big(\sum |r(t)|^{\max(p, m)} \Big)^{1/p_1} \\
&\le C \Big(\sum |r(t)|^p \Big)^{1/p_1} < \infty,
\end{aligned}
$$

since $r \in L^p$. Similarly, $\|r^n\|_{L^{p_2}} < \infty$.

Assume now that $m \ge 1$, $n = 0$. Then condition (8), namely, $\min(mp^{-1}, 1) + 2q^{-1} \ge 3$ implies that $q = 1$ and $m \ge p$. Therefore, again $V_K \to 0$ $(K \to \infty)$ since both b and r^m in (29) are in L^1. This establishes (22). It remains to verify relation (24), namely, that σ_K^2 converges to σ^2, given by (6), as $K \to \infty$. Since $Q^{(1)}_{N,K} + Q^{(2)}_{N,K} = Q_N$ and $\operatorname{Var} Q^{(2)}_{N,K} = \operatorname{Var}, Q^{(1)}_{N,K} - 2\operatorname{Cov}(Q_N, Q^{(1)}_{N,K}) + \operatorname{Var} Q^{(1)}_{N,K}$, then

$$
\begin{aligned}
|\sigma_K^2 - \sigma_{K'}^2| &\le \limsup_N N^{-1} |\operatorname{Var} Q^{(2)}_{N,K} - \operatorname{Var} Q^{(2)}_{N,K'}| \\
&\le N^{-1} \big(2[\operatorname{Var} Q_N \operatorname{Var} Q^{(1)}_{N,K}]^{1/2} + 2[\operatorname{Var} Q_N \operatorname{Var} Q^{(1)}_{N,K'}]^{1/2} \\
&\qquad + \operatorname{Var} Q^{(1)}_{N,K} + \operatorname{Var} Q^{(1)}_{N,K'} \big).
\end{aligned}
$$

Now, $\operatorname{Var} Q^{(1)}_{N,K} \le CNV_K$ by (29), $\operatorname{Var} Q^{(2)}_{N,K} \le CN$ since $Q^{(2)}_{N,K}$ satisfies the CLT [relation (23)], and hence $\operatorname{Var} Q_N \le 2 \operatorname{Var} Q^{(1)}_{N,K} + 2 \operatorname{Var} Q^{(2)}_{N,K} \le CN$ as $N \to \infty$. Since $V_K \to 0$ as $K \to \infty$, we get

$$|\sigma_K^2 - \sigma_{K'}^2| \to 0, \qquad K, K' \to \infty.$$

Thus $\sigma_K^2 \to \sigma^2 < \infty$ and hence (24) holds and the sum on the right-hand side of (6) converges. This establishes the CLT with limiting variance σ^2 given by (6). $\square$

PROOF OF THEOREM 2.3. The assumptions imply that f and $\widehat{b}$ satisfy the assumptions $f(x) \le C|x|^{-\alpha}$, $\hat{b}(x) \le C|x|^{-\beta}$, $x \in [-\pi, \pi]$ of Theorem 2.4 in [14] with the $\alpha = 1-\gamma_1$ and $\beta = 1-\gamma_2$. (For details see the proof in [7], Theorem 3). Moreover, if $0 < \gamma_1, \gamma_2 < 1$, then $r \in L^p$, $b \in L^q$ with $p > [\gamma_1]^{-1}$, $q > [\gamma_2]^{-1}$. Then assumptions (7) and (8) of Theorem 2.2 are satisfied as well and hence the limiting variance can also be represented in the form (6). □

5. Multivariate generalizations. The following result is a multivariate generalization of Theorem 2.2. Let

$$X_t^{(i,l)} = \sum_{u \in \mathrm{Z}} a^{(i,l)}(t-u)\xi_u, \, t \in \mathrm{Z}, \qquad i = 1, \dots, k; \; l = 1, 2$$

be a $2k$ stationary time series with the same mean 0 and finite variance innovations sequence $\{\xi_u, \; u \in \mathrm{Z}\}$ and consider

$$Q_N^{(i)} := \sum_{t,s=1}^{N} b_i(t-s) P_{m_i, n_i}(X_t^{(i,1)}, X_s^{(i,2)}),$$

$i = 1, \dots, k$. Assume $E|\xi_0|^{2(m_i+n_i)} < \infty$, in order to ensure that $Q_N^{(i)}$ has finite second moments for all i. Let $r_{(i,l),(i',l')}(t) = EX_t^{(i,l)} X_0^{(i',l')}$, $1 \le i, i' \le k, 1 \le l, l' \le 2$, denote the cross covariances.

THEOREM 5.1. *Suppose that each quadratic form*

$$Q_N^{(i)} := \sum_{t,s=1}^{N} b_i(t-s) P_{m_i, n_i}(X_t^{(i,1)}, X_s^{(i,2)}), \qquad i = 1, \dots, k$$

satisfies the assumptions

$$\sum_{l_1, l_2, t \in \mathrm{Z}} \Big| b_i(l_1) b_j(l_2) \operatorname{Cov}\Big(P_{m_i, n_i}(X_t^{(i,1)}, X_{t+l_1}^{(i,2)}), P_{m_j, n_j}(X_0^{(j,1)}, X_{l_2}^{(j,2)}) \Big) \Big| < \infty, \qquad 1 \le i, j \le k \tag{30}$$

and

$$\sum_{t \in \mathrm{Z}} |r_{(i,l),(i',l')}(t)|^{m_i+n_i} < \infty, \qquad 1 \le i, \; i' \le k, \; 1 \le l, \; l' \le 2. \tag{31}$$

Then as $N \to \infty$,

$$N^{-1/2}(Q_N^{(1)}, \dots, Q_N^{(k)}) \Rightarrow (Z^{(1)}, \dots, Z^{(k)}), \tag{32}$$

where $(Z^{(1)}, \dots, Z^{(k)})$ is the Gaussian vector with zero mean and cross covariances

$$\begin{aligned} \sigma_{i,j} &\equiv EZ^{(i)} Z^{(j)} \\ &:= \sum_{l_1, l_2, t \in \mathrm{Z}} b_i(l_1) b_j(l_2) \\ &\quad \times \operatorname{Cov}\Big(P_{m_i, n_i}(X_t^{(i,1)}, X_{t+l_1}^{(i,2)}), P_{m_j, n_j}(X_0^{(j,1)}, X_{l_2}^{(j,2)}) \Big). \end{aligned} \tag{33}$$

REMARK. In view of Lemma 4.1, condition (31) is automatically satisfied for any $i = i'$ if $X^{(i,1)} = X^{(i,2)}$ and $b_i(0) \neq 0$.

PROOF OF THEOREM 5.1. It is sufficient to show that

$$S_N = N^{-1/2} \sum_{i=1}^{k} c_i Q_N^{(i)} \Rightarrow S := \sum_{i=1}^{k} c_i Z^{(i)}, \qquad N \to \infty \tag{34}$$

for any real numbers c_i, $i = 1, \ldots, k$. As in the proof of Theorem 2.1, we divide $Q_N^{(i)}$ (1) into two parts:

$$\begin{aligned} Q_N^{(i)} &:= \sum_{t,\, s=1}^{N} b_i(t-s) P_{m,\,n}(X_t^{(i,1)}, X_s^{(i,2)}) \\ &= \sum_{t,\, s=1:\, |t-s|>K}^{N} + \sum_{t,\, s=1:\, |t-s|\le K}^{N} = Q_{N,\,K}^{(i,1)} + Q_{N,\,K}^{(i,2)}, \end{aligned}$$

where $K > 1$ is a fixed constant. Then the multivariate CLT (32) follows from the relations

$$\lim_{N\to\infty} N^{-1} \operatorname{Var} Q_{N,\,K}^{(i,1)} \le \delta(K) \to 0, \qquad K \to \infty,\ i = 1, \ldots, k; \tag{35}$$

$$N^{-1/2} \sum_{i=1}^{k} c_i Q_{N,\,K}^{(i,2)} \Rightarrow \sum_{i=1}^{k} c_i Z_K^{(i)}, \qquad N \to \infty; \tag{36}$$

$$E Z_K^{(i)} Z_K^{(j)} \to E Z^{(i)} Z^{(j)}, \qquad K \to \infty,\ 1 \le i,\ j \le k. \tag{37}$$

Since the cumulants are multilinear, the proof of relations (35), (36) and (37) under assumptions (30) and (31) can be obtained in the same way as that of relations (22), (23) and (24) in the proof of Theorem 2.1, respectively. □

THEOREM 5.2. *The statement of Theorem 5.1 remains true if condition* (30) *is replaced by*

$$\begin{gathered} r_{(i,1),(i,1)} \in L^{p_{(i,1)}}, \qquad r_{(i,2),(i,2)} \in L^{p_{(i,2)}}, \\ b_i \in L^{q_i}, \qquad p_{(i,1)}, p_{(i,2)}, q_i \ge 1, \end{gathered} \tag{38}$$

for $1 \le i \le k$ *and*

$$\min(m_i p_{(i,1)}^{-1}, 1) + \min(n_i p_{(i,2)}^{-1}, 1) + 2q_i^{-1} \ge 3, \qquad 1 \le i \le k. \tag{39}$$

In addition, $a^{(i,1)} \in L^1$ *if* $k > 1$, $m_i = 1$, $n_i = 0$ *and* $a^{(i,2)} \in L^1$ *if* $k > 1$, $m_i = 0$, $n_i = 1$.

REMARK. This extra condition was not required in the "univariate" version, Theorem 2.2, where only one form was considered.

PROOF OF THEOREM 5.2. One has to prove (35), (36) and (37). An argument similar to that of Theorem 2.2 shows that under assumptions (38) and (39), relation (35) is valid. It remains to check limits (36) and (37).

Since the proof of (37) is similar to that of (24) in Theorem 2.2, we only show here that (36) holds.

Set $U_N^{(i)}(l) = \sum_{t=1}^N P_{m_i, n_i}(X_t^{(i,1)}, X_{t+l}^{(i,2)})$, $i = 1, \dots, k$. Since cumulants are multilinear, it is sufficient to show, as in the proof of Theorem 2.2, that

$$\text{cum}(U_N^{(i_1)}(l_1), \dots, U_N^{(i_p)}(l_p)) = o(N^{p/2}) \tag{40}$$

as $N \to \infty$ for any $l_1, \dots, l_p \in \mathbb{Z}$, $1 \le i_1, \dots, i_p \le k$ and $p \ge 3$, and to prove convergence of the cross covariances

$$\begin{aligned} N^{-1} &\text{Cov}(U_N^{(i_1)}(l_1), U_N^{(i_2)}(l_2)) \\ &\to \sigma_{l_1, l_2} \\ &:= \sum_{t \in \mathbb{Z}} \text{Cov}(P_{m,n}(X_t^{(i_1,1)}, X_{t+l_1}^{(i_1,2)}), P_{m,n}(X_0^{(i_2,1)}, X_{l_2}^{(i_2,2)})) \end{aligned} \tag{41}$$

as $N \to \infty$ for any $l_1, l_2 \in \mathbb{Z}$ and $1 \le i_1, \ i_2 \le k$. The convergence of the cumulants (40) can be obtained similarly to relation (25) in Theorem 2.2, and its proof is based on assumption (31).

The proof of (41) is more involved. Clearly, it is sufficient to check that

$$\sum_{t \in \mathbb{Z}} |R(t; l_1, l_2)| < \infty, \tag{42}$$

where

$$\begin{aligned} R(t; l_1, l_2) &:= \text{Cov}(P_{m,n}(X_t^{(i_1,1)}, X_{t+l_1}^{(i_1,2)}), P_{m,n}(X_0^{(i_2,1)}, X_{l_2}^{(i_2,2)})) \\ &= \sum_{\gamma = (V_1, \dots, V_r) \in \Gamma(m_{i_1} + n_{i_1}, m_{i_2} + n_{i_2})} d(\gamma) r_{V_1}(t) \cdots r_{V_r}(t). \end{aligned}$$

Here

$$\begin{aligned} r_V(t) &\equiv r_V(t; l_1, l_2) \\ &= \sum_{u \in \mathbb{Z}} a^{(i_1,1)}(t-u)^{n_1(V)} a^{(i_1,2)} \\ &\qquad \times (t + l_1 - u)^{n_2(V)} a^{(i_2,1)}(-u)^{n_3(V)} a^{(i_2,2)}(l_2 - u)^{n_4(V)} \end{aligned} \tag{43}$$

and $n_1(V) = |V \cap \{(1,1), \dots, (1, m_{i_1})\}|$, $n_2(V) = |V \cap \{(1, m_{i_1}+1), \dots, (1, m_{i_1} + n_{i_1})\}|$, $n_3(V) = |V \cap \{(2,1), \dots, (2, m_{i_2})\}|$, $n_4(V) = |V \cap \{(2, m_{i_2} + 1), \dots, (2, m_{i_2} + n_{i_2})\}|$. Relation (42) will follow if we show that

$$\sum_{t \in \mathbb{Z}} |r_{V_1}(t) \cdots r_{V_r}(t)| < \infty \tag{44}$$

for any $\gamma \in \Gamma(m_{i_1} + n_{i_1}, m_{i_2} + n_{i_2})$. Since $|V_1| + \cdots + |V_r| \equiv |W| = m_{i_1} + n_{i_1} + m_{i_2} + n_{i_2}$, then by Hölder's inequality,

$$\sum_t |r_{V_1}(t) \cdots r_{V_r}(t)| \le \prod_{i=1}^r \left(\sum_t |r_{V_i}(t)|^{|W|/|V_i|} \right)^{|V_i|/|W|}. \tag{45}$$

Set $q_i = \sum_t |r_{V_i}(t)|^{|W|/|V_i|}$, $i = 1, \dots, r$.

If $|V_i| = 2$, ("Gaussian" V_i), namely, $V_i = ((i_1, l), (i_2, l'))$, then $|W|/|V_i| = |W|/2 \geq \min(m_{i_1} + n_{i_1}, m_{i_2} + n_{i_2})$, and therefore,

$$q_i \leq C \sum_t |r_{(i_1, l),(i_2, l')}(t)|^{\min(m_{i_1}+n_{i_1},\, m_{i_2}+n_{i_2})} < \infty$$

by assumption (31).

Recall that V_i is a "triangle" if $|V_i| > 2$ and if it contains only one element from the first or second row of the table W. If V_i is non-Gaussian non-"triangle," then as in the proof of case (a), Lemma 4.1, we have that $q_i < \infty$. Thus, if $\gamma = (V_1, \dots, V_r)$ consists of either Gaussian or non-Gaussian non-"triangle" V_i's, then (44) holds.

Suppose now that γ contains only one "triangle" V_i. (To simplify the notations, set $i = 1$ and suppose that it is the first row of the table W that contains a single element of the "triangle" V_1). In contrast to case (b) of Lemma 4.1, we have to consider three scenarios.

(b1) Suppose all other V_j's are "Gaussian," $j = 2, \dots, r$ and $r \geq 2$. Then $r = m_{i_1} + n_{i_1}$ and there are $m_{i_1} + n_{i_1} - 1$ Gaussian V_j's. Since $r \geq 2$, by the Cauchy inequality,

$$\begin{aligned}
&\sum_{t\in Z} |r_{V_1}(t) \cdots r_{V_r}(t)| \\
&\quad \leq C\Big(\sum_{t\in Z} |r_{V_1}(t)|^2\Big)^{1/2} \Big(\sum_{t\in Z} |r_{V_2}(t) \cdots r_{V_r}(t)|^2\Big)^{1/2} \\
&\quad \leq C\Big(\sum_{t\in Z} |r_{V_1}(t)|^2\Big)^{1/2} \max_{j=2,\dots,r} \Big(\sum_{t\in Z} |r_{V_j}(t)|^{2(m_{i_1}+n_{i_1}-1)}\Big)^{1/2} < \infty
\end{aligned}$$

since $\sum_{t\in Z} |r_{V_1}(t)|^2 < \infty$ for the "triangle" V_1 [see case (b) in the proof of Lemma 4.1], and $\sum_{t\in Z} |r_{V_j}(t)|^{2(m_{i_1}+n_{i_1}-1)} < \infty$ for $j = 2, \dots, r$ "Gaussian" V_j's. Indeed, $2(m_{i_1} + n_{i_1} - 1) \geq m_{i_1} + n_{i_1}$ and for Gaussian V_j, r_{V_j} is a cross covariance, so we can use assumption (31).

(b2) Suppose $r = 1$, that is, γ consists of a single "triangle" V_1. (Assume without loss of generality $m_{i_1} = 1$, $n_{i_1} = 0$.) Then

$$q_1 = \sum_{t\in Z} |r_{V_1}(t)| \leq C \sum_{t,\, u\in Z} |a^{(i_1, 1)}(t-u)| \max(|a^{(i_2, 1)}(u)|^2, |a^{(i_2, 2)}(u)|^2) < \infty$$

using the special assumption $a^{(i_1, 1)} \in L^1$, which holds in this case and the standard assumptions $a^{(i_1, 1)}, a^{(i_1, 2)} \in L^2$.

(b3) Suppose that γ contains more than one "triangle" (say V_1, V_2). Then

$$\begin{aligned}
\sum_{t\in Z} |r_{V_1}(t) \cdots r_{V_r}(t)| &\leq C \sum_{t\in Z} |r_{V_1}(t) r_{V_2}(t)| \\
&\leq C\Big(\sum_{t\in Z} |r_{V_1}(t)|^2\Big)^{1/2} \Big(\sum_{t\in Z} |r_{V_2}(t)|^2\Big)^{1/2} < \infty.
\end{aligned}$$

Relation (42) is now proved. □

In the next result, the weights are different, but the processes are identical.

COROLLARY 5.1. *Suppose that*

$$Q_N^{(i)} := \sum_{t,s=1}^{N} b_i(t-s) P_{m_i,n_i}(X_t, X_s),$$

$i = 1, \dots, k$, *where* $X(t) = \sum_{u \in \mathbb{Z}} a(t-u)\xi_u$, $t \in \mathbb{Z}$ *is a linear process.*

(A1). If $m_i + n_i = 1$ for some $i = 1, \dots, k$, assume

$$a \in L^1 \quad \text{and} \quad \begin{cases} b_j \in L^1, & \text{for all } j = 1, \dots, k \text{ such that } m_i = 0 \text{ or } n_i = 0, \\ b_j \in L^2, & \text{for all } j = 1, \dots, k \text{ such that } m_j \geq 1 \text{ and } n_j \geq 1. \end{cases}$$

(A2). If $m_i + n_i \geq 1$ for all $i = 1, \dots, k$, assume

$$r \in L^p, \qquad b_i \in L^{q_i}, \qquad q_i \geq 1, \ i = 1, \dots, k,$$

and

$$\min(m_i p^{-1}, 1) + \min(n_i p^{-1}, 1) + 2q_i^{-1} \geq 3, \qquad i = 1, \dots, k.$$

Then the multivariate CLT (32) holds.

The proof of the corollary follows from Theorem 5.2.

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DEPARTMENT OF MATHEMATICS
BOSTON UNIVERSITY
111 CUMMINGTON STREET
BOSTON, MASSACHUSETTS 02215
E-MAIL: liudas@math.bu.edu
murad@math.bu.edu