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ON THE RANGE OF $\mathbf{R}^2$ OR $\mathbf{R}^3$-VALUED HARMONIC MORPHISMS

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We prove that, under some general assumptions, the range of any nonconstant harmonic morphism from a simply connected open set U in $\mathbf{R}^n$ to $\mathbf{R}^3$, $n > 3$, cannot avoid three concurrent half-lines, which is an extension to Picard's little theorem. To this end, we will prove two results concerning the windings of Brownian motion around three concurrent half-lines in $\mathbf{R}^3$ and the recurrence of some domains linked with the harmonic morphism.

1. Introduction. Harmonic morphisms between Euclidean spaces (and more generally between Riemannian manifolds) have been introduced by Fuglede [7] in 1978, and some special cases have been studied by Baird and Wood (see, e.g., [1]) and Gudmundsson [8]. Bernard, Campbell and Davie [2] have considered the probabilistic point of view. Let us recall the definition of such objects.

DEFINITION 1.1. A continuous map $f\colon U \to \mathbf{R}^p$, defined on a domain U of $\mathbf{R}^n$, is called a harmonic morphism if, for any open set V in $\mathbf{R}^p$ such that $f^{-1}(V) \neq \emptyset$ and for any harmonic function h on V, the composite function $h \circ f\colon f^{-1}(V) \to \mathbf{R}$ is harmonic.

The following proposition characterizes harmonic morphisms.

PROPOSITION 1.2. *Consider an open set $U \subset \mathbf{R}^n$ and a map in the class $C^2, f\colon U \to \mathbf{R}^p$. The function f is a harmonic morphism if and only if each coordinate f_i is harmonic and if their gradients are orthogonal and have the same norm*:

$$\langle \nabla f_i, \nabla f_j \rangle = \lambda^2(x)\,\delta_{ij}.$$

This proposition leads us to a precise description of harmonic morphisms based on the dimensions of the considered Euclidean spaces. Indeed, Fuglede [7] and Baird and Wood [1] have shown the following.

THEOREM 1.3. (i) *The harmonic morphisms from $\mathbf{R}^n$ to $\mathbf{R}$ are the harmonic functions on $\mathbf{R}^n$* [7].

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(ii) *If* $n < p$, *any harmonic morphism from a domain* $U \subset \mathbf{R}^n$ *to* $\mathbf{R}^p$ *is constant* [7].

(iii) *If* $n > 2$, *any nonconstant harmonic morphism from a domain* $U \subset \mathbf{R}^n$ *to* $\mathbf{R}^n$ *is a constant times an affine orthogonal map on* $\mathbf{R}^n$ [7].

(iv) *The harmonic morphisms from* $U \subset \mathbf{R}^2 \simeq \mathbf{C}$ *to* $\mathbf{C}$ *are the holomorphic or antiholomorphic functions on* U. *More generally, complex-valued holomorphic functions of* n *complex variables are harmonic morphisms on* $\mathbf{C}^n$.

(v) *Any nonconstant harmonic morphism from* $\mathbf{R}^3$ *to* $\mathbf{R}^2$ *is an orthogonal projection from* $\mathbf{R}^3$ *on a two-dimensional subspace, composed by a holomorphic or antiholomorphic function on* $\mathbf{R}^2$ [1].

Harmonic morphisms admit a probabilistic interpretation. It is, in fact, a generalization of a classical result of Lévy [11] that claims that plane Brownian motion is invariant under any conformal transformation. More precisely, Bernard, Campbell and Davie [2] have proved the following.

THEOREM 1.4. *Let* $f\colon U \to \mathbf{R}^p$ *be a continuous map defined on a domain* U *of* $\mathbf{R}^n$ *and let* $(B_t)_{t \geq 0}$ *be a Brownian motion on* $\mathbf{R}^n$, *issued from* $B_0 \equiv b_0 \in U$. *Note* $\tau = \inf\{t\colon B_t \notin U\}$ *and* $\rho(t) = \int_0^t \|\nabla f(B_s)\|^2\, ds$, *for all* $t < \tau$.

The map f *is a harmonic morphism if and only if the paths of the process* $(f(B_t))_{0 \leq t \leq \tau}$ *are Brownian paths on* $\mathbf{R}^p$, *that is, there exists a Brownian motion* $(\overline{B}_s)_{s \geq 0}$ *on* $\mathbf{R}^p$, *issued from* $\overline{B}_0 \equiv f(b_0)$, *such that* $f(B_t) = \overline{B}(\rho(t))$.

In such a context, it is natural to search for information on the range $f(U)$ of a harmonic morphism $f\colon U \to \mathbf{R}^p$, where U is a domain in $\mathbf{R}^n$. In this article, we prove that, under some general assumptions on f and U, the range of any nonconstant $\mathbf{R}^3$-valued harmonic morphism f can not avoid three concurrent half-lines, which is of course an extension to Picard's little theorem, according to which the range of any nonconstant entire function on $\mathbf{C}$ avoids at most one point. Our result has already been announced in [4]. The proof we give of this fact is based on the invariance property of Theorem 1.4, and has been inspired by the probabilistic proof of Picard's little theorem due to [3] (see also [5]). However, transience of Brownian motion in $\mathbf{R}^n$, $n \geq 3$, implies new technical difficulties.

2. Notations and main result. Let us give a few notations: let $h\colon U \to \mathbf{R}$ be a harmonic function on a domain U of $\mathbf{R}^n$. For any unbounded open set $V \subset U$, we note

$$M(r, V, h) = \sup\{|h(x)|; \|x\| \leq r \text{ and } x \in V\}$$

and

$$I(V, h) = \liminf_{r \to \infty} \frac{\ln M(r, V, h)}{\ln r}.$$

Let us recall the definition of a recurrent set [13].

DEFINITION 2.1. A Borelian set C in $\mathbf{R}^n$ is recurrent for the Brownian motion $(B_t)_{t\geq 0}$ in $\mathbf{R}^n$ if and only if, ω-almost surely, for all $A>0$, there exists $\tau(\omega)>A$ such that $B_\tau(\omega)\in C$. Otherwise, the set C is transient.

DEFINITION 2.2. Consider a Borelian set C in $\mathbf{R}^n$ and a Brownian motion $(B_t)_{t\geq 0}$ on $\mathbf{R}^n$ issued from $B_0\equiv b_0\notin C$. The set C is polar if and only if, ω-almost surely, the path $(B_t(\omega))_{t\geq 0}$ does not visit C.

Our main result is the following.

THEOREM 2.3. *Consider a simply connected domain U in $\mathbf{R}^n$ such that $\mathbf{R}^n\setminus U$ is polar. Let $f=(f_1,\dots,f_p)\colon U\to\mathbf{R}^p$, $n>p$, $p\in\{2,3\}$, be a harmonic morphism, and let $(B_t)_{t\geq 0}$ be a Brownian motion in $\mathbf{R}^n$ issued from $B_0\equiv b_0\subset U$. We suppose that*:

(α) *Almost surely, $\int_0^{+\infty}\|\nabla f_1(B_s)\|^2\,ds=+\infty$;*

(β) *The set $f^{-1}(H)$ admits a recurrent connected component for some open half-space H in $\mathbf{R}^p$.*

Then:

(i) *If $p=2$, $f(U)$ avoids at most one point in $\mathbf{R}^2\setminus H$.*

(ii) *If $p=3$, for all $x_0\in\mathbf{R}^3\setminus H$,*

$$\operatorname{card}\left\{u\in\mathbf{S}^2,\ x_0+\mathbf{R}^+u\subset\mathbf{R}^3\setminus\left(f(U)\cup H\right)\right\}\leq 2,$$

where $\mathbf{S}^2$ is the unit sphere in $\mathbf{R}^3$.

REMARK. Assumptions (α) and (β) are always checked if f is a polynomial harmonic morphism.

The hypothesis (β) involves only one coordinate function of the harmonic morphism, which is in fact a harmonic function. We give here a condition, on harmonic functions, that implies hypothesis (β).

THEOREM 2.4. *Let h be a harmonic function on U where U is a domain of $\mathbf{R}^n$ of polar complement. Then any connected component C of $h^{-1}(\mathbf{R}^*_+)$ such that $I(C,h)$ is finite and such that any point in ∂C is regular for C, is recurrent.*

One could think that, for any harmonic function h on $\mathbf{R}^n$, one at least of the connected components of $h^{-1}(\mathbf{R}^*_+)$ or of $h^{-1}(\mathbf{R}^*_-)$ is recurrent. The following example shows this is false.

EXAMPLE. Consider the function $h\colon\mathbf{R}^4\to\mathbf{R}$ defined by

$$h(x,y,z,t)=\cos x\cos y\cos z\,e^{\sqrt{3}\,t}.$$

This function is clearly harmonic on $\mathbf{R}^4$, but no connected component of $h^{-1}(\mathbf{R}^*_+)$ or of $h^{-1}(\mathbf{R}^*_-)$ is recurrent.

3. Proof of Theorem 2.3. As our method can also be used to solve the case $p = 2$, we will study more carefully the case $p = 3$. This theorem is based on two essential facts. We will prove that Brownian paths in $\mathbf{R}^3$ get more and more tangled in their windings around three concurrent half-lines. An argument based on homotopy allows us to conclude.

Notation. Later on in this article, we will note $E = \mathbf{R}^3 \setminus \bigcup_{i=1}^{3} D_i$, where D_1, D_2, D_3 are three concurrent half-lines.

DEFINITION 3.1. Let us note T the border of the infinite trihedral defined by the three concurrent half-lines D_i, $i = 1, 2, 3$, and consider a curve $\Gamma\colon [0, t] \to E$.

(i) We will say that $\Gamma_{[0,t]}$ is unwound in E (around the three half-lines) if there exists a curve $\tilde{\Gamma}\colon [0, 1] \to E$ such that:

(a) $\tilde{\Gamma}(0) = \Gamma(t)$, $\tilde{\Gamma}(1) = \Gamma(0)$;

(b) $\operatorname{card}(\tilde{\Gamma}[0, 1] \cap T) \leq 2$;

(c) $\Gamma_{[0,t]} * \tilde{\Gamma}_{[0,1]}$ is homotopic to a point in E.

Otherwise, we will say Γ is wound in E.

(ii) We will say that a curve $\Gamma\colon [0, +\infty) \to E$ comes wound if, for all t big enough, the curve $\Gamma_{[0,t]}$ is wound.

Then we can state the following result.

THEOREM 3.2. *Let $(\overline{B}_t)_{t \geq 0}$ be an $\mathbf{R}^3$-valued Brownian motion issued from a point $\overline{B}_0 \equiv b_0 \in E$. Then, ω-almost surely, the path $(\overline{B}_t(\omega))_{t \geq 0}$ comes wound around the three concurrent half-lines D_1, D_2, D_3.*

REMARK. As straight lines are polar sets for three-dimensional Brownian motion, the winding of Brownian paths around three half-lines is well defined.

Let us admit for a while Theorem 3.2 and deduce from it Theorem 2.3.

Assume that $f(U)$ avoids three concurrent half-lines D_1, D_2, D_3 included in the half-space $H^-(a, b)$ and that a connected component C of the open set $f^{-1}(H^-(a, b))$ is recurrent. We will build, using Brownian motion on $\mathbf{R}^n$, a closed curve Γ in U whose image by $f\colon U \to E = \mathbf{R}^3 \setminus \bigcup_{i=1}^{3} D_i$ is not homotopic to a point in E (thanks to Theorem 3.2), which is absurd. Indeed, as we have supposed U simply connected, the closed curve Γ is homotopic to a point and its image under the continuous map f should be homotopic to a point in $f(U)$.

Let $(B_t)_{t \geq 0}$ be a Brownian motion on $\mathbf{R}^n$. We will suppose $B_0 \equiv b_0 \in C$ and note:

$$\Omega_0 = \{\omega \in \Omega, (B_t)_{t \geq 0} \text{ does not visit } \mathbf{R}^n \setminus U\}.$$

As $\mathbf{R}^n \setminus U$ is polar, $P(\Omega_0) = 1$.

By Theorem 3.2, the paths of $(f(B_t))_{t\geq 0}$ come wound almost surely around the three half-lines, because the paths of this process are Brownian paths in $\mathbf{R}^3$, and the assumption (α) implies we find whole Brownian paths. Let us note:

$$\Omega_1 = \left\{\omega \in \Omega, \text{ there exists } S_\omega \geq 0, f\left(B_{[0,t]}\right) \text{ is wound for all } t \geq S_\omega\right\}.$$

We have then $P(\Omega_1) = 1$.

Set

$$\Omega_2 = \{\omega \in \Omega, \forall A > 0, \exists \tau(\omega, A) > A, B_\tau \in C\}.$$

As we supposed C recurrent, we have $P(\Omega_2) = 1$, and $P(\Omega_0 \cap \Omega_1 \cap \Omega_2) = 1$.

Fix then $\omega \in \Omega_0 \cap \Omega_1 \cap \Omega_2$. For such an ω, there exists $\sigma = \sigma(\omega) \in \mathbf{R}^+$ such that for all $t \geq \sigma(\omega)$, the path $(f(B_s(\omega)))_{s\in[0,t]}$ is wound in E. We can then choose $\tau(\omega) \geq \sigma(\omega)$ such that $B_\tau(\omega)$ is in C.

The set C is connected and open: it is then arcwise-connected, and there exists a curve γ in C linking $B_\tau(\omega)$ and $B_0(\omega)$. The curve $f(\gamma)$ closes the path $f(B_{[0,\tau]})$ according to the rules of Definition 3.1. By definition of τ, the closed curve $f(B_{[0,\tau]} * \gamma)$ can not be homotopic to a point in $f(U) \subset E$. The curve $B_{[0,\tau]} * \gamma$ forms the announced curve.

4. Proof of Theorem 3.2. Let D_1, D_2, D_3 be three concurrent half-lines in $\mathbf{R}^3$, issued from 0. We note $D_i = \mathbf{R}^+ u_i$ with $\|u_i\| = 1$, $i = 1, 2, 3$. Consider the map

$$\pi: \mathbf{R}^3 \setminus D_1 \to \mathbf{R}^2 \simeq D_1^\perp$$
$$x \to \pi(x) = \frac{x - \langle x, u_1\rangle u_1}{\|x\| - \langle x, u_1\rangle}.$$

LEMMA 4.1. *The map π is a harmonic morphism from $\mathbf{R}^3 \setminus D_1$ to $\mathbf{R}^2$.*

Indeed, π is the composite map of ϕ: $x \to x/\|x\|$ and of the stereographic projection of pole u_1 onto the plane orthogonal to D_1 through 0. The image by ϕ of an $\mathbf{R}^3$-valued Brownian motion is a time change of a spherical Brownian motion and the image by the stereographic projection of a spherical Brownian motion is a time change of a plane Brownian motion: the paths of the image by π of an $\mathbf{R}^3$-valued Brownian motion are plane Brownian paths. Theorem 1.4 allows us to claim that π is a harmonic morphism.

LEMMA 4.2. *We have $\pi(\mathbf{R}^3 \setminus \bigcup_{i=1}^3 D_i) = \mathbf{R}^2 \setminus \{\pi(u_2), \pi(u_3)\}$.*

One can easily check that the inverse image by π of any point in $\mathbf{R}^2$ is a half-line issued from 0, and distinct from D_1, which proves this lemma.

We can now prove Theorem 3.2. To that effect, consider a Brownian motion $(\overline{B}_t)_{t\geq 0}$ in $\mathbf{R}^3$ issued from $\overline{B}_0 \equiv \overline{b}_0 \notin \bigcup_{i=1}^3 D_i$. The process $(\pi(\overline{B}_t))_{t\geq 0})$ is a time change of a Brownian motion on the plane minus $\{\pi(u_2), \pi(u_3)\}$, and the change of time converges almost surely to $+\infty$ as $t \to +\infty$. We will now use a result of McKean [12] (see also [3] and [5]) according to which plane Brownian motion comes wound around two different points.

LEMMA 4.3. *Let* $(B_t)_{t \geq 0}$ *be a plane Brownian motion issued from* $B_0 \equiv b_0$ *and* x_1, x_2, *two distinct points in* $\mathbf{R}^2$, *different from* b_0. *Then,* ω*-almost surely, there exists* $T(\omega) > 0$ *such that for all* $t > T(\omega)$, *the path* $(B_s(\omega))_{0 \leq s \leq t}$ *is wound in* $\mathbf{R}^2 \setminus \{x_1, x_2\}$.

REMARK. One can easily define the winding of any unbounded curve in $\mathbf{R}^2$ minus two points as one did in $\mathbf{R}^3$ minus three half-lines.

Let T be a time such that $(\pi(\overline{B}_s))_{0 \leq s \leq t}$ is wound in $\mathbf{R}^2 \setminus \{\pi(u_2), \pi(u_3)\}$ for all $t > T$, and choose any curve Γ in $\mathbf{R}^3 \setminus \bigcup_{i=1}^3 D_i$ that closes the path $(\overline{B}_s)_{0 \leq s \leq t}$ according to the rules of Definition 3.1. As the curve $\pi(\Gamma)$ closes $\pi(\overline{B}_{[0,t]})$, the plane closed curve $\pi(\overline{B}_{[0,t]} * \Gamma)$ can not be homotopic to a point in $\mathbf{R}^2 \setminus \{\pi(u_2), \pi(u_3)\}$. Hence, the closed curve $\overline{B}_{[0,t]} * \Gamma$ is not homotopic to a point in $\mathbf{R}^3 \setminus \bigcup_{i=1}^3 D_i$, which proves that $\mathbf{R}^3$-valued Brownian motion comes wound around three concurrent half-lines in $\mathbf{R}^3$.

5. Proof of Theorem 2.4. Let us first prove the following lemma.

LEMMA 5.1. *Let* $h: U \to \mathbf{R}$ *be a harmonic function on an open connected set of polar complement and consider a connected component* C *of* $h^{-1}(\mathbf{R}_+^*)$.

If ∂C *is regular for* C *and if* $I(C, h)$ *is finite, the harmonic function* h *admits a continuous extension on the set* $\overline{C}$.

PROOF OF LEMMA 5.1. At first, choose a point x_0 in $\mathring{\overline{C}} \setminus C$. As $I(C, h)$ is finite and $\mathring{\overline{C}} \setminus C$ is polar, the harmonic function h is bounded in a vicinity of x_0 in U. Hence, it admits a limit at that point (see, e.g., [10], page 271]). We can then suppose $C = \mathring{\overline{C}}$.

Consider now, for any $r > 0$, the bounded harmonic function $\tilde{h}_r$ on $C \cap B_r$ (where B_r is the ball of center 0 and radius r) such that

$$\tilde{h}_r = h \quad \text{on } S_r \cap C,$$

$$\tilde{h}_r \equiv 0 \quad \text{on } \partial C \cap B.$$

As ∂C is regular for C, h_r does exist and is unique because C_r is bounded.

We have then $\tilde{h}_r = h$ on ∂C, except maybe on a polar set of $\overline{C}$. By unicity of the solution of the Dirichlet problem on $C \cap B_r$, $h = \tilde{h}_r$ on $\overline{C \cap B_r}$: the harmonic function h is continuous on $\overline{C}$ and we have $h(x) = 0$ for any $x \in \partial C$.

PROOF OF THEOREM 2.4. There is no unicity for the Dirichlet problem for the Laplacian on the domain C: the harmonic function h is nonconstant on C and null on its border. Furthermore, we suppose that h does not grow too fast; the set C has to be big enough, and we will show that it is recurrent.

Let us suppose that C is transient and show that, necessarily, $I(C, h)$ is infinite.

We will use the following lemma, proved by Huber [9] that minorizes the growth of a harmonic function h on a connected component of $h^{-1}(\mathbf{R}_+^*)$:

LEMMA 5.2. *Let $U \subset \mathbf{R}^n$, $n \geq 2$, be an open connected set, $\Gamma = \partial U$ its border and $h: U \to \mathbf{R}$, a subharmonic function such that*

$$\forall\ x \in \Gamma,\ \limsup_{y \to x,\ y \in U} h(y) \leq 0.$$

Then, one of the two following properties is checked:

(i) *We have $h \leq 0$ on U.*

(ii) *There exist two constants $K > 0$ and $r_0 > 0$ such that, for all $r \geq r_0$,*

$$\left(\sup_{x \in U,\, \|x\| = r} h(x) \right)^2 \geq K r^{2-n} \int_{r_0}^{r} \rho^{n-3} \exp\left(2 \int_{r_0}^{\rho} \alpha(s)\ ds/s \right) d\rho, \tag{1}$$

where $\alpha(s)$ is the positive root of the second-degree equation $\alpha(\alpha + n - 2) = \lambda_1(s)$, $\lambda_1(s)$ being the first eigenvalue of the spherical Laplacian on the unit sphere $\mathbf{S}$ in $\mathbf{R}^n$, for the Dirichlet problem on $\mathbf{S} \cap s^{-1} U$.

Let us apply Lemma 5.2 to the harmonic function h and the connected open set C. We have clearly

$$h(x) = 0 \quad \text{for all } x \in \partial C$$

and

$$h(x) > 0 \quad \text{for all } x \in C.$$

Here, inequality (1) becomes

$$M^2(r, C, h) \geq K r^{2-n} \int_{r_0}^{r} \rho^{n-3} \exp\left(2 \int_{r_0}^{\rho} \alpha(s)\ ds/s \right) d\rho$$

for all $r \geq r_0$, K and r_0 being two positive constants.

For all $r \geq 2r_0$, we deduce

$$M^2(r, C, h) \geq K r^{2-n} \left[\int_{r_0}^{r/2} \rho^{n-3} \exp\left(2 \int_{r_0}^{\rho} \alpha(s)\ ds/s \right) d\rho + \int_{r/2}^{r} \rho^{n-3} \exp\left(2 \int_{r_0}^{r/2} \alpha(s)\ ds/s \right) d\rho \right].$$

This inequality implies

$$M^2(r, C, h) \geq \frac{K}{n-2} (1 - 2^{2-n}) \exp\left(2 \int_{r_0}^{r/2} \alpha(s) \frac{ds}{s} \right). \tag{2}$$

Set $K_1 = \frac{1}{2} \ln[K(1 - 2^{2-n})/(n-2)]$.

From the overestimate (2), we infer

$$\frac{\ln M(r,C,h)}{\ln r} \geq \frac{K_1}{\ln r} + \frac{1}{\ln r}\int_{\ln r_0}^{\ln(r/2)} \alpha(e^u)\, du,$$

As C is transient, the area of $\mathbf{S} \cap s^{-1}C$ converges to 0 as s converges to $+\infty$, so that the first eigenvalue of this set converges to $+\infty$. The quantity $\alpha(s)$ converges to $+\infty$ and finally, $I(C, h)$ has to be infinite.

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