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CROSSINGS AND OCCUPATION MEASURES FOR A CLASS OF SEMIMARTINGALES[1]

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We show that

$$\frac{1}{\sqrt{\varepsilon}}\left\{\int_{-\infty}^{\infty} f(u)k_\varepsilon N_\tau^{X_\varepsilon}(u)\,du - \int_0^\tau f(X_t)a_t\,dt\right\}$$

converges in law (as a continuous process) to $c_\psi \int_0^\tau f(X_t)a_t\,dB_t$, where $X_t = \int_0^t a_s\,dW_s + \int_0^t b_s\,ds$, with W a standard Brownian motion, a and b regular and adapted processes, $X_\varepsilon(t) = \int_{-\infty}^{\infty}(1/\varepsilon)\psi((t-u)/\varepsilon)X_u\,du$, ψ a smooth kernel, $N_t^g(u)$ the number of roots of the equation $g(s) = u, s \in (0, t]$, $k_\varepsilon = \sqrt{\pi\varepsilon/2}/\|\psi\|_2$, f a smooth function, B a standard Brownian motion independent of W and c_ψ a constant depending only on ψ.

1. Introduction.

Let $X = \{X_t\colon t \geq 0\}$ be a real-valued continuous semimartingale of the form

$$X_t = \int_0^t a_s\,dW_s + \int_0^t b_s\,ds, \tag{1}$$

where $W = \{W_t\colon t \geq 0\}$ is a standard Brownian motion (BM for short) adapted to a filtration $F = \{F_t\colon t \geq 0\}$, where $F_t \perp \sigma\{W_r - W_s\colon t \leq s \leq r\}\ \forall\, t \geq 0$. Here $a = \{a_t\colon t \geq 0\}$ and $b = \{b_t\colon t \geq 0\}$ are F-adapted processes verifying a certain number of regularity and boundedness conditions to be precised later on. We shall also assume that $a_t > 0$.

The purpose of this paper is to compute the speed of convergence of the normalized number of crossings of regularizations of X to the local time of X.

More precisely, let ψ be a C^∞ kernel, $\psi\colon \mathbb{R} \to \mathbb{R}^+$, with compact support (say $\operatorname{supp}\psi \subset [-1, 1]$), and $\int_{-1}^1 \psi(u)\,du = 1$.

Define

$$X_\varepsilon(t) = \int_{-\infty}^{\infty} \frac{1}{\varepsilon}\psi\left(\frac{t-u}{\varepsilon}\right)X_u\,du = \int_{-1}^{1} \psi(-u)X_{t+\varepsilon u}\,du, \tag{2}$$

where we have extended X by means of $X_t = 0$ if $t < 0$.

A comment on notation: the symbol "$\underset{\varepsilon\to 0^+}{\longrightarrow}$" denotes convergence of real numbers in the ordinary sense, and "$\overset{w}{\underset{\varepsilon\to 0^+}{\Longrightarrow}}$" indicates weak convergence of processes or measures.

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In Azaïs and Wschebor (1995) it is proved that if M is a real-valued and continuous local martingale with bracket $A = \{A_t: t \geq 0\}$ then, almost surely, for any bounded interval I contained in $[0, \infty)$ and any continuous function $f: \mathbb{R} \to \mathbb{R}$, we have

$$k_\varepsilon \int_{-\infty}^{\infty} f(u) N_I^{M_\varepsilon}(u)\, du \underset{\varepsilon \to 0^+}{\longrightarrow} \int_I f(M_t)(\dot{A}_t)^{1/2}\, dt; \tag{3}$$

$N_I^g(u)$ is the number of roots of the equation $g(t) = u$, $t \in I$; $\dot{A} = \{\dot{A}_t: t \geq 0\}$ is the (almost everywhere) derivative of A, and

$$k = \frac{1}{\|\psi\|_2}\sqrt{\frac{\pi}{2}}, \qquad k_\varepsilon = k\sqrt{\varepsilon}, \qquad \|\psi\|_2 = \left(\int_{-1}^{1} \psi(u)^2\, du\right)^{1/2}.$$

We will also denote $N_t^g(u) = N_{(0,\, t]}^g(u)$.

Theorem 1 gives a speed of convergence in (3) for semimartingales of the form (1). Note that in the statement of Theorem 1 neither the centering nor the limit distribution depend on the drift term in (1). The constant c_ψ depends only on the regularizing kernel ψ and not on the process.

Theorem 1 can be used to make inference on the martingale part of X. It also allows measuring the local time of X from the observation of the number of crossings of X_ε. In fact, introduce the modified local time:

$$\hat{L}_I^X(u) = \int_I \frac{1}{a_t} L_{dt}^X(u),$$

where $L_J^X(u)$ is the value at $u \in \mathbb{R}$ of the canonical bicontinuous local time of the continuous martingale M on the interval J [see Revuz and Yor (1991), page 209, (1.6)].

Taking into account that in this case $\dot{A}_t^{1/2} = a_t$, we can rewrite the right-hand term of (3) as

$$\int_I f(X_t) a_t\, dt = \int_{-\infty}^{\infty} f(u) \hat{L}_J^X(u)\, du.$$

To see this, argue as follows:

$$\int_I f(X_t) a_t\, dt = \int_I f(X_t) a_t^2 \frac{1}{a_t}\, dt.$$

If f is nonnegative, consider the measure defined by

$$\nu(J) = \int_{-\infty}^{\infty} f(u) L_J^X(u)\, du = \int_J f(X_t) a_t^2\, dt;$$

then

$$\begin{aligned}\int_I f(X_t) a_t\, dt &= \int_I \frac{1}{a_t} \nu(dt) = \int_I \frac{1}{a_t} \int_{-\infty}^{\infty} f(u) L_{dt}^X(u)\, du \\ &= \int_{-\infty}^{\infty} f(u) \left(\int_I \frac{1}{a_t} L_{dt}^X(u)\right) du = \int_{-\infty}^{\infty} f(u) \hat{L}_J^X(u)\, du.\end{aligned}$$

Then (3) means that, almost surely,

(4) $k_\varepsilon N_I^{X_\varepsilon}(u)\,du \to_{\varepsilon\to 0^+} \hat{L}_I^X(u)\,du$, where convergence takes place as weak convergence of measures on $\mathbb{R}$.

Theorem 1 refers to the speed of convergence in (4) enabling measuring the discrepancy between the approximation and its limit.

Extensions of Theorem 1 to $\mathbb{R}^d$-valued semimartingales will be considered elsewhere. In fact, the general setting consists of the study of the asymptotic behavior (as ε goes to zero) of functionals defined on the smoothed paths $X_\varepsilon(\cdot)$ having the form

$$\int_I F(X_\varepsilon(t), \sqrt{\varepsilon}\dot{X}_\varepsilon(t))\,dt$$

for suitable choices of the function F. Theorem 1 corresponds to $d = 1$ and $F(x, y) = f(x)|y|$.

In the case X is BM, a proof of Theorem 1, based on convergence of moments, has been given in Berzin and León (1994).

2. Main results and examples. In what follows, a and b will be $\mathbb{R}$-valued adapted and continuous processes, with $a > 0$, and such that the following hold:

(A) For every T, $p > 0$, $\sup_{t\in[0,T]} E\{|b_t|^p\} < \infty$.

(B) $\forall\ \varepsilon > 0$, $(a_{s+\varepsilon} - a_s)/\sqrt{\varepsilon} = a_s^*\, Z_{s,\varepsilon} + r_{s,\varepsilon}$, where we have the following:

(i) a^* adapted, $Z_{\bullet,\varepsilon}$ and $r_{\bullet,\varepsilon}$ $F_{\bullet+\varepsilon}$-predictable, $Z_{s,\varepsilon}\perp F_s$.

(ii) For almost every pair $s, t \geq 0$, $t \neq s$, we have the following weak convergence (in $\mathbb{R}^2 \times \mathbb{C}\,([0,\infty))^2$):

$$(Z_{t,\varepsilon}, Z_{s,\varepsilon}, W_\bullet^{\varepsilon,t}, W_\bullet^{\varepsilon,s}) \underset{\varepsilon\to 0}{\overset{w}{\Longrightarrow}} (Z_t, Z_s, W_\bullet^t, W_\bullet^s),$$

where

$$W_\gamma^{\varepsilon,t} = \frac{W_{t+\varepsilon\gamma} - W_t}{\sqrt{\varepsilon}};$$

$\{W_\bullet^t\colon t \geq 0\}$ is a collection of independent Brownian motions, $V_\bullet(t, s) = (Z_t, Z_s, W_\bullet^t, W_\bullet^s)\perp F_\infty$; $V_\bullet(t, s)$ has a symmetric distribution [i.e., $V_\bullet(t, s)$ and $-V_\bullet(t, s)$ have the same distribution] and if $\{s, t\}$ and $\{s', t'\}$ are disjoint, $V_\bullet(s, t)\perp V_\bullet(s', t')$.

(iii) For every $p > 0$, $T > 0$, and some $\delta > 0$,

$$\sup_{s\in[0,T],\,\varepsilon\in[0,\delta]} E(|Z_{s,\varepsilon}|^p) < \infty,$$

$$\sup_{s\in[0,T]} E(|r_{s,\varepsilon}|^p) \underset{\varepsilon\to 0^+}{\longrightarrow} 0,$$

$$\sup_{s\in[0,T]} E(|a_s^*|^p) < \infty.$$

We will consider Brownian semimartingales defined by (1) with a and b as before. In addition, we set

$$\Delta_\varepsilon(t) = X_\varepsilon(t) - X_t, \tag{5}$$

$$X_u^{\varepsilon,t} = \frac{X_{t+\varepsilon u} - X_t}{\sqrt{\varepsilon}}. \tag{6}$$

Observe that if $\operatorname{supp}\psi \subset [-1,0]$, $X_u^{\varepsilon,t} = \int_0^u a_{t+\varepsilon v}\, d_v W_v^{\varepsilon,t} + \sqrt{\varepsilon}\int_0^u b_{t+\varepsilon v}\, dv$. Hence, $X_\bullet^{\varepsilon,t}$ is the solution of the SDE:

$$\mathrm{d}_u X_u^{\varepsilon,t} = a_{t+\varepsilon u} d_u W_u^{\varepsilon,t} + \sqrt{\varepsilon} b_{t+\varepsilon u} du, \qquad u \geq 0,\ X_0^{\varepsilon,t} = 0. \tag{7}$$

Let us denote by C_b^2 the set of real-valued functions with bounded continuous second derivative and set

$$E_\varepsilon(\tau) := \frac{1}{\sqrt{\varepsilon}}\left\{\int_{-\infty}^{\infty} f(u) k_\varepsilon N_\tau^{X_\varepsilon}(u)\, du - \int_o^\tau f(X_t) a_t\, dt\right\}$$

Our main result is the following theorem.

THEOREM 1. *If X is as in (1), $f \in C_b^2$ then*

$$(W_\tau, E_\varepsilon(\tau)) \underset{\varepsilon\to 0}{\overset{w}{\Longrightarrow}} (W_\tau, c_\psi \int_0^\tau f(X_t) a_t\, dB_t) \quad \textit{in } \mathbb{C}([0,\infty))^2,$$

where B is a BM, $B \perp W$, and c_ψ is the constant

$$c_\psi^2 = \int_{-1}^1 \int_{-1}^1 E\left\{\prod_{i=1}^{i=2} H(R_{\gamma_i}, \beta^2(\gamma_i))\psi(-\gamma_i)\right\} d\gamma_2\, d\gamma_1,$$

where $H(x,\theta) := k[2\Phi(x/\theta) - 1]$, Φ is the standard normal distribution, $R_\gamma := \int_0^\gamma \psi(-u)\, dW_u$ and $\beta^2(\gamma) := \int_\gamma^1 \psi^2(-u)\, du$.

EXAMPLE 1 (Diffusions). Consider the diffusion process

$$X_t = \int_0^t \sigma(X_s)\, dW_s + \int_0^t b(X_s)\, ds, \tag{8}$$

where $\sigma > 0$ and assume

$$\sigma(x)^2 + b(x)^2 \leq K(1+x^2) \qquad \forall\, x \in \mathbb{R}, \tag{9}$$

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq L_N |x-y| \qquad \forall\, |x|, |y| \leq N, \qquad \forall\, N \in \mathbb{N}, \tag{10}$$

in which case existence and uniqueness of a strong solution of (8) are guaranteed, with all its moments uniformly bounded over compact intervals. Furthermore, assume that σ belongs to C_b^2.

Denote $a_s = \sigma(X_s)$, $b_s = b(X_s)$. We have $(0 < \theta < 1)$

$$\begin{aligned}\frac{a_{s+\varepsilon} - a_s}{\sqrt{\varepsilon}} &= \sigma'(X_s)\frac{(X_{s+\varepsilon} - X_s)}{\sqrt{\varepsilon}} + \sigma''(X_{s+\theta\varepsilon})\frac{(X_{s+\varepsilon} - X_s)^2}{2\sqrt{\varepsilon}} \\ &= \sigma'(X_s)\sigma(X_s)W_1^{\varepsilon,s} + \sigma'(X_s)\int_0^1 \left[\sigma(X_{s+\varepsilon v}) - \sigma(X_s)\right] d_v W_v^{\varepsilon,s} \\ &\quad + \sigma'(X_s)\sqrt{\varepsilon}\int_0^1 b(X_{s+\varepsilon v})\, d_v W_v^{\varepsilon,s} + \sigma''(X_{s+\theta\varepsilon})\frac{(X_{s+\varepsilon} - X_s)^2}{2\sqrt{\varepsilon}}.\end{aligned}$$

Thus, we have the representation (B) for a, with

$$\begin{aligned} a_s^* &= \sigma'(X_s)\sigma(X_s), \\ Z_{s,\varepsilon} &= W_1^{\varepsilon,s}, \\ r_{s,\varepsilon} &= \sigma'(X_s)\int_0^1 \left[\sigma(X_{s+\varepsilon v}) - \sigma(X_s)\right] d_v W_v^{\varepsilon,s} \\ &\quad + \sigma'(X_s)\sqrt{\varepsilon}\int_0^1 b(X_{s+\varepsilon v})\, d_v W_v^{\varepsilon,s} \\ &\quad + \sigma''(X_{s+\theta\varepsilon})\frac{(X_{s+\varepsilon} - X_s)^2}{2\sqrt{\varepsilon}},\end{aligned}$$

which clearly satisfy all the required conditions.

The statement of Theorem 1 can then be rewritten as

$$\left(W_\tau, \frac{1}{\sqrt{\varepsilon}}\int_{-\infty}^{\infty} f(u)\left[k_\varepsilon N_\tau^{X_\varepsilon}(u) - \frac{L_\tau^X(u)}{\sigma(u)}\right] du\right) \underset{\varepsilon \to 0}{\overset{w}{\Longrightarrow}} \left(W_\tau, c_\psi \int_0^\tau f(X_t)\sigma(X_t)\, dB_t\right)$$

[in $\mathbb{C}([0,\infty)^2)$].

EXAMPLE 2 (Non-Markovian martingales). Consider $X_t = \int_0^t f(W_s)\, dW_s$, where $f\colon \mathbb{R} \to \mathbb{R}$ is a C^3 function such that $\underline{f} = \inf\{f(x)\colon x \in \mathbb{R}\} > 0$, f'' and $f^{(3)}$ are bounded, $f''(0) \neq 0$ and $\|f''\|_\infty < 2\underline{f}$ [e.g., $f(x) = 1 + \mathbb{C}\exp(-x^2)$, with $0 < \mathbb{C} < 1$].

Then X verifies the hypothesis of Theorem 1 with $a_s = f(W_s)$, $b_s \equiv 0$, $a_s^* = f'(W_s)$, $Z_{\varepsilon,s} = W_1^{\varepsilon,s}$. However, X is non-Markovian; hence it is not a diffusion [cf. Nualart and Wschebor (1991), page 106].

EXAMPLE 3 (Smoother integrands). Suppose a satisfies a Hölder condition of the form

$$\sup_{0 \le t \le T-\varepsilon} |a_{t+\varepsilon}(\omega) - a_t(\omega)| \le C_T(\omega)\varepsilon^\alpha \; \left(\alpha > \tfrac{1}{2}\right)$$

for each $T > 0$ and $C_T \in L^p$ for all $p > 0$. Then the process X is included in our framework with $a_t^* = 0$, $Z_{s,\varepsilon} = 0$.

3. Proof of the main result. With no loss of generality, we will restrict the parameter to vary in $[0,1]$. We also may suppose $\operatorname{supp}\psi \subset [-1,0]$ (see Proof of Step 1) and a localization argument implies that we can assume a and b uniformly bounded by a (nonrandom) constant and a bounded away from zero (see Lemma 1).

Throughout the proof, $\mathbb{C}$ will denote a generic positive constant that may change from line to line. We divide the proof into several steps, and include further a series of auxiliary lemmas.

STEP 1. Denote $Z_\varepsilon(\tau) = (1/\sqrt{\varepsilon})\int_0^\tau f(X_t) g^t(\sqrt{\varepsilon}\dot{X}_\varepsilon(t))\,dt$, where $g^t(x) := k|x| - a_t$; then we have the following:

(i) $E_\varepsilon(\tau) - Z_\varepsilon(\tau) \longrightarrow_{\varepsilon\to 0^+} 0$ (in L^2);
(ii) $\{E_\varepsilon - Z_\varepsilon\colon \varepsilon > 0\}$ is $\mathbb{C}([0,1])$-tight.

Hence, E_ε has the same asymptotic distribution as Z_ε.

REMARK 1. It follows from the proof that

$$E\Big[\sup_{0\le\tau\le 1} |E_\varepsilon(\tau) - Z_\varepsilon(\tau)|\Big] \to_{\varepsilon\to 0^+} 0.$$

STEP 2. We can decompose

$$g^t(\sqrt{\varepsilon}\dot{X}_\varepsilon(t)) = a_t \int_0^1 \Phi_{\varepsilon,t}(v)\,d_v W_v^{\varepsilon,t} + R_\varepsilon(t),$$

where

$$\mathbb{R}_\varepsilon(\tau) := \frac{1}{\sqrt{\varepsilon}}\int_0^\tau f(X_t)R_\varepsilon(t)\,dt \underset{\varepsilon\to 0}{\overset{w}{\Longrightarrow}} 0 \quad \text{in } \mathbb{C}([0,1]);$$

$$\Phi_{\varepsilon,t}(v) := k\bigg[2\Phi\bigg(\frac{Y_v^{\varepsilon,t}}{\beta(v)a_t}\bigg) - 1\bigg];$$

$$Y_v^{\varepsilon,t} := \int_0^v \psi(-u)\,d_u X_u^{\varepsilon,t}.$$

REMARK 2. As in Step 1, we obtain $E[\sup_{0\le\tau\le 1}|\mathbb{R}_\varepsilon(\tau)|] \to_{\varepsilon\to 0^+} 0$.

STEP 3. We can decompose: $Z_\varepsilon(\tau) = \int_0^\tau f(X_t)K_\varepsilon(t)a_t\,dW_t + \alpha_\varepsilon(\tau)$, where

$$K_\varepsilon(t) := \int_{\max(0,\,t-\varepsilon)}^t \frac{\Phi_{\varepsilon,v}((t-v)/\varepsilon)}{\varepsilon}dv; \qquad \alpha_\varepsilon \underset{\varepsilon\to 0}{\overset{w}{\Longrightarrow}} 0 \quad \text{in } \mathbb{C}([0,1]).$$

STEP 4. Let $V = \{V_t\colon t\ge 0\}$ be an adapted process such that

$$\sup_{s\in[0,1]} E(|V_s|^p) < \infty \qquad \forall\ p > 0.$$

Then if $V_\varepsilon^*(\tau) := \int_0^\tau V_t K_\varepsilon^2(t)\,dt$, $\hat{V}_\varepsilon(\tau) := \int_0^\tau V_t K_\varepsilon(t)\,dt$, we have

$$V_\varepsilon^* \underset{\varepsilon\to 0}{\overset{w}{\Longrightarrow}} c_\psi \int_0^\tau V_t\,dt \quad \text{in } \mathbb{C}([0,1]); \qquad \hat{V}_\varepsilon \underset{\varepsilon\to 0}{\overset{w}{\Longrightarrow}} 0 \quad \text{in } \mathbb{C}([0,1]).$$

STEP 5. If $S_\varepsilon(\tau) := \int_0^\tau K_\varepsilon(t)\,dW_t$, then $(W, S_\varepsilon) \underset{\varepsilon\to 0}{\overset{w}{\Longrightarrow}} (W, c_\psi B)$ [in $\mathbb{C}\,([0,1])^2$], where B is a BM, $B \perp W$.

STEP 6. If $\hat{Z}_\varepsilon(\tau) := \int_0^\tau f(X_t)a_t\,dS_\varepsilon(t)$, then $(W_\bullet, \hat{Z}_\varepsilon(\cdot)) \underset{\varepsilon\to 0}{\overset{w}{\Longrightarrow}} (W_\bullet, c_\psi B_{\theta(\bullet)})$ [in $\mathbb{C}\,([0,1])^2$], with $\theta(\tau) := \int_0^\tau f(X_t)^2 a_t^2\,dt$.

Hence, from Step 3, $(W_\bullet, Z_\varepsilon(\bullet)) \underset{\varepsilon\to 0}{\overset{w}{\Longrightarrow}} (W_\bullet, c_\psi B_{\theta(\bullet)})$ [in $\mathbb{C}\,([0,1])^2$], with $\theta(\tau) = \int_0^\tau f(X_t)^2 a_t^2\,dt$.

The theorem follows from Steps 1 and 6, which we will prove, with the help of some auxiliary lemmas presented in Section 4.

PROOF OF STEP 1. The formula $\int_{-\infty}^{\infty} u(x)N_I^v(x)\,dx = \int_I u(v(t))|\dot{v}(t)|\,dt$ can be easily checked for $u\colon \mathbb{R}\to\mathbb{R}$ continuous and $v\colon I\to\mathbb{R}$ of class C^1, and I a bounded interval in the line [see, e.g., Nualart and Wschebor (1991), page 88, (2.4)].

Hence,

$$\begin{aligned} E_\varepsilon(\tau) &= \frac{1}{\sqrt{\varepsilon}}\left[\int_0^\tau f(X_\varepsilon(t))\sqrt{\frac{\pi}{2}}\frac{|\sqrt{\varepsilon}\dot{X}_\varepsilon(t)|}{\|\psi\|_2}\,dt - \int_0^\tau f(X_t)a_t\,dt\right] \\ &= Z_\varepsilon(\tau) + k\int_0^\tau [f(X_\varepsilon(t)) - f(X_t)]|\dot{X}_\varepsilon(t)|\,dt. \end{aligned}$$

Applying Lemma 3c, we deduce that (i) and (ii) hold, which concludes the proof of this step.

PROOF OF STEP 2. First we will prove tightness. Set

$$G^t(x,\theta) = E\{g^t(x+\sqrt{\theta}a_t N)/F_t\} = k\int_{-\infty}^{\infty}\left|x+\sqrt{\theta}a_t s\right|\phi(s)\,ds - a_t,$$

where N is a standard normal random variable, $N\perp F_\infty$, $x\in\mathbb{R}$, $\theta>0$ and ϕ stands for the standard normal density.

G^t is the $\mathbb{C}^\infty$ ($\mathbb{R}\times(0,\infty)$) solution of

$$D_\theta G^t = \frac{a_t^2}{2}D_{xx}^2 G^t; \qquad G^t(x, 0^+) = g^t(x). \tag{11}$$

Denoting by Φ, the standard normal distribution, we have

$$D_x G^t(x,\theta) = k\left[2\Phi\left(\frac{x}{\sqrt{\theta}a_t}\right) - 1\right], \tag{12}$$

$$D_{xx}^2 G^t(x,\theta) = \frac{2k}{\sqrt{\theta}}\phi\left(\frac{x}{\sqrt{\theta}a_t}\right). \tag{13}$$

Note that $D_x G^t(x,\theta)$ and $\sqrt{\theta}D_{xx}^2 G^t(x,\theta)$ are continuous and bounded. In addition, $H(x,\theta) := D_x^t(a_t x,\theta)$ and $J(x,\theta) := a_t D_{xx}^2 G^t(a_t x,\theta)$ do not depend on t, $H(\cdot,\theta)$ is odd and $J(\cdot,\theta)$ is even.

Define $Y_\gamma^{\varepsilon,t} = \int_0^\gamma \psi(-u)\,d_u X_u^{\varepsilon,t}$. Applying Itô's formula to

$$\eta_\gamma{}^{\varepsilon,t} = G^t(Y_\gamma^{\varepsilon,t}, \beta^2(\gamma)),$$

and noting that $\sqrt{\varepsilon}\dot{X}_\varepsilon(t) = Y_1^{\varepsilon,t}$, we get

$$\begin{aligned} g^t\big(\sqrt{\varepsilon}\dot{X}_\varepsilon(t)\big) &= \eta_1{}^{\varepsilon,t} - \eta_0{}^{\varepsilon,t} \\ &= \int_0^1 D_x G^t(Y_\gamma^{\varepsilon,t}, \beta^2(\gamma))\,d_\gamma Y_\gamma^{\varepsilon,t} \\ &\quad + \int_0^1 D_\theta G^t(Y_\gamma^{\varepsilon,t}, \beta^2(\gamma))\left(\frac{d\beta^2(\gamma)}{d\gamma}\right)d\gamma \\ &\quad + \frac{1}{2}\int_0^1 D_{xx}^2 G^t(Y_\gamma^{\varepsilon,t}, \beta^2(\gamma))\,d_\gamma\langle Y^{\varepsilon,t}, Y^{\varepsilon,t}\rangle_\gamma. \end{aligned} \tag{14}$$

Using (7), (11) and (14) we obtain

$$\begin{aligned} g^t\big(\sqrt{\varepsilon}\dot{X}_\varepsilon(t)\big) &= \int_0^1 D_x G^t(Y_\gamma^{\varepsilon,t}, \beta^2(\gamma))\psi(-\gamma)a_t\,d_\gamma W_\gamma^{\varepsilon,t} \\ &\quad + \int_0^1 D_x G^t(Y_\gamma^{\varepsilon,t}, \beta^2(\gamma))\psi(-\gamma)\big[a_{t+\varepsilon\gamma} - a_t\big]\,d_\gamma W_\gamma^{\varepsilon,t} \\ &\quad + \tfrac{1}{2}\int_0^1 D_{xx}^2 G^t(Y_\gamma^{\varepsilon,t}, \beta^2(\gamma))\psi^2(-\gamma)\big[a_{t+\varepsilon\gamma} - a_t\big]\big[a_{t+\varepsilon\gamma} + a_t\big]\,d\gamma \\ &\quad + \sqrt{\varepsilon}\int_0^1 D_x G^t(Y_\gamma^{\varepsilon,t}, \beta^2(\gamma))\psi(-\gamma)b_{t+\varepsilon\gamma}\,d\gamma \\ &= a_t\int_0^1 \Phi_{\varepsilon,t}(\gamma)\,d_\gamma W_\gamma^{\varepsilon,t} + R_\varepsilon(t), \end{aligned}$$

with

$$\Phi_{\varepsilon,t}(\gamma) = D_x G^t(Y_\gamma^{\varepsilon,t}, \beta^2(\gamma))\psi(-\gamma); \tag{15}$$

$$\begin{aligned} R_\varepsilon(t) &= \int_0^1 D_x G^t(Y_\gamma^{\varepsilon,t}, \beta^2(\gamma))\psi(-\gamma)\big[a_{t+\varepsilon\gamma} - a_t\big]\,d_\gamma W_\gamma^{\varepsilon,t} \\ &\quad + \tfrac{1}{2}\int_0^1 D_{xx}^2 G^t(Y_\gamma^{\varepsilon,t}, \beta^2(\gamma))\psi^2(-\gamma)\big[a_{t+\varepsilon\gamma} - a_t\big]\big[a_{t+\varepsilon\gamma} + a_t\big]\,d\gamma \\ &\quad + \sqrt{\varepsilon}\int_0^1 D_x G^t(Y_\gamma^{\varepsilon,t}, \beta^2(\gamma))\psi(-\gamma)b_{t+\varepsilon\gamma}\,d\gamma. \end{aligned} \tag{16}$$

We have

$$\mathbb{R}_\varepsilon(\tau) = A_\varepsilon^1(\tau) + A_\varepsilon^2(\tau) + A_\varepsilon^3(\tau) \tag{17}$$

where

$$A_\varepsilon^1(\tau) = \int_0^\tau\int_0^1 f(X_t)D_x G^t(Y_\gamma^{\varepsilon,t}, \beta^2(\gamma))\psi(-\gamma)\frac{[a_{t+\varepsilon\gamma} - a_t]}{\sqrt{\varepsilon}}\,d_\gamma W_\gamma^{\varepsilon,t}\,dt; \tag{18}$$

$$A^2_\varepsilon(\tau) = \frac{1}{2}\int_0^\tau \int_0^1 f(X_t) D^2_{xx} G^t(Y^{\varepsilon,t}_\gamma, \beta^2(\gamma))\psi^2(-\gamma) \times \frac{[a_{t+\varepsilon\gamma} - a_t]}{\sqrt{\varepsilon}}[a_{t+\varepsilon\gamma} + a_t]\, d\gamma\, dt; \tag{19}$$

$$A^3_\varepsilon(\tau) = \int_0^\tau \int_0^1 f(X_t) D_x G^t(Y^{\varepsilon,t}_\gamma, \beta^2(\gamma))\psi(-\gamma) b_{t+\varepsilon\gamma}\, d\gamma\, dt. \tag{20}$$

We will prove in what follows that $\{A^i_\varepsilon\colon \varepsilon > 0\}$ is $\mathbb{C}$ ([0, 1])-tight for $i = 1, 2, 3$.

Using Hölder's inequality and Lemma 2, we see that the integrand in (18) is bounded in L^p for each $p > 0$; applying Jensen and the Burkholder–Davis–Gundy inequality [cf. Revuz and Yor (1991), page 152] we obtain

$$E\{[A^1_\varepsilon(\tau') - A^1_\varepsilon(\tau)]^4\} \le \mathbb{C}\, |\tau' - \tau|^3, \tag{21}$$

which proves tightness for $i = 1$. The case $i = 3$ is even easier. For the case $i = 2$, it suffices to remark that $D^2_{xx}(x, \beta^2(\gamma))\psi^2(-\gamma)$ is bounded by $\mathbb{C}\, (\psi^2(-\gamma)/\beta(\gamma))$ and that

$$\int_0^1 \frac{\psi^2(-\gamma)}{\beta(\gamma)}\, d\gamma = \int_0^{\|\psi\|_2^2} \frac{1}{\sqrt{u}}\, du < \infty$$

and tightness follows.

For the convergence to zero in L^2, note that

$$\begin{aligned} A^1_\varepsilon(\tau) &= \int_0^\tau \int_t^{t+\varepsilon} f(X_t) D_x G^t\Bigl(Y^{\varepsilon,t}_{((v-t)/\varepsilon)}, \beta^2\Bigl(\Bigl(\frac{v-t}{\varepsilon}\Bigr)\Bigr)\Bigr) \\ &\qquad \times \psi\Bigl(-\Bigl(\frac{v-t}{\varepsilon}\Bigr)\Bigr)\frac{[a_v - a_t]}{\varepsilon}\, dW_v\, dt \\ &= \int_0^\tau I^1_\varepsilon(t)\, dt. \end{aligned}$$

Because of the martingale property of the stochastic integral it is clear that $I^1_\varepsilon(t),\ I^1_\varepsilon(s)$ are uncorrelated for $|t - s| > \varepsilon$ and it follows that $E\{[A^1_\varepsilon(\tau)]^2\} = O(\varepsilon)$.

Now

$$Y^{\varepsilon,t}_\gamma = a_t R^{\varepsilon,t}_\gamma + \sqrt{\varepsilon}\, O_{L^p}(1)\ \forall\ p > 0, \tag{22}$$

where $T(\varepsilon, t) = O_{L^p}$ means $\sup_{t\in[0,1],\, \varepsilon>0} E\{|T(\varepsilon, t)|^p\} < \infty$, and $R^{\varepsilon,t}_\gamma = \int_0^\gamma \psi(-u)\, d_u W^{\varepsilon,t}_u$.

(23) For almost every pair $s, t > 0$, $s \neq t$, $(R^{\varepsilon,s}_\gamma, Z_{s,\varepsilon}, R^{\varepsilon,t}_\gamma, Z_{t,\varepsilon}) \underset{\varepsilon\to 0}{\overset{w}{\Longrightarrow}} (R^s_\gamma, Z_s, R^t_\gamma, Z_s)$, where $\{R^s_\bullet\colon s \in [0, 1]\}$ are independent copies of $R_\bullet$ (defined in the statement of Theorem 1), $(R^s_\bullet, Z_s, R^t_\bullet, Z_t) \perp W$, $(R^s_\bullet, Z_s) \perp (R^t_\bullet, Z_t)$, $(R^s_\bullet, Z_s) \perp F_s$, $(R^t_\bullet, Z_t)$ have symmetric distributions.

Equation (22) follows from (8), the definition of $Y_\gamma^{\varepsilon,t}$, $R_\gamma^{\varepsilon,t}$, and condition (B)(iii), (23) follows from condition (B)(ii).

Set $c_t = a_t a_t^*$. Since the integrands in (19), (20) are $O_{L^p}(1)\ \forall\ p > 0$, applying dominated convergence and (22), our problem reduces to show that, for all $\gamma, \gamma' > 0$, and almost every pair $s, t > 0,\ s \neq t$,

$$\lim_{\varepsilon\to 0} E\{f(X_t)f(X_s)J(R_\gamma^{\varepsilon,t}, \beta^2(\gamma))J(R_{\gamma'}^{\varepsilon,s}, \beta^2(\gamma'))c_t c_s Z_{t,\varepsilon} Z_{s,\varepsilon}\} = 0, \tag{24}$$

$$\lim_{\varepsilon\to 0} E\{f(X_t)f(X_s)H(R_\gamma^{\varepsilon,t}, \beta^2(\gamma))H(R_{\gamma'}^{\varepsilon,s}, \beta^2(\gamma'))b_t b_s\} = 0. \tag{25}$$

Assume that $s > t$ are such that (23) holds and take ε so that $0 < \varepsilon < s - t$. Conditioning on F_s and using that $(R_{\gamma'}^{\varepsilon,s}, Z_{s,\varepsilon}\})\perp F_s$, we reduce the problem to show that

$$\lim_{\varepsilon\to 0} E\{J(R_{\gamma'}^{\varepsilon,s}, \beta^2(\gamma'))Z_{s,\varepsilon}\} = 0 = \lim_{\varepsilon\to 0} E\{H(R_{\gamma'}^{\varepsilon,s}, \beta^2(\gamma'))\}.$$

From (23) and uniform integrability it suffices to prove that

$$E\{J(R_\gamma^t, \beta^2(\gamma))Z_t\} = E\{H(R_\gamma^t, \beta^2(\gamma))\} = 0,$$

which is obvious by the symmetry of the distribution of (R_γ^t, Z_t) and the fact that $J(\cdot, \theta)$ (resp. $H(\cdot, \theta)$) is even (resp. odd).

PROOF OF STEP 3. Replacing $g^t(\sqrt{\varepsilon}\dot{X}_\varepsilon(t))$ by the formula in Step 2 we obtain

$$Z_\varepsilon^\tau = \int_0^\tau \int_0^1 f(X_t)a_t \frac{\Phi_{\varepsilon,t}(v)}{\sqrt{\varepsilon}}\, d_v W_v^{\varepsilon,t}\, dt + \frac{1}{\sqrt{\varepsilon}} \int_0^\tau f(X_t)R_\varepsilon(t)\, dt.$$

For $\varepsilon > 0$ fixed and every $p > 0$, it is obvious that the integrand in the first term of the right-hand member is $O_{L^p}(1)$; hence the Fubini-type Lemma 4 shows that

$$\begin{aligned}
\int_0^\tau \int_0^1 & f(X_t)a_t \frac{\Phi_{\varepsilon,t}(v)}{\sqrt{\varepsilon}}\, d_v W_v^{\varepsilon,t}\, dt \\
&= \int_0^\tau \int_t^{t+\varepsilon u} f(X_t)a_t \frac{\Phi_{\varepsilon,t}((u-t)/\varepsilon)}{\varepsilon}\, dW_u\, dt \\
&= \int_0^{\tau+\varepsilon} \int_{\max(0,u-\varepsilon)}^{\min(u,\tau)} f(X_t)a_t \frac{\Phi_{\varepsilon,t}((u-t)/\varepsilon)}{\varepsilon}\, dt\, dW_u.
\end{aligned}$$

Define

$$Q_\varepsilon(u,\tau) = \int_{\max(0,u-\varepsilon)}^{\min(u,\tau)} f(X_t)a_t \frac{\Phi_{\varepsilon,t}((u-t)/\varepsilon)}{\varepsilon}\, dt, \tag{26}$$

$$K_\varepsilon(u) = \int_{\max(0,u-\varepsilon)}^{u} \frac{\Phi_{\varepsilon,t}((u-t)/\varepsilon)}{\varepsilon}\, dt. \tag{27}$$

By (15) and Lemma 2, it follows that

$$\sup_{u\in[0,1],\,\tau\in[0,1]} E\{|Q_\varepsilon(u,\tau)|^p\} < \infty, \quad \sup_{u\in[0,1]} E\{|K_\varepsilon(u)|^p\} < \infty. \tag{28}$$

From this, the continuity of X, a, the Burkholder–Davies–Gundy inequality and Lemma 2, we obtain

$$\int_{\max(0,\,u-\varepsilon)}^{\min(u,\,\tau)} \big[f(X_u)a_u K_\varepsilon(u) - Q_\varepsilon(u,\tau)\big]\,dW_u = o_{L^p}(1) \tag{29}$$

[indeed, it is an $O_{L^p}(\sqrt{\varepsilon})$) $\forall\ p > 0$];

$$\int_\tau^{\tau+\varepsilon} Q_\varepsilon(u,\tau)\,dW_u = O_{L^p}(\varepsilon) \qquad \forall\ p > 0. \tag{30}$$

After Step 2, (29) and (30), Step 3 is proved.

Proof of Step 4. As a consequence of (28) and Jensen's inequality, both V^* and $\hat{V}$ are tight. Equations (22),(23) also imply

$$E\{\hat{V}_\varepsilon(\tau)\} \underset{\varepsilon\to 0^+}{\longrightarrow} c_\psi^2 \int_0^\tau E\{V_t\}\,dt, \tag{31}$$

$$E\{\hat{V}_\varepsilon^2(\tau)\} \underset{\varepsilon\to 0^+}{\longrightarrow} c_\psi^4 E\left\{\left[\int_0^\tau V_t\,dt\right]^2\right\}, \tag{32}$$

$$E\left\{\hat{V}_\varepsilon(\tau)\int_0^\tau V_t\,dt\right\} \underset{\varepsilon\to 0^+}{\longrightarrow} c_\psi^2 E\left\{\left[\int_0^\tau V_t\,dt\right]^2\right\}, \tag{33}$$

$$E\{V_\varepsilon^*(\tau)^2\} \underset{\varepsilon\to 0^+}{\longrightarrow} 0; \tag{34}$$

and Step 4 follows.

Proof of Step 5. Apply Step 4 with $V = 1$ and Rebolledo's theorem for convergence of martingales [cf. Revuz and Yor (1991), page 478].

Proof of Step 6. Consider $P_t := f(X_t)a_t$ and

$$P_t^N := \sum_{i=0}^{i=N-1} f(X_{i\tau/N})a_{i\tau/N}\,\mathbb{1}_{[i\tau/N,\,(i+1)\tau/N)}(t).$$

It follows from Step 5 that

$$\int_0^\tau P_t^N\,dS_\varepsilon(t) \underset{\varepsilon\to 0}{\overset{w}{\Longrightarrow}} c_\psi \int_0^\tau P_t^N\,dB_t \quad \text{in } \mathbb{C}\,([0,1]). \tag{35}$$

Step 4 applied to $(P_t - P_t^N)^2$ shows that

$$\lim_{N\to\infty}\lim_{\varepsilon\to 0}\int_0^\tau (P - P^N)\,dS_\varepsilon(t) = 0 \quad \text{in } \mathbb{C}\,([0,1]). \tag{36}$$

Since

$$\int_0^\tau P_t^N \, dB_t \underset{N\to\infty}{\overset{w}{\Longrightarrow}} \int_0^\tau P_t \, dB_t$$

and by (35), (36), Step 6 follows and the theorem is proved. □

4. Auxiliary lemmas.

LEMMA 1. *If a satisfies* (A), (B) *and φ: $\mathbb{R} \to \mathbb{R}$ is a bounded C^∞ function, then $\varphi \circ a$ satisfies* (A), (B).

For the proof, use Taylor's expansion.

LEMMA 2. *Let X be as in (1), with a and b uniformly bounded by a (nonrandom) constant. Then, $\forall\ p \geq 2$ we have the following:*

(a) $E\{\sup_{t\in[0,1]} |X_t|^p\} < \infty$.
(b) $E\{\sup_{t\in[0,1]} |X_\varepsilon(t)|^p\} < \infty$.
(c) $E\{\sup_{t\in[0,1]} |\dot{X}_\varepsilon(t)|^p\} = O(\varepsilon^{-p/2})$.
(d) $E\{\sup_{t\in[0,1]} |\Delta_\varepsilon(t)|^p\} = O(\varepsilon^{p/2})$.

For the proof, use the Burkholder–Davis–Gundy inequality.

LEMMA 3. *Let $V = \{V_t: t \geq 0\}$ be a real-valued adapted process such that $\sup_{t\in[0,1]} E\{|V_t|^p\} < \infty\ \forall\ p > 0$.*

If X is as in (1), with a and b uniformly bounded by a (nonrandom) constant, we have, for $0 < \varepsilon < 1$, $0 < h < 1$,

(a) $$\sup_{0\leq t\leq 1-h} E\left[\left\{\int_t^{t+h} \Delta_\varepsilon^2(s)|\dot{X}_\varepsilon(s)|V_s \, ds\right]^2\right\} \leq \mathbb{C}h^2\varepsilon,$$

(b) $$\sup_{0\leq t\leq 1-h} E\left[\left\{\int_t^{t+h} \Delta_\varepsilon(s)|\dot{X}_\varepsilon(s)|V_s \, ds\right]^2\right\} \leq \mathbb{C}\sqrt{\varepsilon}h^{3/2},$$

(c) *if $f \in C_b^2(\mathbb{R})$, then*

$$\sup_{0\leq t\leq 1-h} E\left\{\left[\int_t^{t+h} f(X_\varepsilon(s)) - f(X_s))|\dot{X}_\varepsilon(s)| \, ds\right]^2\right\} \leq \mathbb{C}\sqrt{\varepsilon}h^{3/2}.$$

PROOF. (a) Apply Lemma 2.

(b) Observe that

$$\begin{aligned}\sqrt{\varepsilon}\dot{X}_\varepsilon(s) &= \int_0^1 \psi(-u)a_{s+\varepsilon u} \, d_u W_u^{\varepsilon,s} + \sqrt{\varepsilon}\int \psi(-u)b_{s+\varepsilon u} \, du \\ &= a_s \int_0^1 \psi(-u) \, d_u W_u^{\varepsilon,s} + O_{L^p}(\varepsilon) \qquad \forall\ p > 0;\end{aligned} \tag{37}$$

$$\begin{aligned}\frac{\Delta_\varepsilon(s)}{\sqrt{\varepsilon}} &= \int_0^1 \psi(-u)\int_0^u a_{s+\varepsilon v}\, d_v W_v^{\varepsilon,s}\, du + \sqrt{\varepsilon}\int_0^1 \psi(-u)\int_0^u b_{s+\varepsilon v}\, d_v\, du \\ &= a_s \int_0^1 \psi(-u) W_u^{\varepsilon,s}\, du + O_{L^p}(\varepsilon) \qquad \forall\ p > 0.\end{aligned} \tag{38}$$

Set $H_\varepsilon(s,r) = E\{\Delta_\varepsilon(s)|\dot{X}_\varepsilon(s)|V_s \Delta_\varepsilon(r)|\dot{X}_\varepsilon(r)|V_r\}$.

Compute the second moment as follows:

$$\begin{aligned} &E\left\{\left[\int_t^{t+h} \Delta_\varepsilon(s)|\dot{X}_\varepsilon(s)|V_s\, ds\right]^2\right\} \\ &\quad = \int_t^{t+h}\int_t^{t+h} E\{\Delta_\varepsilon(s)|\dot{X}_\varepsilon(s)|V_s\Delta_\varepsilon(r)|\dot{X}_\varepsilon(r)|V_r\Big\}\, dr\, ds \\ &\quad = \int_{\{t\le r,\, s\le t+h,\, |s-r|<\varepsilon\}} H_\varepsilon(s,r)\, dr\, ds + 2\int_t^{t+h}\int_{s+\varepsilon}^{t+h} H_\varepsilon(s,r)\, dr\, ds \\ &\quad = (I) + 2(II).\end{aligned}$$

Taking into account that the integrand H_ε is bounded, it is trivial to observe that

$$(I) \le \mathbb{C}h \min\{\varepsilon, h\} \le \mathbb{C}\sqrt{\varepsilon}h^{3/2}. \tag{39}$$

For the second term, $A_{s,r} = a_s^2 V_s a_r^2 V_r$, and using (37), we deduce

$$\begin{aligned} H_\varepsilon(s,r) = E\Big\{& A_{s,r}\left|\int_0^1 \psi(-u)\, d_u W_u^{\varepsilon,s}\right| \int_0^1 \psi(-u) W_u^{\varepsilon,s}\, du \\ &\times \left|\int_0^1 \psi(-u)\, d_u W_u^{\varepsilon,r}\right| \int_0^1 \psi(-u) W_u^{\varepsilon,r}\, du\Big\} + O(\sqrt{\varepsilon}).\end{aligned} \tag{40}$$

Since $s+\varepsilon \le r$, conditioning to F_r and using the independence of the Brownian increments, we get

$$H_\varepsilon(s,r) = P(s,r) + O(\sqrt{\varepsilon})$$

with

$$\begin{aligned} P(s,r) = E\Big\{& A_{s,r}\left|\int_0^1 \psi(-u)\, d_u W_u^{\varepsilon,s}\right| \int_0^1 \psi(-u) W_u^{\varepsilon,s}\, du\Big\} \\ &\times E\left\{\left|\int_0^1 \psi(-u)\, d_u W_u^{\varepsilon,r}\right| \int_0^1 \psi(-u) W_u^{\varepsilon,r}\, du\right\}.\end{aligned} \tag{41}$$

Since $(\int_0^1 \psi(-u)\, d_u W_u^{\varepsilon,r}, \int_0^1 \psi(-u) W_u^{\varepsilon,r}\, du)$ is a centered Gaussian vector, it follows by symmetry that

$$E\left\{\left|\int_0^1 \psi(-u)\, d_u W_u^{\varepsilon,r}\right| \int_0^1 \psi(-u) W_u^{\varepsilon,r}\, du\right\} = 0. \tag{42}$$

By (41) and (42) we deduce

$$(II) \le \mathbb{C}\sqrt{\varepsilon}h^2 \le \mathbb{C}\sqrt{\varepsilon}h^{3/2}. \tag{43}$$

This concludes the proof of part (b).

(c) Use Taylor's formula, apply (b) to the linear term and (a) to the quadratic one. □

LEMMA 4. *Let* $\{K(t,s): t, s \in [0,1]\}$ *be a real-valued random process such that*

(a) $$\sup_{t,s\in[0,1]} E\{|K(t,s)|^p\} < \infty \qquad \forall\ p > 0,$$

(b) $$\int_0^1 K(t,s)\,ds \text{ is predictable,}$$

(c) $$\int_0^1 K(t,s)\,dW_t \text{ is measurable.}$$

Then

$$\int_0^\tau \int_0^1 K(t,s)\,ds\,dW_t = \int_0^1 \int_0^\tau K(t,s)\,dW_t\,ds.$$

The proof is an analogue to Lemma 1.4.1 of Ikeda and Watanabe (1981).

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