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THE STOCHASTIC WAVE EQUATION IN TWO SPATIAL DIMENSIONS

BY ROBERT C. DALANG[1] AND N. E. FRANGOS[2]

Ecole Polytechnique Fédérale and Athens University of Economics and Business

We consider the wave equation in two spatial dimensions driven by space–time Gaussian noise that is white in time but has a nondegenerate spatial covariance. We give a necessary and sufficient integral condition on the covariance function of the noise for the solution to the linear form of the equation to be a real-valued stochastic process, rather than a distribution-valued random variable. When this condition is satisfied, we show that not only the linear form of the equation, but also nonlinear versions, have a real-valued process solution. We give stronger sufficient conditions on the spatial covariance for the solution of the linear equation to be continuous, and we provide an estimate of its modulus of continuity.

1. Introduction.

The wave equation subject to random excitation in one spatial dimension, written

$$\frac{\partial^2 u}{\partial t^2}(t,x) - \frac{\partial^2 u}{\partial x^2}(t,x) = a(t,x,u)\dot{G}(t,x) + b(t,x,u), \qquad t > 0,\ x \in \mathbb{R},$$

has been studied by many authors [5–7, 10, 12, 16, 19, 22]. In this equation, the functions $a(t,x,u)$ and $b(t,x,u)$ satisfy certain smoothness and growth conditions, and $\dot{G}(t,x)$ is generally white noise, or sometimes Lévy noise.

There are fewer results concerning the wave equation driven by random noise in two (or more) dimensions:

$$\frac{\partial^2 u}{\partial t^2}(t,x) - \Delta\, u(t,x) = a(t,x,u)\dot{F}(t,x) + b(t,x,u), \qquad t > 0,\ x = (x_1,x_2) \in \mathbb{R}^2, \tag{1}$$

where $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$. One reason for this is that if $\dot{F}(t,x)$ is white noise, even the linear equation ($a \equiv 1$, $b \equiv 0$) has no solution in the space of real-valued measurable stochastic processes (see [19]). Given that white noise can be viewed as a random variable with values in a space of distributions, the linear equation has of course a distribution-valued solution (see [21]). However, the study of distribution-valued processes is technically demanding and is not readily amenable to numerical calculations. Furthermore, within

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this framework, there is no convenient way even to formulate (the nonlinear form of) equation (1) when $a(\cdot)$ and $b(\cdot)$ are not constant, although there have been efforts in this direction [2, 11]. It is therefore natural to investigate other types of random noise that will produce solutions that are real-valued stochastic processes.

One way to achieve this is to consider random noises $\dot{F}(t, x)$ that are smoother than white noise, namely, a Gaussian noise that is "white noise in time but has smooth spatial covariance," that is, a covariance of the form

$$E(\dot{F}(t, x)\dot{F}(s, y)) = \delta(t-s)f(\|x-y\|). \tag{2}$$

In this equation, $\delta(\cdot)$ denotes the Dirac delta function. The case $f(r) = \delta(r)$ would correspond to the case of space–time white noise, and in this case one has (formally) $f(0) = +\infty$, but functions f with greater regularity can also be considered in order to smooth out the noise $\dot{F}(t, x)$.

Smoother random noises are particularly interesting in view of the fact that in many physical applications, spatial correlations are of a much larger order of magnitude than time correlations, and there are some examples in the literature in which correlations of type (2) seem to provide a better model than white noise [4, 13].

The case where the function $f\colon \mathbb{R}_+ \to \mathbb{R}$ is bounded has been considered in [14]. However, with regard to the questions of existence and uniqueness of a real-valued process solution to (1), this assumption on f is clearly too strong. In order to be able to model the widest possible range of situations, one would like weaker assumptions on f that ensure that (1) has a process solution, and the main result of this paper provides a necessary and sufficient condition for the linear form of (1) to have such a solution. The condition is formulated as an integral test on the function f (see Theorem 1). When this condition is satisfied, not only the linear equation but also the nonlinear forms of (1) have a process solution (see Theorem 2). The solution is jointly measurable, but may not be continuous. We give a stronger sufficient condition on f for the solution of the linear equation to be continuous and provide an estimate of its modulus of continuity (see Theorem 3).

The paper is organized as follows. In Section 2, we give a rigorous formulation of equation (1) using Walsh's theory of martingale measures [19] and some examples of Gaussian noises of the form (2) with unbounded f. In Section 3, we state Theorem 1 and prove the necessity of our integral condition on f. In Section 4, we show that under this condition both the linear and nonlinear forms of (1) have a process solution. Finally, we give the sufficient condition for the process solution of the linear equation to have a continuous version, and the estimate of its modulus of continuity.

2. Preliminaries.

We shall be working with a Gaussian process indexed by a family of test functions. In order to use the theory of stochastic partial differential equations (s.p.d.e.'s) developed in [19], we need to construct from this process a worthy martingale measure. We detail the construction in this section.

Recall that $\mathscr{D}(\mathbb{R}^3)$ is the topological vector space of functions φ in $C_0^\infty(\mathbb{R}^3)$ with a topology that corresponds to the following notion of convergence ([1], page 19): $\varphi_n \to \varphi$ if and only if the following two conditions hold:

1. there is a compact subset K of $\mathbb{R}^3$ such that supp$(\varphi_n - \varphi) \subset K$, for all n;
2. $\lim_{n\to\infty} D^a \varphi_n = D^a \varphi$ uniformly on K for each multiindex a.

Let $F = (F(\varphi),\ \varphi \in \mathscr{D}(\mathbb{R}^3))$ be an $L^2(\Omega, \mathscr{F}, P)$-valued mean zero Gaussian process with covariance functional of the form

$$(\varphi, \psi) \to E(F(\varphi)F(\psi)) = \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy\ \varphi(t, x) f(\|x - y\|)\psi(t, y), \tag{3}$$

where $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ is continuous on $\mathbb{R}_+ \setminus \{0\}$, and $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^2$. Formula (3) is the rigorous meaning of (2).

If f is unbounded near 0, the functional in (3) need not even be finite-valued. We have the following result.

PROPOSITION 1. *For the map (3) to be finite-valued it is necessary and sufficient that*

$$\int_{0^+} f(r) r\, dr < \infty \tag{4}$$

(i.e., the integral over any small interval $[0, r_0]$*, with* $r_0 > 0$*, is finite).*

PROOF. Fix $\varepsilon > 0$ and $t_0 > \varepsilon$. Assume that $\varphi \in \mathscr{D}(\mathbb{R}^3)$ is such that $\varphi \geq 0$, and $\varphi \geq 1$ on $B(t_0, \varepsilon) \times B(x_0, \varepsilon)$, where $B(x_0, \varepsilon)$ denotes the ball centered at t_0 with radius ε. Then

$$\begin{aligned} E(F(\varphi)^2) &= \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy\ \varphi(t, x) f(\|x - y\|)\varphi(t, y) \\ &\geq \int_{t_0-\varepsilon}^{t_0+\varepsilon} dt \int_{B(x_0,\, \varepsilon)} dx \int_{B(x_0,\, \varepsilon)} dy\ f(\|x - y\|) \\ &= 2\varepsilon \int_{B(x_0,\, \varepsilon)} dx \int_{B(x_0,\, \varepsilon)} dy\ f(\|x - y\|). \end{aligned}$$

Using the change of variables $u = x + y$, $v = x - y$, which has Jacobian 4, we get

$$E(F(\varphi)^2) \geq C'_\varepsilon \int_{B(2x_0,\, 2\varepsilon)} du \int_{B(0,\, 2\varepsilon)} dv\ f(\|v\|) \geq C_\varepsilon \int_0^{2\varepsilon} f(r) r\, dr.$$

Therefore, if (3) is finite-valued, then (4) holds. Conversely, similar arguments show that, for $\varphi \in \mathscr{D}(\mathbb{R}^3)$, $E(F(\varphi)^2) \leq C_\varphi \int_{0+} f(r) r\, dr$. $\square$

REMARK 1. The passage to polar coordinates in the last step shows that

$$\int_{B(0,\, \varepsilon)} f(\|v\|)\, dv < +\infty \quad \Leftrightarrow \quad \int_{0^+} f(r) r\, dr < +\infty.$$

A $\mathscr{D}'(\mathbb{R}^3)$-valued version of F. A direct calculation, using (3), of the L^2-norm of the difference between $F(a\varphi + b\psi)$ and $aF(\varphi) + bF(\psi)$ shows that this L^2-norm is zero, and therefore $(F(\varphi),\ \varphi \in \mathscr{D}(\mathbb{R}^3))$ is a *random linear functional* in the terminology of [19], page 332. According to [19], Corollary 4.2, F will have a version with values in $\mathscr{D}'(\mathbb{R}^3)$ provided $\varphi \mapsto F(\varphi)$ is continuous in probability. We shall show that, in fact, this map is continuous in $L^2(P)$. Indeed, if $\varphi_n \to 0$ in $\mathscr{D}(\mathbb{R}^3)$, then

$$E(F(\varphi_n)^2) = \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy\ \varphi_n(t,x) f(\|x-y\|) \varphi_n(t,y) \to 0$$

by the dominated convergence theorem, because all the φ_n have support in some fixed compact set, $\varphi_n \to 0$ uniformly and $(x, y) \mapsto f(\|x-y\|)$ is integrable over compact sets by Proposition 1.

Therefore, F has a version with values in $\mathscr{D}'(\mathbb{R}^3)$, and so without loss of generality we make the following assumption throughout the remainder of this paper.

ASSUMPTION. Condition (4) holds and F takes values in $\mathscr{D}'(\mathbb{R}^3)$.

Extending F to a worthy martingale measure. If A is a rectangle in $\mathbb{R}^3$ and $(\varphi_n) \subset \mathscr{D}(\mathbb{R}^3)$ is such that, for all n, supp $\varphi_n \subset K$, where K is a fixed compact set, and $\varphi_n \to I_A$, then it is not difficult to check, as above, that as $n, m \to \infty$,

$$E[(F(\varphi_n) - F(\varphi_m))^2] = E(F(\varphi_n - \varphi_m)^2) \to 0.$$

So $(F(\varphi_n), n \in \mathbb{N})$ is a Cauchy sequence in $L^2(P)$, whose limit does not depend on the choice of (φ_n). We call the limit $F(A)$. The same can be done for finite unions of rectangles, and this defines an additive set function $A \mapsto F(A)$ on finite unions A of rectangles, such that

$$E(F(A)F(B)) = \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy\ I_A(t,x) f(\|x-y\|) I_B(t,y). \tag{5}$$

If $A_n \downarrow \varnothing$, then $E(F(A_n)^2) \to 0$, so the above defined set function is countably additive. In particular, $E(F(A)^2) = \nu(A)$, where

$$\nu(A) = \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy\ I_A(t,x) f(\|x-y\|) I_B(t,y). \tag{6}$$

Therefore, condition (2.1) of [19], page 286, is satisfied, and so the map $A \mapsto F(A)$ defines a σ-finite L^2-valued measure.

We set

$$M_t(A) = F([0,t] \times A), \tag{7}$$

and let

$$\mathscr{F}_t^0 = \sigma(M_s(A),\ s \le t,\ A \in \mathscr{B}_b(\mathbb{R}^2)), \qquad \mathscr{F}_t = \mathscr{F}_t^0 \vee \mathscr{N},$$

where $\mathscr{B}_b(\mathbb{R}^2)$ denotes the bounded Borel subsets of $\mathbb{R}^2$ and $\mathscr{N}$ is the σ-field generated by P-null sets. It is straightforward to check that

$$(M_t(A),\ \mathscr{F}_t,\ t \geq 0,\ A \in \mathscr{B}_b(\mathbb{R}^2))$$

is a martingale measure according to [19], page 287. Because the martingale $(M_t(A),\ t \geq 0)$ is Gaussian, the mutual variation

$$\bar{Q}_t(A, B) = \langle M(A), M(B) \rangle_t \tag{8}$$

is deterministic and equal to $E(M_t(A)M_t(B))$, that is,

$$\bar{Q}_t(A, B) = t \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy\ I_A(x) f(\|x - y\|) I_B(y). \tag{9}$$

Let

$$\begin{aligned} Q(A \times B \times]s, t]) &= \bar{Q}_t(A, B) - \bar{Q}_s(A, B) \\ &= (t - s) \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy\ I_A(x) f(\|x - y\|) I_B(y). \end{aligned} \tag{10}$$

Clearly, Q is a positive definite measure on $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}_+$ ([19], page 290). Therefore, according to [19], page 291, the martingale measure M is worthy with dominating measure $K \equiv Q$. By construction,

$$F(\varphi) = \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^2} dx\ \varphi(t, x) M(dt, dx), \tag{11}$$

where the integral is the stochastic integral defined in [19].

Predictable processes. We now examine the class of processes $X = (X(t, x))$ for which the stochastic integral $\int X\, dM$ is defined. Recall [19] that an *elementary process* is a process X such that

$$X(t, x; \omega) = 1_{]s_1, s_2]}(t)\, 1_A(x)\, Y(\omega),$$

where $0 < s_1 < s_2$, $A \in \mathscr{B}_b(\mathbb{R}^2)$ and Y is a bounded and $\mathscr{F}_{s_1}$-measurable random variable.

Let $\mathscr{S}$ denote the set of elementary processes. If $(X(t, x))$ is such that

$$E\left(\int_{\mathbb{R}_+} dt \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy\, X(t, x) f(\|x - y\|) X(t, y) \right) < +\infty,$$

then we denote the square root of this quantity by $\|X\|_f$. Let $\mathscr{H}$ be the set of all jointly measurable processes X such that $\|X\|_f < +\infty$. Note that $\mathscr{S} \subset \mathscr{H}$ and let $\mathscr{P}_M$ be the closure of $\mathscr{S}$ in $\mathscr{H}$ for $\|\cdot\|_f$. Processes in $\mathscr{P}_M$ are termed *predictable processes.* According to [19], $\int X\, dM$ is defined for all $X \in \mathscr{P}_M$. The following proposition gives an easily checkable sufficient condition for a process to belong to $\mathscr{P}_M$. The hypotheses are chosen for ease of later use.

PROPOSITION 2. *Suppose a process $X = (X(t, x),\ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2)$ has the following four properties:*

(a) *for all (t, x), $X(t, x)$ is $\mathscr{F}_t$-measurable;*

(b) *$(t, x; \omega) \mapsto X(t, x; \omega)$ is $\mathscr{B}(\mathbb{R}_+ \times \mathbb{R}^2) \times \mathscr{F}$-measurable;*

(c) *for all* (t, x), $E(X(t, x)^2) < +\infty$ *and the function* $(t, x) \mapsto X(t, x)$ *from* $\mathbb{R}_+ \times \mathbb{R}^2$ *into* $L^2(\Omega, \mathscr{F}, P)$ *is continuous;*

(d) *there are* $t_0 > 0$ *and a compact set* $K \subset \mathbb{R}^2$ *such that*

$$E\left(\int_0^{t_0} dt \int_K dx \int_K dy\, X(t, x)\, f(\|x - y\|)\, X(t, y)\right) < +\infty.$$

Then $X\, 1_{[0,t_0]\times K}$ *belongs to* $\mathscr{P}_M$.

PROOF. Fix $\varepsilon > 0$ and choose $n \in \mathbb{N}$ large enough so that, for all $s, t \in [0, t_0]$ and $x, y \in K$,

$$|t - s| + \|x - y\| \le \frac{2t_0}{n} \quad \Rightarrow \quad \|X(t, x) - X(t, y)\|_{L^2(P)} < \varepsilon.$$

Set $t_j = jt_0/n$ and let (K_ℓ) be a finite family of disjoint subsets of K of diameter $< t_0/n$ such that $\bigcup_\ell K_\ell = K$. Fix $x_\ell \in K_\ell$ and set

$$X_n(t, x) = \sum_{j=0}^{n-1} \sum_\ell X(t_j, x_\ell)\, 1_{]t_j, t_{j+1}]}(t)\, 1_{K_\ell}(x).$$

Clearly, $X_n \in \mathscr{P}_M$, and $\|X_n - X\, 1_{[0,t_0]\times K}\|_f^2$ is equal to

$$\begin{aligned} E\Bigg(\int_0^{t_0} dt \int_K dx \int_K dy\, (X_n(t, x) - X(t, x)) f(\|x - y\|)(X_n(t, y) - X(t, y))\Bigg) \\ = \sum_\ell \sum_m \int_{K_\ell} dx \int_{K_m} dy\, f(\|x - y\|) \int_{t_j}^{t_{j+1}} dt\, E\Big((X(t_j, x_\ell) - X(t, x)) \\ \times (X(t_j, x_m) - X(t, y))\Big). \end{aligned}$$

The Cauchy–Schwarz inequality implies that the expectation is less than or equal to

$$\|X(t_j, x_\ell) - X(t, y)\|_{L^2(P)} \cdot \|X(t_j, x_m) - X(t, y)\|_{L^2(P)},$$

and so

$$\|X_n - X\, 1_{[0,\, t_0]\times K}\|_f^2 \le \varepsilon^2 \int_0^{t_0} dt \int_K dx \int_K dy\, f(\|x - y\|).$$

Therefore, $X \in \mathscr{P}_M$. □

REMARK 2. Given a function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ which is continuous, positive in some neighborhood of 0 and which satisfies (4), a natural question is whether a Gaussian process F exists with covariance given by (3). This occurs if and only if the functional $J(\cdot, \cdot)$ defined by

$$J(\varphi, \psi) = \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy\ \varphi(t, x) f(\|x - y\|) \psi(t, y) \tag{12}$$

is nonnegative definite or, equivalently, the spectrum of the operator

$$\psi \to \left[x \to \int_{\mathbb{R}^2} f(\|x-y\|)\ \psi(y)\, dy \right] \tag{13}$$

is contained in $\mathbb{R}_+$ (see [9]). It is not our objective to address this functional analysis question here. A sufficient (and necessary, see [18], page 131) condition, which is related to the Bochner–Herglotz–Weyl theorem, is that $u \mapsto f(\|u\|)$ be the Fourier transform of a nonnegative measure $d\lambda$ on $\mathbb{R}^2$, that is,

$$f(\|u\|) = \int_{\mathbb{R}^2} e^{-i\,u\cdot x}\, d\lambda(x). \tag{14}$$

Examples. There are several interesting examples of Gaussian random fields that satisfy conditions (3) and (4). For example, the function

$$f(u) = u^{-\alpha}, \qquad u > 0, \text{ with } 0 < \alpha < 2, \tag{15}$$

satisfies (4). Moreover, with this choice of f, the functional J defined in (12) is nonnegative definite. Indeed, in this case, the operator defined in (13) is then a Riesz potential, and according to [17], Chapter V, Section 1, Lemma 2(b),

$$J(\varphi,\psi) = \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^2} dx\ \hat{\varphi}(t,x)\ \frac{1}{(2\pi\|x\|)^{2-\alpha}}\ \hat{\psi}(t,x),$$

where $\hat{\varphi}$ denotes the Fourier transform in the x-variable for fixed t. Therefore, $J(\varphi,\varphi)$ is clearly greater than or equal to 0, for all φ. The Gaussian random field with covariance given by (3), in which f is defined as in (15), therefore satisfies (3) and (4).

Another interesting example of a "generalized" Gaussian process with covariance of the form (3) is what one might call a *Brownian free field,* that is, a process which at each time t is a spatial *free field* [15], and at distinct times these fields are independent. Such a process $G(t,x_1,x_2)$ satisfies the s.p.d.e.

$$(m^2 - \Delta_x)^{1/2} G(t,x_1,x_2) = \dot{W}(t,x_1,x_2),$$

where $\dot{W}$ is a space–time white noise, $m > 0$ and $\Delta_x = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ (see [3]). The covariance functional of this process is

$$E(G(\varphi)G(\psi)) = \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy\ \varphi(t,x) G_m(\|x-y\|)\psi(t,y),$$

where $G_m(r) = K_0(r)m/(2\pi)$, and K_0 is a modified Bessel function. For $r \downarrow 0$, $K_0(r) \sim \log(1/r)$, so condition (4) is satisfied. We note that G_m is the kernel of $(m-\Delta)^{-1}$, that is, the fundamental solution of the equation $(m-\Delta)u = g$.

3. The linear case. The linear wave equation driven by F is as follows:

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) = \dot{F}(t, x), \\ &u(0, x) = 0, \qquad\qquad t > 0,\ x \in \mathbb{R}^2. \\ &\frac{\partial u}{\partial t}(0, x) = 0, \end{aligned} \tag{16}$$

The excitation $\dot{F}$ is assumed to be a Gaussian random field whose covariance function is given by (3) and f is such that condition (4) is satisfied. Because we are interested in the relation between the solution and the noise $\dot{F}$, we do not consider nonzero initial conditions, although this could be done with minimal additional effort.

If $\dot{F}(t, x)$ were smooth, then the solution (see, e.g., [21]) would be

$$u(t, x) = \int_0^t \int_{\mathbb{R}^2} S(t-s, x-y)\dot{F}(s, y)\, dy\, ds, \tag{17}$$

where

$$S(s, y) = \frac{1}{2\pi}\, (s^2 - \|y\|^2)^{-1/2}\, I_{\{\|y\|<s\}}.$$

When $\dot{F}(t, x)$ is a distribution, rather than a smooth function, (16) can be interpreted rigorously in the sense of distributions. Thinking of $\dot{F}$ as the distribution

$$\varphi \mapsto \dot{F}(\varphi) = \int_{\mathbb{R}_+ \times \mathbb{R}^2} \varphi(t, x) F(dt, dx), \tag{18}$$

a solution to (16) in the sense of distributions is a distribution $\varphi \mapsto \langle u, \varphi\rangle$ with support in $\mathbb{R}_+ \times \mathbb{R}^2$ such that

$$\left\langle u, \frac{\partial^2 \varphi}{\partial t^2} - \left(\frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2}\right)\right\rangle = \dot{F}(\varphi).$$

The unique distribution-valued solution u to this equation is given in [21], Theorem 4, and is defined by

$$\langle u, \varphi\rangle = \int_0^\infty dr \int_{\mathbb{R}^2} dv\ S(r, v)\dot{F}(\varphi_{r,v}), \tag{19}$$

where $\varphi_{r,v}(s, y) = \varphi(r+s, v+y)$. To see how this formula arises, we start from (17) and proceed for a moment as though $\dot{F}$ were a smooth function. Multiply (17) by a test function φ, and integrate:

$$\begin{aligned} &\int_{\mathbb{R}_+ \times \mathbb{R}^2} \varphi(t, x)\, u(t, x)\, dt\, dx \\ &\qquad = \int_0^\infty dt \int_{\mathbb{R}^2} dx\ \varphi(t, x) \int_0^t \int_{\mathbb{R}^2} S(t-s, x-y)\dot{F}(s, y)\, dy\, ds. \end{aligned}$$

For fixed t and x, use the change of variables $r = t - s$, $v = x - y$, to see that this equals

$$\int_0^\infty dt \int_{\mathbb{R}^2} dx\ \varphi(t, x) \int_0^t dr \int_{\mathbb{R}^2} dv\ S(r, v)\dot{F}(t - r, x - v)$$
$$= \int_0^\infty dr \int_{\mathbb{R}^2} dv\ S(r, v) \int_r^\infty dt \int_{\mathbb{R}^2} dx\ \varphi(t, x)\dot{F}(t - r, x - v).$$

For fixed r and v, use the change of variables $s = t - r$, $y = x - v$, to see that this equals

$$\int_0^\infty dr \int_{\mathbb{R}^2} dv\ S(r, v) \int_0^\infty ds \int_{\mathbb{R}^2} dy\ \varphi(r + s, v + y)\dot{F}(s, y),$$

which would be formula (19) if $\dot{F}$ were smooth.

Given that $F \in \mathscr{D}'(\mathbb{R}^3)$ and interpreting $\dot{F}(\varphi)$ as in (18), formula (19) defines the distribution-valued solution of (16). Note that the expression (19) for the solution to (16) is valid regardless of the choice of the covariance functional of F.

We are interested in knowing when the distribution-valued random variable u defined by (19) is in fact associated with a real-valued stochastic process. When this occurs, can one include nonlinear terms in (16) and still get a real-valued solution? We address the first question here and the second one in the next section.

The main result of this section is the following.

THEOREM 1. *Let u be the distribution-valued solution to the linear wave equation (16) given by formula (19). In order that there exist a jointly measurable process $X: (t, x, \omega) \mapsto X(t, x; \omega)$ that is locally mean-square bounded and such that a.s., for all $\varphi \in \mathscr{D}(\mathbb{R}^3)$,*

$$\langle u, \varphi \rangle = \int_{\mathbb{R}_+ \times \mathbb{R}^2} X(t, x)\varphi(t, x)\, dt\, dx, \tag{20}$$

it is necessary and sufficient that

$$\int_{0^+} f(r) \ln\left(\frac{1}{r}\right) r\, dr < +\infty. \tag{21}$$

REMARK 3. Condition (21) is satisfied for the examples of Gaussian random fields mentioned at the end of Section 2.

The proof of Theorem 1 relies on two technical lemmas.

LEMMA 1. *Fix $t_0 > 0$. For any smooth functions f and g, we have*

$$\begin{aligned}\int_{\|y\|<\|x\|<t_0} dy\, dx\ f(\|x - y\|)g(\|x\|^2 - \|y\|^2)\\ = \pi \int_0^{2t_0} dr\ rf(r) \int_r^{2t_0} dw\ g(rw - r^2)(4t_0^2 - w^2)^{1/2},\end{aligned} \tag{22}$$

and

$$\int_{\|y\|<\|x\|<t_0} dy\ \frac{dx}{\|x\|}\ f(\|x-y\|)g(\|x\|^2-\|y\|^2)$$

(23)
$$= 4\pi\int_0^{2t_0} dr\ rf(r)\int_r^{2t_0} dw\ g(rw-r^2)$$
$$\times\left(\ln\left(2t_0+(4t_0^2-w^2)^{1/2}\right)-\ln w\right).$$

PROOF. For fixed x, let $z = x - y$ be a change of variables in y. Then the left-hand side of (22) is equal to

$$\int_{\|x\|<t_0} dx\int_{\|x-z\|<\|x\|} dz\ f(\|z\|)g(2\langle x,z\rangle-\|z\|^2), \tag{24}$$

where $\langle x, z\rangle$ denotes the Euclidean inner product of x and z.

For fixed $x = (x_1, x_2)$, assume $x_1 = \|x\|\cos\theta_0$, $x_2 = \|x\|\sin\theta_0$, and consider the change of variables $(z_1, z_2) \mapsto (r, \theta)$ defined by $z_1 = r\cos(\theta-\theta_0)$, $z_2 = r\sin(\theta-\theta_0)$. Then $\|x-z\| < \|x\|$ is equivalent to $0 \le r \le 2\|x\|$, $|\cos\theta| \ge r/(2\|x\|)$ and $|\theta| \le \pi/2$. Therefore the left-hand side of (22) is equal to

$$2\int_{\|x\|<t_0} dx\int_0^{2\|x\|} dr\ rf(r)\int_0^{\cos^{-1}(r/(2\|x\|))} d\theta\ g(2r\|x\|\cos\theta-r^2). \tag{25}$$

Now do the change of variables $w = 2\|x\|\cos\theta$, for which

$$d\theta = -(4\|x\|^2-w^2)^{-1/2}\,dw,$$

to see that the inner integral in (25) is equal to

$$\int_r^{2\|x\|} dw\ (4\|x\|^2-w^2)^{-1/2}g(rw-r^2).$$

Changing the order of integration in (25), we see that it is equal to

$$2\int_0^{2t_0} dr\ rf(r)\int_r^{2t_0} dw\ g(rw-r^2)\int_{w/2<\|x\|<t_0} dx\,(4\|x\|^2-w^2)^{-1/2}. \tag{26}$$

Passing to polar coordinates in the last integral makes it easy to calculate. The last expression is equal to

$$\pi\int_0^{2t_0} dr\ rf(r)\int_r^{2t_0} dw\ g(rw-r^2)(4t_0^2-w^2)^{1/2},$$

which proves (22).

The proof of (23) is similar to that of (22), the only difference is that instead of leading to (26), the left-hand side of (23) becomes

$$2\int_0^{2t_0} dr\ rf(r)\int_r^{2t_0} dw\ g(rw-r^2)\int_{w/2<\|x\|<t_0}\frac{dx}{\|x\|}\ (4\|x\|^2-w^2)^{-1/2}.$$

Passing to polar coordinates, the inner integral becomes

$$2\pi\int_{w/2}^{t_0} ds\ (4s^2-w^2)^{-1/2} = \ln\left(2s+\sqrt{4s^2-w^2}\right)\Big|_w^{2t_0},$$

which yields formula (23). □

LEMMA 2. (a) *For small* $t_0 > 0$, *there exist positive constants* C_1 *and* C_2 *such that, for* $0 < t < t_0$,

$$\int_0^t f(r)\left(C_1 t^2 \ln\left(\frac{1}{r}\right) - C_2\right) r\, dr \leq \int_0^t ds \int_{\|x\|<s} dx \int_{\|y\|<s} dy\ S(s,x) f(\|x-y\|) S(s,y).$$

(b) *For all* $t_0 > 0$, *there exist positive constants* C_3 *and* C_4 *such that, for* $0 < t < t_0$,

$$\int_0^t ds \int_{\|x\|<s} dx \int_{\|y\|<s} dy\ S(s,x) f(\|x-y\|) S(s,y) \leq C_3 \int_0^{2t} f(r)\left(\ln\left(\frac{1}{r}\right) + C_4\right) r\, dr.$$

PROOF. (a) Observe that, for $t = t_0$, the triple integral in (a) is equal to

$$2 \int_{\|y\|<\|x\|<t_0} dx\, dy\ f(\|x-y\|) \int_{\|x\|}^{t_0} dt\ S(t,x) S(t,y). \tag{27}$$

Omitting the constant $1/(2\pi)$ in the definition of $S(t,x)$, the inner integral is equal to

$$\begin{aligned} &\int_{\|x\|}^{t_0} dt\, (t^4 - (\|x\|^2 + \|y\|^2) t^2 + \|x\|^2\, \|y\|^2)^{-1/2} \\ &\quad = \int_{\|x\|^2}^{t_0^2} \frac{ds}{2}\ s^{-1/2} (s^2 - (\|x\|^2 + \|y\|^2) s + \|x\|^2\, \|y\|^2)^{-1/2}, \end{aligned} \tag{28}$$

where we have used the change of variables $t^2 = s$. Replacing $s^{-1/2}$ by $1/t_0$ and using the fact that

$$\int (s^2 + as + b)^{-1/2}\, ds = \ln\left(a + 2s + 2\sqrt{s^2 + as + b}\right), \tag{29}$$

we see that the integral in (28) is greater than or equal to

$$\begin{aligned} &\frac{1}{2t_0}\Big[\ln\Big(-(\|x\|^2 + \|y\|^2) + 2t_0^2 + 2\sqrt{t_0^4 - (\|x\|^2 + \|y\|^2) t_0^2 + \|x\|^2 \|y\|^2}\Big) \\ &\qquad\qquad - \ln(\|x\|^2 - \|y\|^2)\Big] \\ &\quad = \frac{1}{2t_0}\Big[\ln\Big(\Big(\sqrt{t_0^2 - \|x\|^2} + \sqrt{t_0^2 - \|y\|^2}\Big)^2\Big) - \ln(\|x\|^2 - \|y\|^2)\Big]. \end{aligned}$$

Applying the inequality $(a+b)^2 \geq a^2 + b^2$ for $a, b \geq 0$, we therefore conclude that the triple integral in (a) is greater than or equal to

$$\begin{aligned} \frac{1}{4\pi^2 t_0} \int_{\|y\|<\|x\|<t_0} dx\, dy\ f(\|x-y\|)\Big(&\ln(2t_0^2 - \|x\|^2 - \|y\|^2) \\ &- \ln(\|x\|^2 - \|y\|^2)\Big). \end{aligned} \tag{30}$$

For small t_0, both logarithms are negative. The integral in (30) is the sum of two terms. Since $t_0^2 - \|y\|^2 \geq 0$, the first term becomes smaller if the first logarithm is replaced by $\ln(t_0^2 - \|x\|^2)$. We then apply the change of variables $z = x - y$ (x fixed) to see that the first term from (30) is greater than or equal to

$$\frac{1}{4\pi^2 t_0} \int_{\|x\|<t_0} dx \int_{\|x-z\|<\|x\|} dz\ f(\|z\|) \ln(t_0^2 - \|x\|^2)$$
$$= \frac{1}{4\pi^2 t_0} \int_{\|z\|<2t_0} dz\ f(\|z\|) \int_{\|x-z\|<\|x\|<t_0} dx\ \ \ln(t_0^2 - \|x\|^2).$$

Since the integrand is negative if t_0 is small enough, the inner integral becomes smaller if we enlarge the domain of integration, so it is greater than

$$\int_{\|x\|<t_0} dx\ \ \ln(t_0^2 - \|x\|^2).$$

Passing to polar coordinates with $r = \|x\|$ and changing variables to $s = r^2$, we conclude that this integral is greater than $-C_2 > -\infty$. It follows that, in order to complete the proof of the first inequality in the lemma, it suffices to compare the second term from (30) with the first term on the left hand-side of this inequality, more precisely, to show that

$$\begin{aligned} &\int_{\|y\|<\|x\|<t_0} dx\, dy\ f(\|x-y\|)(-\ln(\|x\|^2 - \|y\|^2)) \\ &\qquad \geq C_1 t_0^2 \int_0^{t_0} f(r) \ln\left(\frac{1}{r}\right) r\, dr. \end{aligned} \tag{31}$$

Using (22), we see that the left-hand side of (31) is equal to

$$\int_0^{2t_0} dr\, rf(r) \int_r^{2t_0} dw\ (-\ln r - \ln(w - r))(4t_0^2 - w^2)^{1/2}.$$

For t_0 small enough, removing the second logarithm makes the expression smaller, so it is greater than or equal to

$$\int_0^{2t_0} dr\, f(r) r(-\ln r) \int_r^{2t_0} dw\, (4t_0^2 - w^2)^{1/2}.$$

Since the integrand is nonnegative, the expression becomes smaller if we only integrate over r in $[0, t_0]$. In this case, the inner integral is greater than or equal to

$$\int_{t_0}^{2t_0} dw\, (4t_0^2 - w^2)^{1/2} > C\ t_0^2.$$

We therefore conclude that (31) is valid, completing the proof of (a).

(b) Replace $s^{-1/2}$ in (28) by $\|x\|^{-1}$ to see that the integral in (28) is less than or equal to

$$\frac{1}{2\|x\|}\left(\ln\left(\left(\sqrt{t_0^2 - \|x\|^2} + \sqrt{t_0^2 - \|y\|^2}\right)^2\right) - \ln(\|x\|^2 - \|y\|^2)\right).$$

Therefore the triple integral in (b) is less than or equal to

$$\int_{\|y\|<\|x\|<t_0} dy\ \frac{dx}{2\|x\|}\ f(\|x-y\|)(\ln(4t_0^2)-\ln(\|x\|^2-\|y\|^2)).$$

By (23), this is equal to

$$
\begin{aligned}
&\tfrac{1}{2}\int_0^{2t_0} dr\ rf(r)\int_r^{2t_0} dw\ (\ln(4t_0^2)-\ln(rw-r^2))\\
&\qquad\times\Big(\ln\Big(2t_0+\sqrt{4t_0^2-w^2}\Big)-\ln w\Big)\\
&\quad\le \tfrac{1}{2}\int_0^{2t_0} dr\ rf(r)\int_r^{2t_0} dw\ (\ln(4t_0^2)-\ln r-\ln(w-r))\\
&\qquad\times(\ln(4t_0)-\ln w).
\end{aligned}
\tag{32}
$$

Because $-\ln w\le-\ln r$, the integrand is less than or equal to

$$\ln\Big(\frac{1}{r}\Big)(\ln(4t_0)-\ln w)+(\ln(4t_0^2)-\ln(w-r))(\ln(4t_0)-\ln r).$$

Using the fact that the antiderivative of $-\ln x$ is $x-x\ln x$ and that $x-x\ln x\le 1$, for all $x>0$, (32) is less than or equal to

$$C_3\int_0^{2t_0} dr\ rf(r)\Big(\ln\Big(\frac{1}{r}\Big)+C_4\Big). \qquad\square$$

PROOF OF THEOREM 1 (Necessity). Assume that there exists a process X satisfying the conditions in Theorem 1. Then $E(\langle u,\varphi\rangle^2)$ can be computed in two different ways: from (20), yielding

$$E(\langle u,\varphi\rangle^2)=\int_0^\infty dt\int_{\mathbb{R}^2} dx\int_0^\infty ds\int_{\mathbb{R}^2} dy\,\varphi(t,x)\varphi(s,y)E(X(t,x)X(s,y)). \tag{33}$$

Note that the function $g(t,x,s,y)=E(X(t,x)X(s,y))$ is locally integrable, since for any two compacts subsets C and C' of $\mathbb{R}\times\mathbb{R}^2$,

$$
\begin{aligned}
&\int_C dt\,dx\int_{C'} ds\,dy\ E(X(t,x)X(s,y))\\
&\quad\le\int_C dt\,dx\int_{C'} ds\,dy\ \big[E(X(t,x)^2)E(X(s,y)^2)\big]^{1/2}\\
&\quad<\infty,
\end{aligned}
$$

because $X(t,x)$ is locally mean-square bounded. The expectation $E(\langle u,\varphi\rangle)^2)$ can also be computed from (19), yielding

$$
\begin{aligned}
E(\langle u,\varphi\rangle^2)&=E\Big(\int_0^\infty dr\int_{\mathbb{R}^2} du\ S(r,u)F(\varphi_{r,u})\\
&\qquad\times\int_0^\infty ds\int_{\mathbb{R}^2} dv\ S(s,v)F(\varphi_{s,v})\Big)\\
&=\int_0^\infty dr\int_0^\infty ds\int_{\mathbb{R}^2} du\int_{\mathbb{R}^2} dv\ S(r,u)S(s,v)\\
&\qquad\times E(F(\varphi_{r,u})F(\varphi_{s,v})).
\end{aligned}
\tag{34}
$$

However, by (3), the expectation inside the integral is equal to

$$\int_0^\infty dt \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy\ \varphi(r+t, u+x) f(\|x-y\|) \varphi(s+t, v+y). \tag{35}$$

Replace φ by φ_n in (33), (34) and (35), where φ_n is chosen as follows. Fix $\psi \in C_0^\infty(\mathbb{R}^3, \mathbb{R})$ to be such that $\psi \geq 0$, $\psi(x) = 0$ if $|x| > 1$, and $\int_{\mathbb{R}^3} \psi = 1$. Then $\varphi_n(t, x) = n^3 \psi(n(t - t_0, x - x_0))$. Apply Fubini's theorem to (34) and (35) to see that $E(\langle u, \varphi_n \rangle^2)$ is equal to

$$\begin{aligned} &\iiint dt\, dx\, dy\ f(\|x-y\|) \iint dr\, du\ S(r, u) \varphi_n(r+t, u+x) \\ &\qquad \times \iint ds\, dv\ S(s, y) \varphi_n(s+t, y+v). \end{aligned} \tag{36}$$

As $n \to \infty$, the integrand in (36) converges pointwise a.e. to

$$f(\|x-y\|) S(t_0 - t, x_0 - x) S(t_0 - t, x_0 - y),$$

while by the Lebesgue differentiation theorem (in an extended form [20], Chapter 7, Exercise 2), for a.a. (t_0, x_0), (33) converges to $g(t_0, x_0, t_0, x_0) = E(X(t_0, x_0)^2) < \infty$. Applying Fatou's lemma to the nonnegative functions above, we conclude that

$$\begin{aligned} &\int_0^\infty dt \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy\ S(t_0 - t, x_0 - x) f(\|x-y\|) S(t_0 - t, x_0 - y) \\ &\qquad \leq g(t_0, x_0, t_0, x_0) < \infty. \end{aligned} \tag{37}$$

But now a simple change of variables shows that the left-hand side of (37) is equal to

$$\int_0^{t_0} dt \int_{\|x\|<t} dx \int_{\|y\|<t} dy\ S(t, x) f(\|x-y\|) S(t, y). \tag{38}$$

Thus the existence of a process solution implies that (38) is finite for a.a. $t_0 \in \mathbb{R}_+$. Therefore the inequality in Lemma 2(a) finishes the proof of necessity. □

We now consider the sufficiency portion of the statement in Theorem 1. Suppose condition (21) holds. The formula

$$X(t, x) = \int_0^t \int_{\mathbb{R}^2} S(t-s, x-y) F(dt, dx), \qquad t > 0,\ x \in \mathbb{R}^2,$$

is the natural candidate for the solution of (16). The integral in this formula is the integral of $(s, y) \mapsto S(t-s, x-y)$ with respect to the martingale measure M defined in Section 2. For each fixed (t, x), this integral is well defined since (21) and Lemma 2 imply that

$$\int_0^t ds \int_{\mathbb{R}^2} dy \int_{\mathbb{R}^2} dz\, S(s, y)\, f(\|y-z\|)\, S(s, z) < \infty. \tag{39}$$

However, to show that this family of random variables actually gives a solution to (16) in the sense of distributions, it is necessary to establish that, for almost all ω, the function $(t, x) \mapsto X(t, x; \omega)$ is measurable. For this, we need to show that the process $(X(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2)$ has a jointly measurable version. This is the case if the map $(t,x) \mapsto X(t,x)$ from $\mathbb{R}_+ \times \mathbb{R}^2$ into the space of random variables is continuous in probability ([8], Chapter IV, Théorème 30) or, equivalently, since this is a Gaussian process, is continuous in L^2. Proving this requires some bounds on the L^2-norm of increments of X. These bounds are the same as those needed for the nonlinear wave equation, so we proceed directly to this situation.

4. The nonlinear case. In this section, we consider the following equation:

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) = \alpha(u)\dot{F}(t,x), \\ &u(0,x) = 0, \qquad\qquad t > 0,\ x \in \mathbb{R}^2. \\ &\frac{\partial u}{\partial t}(0,x) = 0, \end{aligned} \tag{40}$$

The right-hand side of the equation could contain an additional additive term, or the initial conditions could be functions, but it is not our objective to consider the most general equation.

We assume that $u \mapsto \alpha(u)$ is globally Lipschitz with Lipschitz constant K, and this implies the linear growth condition

$$|\alpha(u)| \le K(1 + |u|). \tag{41}$$

THEOREM 2. *Under the above assumption on $\alpha(\cdot)$, if (21) holds, then there exists $t_0 > 0$ and a unique jointly measurable L^2-continuous process $(X(t,x),\ (t,x) \in [0,t_0] \times \mathbb{R}^2)$ that is a solution to (40).*

We recall that this theorem contains as a special case ($\alpha \equiv 1$) the sufficiency statement of Theorem 1. In this case, t_0 can be taken equal to $+\infty$. In the nonlinear case, it is not clear whether the solution exists for all time, and there is some reason to believe that this may not be the case in general [see the comments following (46) in the proof of the theorem].

PROOF OF THEOREM 2. The proof of Theorem 2 is based on a standard Picard iteration scheme. Define

$$X^{(0)}(t,x) = \int_0^t \int_{\mathbb{R}^2} S(t-s, x-y) F(ds,dy), \qquad t > 0,\ x \in \mathbb{R}^2,$$

and, for $n \ge 0$ and assuming that $X^{(n)}$ has been defined, set

$$X^{(n+1)}(t,x) = \int_0^t \int_{\mathbb{R}^2} S(t-s, x-y)\alpha(X^{(n)}(s,y)) F(ds,dy).$$

It is of course necessary to make sure that the two stochastic integrals above are well defined. Recalling the form of the dominating measure of $F(ds,dy)$ [see the lines following (10)], the first is well defined by (39), and Lemma 2(b) even implies that $E(X^{(0)}(t,x)^2)$ is bounded over compact sets. Furthermore, $X^{(0)}(t,x)$ is $\mathscr{F}_t$-measurable for all x. Assume by induction that $E(X^{(n)}(t,x)^2)$ is bounded when (t,x) runs over any fixed compact set and that $X^{(n)}(t,x)$ is $\mathscr{F}_t$-measurable for all x. To see that the second integral is well defined, observe by Lemma 3 below and the considerations that precede this lemma that $(t,x;\omega)\mapsto X^{(n)}(t,x;\omega)$ is jointly measurable and that the conditions (a)–(c) of Proposition 2 are satisfied. We check condition (d) of this proposition. Observe that

$$\begin{aligned} E\bigg[\iiint S(t-s,x-y)\alpha(X^{(n)}(s,y))f(\|y-z\|) \\ \times S(t-s,x-z)\alpha(X^{(n)}(s,z))\bigg]\,dy\,dz\,ds \\ = \iiint S(t-s,x-y)\,f(\|y-z\|)\,S(t-s,x-z) \\ \times E\big[\alpha(X^{(n)}(s,y))\alpha(X^{(n)}(s,z))\big]\,dy\,dz\,ds. \end{aligned}$$

Using the Cauchy–Schwarz inequality, we conclude by (41) and the induction hypothesis that the expectation is bounded. It follows that $E(X^{(n+1)}(t,x)^2)$ is bounded over compact sets. This proves that the sequence $(X^{(n)})$ is well defined.

In the linear case ($\alpha\equiv 1$), it suffices to consider $X^{(0)}$ and it is not necessary to introduce the sequence $(X^{(n)},\ n\geq 0)$, but in the nonlinear case we must prove that this sequence converges. For this, let

$$C(t,x)=\{(s,y)\colon 0\leq s\leq t,\ \|y-x\|<t-s\}.$$

We first construct $t_0>0$ and the solution in $C(t_0,0)$, but the same method gives the solution in $C(t_0,x)$, for any $x\in\mathbb{R}^2$, and therefore in $[0,t_0]\times\mathbb{R}^2$. Define

$$M_n(t)=\sup_{(s,y)\in C(t,0)} E\big[(X^{(n+1)}(s,y)-X^{(n)}(s,y))^2\big]. \tag{42}$$

Note that $t\mapsto M_n(t)$ is nondecreasing. Fix $(s,y)\in C(t,0)$. Then the expectation on the right-hand side of (42) is equal to

$$E\bigg(\int_0^s\int_{\mathbb{R}^2} S(s-u,y-z)\{\alpha(X^{(n)}(u,z))-\alpha(X^{(n-1)}(u,z))\}F(du,dz)\bigg)^2.$$

Use the dominating measure of F to see that this is less than or equal to

$$\begin{aligned} E\bigg[\int_0^s\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} S(s-u,y-z)\{\alpha(X^{(n)}(u,z))-\alpha(X^{(n-1)}(u,z))\}f(\|z-z'\|) \\ \times S(s-u,y-z')\{\alpha(X^{(n)}(u,z'))-\alpha(X^{(n-1)}(u,z'))\}\,dz\,dz'\,du\bigg]. \end{aligned}$$

By the Lipschitz condition on $\alpha(\cdot)$, this is less than or equal to

$$\begin{aligned}(43)\qquad & K^2 \int_0^s \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} S(s-u,\, y-z) f(\|z-z'\|) S(s-u,\, y-z') \\ & \quad \times E\big[|X^{(n)}(u,z) - X^{(n-1)}(u,z)| \\ & \qquad \times |X^{(n)}(u,z')) - X^{(n-1)}(u,z')|\big]\, dz\, dz'\, du.\end{aligned}$$

Apply the Cauchy–Schwarz inequality to the expectation and note that $(u, z) \in C(u + \|z\|, 0)$, and so (43) is less than or equal to

$$\begin{aligned}& K^2 \int_0^s \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} S(s-u,\, y-z)\, f(\|z-z'\|)\, S(s-u,\, y-z') \\ & \qquad \times [M_{n-1}(u+\|z\|) M_{n-1}(u+\|z'\|)]^{1/2}\, dz\, dz'\, du.\end{aligned}$$

We do the change of variables $v = s-u$, $\zeta = y-z$, $\zeta' = y-z'$ to see that the expectation on the right-hand side of (42) is bounded by

$$\begin{aligned}(44)\qquad & K^2 \int_0^s dv \int_{\|\zeta\|<v} d\zeta \int_{\|\zeta'\|<v} d\zeta'\ S(v,\zeta) f(\|\zeta-\zeta'\|) S(v,\zeta') \\ & \qquad \times [M_{n-1}(s-v+\|y-\zeta\|) M_{n-1}(s-v+\|y-\zeta'\|)]^{1/2}.\end{aligned}$$

We now note that $s - v + \|y-\zeta\| \le t - v + \|\zeta\|$, because $(s, y) \in C(t, 0)$ and therefore $\|y-\zeta\| - \|\zeta\| \le \|y\| \le t-s$. The integrand in (44) therefore becomes larger if $s - v + \|y-\zeta\|$ is replaced by $t - v + \|\zeta\|$, and if the s in the upper bound of the leftmost integral is then replaced by t. We conclude that

$$\begin{aligned}(45)\qquad M_n(t) \le {} & K^2 \int_0^t dv \int_{\|z\|<v} dz \int_{\|z'\|<v} dz'\ S(v,z) f(\|z-z'\|) S(v,z') \\ & \qquad \times [M_{n-1}(t-v+\|z\|) M_{n-1}(t-v+\|z'\|)]^{1/2}.\end{aligned}$$

Permute the order of integration and use the symmetry of the integrand in z and z', along with the monotonicity of $M_{n-1}(\cdot)$ to see that this is less than or equal to

$$2K^2 \int_{\|z'\|<\|z\|<t} dz\, dz'\, f(\|z-z'\|) \int_{\|z\|}^t dv\ S(v,z) S(v,z') M_{n-1}(t-v+\|z\|).$$

Apply the change of variables $u = t - v + \|z\|$ to the inner integral. This does not change the interval of integration, so after another permutation of the order of integration we conclude that

$$\begin{aligned}(46)\qquad M_n(t) \le 2K^2 \int_0^t du\, M_{n-1}(u) \int_{\|z'\|<\|z\|<u} & dz\, dz'\, S(t-u+\|z\|, z) \\ & \times f(\|z-z'\|) S(t-u+\|z'\|, z').\end{aligned}$$

If the inner integral could be bounded by $(t-u)^a$ for some $a > -1$, then we could apply Lemma 3.3 of [19] to get existence of a solution to (40) for all time. However, tedious calculations show that in general the inner integral can be of order $(t-u)^{-1}$.

To get existence in the region $C(t_0, 0)$, note that (46) implies that

$$M_n(t) \leq 2K^2 M_{n-1}(t) \int_0^t du \int_{\|z'\|<\|z\|<u} dz\, dz'\, S(t-u+\|z\|, z) \times f(\|z-z'\|) S(t-u+\|z\|, z'),$$

and reversing the steps above shows that the integral on the right-hand side is equal to

$$\int_0^t dv \int_{\|z'\|<\|z\|<v} dz\, dz'\, S(v, z) f(\|z-z'\|) S(v, z'),$$

which, by hypothesis and Lemma 2(b), is less than or equal to $\lambda < 1$ if $t \leq t_0$ and t_0 is chosen small enough.

In conclusion, there exists $t_0 > 0$ and $0 < \lambda < 1$ such that

$$M_n(t_0) \leq \lambda\, M_{n-1}(t_0). \tag{47}$$

Therefore $M_n(t_0) < \lambda^n M_1(t_0) < \infty$, so in fact $(X^{(n)}(t, x),\ n \in \mathbb{N})$ converges uniformly over $(t, x) \in C(t_0, 0)$ in L^2 to a limit that we denote $X(t, x)$.

In order to check that X has a jointly measurable version, we must show that it is continuous in L^2, as we pointed out at the end of the previous section. Once this is done, it is not difficult to see that the process $(X(t, x),\ (t, x) \in [0, t_0] \times \mathbb{R}^2)$ so defined is indeed a solution of (40). We note that the same calculations that led to (47) also can be used to establish uniqueness of the solution. This standard argument is left to the reader.

Since the convergence of $X^{(n)}$ to X is uniform in L^2, it suffices in fact to show that each $X^{(n)}$ is L^2-continuous.

LEMMA 3. *For $t_0 > 0$ and small $h_0 > 0$, there is a sequence (K_n) of positive constants and $C > 0$ such that, for $0 \leq h \leq h_0$, $0 \leq t \leq t_0$, $x, y \in \mathbb{R}^2$ with $\|x - y\| = h$,*

$$E[(X^{(n)}(t, x) - X^{(n)}(t+h, x))^2] \quad \textit{and} \quad E[(X^{(n)}(t, x) - X^{(n)}(t, y))^2] \tag{48}$$

are both no greater than the sum of the following two terms:

$$K_n \int_0^{2t} dr\ f(r) r \int_r^{2t} dw\ \ln\left(1 + \frac{Ch^{1/2}}{r(w-r)}\right)(\ln(4t) - \ln w) \tag{49}$$

and

$$K_n\left(h + \int_0^{2h} dr\ f(r) r \int_r^{2h} dw\ \ln\left(\frac{1}{r(w-r)}\right)(\ln(4t) - \ln w)\right). \tag{50}$$

We assume the lemma for the moment and complete the proof of Theorem 2. Note that, as $h \downarrow 0$, the integrand in (49) converges pointwise to 0. Moreover, for $0 < r < 2t$ and $r < w < 2t$, the inequality $1 \leq 4t^2/(r(w-r))$ holds, so the integrand in (49) is dominated for small $h > 0$ by the function

$$f(r)r \ln\left(\frac{C'}{r(w-r)}\right)(\ln(4t) - \ln w)$$
$$= f(r)r\,(\ln(C') - \ln r - \ln(w-r))(\ln(4t) - \ln w).$$

As shown in the lines following (32), under condition (21), this function is integrable over the set of (r, w) such that $0 < r < 2t$ and $r < w < 2t$. On the one hand, this allows us to conclude from the dominated convergence theorem that (49) converges to 0 as $h \downarrow 0$, and that (50) also converges to 0 as $h \downarrow 0$.

Since the bounds in (49) and (50) are uniform in x, the lemma implies that $X^{(n)}$ is L^2-continuous, and therefore the solution X of (40) is too. □

The proof of Lemma 3 requires the following estimate.

LEMMA 4. *Suppose* $0 < c < t^2$ *and* $a < b < c$ *(*a *and* b *may be negative). Then*

$$\int_{\sqrt{c}}^{t} \left[(s^2-b)^{-1/2} - (s^2-a)^{-1/2}\right](s^2-c)^{-1/2} ds \leq \frac{1}{2\sqrt{c}} \ln\left(1 + \frac{b-a}{c-b}\right).$$

PROOF. The left-hand side of the inequality is equal to

$$\int_{\sqrt{c}}^{t} \left[(s^4 - (b+c)s^2 + bc)^{-1/2} - (s^4 - (a+c)s^2 + ac)^{-1/2}\right] ds.$$

Using the change of variables $u = s^2$, for which $ds = \frac{1}{2}\, u^{-1/2}\, du$, we see that this is less than or equal to

$$\frac{1}{2\sqrt{c}} \int_{c}^{t^2} \left[(u^2 - (b+c)u + bc)^{-1/2} - (u^2 - (a+c)u + ac)^{-1/2}\right] du.$$

By (29), this is equal to the difference of the values of the antiderivative

$$\frac{1}{2\sqrt{c}}\Big[\ln\Big(-(b+c) + 2u + 2\sqrt{u^2 - (b+c)u + bc}\Big)$$
$$- \ln\Big(-(a+c) + 2u + 2\sqrt{u^2 - (a+c)u + ac}\Big)\Big]$$

at the bounds t^2 and c. For $u \geq c$,

$$u^2 - (a+c)u + ac \geq u^2 - (b+c)u + bc;$$

therefore, plugging $u = t^2$ into the antiderivative yields a negative number. By omitting this term, we get a larger value. Plugging $u = c$ into the antiderivative yields, after some simplification,

$$-\frac{1}{2\sqrt{c}}[\ln(c-b) - \ln(c-a)] = \frac{1}{2\sqrt{c}} \ln\left(1 + \frac{b-a}{c-b}\right). \qquad \square$$

PROOF OF LEMMA 3. We begin with the time increments. Note that $X^{(n)}(t, x) - X^{(n)}(t+h, x)$ is equal to

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}^2} (S(t-s, x-y) - S(t+h-s, x-y))\, \alpha(X^{(n-1)}(s, y)) F(ds, dy),$$

so its mean square is no greater than

$$E\Big[\int_{\mathbb{R}_+} ds \int_{\mathbb{R}^2} dy \int_{\mathbb{R}^2} dz\, (S(t-s, y) - S(t+h-s, y))\alpha(X^{(n-1)}(s, y)) \\ \times f(\|y-z\|)(S(t-s, z) - S(t+h-s, z))\alpha(X^{(n-1)}(s, z))\Big].$$

Apply Fubini's theorem to bring the expectation inside the integrals, use the Cauchy–Schwarz inequality and the linear growth condition on $\alpha(\cdot)$, to see that this is less than or equal to

$$K^2 \int_{\mathbb{R}_+} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (S(s, y) - S(s+h, y))\, f(\|y-z\|) \\ \times (S(s, z) - S(s+h, z))\, dy\, dz\, ds, \tag{51}$$

where the constant K is bounded by a finite multiple of

$$\sup_{(s,\, y) \in C(t+1,\, x)} E\big(\big(X^{(n-1)}(s, y)\big)^2\big).$$

It therefore appears that the nonlinearity $\alpha(\cdot)$ only changes the estimate of the L^2-norm of time increments by a constant factor. The same thing occurs when considering space increments. We assume therefore that $\alpha \equiv 1$ and focus on the deterministic integral in (51), which is equal to

$$E\big(\big(X^{(0)}(t, x) - X^{(0)}(t+h, x)\big)^2\big).$$

The difference $X^{(0)}(t, x) - X^{(0)}(t+h, x)$ is the sum of three terms:

$$Y_1 = \int_0^t \int_{\|x-y\|<t-s} (S(t-s, x-y) - S(t+h-s, x-y)) F(ds, dy),$$

$$Y_2 = -\int_0^t \int_{t-s<\|x-y\|<t+h-s} S(t+h-s, x-y) F(ds, dy),$$

$$Y_3 = \int_t^{t+h} \int_{\|x-y\|<t+h-s} S(t+h-s, x-y) F(ds, dy).$$

Therefore, the mean square of $X^{(0)}(t, x) - X^{(0)}(t+h, x)$ is bounded by 3 times the sum of the mean squares of each of these terms. We bound each separately.

First term. The mean square $E(Y_1^2)$ is equal to

$$\int_0^t ds \int_{\|y\|<s} dy \int_{\|z\|<s} dz \big(S(s, y) - S(s+h, y)\big) f(\|y-z\|) \big(S(s, z) - S(s+h, z)\big).$$

Replacing $S(s+h,y)$ by 0 in the first factor makes the integral larger. By permuting the order of the integrals, this expression is less than or equal to

$$
\begin{aligned}
(52)\quad & 2\int_{\|z\|<\|y\|<t} dy\,dz\, f(\|y-z\|)\int_{\|y\|}^{t} ds\,(s^2-\|y\|^2)^{-1/2}\\
&\qquad\qquad \times\big((s^2-\|z\|^2)^{-1/2}-(s^2+2ht+h^2-\|z\|^2)^{-1/2}\big).
\end{aligned}
$$

Apply Lemma 4 with $a=\|z\|^2-2ht-h^2$, $b=\|z\|^2$ and $c=\|y\|^2$ to see that this is less than or equal to

$$
(53)\qquad \int_{\|z\|<\|y\|<t} dz\,dy\, f(\|y-z\|)\frac{1}{\|y\|}\ln\biggl(1+\frac{2ht+h^2}{\|y\|^2-\|z\|^2}\biggr).
$$

By (23), this is equal to

$$
\int_0^{2t} dr\ rf(r)\int_0^{2t} dw\ln\biggl(1+\frac{2ht+h^2}{rw-r^2}\biggr)\biggl(\ln\Bigl(2t+\sqrt{4t^2-w^2}\Bigr)-\ln w\biggr).
$$

Bounding $2ht+h^2$ by $Ch^{1/2}$ and $2t+\sqrt{4t^2-w^2}$ by $4t$ yields the desired inequality.

Second term. The mean square $E(Y_2^2)$ is equal to

$$
\int_0^t ds\int_{s<\|y\|<s+h} dy\int_{s<\|z\|<s+h} dz\ S(s+h,y)f(\|y-z\|)S(s+h,z).
$$

Permuting the order of the integrals and doing the change of variables $s+h=u$, this is equal to

$$
2\int_{\|y\|-h<\|z\|<\|y\|<t+h} dy\,dz\ f(\|y-z\|)\int_{\|y\|\vee h}^{(\|z\|+h)\wedge(t+h)} du\ S(u,y)S(u,z).
$$

With the change of variables $v=u^2$, we see that this is less than or equal to

$$
\begin{aligned}
(54)\quad & \int_{\|y\|-h<\|z\|<\|y\|<t+h} dy\,dz\ \frac{f(\|y-z\|)}{\|y\|}\\
&\qquad \times\int_{\|y\|^2}^{(\|z\|+h)^2} dv\ (v^2-(\|y\|^2+\|z\|^2)v+\|y\|^2\|z\|^2)^{-1/2}.
\end{aligned}
$$

Using (29), the inner integral is equal to

$$
\begin{aligned}
&\ln\Bigl[-\|y\|^2-\|z\|^2+2(\|z\|+h)^2\\
&\qquad +2\Bigl\{(\|z\|+h)^4-(\|y\|^2+\|z\|^2)(\|z\|+h)^2\\
&\qquad\qquad\qquad +\|y\|^2\|z\|^2)\Bigr\}^{1/2}\Bigr]-\ln(\|y\|^2-\|z\|^2)).
\end{aligned}
$$

Replace $\|z\|$ by $\|y\|$ in the third square and in the first term under the root, and $\|z\|+h$ by $\|y\|$ in the second factor of the negative term under the root, to see that this is less than or equal to

$$\ln\big[\|y\|^2-\|z\|^2+4h\|y\|+2h^2+2\{(\|y\|+h)^4-\|y\|^4\}^{1/2}\big]-\ln(\|y\|^2-\|z\|^2)$$
$$\leq \ln\biggl(1+\frac{C\,h^{1/2}}{\|y\|^2-\|z\|^2}\biggr),$$

for some constant C that depends on t_0. Going through the steps following (53) completes the estimate for the second term.

Third term. The mean square $E(Y_3^2)$ is equal to

$$\int_0^h ds\int_{\|y\|<s}dy\int_{\|z\|<s}dz\;S(s,y)f(\|y-z\|)S(s,z)$$
$$\leq 2\int_{\|z\|<\|y\|<h}dy\,dz\;\frac{f(\|y-z\|)}{\|y\|}$$
$$\times\int_{\|y\|^2}^{h^2}dv\;(v^2-(\|y\|^2+\|z\|^2)v+\|y\|^2\|z\|^2)^{-1/2},$$

the inequality having been obtained as in (54). Applying (29), we see after some manipulation that the inner integral is equal to

$$\ln\big(((h^2-\|y\|^2)^{1/2}+(h^2-\|z\|^2)^{1/2})^2\big)-\ln(\|y\|^2-\|z\|^2)\leq C-\ln(\|y\|^2-\|z\|^2),$$

so $E(Y_3^2)$ is bounded by

$$2\int_{\|z\|<\|y\|<h}dy\,dz\;\frac{f(\|y-z\|)}{\|y\|}$$
$$(55)\qquad +2\int_{\|z\|<\|y\|<h}dy\,dz\,\frac{f(\|y-z\|)}{\|y\|}(-\ln(\|y\|^2-\|z\|^2)).$$

The first integral is bounded by

$$\int_{\|y\|<h}\frac{dy}{\|y\|}\int_{\|z\|<2h}dz\;f(\|z\|)\leq h\int_{\|z\|<2h}dz\;f(\|z\|)\leq Kh,$$

and, by (23), the second integral is equal to

$$\int_0^{2h}dr\;f(r)r\int_r^{2h}dw\,(-\ln(rw-r^2))\big(\ln\big(2h+\sqrt{4h^2-w^2}\big)-\ln w\big).$$

Replacing $\ln(2h+\sqrt{4h^2-w^2})$ by $4t$, this completes the estimate for the third term.

We now look at the spatial increments. By spatial homogeneity of the covariance of u, it suffices to consider the case $y=0$ and $\|x\|=h$. Note that

$$u(t,0)=\int_0^t ds\int_{\|y\|<t-s}S(t-s,y)\,F(ds,dy),$$

$$u(t,x)=\int_0^t ds\int_{\|y-x\|<t-s}S(t-s,x-y)\,F(ds,dy).$$

Therefore, $u(t,0) - u(t,x) = Z_1 + Z_2 + Z_3$, where

$$Z_1 = \int_0^{t-\|x\|/2} \int_{\substack{\|y\|<t-s \\ \|y-x\|<t-s}} \big(S(t-s,y) - S(t-s,x-y)\big) F(ds,dy),$$

$$Z_2 = \int_0^t \int_{\substack{\|y\|<t-s \\ \|y-x\|>t-s}} S(t-s,y)\, F(ds,dy),$$

$$Z_3 = -\int_0^t \int_{\substack{\|y\|>t-s \\ \|y-x\|<t-s}} S(t-s,x-y)\, F(ds,dy).$$

We observe that $E(Z_2^2) = E(Z_3^2)$, and

$$\begin{aligned} E(Z_2^2) &= \int_0^t ds \int_{\substack{\|y\|<s \\ \|y-x\|>s}} dy \int_{\substack{\|z\|<s \\ \|z-x\|>s}} dz\, S(s,y)\, f(\|y-z\|)\, S(s,z) \\ &\le \int_0^t ds \int_{s-\|x\|<\|y\|<s} dy \int_{s-\|x\|<\|z\|<s} dz\ S(s,y)\, f(\|y-z\|)\, S(s,z). \end{aligned}$$

However, this last expression is the sum of two terms: the integral over $[0, \|x\|]$ and the integral over $[\|x\|, t]$. The first term is of the same form as the “third term” of the time increments, and the other is of the same form as the “second term” of the time increments. Therefore $E(Z_2^2) + E(Z_3^2)$ is indeed bounded by the sum of the expressions in (49) and (50).

We now compute $E(Z_1^2)$. Clearly, this is equal to

$$\begin{aligned} \int_{\|x\|/2}^t ds \int_{\substack{\|y\|<s \\ \|y-x\|<s}} dy \int_{\substack{\|z\|<s \\ \|z-x\|<s}} dz\, \big(S(s,y) - S(s,x-y)\big) \\ \times f(\|y-z\|)\, \big(S(s,z) - S(s,x-z)\big), \end{aligned}$$

which, after permuting the order of the integrals, is equal to

$$\begin{aligned} &\int_{\substack{\|y\|<t \\ \|y-x\|<t}} dy \int_{\substack{\|z\|<t \\ \|z-x\|<t}} dz\, f(\|y-z\|) \\ &\quad \times \int_{\max(\|y\|,\|z\|,\|y-x\|,\|z-x\|)}^t ds \big(S(s,y) - S(s,x-y)\big)\big(S(s,z) - S(s,x-z)\big). \end{aligned}$$

The domain of integration is the union of four disjoint regions:

$$\begin{aligned} D_1 &= \big\{(y,z)\colon \|y-x\| < \|y\| < t,\ \|z-x\| < \|z\| < t\big\}, \\ D_2 &= \big\{(y,z)\colon \|y\| < \|y-x\| < t,\ \|z\| < \|z-x\| < t\big\}, \\ D_3 &= \big\{(y,z)\colon \|y-x\| < \|y\| < t,\ \|z\| < \|z-x\| < t\big\}, \\ D_4 &= \big\{(y,z)\colon \|y\| < \|y-x\| < t,\ \|z-x\| < \|z\| < t\big\}. \end{aligned}$$

The change of variables $s' = s$, $y' = y - x$, $z' = z - x$, shows that the integrals over D_1 and D_2 are equal, as are the integrals over D_3 and D_4.

Since $f(\|y-z\|) \geq 0$, we see that the integral over D_3 is negative. Therefore, we only need to consider the integral over D_1. On D_1,

$$\max(\|y\|, \|z\|, \|y-x\|, \|z-x\|) = \max(\|y\|, \|z\|),$$

so the integral over D_1 is equal to the integral over $D_1 \cap \{\|z\| < \|y\|\}$ plus the integral over $D_1 \cap \{\|y\| < \|z\|\}$. However, both of these integrals are equal, so the integral over D_1 is less than or equal to

$$2 \int_{D_1 \cap \{\|z\| < \|y\|\}} dy\, dz\, f(\|y-z\|) \int_{\|y\|}^{t} ds\, \big(S(s,z) - S(s, z-x)\big) S(s,y).$$

Apply Lemma 4 with $a = \|z-x\|^2$, $b = \|z\|^2$, and $c = \|y\|^2$, to see that this is less than or equal to

$$\int_{D_1 \cap \{\|z\| < \|y\|\}} dy\, dz\, \frac{f(\|y-z\|)}{\|y\|} \ln\left(1 + \frac{\|z\|^2 - \|z-x\|^2}{\|y\|^2 - \|z\|^2}\right),$$

so we conclude that

$$E(Z_1^2) \leq \int_{D_1 \cap \{\|z\| < \|y\|\}} dy\, dz\, \frac{f(\|y-z\|)}{\|y\|} \ln\left(1 + \frac{2\langle x, z\rangle - \|x\|^2}{\|y\|^2 - \|z\|^2}\right).$$

Because $\|z-x\|^2 < \|z\|^2$ on D_1, we see that $2\langle x, z\rangle - \|x\|^2 > 0$. Using the Cauchy–Schwarz inequality, the last integral is less than or equal to

$$\int_{\|z\| < \|y\| < t} dy\, dz\, \frac{f(\|y-z\|)}{\|y\|} \ln\left(1 + \frac{2\|x\| t + \|x\|^2}{\|y\|^2 - \|z\|^2}\right).$$

This is exactly the integral in (53) with h replaced by $\|x\|$. Proceeding in the same way yields the desired inequality. $\square$

We now give a sufficient condition on the covariance function for the process solution to the linear equation (16) provided in Theorems 1 and 2 to be continuous.

THEOREM 3. *Assume there is $a > 0$ such that*

$$\int_{0+} f(r) r^{1-a}\, dr < \infty,$$

and let X be the process solution to (16) constructed in Theorems 1 and 2. Then X has a continuous version, and, in fact, a version that is Hölder continuous with exponent b, for any $b \in \,]0, a/4[$.

PROOF. The solution X to (16) that we constructed in Theorem 2 satisfies the equation

$$X(t,x) = \int_0^t \int_{\mathbb{R}^2} S(t-s, x-y) F(ds, dy).$$

By Lemma 3, the mean square of the increments of X within a compact region of $\mathbb{R}_+ \times \mathbb{R}^2$ are bounded by the expressions in (49) and (50).

For each $b \in]0,1]$, there is a constant c such that, for all $x > 0$, $\ln(1+x) \le c\,x^b$. We conclude that the expression in (49) (with $n = 0$) is less than or equal to

$$c\,K_0\,C^b\,h^{b/2}\int_0^{2t} dr\ f(r)r\int_r^{2t} dw\ (r(w-r))^{-b}(\ln(4t) - \ln w). \tag{56}$$

Replace $-\ln w$ by $-\ln r$ in the last factor to see that the inner integral is less than or equal to

$$r^{-b}(\ln(4t) - \ln r)\frac{(2t-r)^{1-b}}{1-b} \le \frac{(2t)^{1-b}}{1-b}\,(\ln(4t) - \ln r)\,r^{-b},$$

and therefore (56) is less than or equal to

$$C'h^{b/2}\int_0^{2t} dr\ f(r)r^{1-b}(\ln(4t) - \ln r).$$

If b is chosen in $]0,a[$, then, for small r, $(\ln(4t) - \ln r)\,r^{1-b}$ is no greater than r^{1-a}, and therefore (56) is less than or equal to

$$C'h^{a/2}\int_0^{2t} dr\ f(r)r^{1-a} \le C''h^{a/2}. \tag{57}$$

Now consider (50). For small $h > 0$, the integral in (50) (with $n = 0$) is less than or equal to

$$\begin{aligned} K_0\int_0^{2h} dr\ f(r)r\int_r^{2h} dw\ &(-\ln r - \ln(w-r))(C - \ln r) \\ = K_0\int_0^{2h} dr\ f(r)r\Big[&-(C-\ln r)(2h-r)\ln r \\ &+ (C-\ln r)\int_r^{2h}(-\ln(w-r))\,dw\Big]. \end{aligned}$$

For small $h > 0$, the first term in the brackets is less than or equal to $2h\,r^{-a}$, and the inner integral is equal to

$$2h - r - (2h-r)\ln(2h-r) \le Ch^a. \tag{58}$$

From (56) and (58), we conclude that there are $t_0 > 0$ and $h_0 > 0$ such that, for $0 \le h \le h_0$, $0 \le t \le t_0$ and $x, y \in \mathbb{R}^2$ with $\|x - y\| = h$, the mean squares of

$$X(t,x) - X(t+h,x) \quad \text{and} \quad X(t,x) - X(t,y)$$

are both bounded by $Ch^{a/2}$. Because these increments are Gaussian random variables, their pth moments are bounded by $C^p h^{pa/4}$. According to the Kolmogorov continuity theorem ([19], Corollary 1.2), X has a continuous version, and, in fact, a version that is Hölder-continuous with exponent b, for any $b \in]0, a/4[$. □

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DÉPARTEMENT DE MATHÉMATIQUES
ECOLE POLYTECHNIQUE FÉDÉRALE
1015 LAUSANNE
SWITZERLAND
E-MAIL: dalang@math.epfl.ch

DEPARTMENT OF STATISTICS
ATHENS UNIVERSITY OF ECONOMICS AND BUSINESS
104 34 ATHENS
GREECE