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UNUSUALLY LARGE VALUES FOR SPECTRALLY POSITIVE STABLE AND RELATED PROCESSES[1]

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Two classes of processes are considered. One is a class of spectrally positive infinitely divisible processes which includes all such stable processes. The other is a class of processes constructed from the sequence of partial sums of independent identically distributed positive random variables. A condition analogous to regular variation of the tails is imposed. Then a large deviation principle and a Strassen-type law of the iterated logarithm are presented. These theorems focus on unusually large values of the processes. They are expressed in terms of Skorokhod's M_1 topology.

1. Introduction. The purpose of this paper is to present a large deviation principle (LDP) and an analogue of Strassen's law of the iterated logarithm (LIL) for two classes of stochastic processes. The LDP and LIL provide information about the likelihood of unusually large values of the processes. The first class of processes consists of certain spectrally positive infinitely divisible (inf. div.) processes, including all spectrally positive stable processes. The second is a class of processes obtained from sequences of partial sums (p. sums) of independent identically distributed positive random variables. The results and the proofs for the two types of processes are very similar, so it is efficient to consider them together. Likewise, there are close ties between the proofs of the LDP and the LIL; probability estimates for the former are used in conjunction with the Borel–Cantelli lemma to prove the latter.

The inf. div. processes and p. sums processes considered here are represented as integrals with respect to planar point processes. Such representations permit us to apply limit theorems proved for the point processes in O'Brien and Vervaat (1996). These theorems are summarized in Section 3. If the representations were continuous, the results of the present paper would be trivial. As it is, we must work hard to reduce the problems to the point process results.

We now describe the two classes of processes, beginning with the inf. div. case. Let ν be a measure on $(0, \infty]$ with $\nu(\{\infty\}) = 0$ and $0 < \nu([x, \infty]) < \infty$ for all $x > 0$. Some further restrictions, including an analogue of regular variation, will be imposed on ν later. Let $E := [0, \infty) \times (0, \infty]$. We generally use the symbol t for the first (horizontal or time) coordinate of E and x or y for the second. Next, let Φ be the Poisson point process (random measure) on E with intensity $dt\nu(dx)$.

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For $A > 0.5$ such that

$$\int_0^1 y^{2A}\nu(dy) < \infty \tag{1.1}$$

and for $c \in \mathbb{R}$, we define a process S in $D[0, \infty)$ by

$$\begin{aligned} S(t) &:= \int_0^\infty \int_0^t y^A[\Phi(ds, dy) - \mathbf{1}_{(0,1]}(y)ds\nu(dy)] + ct \\ &= \int_0^\infty y^A[\Phi((0, t], dy) - \mathbf{1}_{(0,1]}(y)t\nu(dy)] + ct. \end{aligned} \tag{1.2}$$

Here, $\mathbf{1}_{(0,1]}$ denotes the usual indicator function and $\Phi([0, t], \cdot)$ is the measure on $(0, \infty]$ given by $\Phi([0, t], B) = \Phi([0, t] \times B)$. We can write S as a single integral since the integrand in the double integral depends only on y. Also, (1.1) guaranties that, with probability one (wp1), S is finite everywhere and is in $D[0, \infty)$. Note that S depends implicitly on A.

If $\int_0^1 y^A\nu(dy) < \infty$, then S can be written more simply as

$$S(t) = \int_0^\infty y^A\Phi((0, t], dy) + c_1 t. \tag{1.3}$$

Similarly, if $\int_1^\infty y^A\nu(dy) < \infty$, we may write

$$S(t) = \int_0^\infty y^A[\Phi(0, t], dy) - t\nu(dy)] + c_2 t. \tag{1.4}$$

An important special case arises by choosing $\nu([x, \infty]) = x^{-1}$ for all $x > 0$, so that $\nu(dy) = y^{-2}dy$. Then (1.2) with $A = 1$, (1.3) with $A > 1$ and (1.4) with $1/2 < A < 1$ give the usual Itô (1942) representations of the spectrally positive stable processes, with stability index A^{-1}. The theorems in this paper are broad enough to include these processes. We note however that a nonrandom translation term is required for stable processes of index 1 and, for S as in (1.4), the drift term $c_2 t$ sometimes must be 0.

We now define our p. sums processes in a similar way. Let ν be a probability measure on $[0, \infty]$ with $\nu(\{\infty\}) = 0$ and $\nu([x, \infty]) > 0$ for all $x \in (0, \infty)$. Let $(X_n)_{n=1}^\infty$ be a sequence of independent random variables, all with distribution ν. Let Φ be the *observation process* of (X_n), namely the random measure on E defined by

$$\Phi(B) := \text{the cardinality of } \{n \in \mathbb{N}\colon (n, X_n) \in B\}.$$

Define S in this case by

$$S(t) := \sum_{i=1}^{\lfloor t \rfloor}(X_i^A - E[X_i^A \mathbf{1}_{(0,1]}(X_i)]) + c\lfloor t \rfloor \tag{1.5}$$

$$= \int_0^\infty y^A[\Phi([0, t], dy) - \lfloor t \rfloor \mathbf{1}_{(0,1]}(y)\nu(dy)] + c\lfloor t \rfloor. \tag{1.6}$$

Here, $\lfloor \cdot \rfloor$ denotes the greatest integer function.

For both the inf. div. case and the p. sums case, the paths of S are in $D[0,\infty)$. Our main results, however, do not hold for the usual Skorokhod (1956) J_1 topology. Indeed, we mainly use a variation of the slightly coarser M_1 topology, also considered in Skorokhod (1956). These topologies are discussed in Section 2.

Our LDPs involve a *speed function* α: $[1,\infty)\mapsto(0,1]$ such that $\alpha(1)=1$ and $\alpha(\gamma)\to 0$ as $\gamma\to\infty$. We will generally write α for $\alpha(\gamma)$ and β or $\beta(\gamma)$ for $(\alpha(\gamma))^{-1}$. Our LILs are related to the LDPs with the specific speed function $\alpha(\gamma)=(\log\log\gamma)^{-1}$ for large γ. All limits, superior limits and the like will be understood to be as $\gamma\to\infty$, unless indicated otherwise.

BASIC HYPOTHESES 1.1. The situation described above will be assumed throughout (for either the inf. div. case or the p. sums case), and in addition the following conditions will be assumed for all $x>1$:

$$\lim[\gamma\nu((\gamma x^{\beta},\infty))]^{\alpha}=x^{-1}; \tag{1.7}$$

$$(\gamma x^{\beta})^{-2A}\int_1^{\gamma x^{\beta}} y^{2A}\nu(dy)=O(\gamma^{-1}); \tag{1.8}$$

and one of the following two conditions:

$$\limsup\left|\int_1^{\gamma x^{\beta}} y^{A}\gamma^{1-A}\nu(dy)-g(\gamma)\right|^{\alpha}\le x^{A} \tag{1.9}$$

for some function g: $[1,\infty)\mapsto\mathbb{R}$ and $c=0$, $A\ge 1$ or $\alpha\log\gamma\to 0$; or

$$\limsup\left(\int_{\gamma x^{\beta}}^{\infty} y^{A}\gamma^{1-A}\nu(dy)\right)^{\alpha}\le 1 \tag{1.10}$$

and $\alpha\log\gamma\to 0$ or $c=-\int_1^{\infty} y^{A}\nu(dy)$ [which is finite by (1.10)].

Now let $\hat{S}_{\gamma}$ denote the process $t\mapsto\gamma^{-A}S(\gamma t)-tg(\gamma)$ when (1.9) is assumed and $t\mapsto\gamma^{-A}S(\gamma t)$ when (1.10) is assumed. In the p. sums case, we modify the version for (1.9) to $t\mapsto\gamma^{-A}S(\gamma t)-\gamma^{-1}\lfloor\gamma t\rfloor g(\gamma)$.

REMARKS 1.2. (a) Suppose $\nu((x,\infty))=x^{-1}$ for all x, so that S is stable. Then, for any α, (1.7) and (1.8) hold. Also (1.9) holds with $g=0$ when $A>1$, (1.9) holds with $g(\gamma)=\log\gamma$ when $A=1$ and (1.10) holds when $A<1$. The supplementary assumptions following (1.9) and (1.10) are needed only to deal with the term ct.

(b) Equation (1.7) comes from the LDPs for Φ (cf. Section 3) and is actually necessary and sufficient for those LDPs to hold. It implies that for any $\varepsilon>0$,

$$\gamma^{-1-\varepsilon}<\nu((\gamma,\infty))<\gamma^{-1+\varepsilon} \tag{1.11}$$

for γ sufficiently large. Suppose $\alpha\log\gamma\to 0$. Then (1.11) is enough to show (by integration by parts) that (1.8) holds for all A, that (1.9) holds with $g\equiv 0$ if $A>1$ and that (1.10) holds if $A<1$. On the other hand, (1.9) with $g\equiv 0$ implies $A\ge 1$ or $\alpha\log\gamma\to 0$, while (1.10) implies $A\le 1$. Incidently, (1.11) implies that our results are disjoint from those for processes which have finite moment-generating functions, such as the results of Lynch and Sethuraman (1987).

(c) In general, the faster $\alpha \to 0$, the larger the class of ν's which satisfy the above assumptions. For our LIL we need α to behave something like $(\log\log\gamma)^{-1}$, which means the class is rather small.

We recall from Schilder (1966) and Strassen (1964) that to get an LDP or LIL for Brownian motion or related partial sums, we normalize in a way that slightly perturbs the normalization used to get invariance or weak convergence. The same applies in our situation, although the perturbation manifests itself in the form of an exponent. Invariance is obtained by considering $\hat{S}_\gamma$. For our LDP, we consider the processes $(\hat{S}_\gamma \vee 1)^\alpha$. The truncation at 1 serves in the first instance to make sure this process is well defined (since $\hat{S}_\gamma$ can be negative). We investigate values of $\hat{S}_\gamma$ which exceed 1.

We now establish our terminology for LDPs. If P is a probability measure and $\delta \in (0, 1]$, define P^δ by $P^\delta(A) = (P(A))^\delta$. If D is a topological space and R: $D \to [0, \infty]$ is lower semicontinuous (lsc), define a set function e^{-R} on the subsets of D by $e^{-R}(B) = \sup\{e^{-R(x)}: x \in B\}$. If $(P_\gamma)_{\gamma \geq 1}$ is a family of Borel probability measures on D, α is a speed function and R: $D \mapsto [0, \infty]$ is lsc, then a (*full*) LDP is an assertion that

$$P_\gamma^\alpha \Rightarrow e^{-R}, \tag{1.12}$$

where (1.12) is interpreted (as with weak convergence) as (1.13) and (1.14):

$$\limsup P_\gamma^\alpha(F) \leq e^{-R}(F) \text{ for all closed } F, \tag{1.13}$$

$$\liminf P_\gamma^\alpha(G) \geq e^{-R}(G) \text{ for all open } G \tag{1.14}$$

We do not assume a priori that the *rate function* R is lower compact, that is, that $\{x \in D: R(x) \leq r\}$ is compact for all $r < \infty$. A *vague* LDP is an assertion that (1.13) holds for all *compact* F and (1.14) holds as stated; in this case, we write

$$P_\gamma^\alpha \to_v e^{-R}.$$

It is convenient to extend $D[0, \infty)$ by allowing paths to take the value $+\infty$. We then define two subspaces of this extended $D[0, \infty)$,

$$D := \{\xi \in D[0, \infty): 1 \leq \xi(t) \leq \infty \text{ for all } t \in [0, \infty)\}, \tag{1.15}$$

$$D^\uparrow := \{\xi \in D: \xi \text{ is nondecreasing}\} \tag{1.16}$$

The main reason that the M_1 topology is more useful than the J_1 topology for the current study is that $D^\uparrow$ is M_1-compact but not J_1-compact. Details are given in Section 2.

THEOREM 1.3. *For $A > 0.5$ and $\gamma \geq 1$, let $\hat{S}_\gamma$ be a process as specified in Basic Hypotheses* 1.1, *either for the inf. div. or the p. sums case. Let P_γ denote the distribution of $(\hat{S}_\gamma \vee 1)^\alpha$. Then*

$$P_\gamma^\alpha \Rightarrow e^{-R} \tag{1.17}$$

in the M_1 topology, where $R\colon D \mapsto [0, \infty]$ is the M_1-lower compact function

$$R(\xi) := \begin{cases} A^{-1} \displaystyle\sum_{x \in \mathrm{Range}(\xi)} \log x, & \text{if } \xi \in D^\uparrow, \\ \infty, & \text{otherwise}. \end{cases} \tag{1.18}$$

Further, in the J_1 topology, the corresponding vague LDP holds, but the rate function R is not J_1-lower compact.

REMARKS 1.4. (a) Clearly, $R(\xi) = \infty$ if ξ takes infinitely many distinct values above any $x > 1$ or if $\xi(t) = \infty$ for any $t \in [0, \infty)$. (b) It is not known whether a full LDP holds for the J_1 topology. In any case, the rate function of any full or vague LDP for the J_1 topology must be R by the uniqueness of vague rate functions [cf. O'Brien (1996), Section 2]. Since R is not J_1-lower compact, a full LDP cannot be proved by using a tightness argument.

It is often the case that LDPs have an associated Strassen-type LIL, for which the set of limit points coincides with the set on which the LDP rate function is at most 1. That turns out to be the case here. We will use the following terminology.

Let $(x_\gamma)_{\gamma \geq 1}$ be a family of elements in a Hausdorff space. We say (x_γ) is *relatively compact* (*as* $\gamma \to \infty$) if there is a compact set K such that, for every open $G \supset K$, x_γ is in G for all sufficiently large γ. We then call K an *outer limit* for (x_γ). An element x is called a *limit point* of (x_γ) if for each open $G \ni x$, the set $\{\gamma\colon x_\gamma \in G\}$ is unbounded. If K is an outer limit for (x_γ), then of course the set of limit points of (x_γ) is a compact subset of K. Now let (X_γ) be a family of (Borel) random elements. A compact set K is called a *Strassen outer limit* of (X_γ) if K is an outer limit for (X_γ) wp1, and x is a *Strassen limit point* for (X_γ) if x is a limit point of (X_γ) wp1. Finally, a *Strassen LIL* for (X_γ) with set of limit points K is an assertion that K is a Strassen outer limit for (X_γ) and every $x \in K$ is a Strassen limit point for (X_γ). In this case we write

$$X_\gamma \to K \quad \text{wp } 1. \tag{1.19}$$

THEOREM 1.5. *Let $\hat{S}_\gamma$ be a process as specified in Basic Hypotheses* 1.1. (a) *Assume β is nondecreasing and satisfies the following*:

$$\limsup \beta(e\gamma)/\beta(\gamma) < \infty \tag{1.20}$$

and

$$\sum_{k=1}^{\infty} \delta^{\beta(e^k)} < \infty \tag{1.21}$$

for some $\delta > 0$. *Then*, *in the* M_1 *topology*, $D^{\uparrow}$ *is a Strassen outer limit for* $(\hat{S}_{\gamma} \vee 1)^{\alpha}$. (b) *Suppose more specifically that*

$$\beta(\gamma) = \log\log\gamma \tag{1.22}$$

for large γ. *Then*, *also in the* M_1 *topology*, *wp1*,

$$\begin{aligned}(\hat{S}_{\gamma} \vee 1)^{\alpha} &\to \{\xi \in D^{\uparrow}\colon R(\xi) \le 1\}\\ &= \Big\{\xi \in D^{\uparrow}\colon \sum_{x\in\mathrm{Range}(\xi)} \log x \le A\Big\}.\end{aligned} \tag{1.23}$$

REMARKS 1.6. Pakshirajan and Vasudeva (1981) proved Theorem 1.5(b) for the case where S is a spectrally positive stable process with $A > 1$. The specific very slowly increasing choice of β in (1.22) means in particular that hypothesis (1.7) is quite restrictive. We can relax (1.22) slightly: we need that β is nondecreasing and that $\sum_{k=1}^{\infty} \exp\{-r\beta(e^k)\}$ is finite if $r > 1$ and infinite if $r < 1$. We do not provide a proof of this extension because we would first have to generalize the results of O'Brien and Vervaat (1996) in a similar way. We note that the generalization would result in a "LIL" without any iterated logarithms. Finally, we note that Theorem 1.5 is not valid for the J_1 topology since the set $\{\xi \in D^{\uparrow}\colon R(\xi) \le 1\}$ is not compact.

Our final result is a simple corollary of Theorems 1.3 and 1.5. Given a process S of one of the types described above, define a new process S^* in $D[0,\infty)$ by

$$S^*(t) = \sup_{s\le t} S(s). \tag{1.24}$$

Also, define $\hat{S}^*_{\gamma}$ to be the analogue of $\hat{S}_{\gamma}$ for S^* and let P^*_{γ} be corresponding analogue of P_{γ}.

THEOREM 1.7. *If* $\hat{S}_{\gamma}$ *satisfies the hypotheses of Theorem* 1.3, *then also*

$$P^{*\alpha}_{\gamma} \Rightarrow e^{-R}$$

in the M_1 *topology*. *If the hypotheses of Theorem* 1.5 *hold*, *then the conclusions hold also for* $\hat{S}^*_{\gamma}$.

Theorem 1.7 is an immediate consequence of Theorems 1.3 and 1.5, combined with the fact that the function $f\colon D \mapsto D$ given by $(f\xi)(t) = \sup_{s\le t} \xi(t)$ is continuous relative to the M_1 topology.

2. Two topologies. In this section, we discuss our two topologies for the space D defined in (1.15). These are modified versions of Skorokhod's M_1 and J_1 topologies. The space D is a metric space under both topologies. In the proofs of our main theorems, we work with the specific metrics and related pseudometrics described below.

Given $\xi \in D$, we extend ξ to an element of $D[-1,\infty)$ by defining

$$\xi(t) = 1 \quad \text{for } -1 \le t < 0. \tag{2.1}$$

With this extension we arrange for both our topologies that

$$1 + \mathbf{1}_{[n^{-1},\infty)} \to 1 + \mathbf{1}_{[0,\infty)} \quad \text{as } n \to \infty. \tag{2.2}$$

With the usual M_1 and J_1 topologies, (2.2) fails.

We now consider the (modified) M_1 topology. We define the *extended graph* $EG(\xi)$ of $\xi \in D$ by extending ξ as in (2.1) and then setting

$$\begin{aligned} EG(\xi) := {} & ([-1,0) \times \{1\}) \\ & \cup \big\{(t,x) \in [0,\infty) \times [1,\infty] \colon \xi(t-) \le x \le \xi(t) \text{ or} \\ & \qquad\qquad \xi(t-) \ge x \ge \xi(t)\big\}. \end{aligned} \tag{2.3}$$

Since $\xi \in D[-1,\infty)$, $EG(\xi)$ is homeomorphic to $[0,\infty)$. A *parametrization* of ξ is a continuous bijection $\xi^*\colon [-1,\infty) \longmapsto EG(\xi)$ with $\xi^*(-1) = (-1,1)$ and, denoting $\xi^*(t)$ as the vector $(\xi_1^*(t), \xi_2^*(t))$, with

$$|\xi_1^*(t) - t| < 1 \quad \text{for all } t \in [-1,\infty). \tag{2.4}$$

Parametrizations can be shown to exist. For $t > 0$, we define a pseudometric d_t on $\{\xi \in D\colon \xi \text{ is locally bounded}\}$ by

$$d_t(\xi,\zeta) := \inf \sup_{-1 \le s \le t} (|\xi_1^*(s) - \zeta_1^*(s)| + (|\xi_2^*(s) - \zeta_2^*(s)|), \tag{2.5}$$

where the infimum is over all parametrizations of ξ and ζ. We next define a pseudometric d_t' on all of D by

$$d_t'(\xi,\zeta) := d_t(2 - (1/\xi), 2 - (1/\zeta)). \tag{2.6}$$

Finally, we define a metric d' on D by

$$d'(\xi,\zeta) := \sum_{n=1}^{\infty} 2^{-n} d_n'(\xi,\zeta). \tag{2.7}$$

Note that $d_t' \le \min(d_t, 1)$ and that, for all n, $d_n' \le 2^n d'$ and $d' \le d_n' + 2^{-n}$. Also, d_n' and d_n give rise to the same topologies.

We now consider our modified J_1 topology. We again extend $\xi \in D$ as in (2.1). A time-shift is a continuous bijection $\lambda\colon [-1,\infty) \mapsto [-1,\infty)$. For $t > 0$ we define the following pseudometric $d_{J,t}$ on $\{\xi \in D\colon \xi(s) < \infty \text{ for all } s\}$:

$$\begin{aligned} d_{J,t}(\xi,\zeta) := {} & \inf_{\lambda} \sup_{-1 \le s \le t} \{|\xi(s) - \zeta(\lambda(s))| + |s - \lambda(s)|\} \\ & + \inf_{\lambda} \sup_{-1 \le s \le t} \{|\zeta(s) - \xi(\lambda(s))| + |s - \lambda(s)|\}, \end{aligned}$$

where the infima are over all time-shifts λ. We use the symmetrized version in order to handle end effects near t. We also define $d_{J,t}'$ and d_J' in analogy to (2.6) and (2.7).

DEFINITION 2.1. The M_1 and J_1 topologies on D are the topologies generated by the metrics d' and d'_J, respectively.

It can be shown that the M_1 topology is coarser than the J_1 topology. A simple example that displays a key difference between the topologies is that

$$(1+\mathbf{1}_{[n^{-1},\infty)}+\mathbf{1}_{[2n^{-1},\infty)})\to 1+2\mathbf{1}_{[0,\infty)}\quad\text{as } n\to\infty \tag{2.8}$$

for the M_1 topology, but the sequence has no limit for the J_1 topology. We also note that the following sequences all diverge for both topologies:

$$(1+\mathbf{1}_{[n^{-1},2n^{-1})}),(1+\mathbf{1}_{[0,n^{-1})})\quad\text{and}\quad(1+\mathbf{1}_{[1-2n^{-1},1-n^{-1})}+\mathbf{1}_{[1,\infty)}).$$

LEMMA 2.2. *For $t\geq 0$ and $x\geq 1$, the set $\{\xi\in D\colon \xi(t)>x\}$ is Borel measurable for both topologies; further, the two Borel σ-algebras coincide.*

Lemma 2.2 can be proved by methods similar to those of Billingsley (1968), Chapter 3. Our short proof of the next result avoids the issue of whether D with the M_1 topology is Polish.

LEMMA 2.3. *Every Borel probability measure on D is tight relative to the M_1 topology.*

PROOF. This is known for the J_1 topology, and all sets which are J_1-compact are M_1-compact. □

LEMMA 2.4. *The set $D^{\uparrow}$ is M_1-compact but not J_1-compact.*

PROOF. For $\xi_n,\ \xi\in D^{\uparrow}$, $\xi_n\to\xi$ iff $\xi_n(t)\to\xi(t)$ at continuity points of ξ. Thus, $D^{\uparrow}$ with the M_1 topology is homeomorphic to the set of subprobability measures on $[0,\infty)$, with the vague topology. This topology is known to be compact. The example in (2.8) shows that $D^{\uparrow}$ is not J_1-compact. □

3. Large deviations and the point process Φ. In this section, we recall a few facts about LDPs and Φ. We will use the notation and ideas given near (1.12). We begin with a variation of the usual notion of tightness. Let D' be any metric space, let $(P_\gamma)_{\gamma\geq 1}$ be any family of tight Borel probability measures on D' and let α be a speed function.

DEFINITION 3.1. We say $(P_\gamma^\alpha)_{\gamma\geq 1}$ is *damped* if for any $\varepsilon>0$ there is a compact $K\subset D'$ such that, for all open $G\supset K$,

$$\limsup_{\gamma\to\infty} P_\gamma^\alpha(G^c)<\varepsilon. \tag{3.1}$$

We say (P_γ^α) is *tight* if for all $\varepsilon>0$ there is a compact $K\subset D'$ such that $P_\gamma^\alpha(K^c)<\varepsilon$ for all $\gamma\geq 1$.

It is obvious that tightness implies dampedness. The converse for metric spaces was proved in O'Brien [(1996), Theorem 3.3] for the case where $\gamma \to \infty$ through a sequence. [Some related results for weak convergence are given, for example, in Ethier and Kurtz (1986).]

LEMMA 3.2. *With P_γ and α as above and $R: D \longmapsto [0,\infty]$ lsc, suppose that*

$$P_\gamma^\alpha \to_v e^{-R} \tag{3.2}$$

and that $(P_\gamma^\alpha)_{\gamma\geq 1}$ is damped. Then R has compact level sets and

$$P_\gamma^\alpha \Rightarrow e^{-R}. \tag{3.3}$$

PROOF. It suffices to prove (3.3) for every sequence of values of γ which goes to ∞. Then dampedness implies tightness, which with (3.2) implies (3.3). □

In Section 4, we will prove Theorem 1.3 by verifying dampedness and the corresponding vague LDP. We remind the reader of the following facts.

LEMMA 3.3. (a) *To prove* (1.14) *for all open G, it suffices to prove that for all $\xi \in D$ and open $G \ni \xi$,*

$$\liminf P_\gamma^\alpha(G) \geq e^{-R(\xi)}. \tag{3.4}$$

(b) *To prove* (1.13) *for all compact sets it suffices to prove that for all $\xi \in D$ and $\varepsilon > 0$ there is a neighborhood N of ξ such that*

$$\limsup P_\gamma^\alpha(N) < e^{-R(\xi)} + \varepsilon. \tag{3.5}$$

We will need the following results from O'Brien and Vervaat (1996) about the random measure Φ, as defined in Section 1.

First, let $\mathscr{N}$ denote the space of $\{\infty, 0, 1, 2, \ldots\}$-valued regular Borel measures on E, endowed with the vague topology. [We say μ is regular if $\mu(K) = \inf\{\mu(G): G \text{ open}, G \supset K\}$ for all compact K and $\mu(B) = \sup\{\mu(K): K \text{ compact}, K \subset B\}$ for all Borel B.] A subbase for the vague topology is the collection of all sets of the form $\{\mu \in \mathscr{N}: \mu(G) > r\}$ or $\{\mu \in \mathscr{N}: \mu(K) < r\}$ for $r > 0$, G open in E and K compact in E. Given $\mu \in \mathscr{N}$, a speed function α and $\gamma \geq 1$, we define μ_γ by

$$\begin{aligned}\mu_\gamma(B) &= \mu(\{(t,x) \in E: (t\gamma^{-1}, (x\gamma^{-1})^\alpha) \in B\})\\ &= \mu(\{(\gamma t, \gamma x^\beta): (t,x) \in B\}).\end{aligned}$$

Since Φ is in $\mathscr{N}$ wp1, we can consider the family $(\Phi_\gamma)_{\gamma\geq 1}$. Let $P_{\Phi,\gamma}$ denote the distribution of the restriction of Φ_γ to $[0,\infty) \times (1,\infty]$, and let $\mathscr{N}_1$ denote the set of restrictions of elements of $\mathscr{N}$ to the same set. We also endow $\mathscr{N}_1$ with the vague topology. Conveniently, $\mathscr{N}$ and $\mathscr{N}_1$ are compact.

THEOREM 3.4. *Under Basic Hypotheses* 1.1, *we have the LDP*

$$P^{\alpha}_{\Phi,\gamma} \Rightarrow \exp(-R_{\Phi}),$$

where $R_{\Phi}\colon \mathscr{N}_1 \to [0,\infty]$ *is the lower compact function given by*

$$\begin{aligned} R_{\Phi}(\mu) &= \int_0^{\infty}\int_1^{\infty} \log y\mu(dt, dy) \\ &= \sum_{t\geq 0,\, y>1} \log y\mu(\{(t,y)\}). \end{aligned}$$

In particular, for $t > 0$, $x > 1$ and $r \geq 1$,

$$\begin{aligned} &\limsup P^{\alpha}[\Phi([0,\gamma t]\times[\gamma x^{\beta},\infty]) \geq r] \\ &\quad = \limsup P^{\alpha}[\Phi_{\gamma}([0,t]\times[x,\infty]) \geq r] \\ &\quad \leq \exp(-r\log x). \end{aligned} \tag{3.6}$$

We also have the LIL associated with Theorem 3.4. Let Φ'_{γ} denote the restriction of Φ_{γ} to $[0,\infty)\times(1,\infty]$.

THEOREM 3.5. *Under the above conditions and with* $\beta(\gamma) = \log\log\gamma$ *for large* γ,

$$\Phi'_{\gamma} \to \{\mu \in \mathscr{N}_1\colon R_{\Phi}(\mu) \leq 1\} \ wp\ 1.$$

4. Proof of Theorem 1.3. We concentrate on the p. sums case with (1.9), except in Remark 4.3. The inf. div. case is similar. For $t > 0$, $x > 1$ and $\gamma \geq 1$, let

$$I_1(x,\gamma,t) := \int_{\gamma x^{\beta}}^{\infty} \gamma^{-A} y^A \Phi([0,\gamma t], dy), \tag{4.1}$$

$$I_2(x,\gamma,t) := \int_0^{\gamma x^{\beta}} \gamma^{-A} y^A [\Phi([0,\gamma t], dy) - \lfloor \gamma t\rfloor \nu(dy)]. \tag{4.2}$$

Then

$$\begin{aligned} \hat{S}_{\gamma}(t) &= I_1(x,\gamma,t) + I_2(x,\gamma,t) \\ &\quad + \int_1^{\gamma x^{\beta}} \gamma^{-A} y^A \lfloor \gamma t\rfloor \nu(dy) - \gamma^{-1}\lfloor \gamma t\rfloor g(\gamma) + c\gamma^{-A}\lfloor \gamma t\rfloor. \end{aligned} \tag{4.3}$$

If we assume (1.10) instead of (1.9), much the same proof works, with the third and fourth terms in (4.3) together replaced by

$$-\int_{\gamma x^{\beta}}^{\infty} \gamma^{-A} y^A \lfloor \gamma t\rfloor \nu(dy).$$

We will show in Lemma 4.4 that in an appropriate sense $I_1(x,\gamma,t)$ is the dominant term on the right side of (4.3).

LEMMA 4.1. *Let $\varepsilon > 0$. There exists $r_0 > 0$ such that*

$$W := |((w+v)\vee 1)^r - (w\vee 1)^r| \le 2\varepsilon \tag{4.4}$$

for all $r \in (0, r_0)$, $w \in \mathbb{R}$ and $v \in \mathbb{R}$ with $|v|^r < 1+\varepsilon$.

PROOF. Obviously, we may assume $v > 0$. If $w + v \le 1$, then $W = 0$. If $w \le 1 < w + v$, then $W \le (1 + (1+\varepsilon)^{1/r})^r - 1 \le 2^r(1+\varepsilon) - 1 < 2\varepsilon$ for small $r > 0$. If $w > 1$, then $W = (w+v)^r - w^r$, which is decreasing in $w \ge 1$ for fixed $v > 0$ and $r \in (0,1)$. Thus $W \le (1+v)^r - 1$, which is less than 2ε by the previous case with $w = 1$. □

LEMMA 4.2. *For $T > 0$ and $v > x > 1$, the following hold*:

$$\limsup \sup_{0\le t\le T} |c\gamma^{1-A}t|^\alpha \le 1, \tag{4.5}$$

$$\limsup \sup_{0\le t\le T} \left| \int_1^{\gamma x^\beta} \gamma^{1-A} t y^A \nu(dy) - tg(\gamma)\right|^\alpha \le x^A, \tag{4.6}$$

$$P^\alpha\Big[\sup_{0\le t\le T} |I_2(x,\gamma,t)|^\alpha > v^A\Big] \to 0. \tag{4.7}$$

PROOF. Formula (4.6) follows from (1.9) and (4.5) holds since $c = 0$, $A \ge 1$ or $\alpha \log\gamma \to 0$. We now prove (4.7). For now, fix $x > 1$ and $\gamma \ge 1$ and write n for $\lfloor \gamma t\rfloor$, N for $\lfloor \gamma T\rfloor$ and $I(n)$ for $I_2(x,\gamma,t)$. Let $Y_k = X_k$ if $X_k \le \gamma x^\beta$, $Y_k = 0$ otherwise. Then

$$I(n) = \sum_{k=1}^n \gamma^{-A}(Y_k^A - EY_k^A). \tag{4.8}$$

Since Y_k is bounded, the following are finite for $\theta \in \mathbb{R}$:

$$\begin{aligned} L(\theta) &:= \log E[\exp\{\theta\gamma^{-A}(Y_k^A - EY_k^A)\}];\\ M_\theta(n) &:= \exp\{\theta I(n) - nL(\theta)\}. \end{aligned} \tag{4.9}$$

Also M_θ is a martingale in n and

$$E(M_\theta(n))^2 = \exp\{nL(2\theta) - 2nL(\theta)\}. \tag{4.10}$$

By Jensen's inequality, $L(\theta) \ge 0$ for all θ. Henceforth, we consider only the values $\theta = \pm(2x^{A\beta})^{-1}$. From (4.9), the fact that $e^w \le 1 + w + w^2$ for $|w| \le 1$, (1.1) and (1.8), we obtain, with $w := 2\theta\gamma^{-A}(y^A - EY^A)$,

$$\begin{aligned} L(2\theta) &\le \log \int_0^{\gamma x^\beta} [1 + w + w^2]\nu(dy)\\ &= \log \int_0^{\gamma x^\beta} [1 + w^2]\nu(dy) \le \int_0^{\gamma x^\beta} w^2 \nu(dy)\\ &\le \gamma^{-2A}\int_0^{\gamma x^\beta} x^{-2A\beta} y^{2A}\nu(dy) = O(\gamma^{-1}). \end{aligned} \tag{4.11}$$

Applying Markov's inequality, Doob's L^2-inequality for martingales, (4.10) and (4.11), we obtain for $\theta = (2x^{A\beta})^{-1}$,

$$\begin{aligned}
&P^\alpha\Big[\sup_{0\le t\le T} I_2(x,\gamma,t) > v^{A\beta}\Big]\\
&\quad = P^\alpha\Big[\sup_{0\le n\le N} I(n) > v^{A\beta}\Big]\\
&\quad \le P^\alpha\Big[\sup_{0\le n\le N} (M_\theta(n))^2 > \exp\{2\theta v^{A\beta} - 2NL(\theta)\}\Big]\\
&\quad \le \Big(E\Big[\sup_{0\le n\le N} (M_\theta(n))^2\Big]\exp\{-2\theta v^{A\beta} + 2NL(\theta)\}\Big)^\alpha\\
&\quad \le (4E(M_\theta(N))^2 \exp\{-2\theta v^{A\beta} + 2NL(\theta)\})^\alpha\\
&\quad \le (4\exp\{-2\theta v^{A\beta} + NL(2\theta)\})^\alpha\\
&\quad \le 4\exp\{-\alpha((v/x)^{A\beta} + \gamma TL(2\theta))\} \to 0.
\end{aligned} \tag{4.12}$$

Similarly, with $\theta = -(2x^{A\beta})^{-1}$,

$$\begin{aligned}
&P^\alpha\Big[\sup_{0\le n\le T} (-I(n)) > v^{A\beta}\Big]\\
&\quad \le P^\alpha\Big[\sup_{0\le t\le N} (M_{-\theta}(n))^2 > \exp\{-2\theta v^{A\beta} + 2NL(-\theta)\}\Big]\\
&\quad \le (4E(M_{-\theta}(N))^2 \exp\{-2\theta v^{A\beta} + 2NL(-\theta)\})^\alpha\\
&\quad \le (4\exp\{-2\theta v^{A\beta} + 2NL(-2\theta)\}^\alpha\\
&\quad \le 4\exp\{-\alpha((v/x)^{A\beta} + \gamma TL(2\theta))\} \to 0.
\end{aligned}$$

Combining this with (4.12), we get (4.7). □

REMARK 4.3. In the proof for the inf. div. case, we get a similar result using a continuous-time martingale. To summarize, we have $E[\exp(\theta I_2(x,\gamma,t))] = \exp(tL(\theta))$, where, now,

$$L(\theta) := \int_0^{\gamma x^\beta} [\exp\{\theta\gamma^{-A} y^A\} - 1 - \theta\gamma^{-A} y^A]\nu(dy).$$

By (1.1), $0 \le L(\theta) < \infty$ for all $\theta \in \mathbb{R}$. We then take

$$M_\theta(t) := \exp\{\theta I_2(x,\gamma,t) - tL(\theta)\}.$$

Then M_θ is a martingale and (4.10) holds for positive *real* n. Taking $\theta = \pm(2x^{A\beta})^{-1}$ as before, we again have

$$L(2\theta) \le \int_0^{\gamma x^\beta} (2\theta\gamma^{-A} y^A)^2 \nu(dy) = O(\gamma^{-1}).$$

We now return to the sequence case. Throughout Sections 4 and 5, we will say a property holds "for sufficiently small $x > 1$" if there exists an $x_0 > 1$ such that the property holds for all $x \in (1, x_0)$. We have the following key results. The (pseudo-)metrics were defined in Section 2.

LEMMA 4.4. *For all* $\rho > 0$,

$$P^\alpha[d'((\hat{S}_\gamma \vee 1)^\alpha, (I_1(x, \gamma, \cdot) \vee 1)^\alpha) > \rho] \to 0$$

for sufficiently small $x > 1$. *For all* $\rho > 0$ *and* $T > 0$,

$$P^\alpha[d_T((\hat{S}_\gamma \vee 1)^\alpha, (I_1(x, \gamma, \cdot) \vee 1)^\alpha) > \rho] \to 0$$

for sufficiently small $x > 1$. *The same results hold for the* J_1 *topology*.

PROOF. By the relationships between (pseudo-)metrics and by Lemmas 4.1 and 4.2, we have for all $T > 0$, $\rho > 0$ and $\varepsilon > 0$ that

$$\begin{aligned} P^\alpha&[d'_T((\hat{S}_\gamma \vee 1)^\alpha, (I_1(x, \gamma, \cdot) \vee 1)^\alpha) > \rho] \\ &\le P^\alpha[d_T((\hat{S}_\gamma \vee 1)^\alpha, (I_1(x, \gamma, \cdot) \vee 1)^\alpha) > \rho] \\ &\le P^\alpha\Big[\sup_{0 \le t \le T+1} |(\hat{S}_\gamma(t) \vee 1)^\alpha - (I_1(x, \gamma, t) \vee 1)^\alpha| > \rho/2\Big] < \varepsilon \end{aligned}$$

eventually, for sufficiently small $x > 1$. By (2.7), for all $\rho > 0$ and $\varepsilon > 0$, $P^\alpha[d'((\hat{S}_\gamma \vee 1)^\alpha, (I_1(x, \gamma, t) \vee 1)^\alpha) > 2\rho] < \varepsilon$ eventually, for sufficiently small $x > 1$. The same argument works for the J_1 topology. $\square$

LEMMA 4.5. *With* P_γ *defined as in Theorem* 1.3, (P^α_γ) *is* M_1-*damped*.

PROOF. This follows from Lemma 4.4 and the fact that $(I_1(x, \gamma, \cdot) \vee 1)^\alpha$ is in the compact set $D^\uparrow$. $\square$

In view of Lemmas 3.2 and 4.5, we may prove our LDP by verifying the local conditions given in Lemma 3.3. We begin with the upper bound. Every M_1-neighborhood of ξ is also a J_1-neighborhood, so the following argument also yields the upper bound for the vague LDP in the J_1 topology. First assume $\xi \notin D^\uparrow$. By Lemma 4.4, we have for $\rho < d'(\xi, D^\uparrow)$ and $G := \{\eta \in D: d'(\xi, \eta) < \rho\}$ that $P^\alpha[(\hat{S}_\gamma \vee 1)^\alpha \in G] \to 0$. Thus we need only consider $\xi \in D^\uparrow$.

Suppose now that $\xi \in D^\uparrow$ and $\xi(t) < \infty$ for all $t \in [0, \infty)$, so that ξ is locally bounded. We will show that for all $\eta > 0$, there exist $T > 0$ and $\rho > 0$ such that

$$\limsup P^\alpha[d_T(\hat{S}_\gamma \vee 1)^\alpha, \xi) < \rho/2] \le \exp(-R(\xi)) + \eta. \tag{4.13}$$

In view of Lemma 4.4, it suffices to show that for all $\eta > 0$ there exist $T > 0$ and $\rho > 0$ such that

$$\limsup P^\alpha(E_0(\gamma)) \le \exp(-R(\xi)) + \eta, \tag{4.14}$$

for sufficiently small $x > 1$, where the event $E_0(\gamma)$ is given by

$$E_0(\gamma) := [d_T((I_1(x,\gamma,\cdot) \vee 1)^\alpha, \xi) < \rho]. \tag{4.15}$$

If $\xi \equiv 1$, then $R(\xi) = 0$ so (4.14) holds. Thus we may suppose there exist $m \ge 1$ and real numbers $0 \le t_1 < \cdots < t_m$ and $1 < x_1 < \cdots < x_m < \infty$ such that $\xi(t_i) = x_i^A$, $i = 1, 2, \ldots, m$.

Choose r such that $rAx_i^{A-1} > 1, i = 1, 2, \ldots, m$. Since $(x_i - r\varepsilon)^A = x_i^A - rAx_i^{A-1}\varepsilon + o(\varepsilon)$ as $\varepsilon \to 0$, we have

$$p_i := (x_i^A - \varepsilon)(x_i - r\varepsilon)^{-A} > 1, \qquad i = 1, 2, \ldots, m \tag{4.16}$$

for sufficiently small $\varepsilon > 0$. Let $x \in (1, x_1)$. Then choose $\delta > 0$ with $\delta < \min\{x_1 - x, x_2 - x_1, \ldots, x_m - x_{m-1}\}$. Then, suppose $\varepsilon > 0$ is also sufficiently small that all the following hold: $x_1 - r\varepsilon > 1$; $x_1^A - \varepsilon > 1$; $x_i > x_{i-1} + r\varepsilon + \delta$, $i = 2, 3, \ldots, m$; $(x_i + \delta)^A > x_i^A + 2\varepsilon$, $i = 1, 2, \ldots, m$; $\prod_{i=1}^m (x_i - r\varepsilon)^{-1} \le (\prod_{i=1}^m x_i)^{-1} + \eta$ and $\varepsilon < 1$. Next, let $T = t_m + 1$. Since $\xi \in D$, we may choose $\rho \in (0, \varepsilon)$ such that $\xi(t_i + 2\rho) < x_i^A + \varepsilon$, $i = 1, \ldots, m$. Finally, by making $x > 1$ smaller if necessary, we may assume that $x < 1 + \rho$ and $x < x_1 - r\varepsilon$.

We will prove that

$$\limsup P^\alpha(E_0(\gamma)) \le \prod_{i=1}^m x_i^{-1} + \eta. \tag{4.17}$$

This will imply (4.14), since the infimum of $\prod_{i=1}^m x_i^{-1}$ in (4.17), over all possible choices of m and $x_1, \ldots, x_m$, is $\exp(-R(\xi))$.

If $E_0(\gamma)$ occurs, then each of the points (t_i, x_i^A) and $(t_i + 2\rho, \xi(t_i + 2\rho))$ must lie within distance ρ of some point on the extended graph of $(I_1(x, \gamma, \cdot) \vee 1)^\alpha$. Since each function is in $D^\uparrow$, this implies that, for $i = 1, 2, \ldots, m$,

$$x_i^A - \varepsilon < \xi(t_i) - \rho < (I_1(x, \gamma, t_i + \rho))^\alpha < \xi(t_i + 2\rho) + \rho < x_i^A + 2\varepsilon. \tag{4.18}$$

If $E_0(\gamma)$ occurs, it must occur in conjunction with at least one of the following three events:

$$E_1(\gamma) := \bigcup_{i=1}^m [\Phi([0, \gamma(t_i + \rho)] \times [\gamma(x_i + \delta)^\beta, \infty]) > 0], \tag{4.19}$$

$$E_2(\gamma) := \bigcup_{i=1}^m [\Phi([0, \gamma(t_i + \rho)] \times [\gamma(x_i - r\varepsilon)^\beta, \infty]) = 0], \tag{4.20}$$

$$E_3(\gamma) := \bigcap_{i=1}^m [\Phi([0, \gamma(t_i + \rho)] \times [\gamma(x_i - r\varepsilon)^\beta, \gamma(x_i + \delta)^\beta]) \ge 1]. \tag{4.21}$$

If $E_1(\gamma)$ occurs, then for some i, $(I_1(x,\gamma,t_i+\rho))^\alpha > (x_i+\delta)^A > x_i^A+2\varepsilon$, which is inconsistent with (4.18). Thus $P[E_0(\gamma)\cap E_1(\gamma)]=0$. By (4.18),

$$
\begin{aligned}
&E_0(\gamma)\cap E_2(\gamma)\\
&\quad\subset \bigcup_{i=1}^m \left[\int_{\gamma x^\beta}^{\gamma(x_i-r\varepsilon)^\beta} \gamma^{-A} y^A \Phi([0,\gamma(t_i+\rho)],dy) > (x_i^A-\varepsilon)^\beta\right]\\
&\quad\subset \bigcup_{i=1}^m \left[\int_{\gamma x^\beta}^{\infty} \gamma^{-A} [\gamma(x_i-r\varepsilon)^\beta]^A \Phi([0,\gamma(t_i+\rho)],dy) > (x_i^A-\varepsilon)^\beta\right]\\
&\quad\subset \bigcup_{i=1}^m \left[\Phi([0,\gamma(t_i+\rho)]\times[\gamma x^\beta,\infty]) > p_i^\beta\right].
\end{aligned}
\tag{4.22}
$$

By (3.6) and the fact that $p_i^\beta \to \infty$,

$$P^\alpha[E_0(\gamma)\cap E_2(\gamma)] \to 0. \tag{4.23}$$

It now follows that

$$\limsup P^\alpha[E_0(\gamma)] = \limsup P^\alpha[E_0(\gamma)\cap E_3(\gamma)] \le \limsup P^\alpha[E_3(\gamma)].$$

By (3.6) extended to m disjoint rectangles, we conclude (4.17) since

$$\limsup P^\alpha[E_3(\gamma)] \le \prod_{i=1}^m (x_i-r\varepsilon)^{-1} \le \prod_{i=1}^m x_i^{-1}+\eta. \tag{4.24}$$

We now sketch a parallel argument for the case where $\xi(t)=\infty$ for some $t\in[0,\infty)$, so that $R(\xi)=\infty$. Take $T>t+1$. It suffices to show that, for any $x>1$ and for large enough z,

$$\limsup P^\alpha[(I_1(x,\gamma,T))^\alpha > (2z)^A] < 2/z. \tag{4.25}$$

By (3.6),

$$\limsup P^\alpha[\Phi([0,\gamma T]\times[\gamma z^\beta,\infty])>0] \le z^{-1}. \tag{4.26}$$

Also,

$$
\begin{aligned}
&\limsup P^\alpha[\Phi([0,\gamma T]\times[\gamma z^\beta,\infty])=0 \text{ and } (I_1(x,\gamma,T))^\alpha > (2z)^A]\\
&\quad\le \limsup P^\alpha\left[\int_{\gamma x^\beta}^{\gamma z^\beta} \gamma^{-A} y^A \Phi([0,\gamma T],dy) > (2z)^{A\beta}\right]\\
&\quad\le \limsup P^\alpha[\Phi([0,\gamma T]\times[\gamma x^\beta,\infty]) > 2^{A\beta}] = 0.
\end{aligned}
\tag{4.27}
$$

Then (4.25) follows from (4.26) and (4.27). This completes the upper bound half of our LDP.

By Lemma 3.3(a), the lower bound half can be proved by showing that for each $\xi\in D$ and each neighborhood N of ξ, we have

$$\liminf P^\alpha[(\hat{S}_\gamma \vee 1)^\alpha \in N] \ge \exp(-R(\xi)). \tag{4.28}$$

We do this for neighborhoods in the J_1 topology as well as just the M_1 topology, thereby proving also the lower bound half of the vague LDP for the J_1 tolopogy.

Obviously we need only consider ξ for which $R(\xi) < \infty$, so we may assume that $\xi \in D^{\uparrow}$, ξ is everywhere finite, and that ξ takes only finitely many values above any $x > 1$. Given such a ξ, define $\xi_n \in D^{\uparrow}$ by

$$\xi_n(t) = \begin{cases} \xi(t), & \text{if } \xi(t) \geq 1 + n^{-1}, \\ 1 + n^{-1}, & \text{otherwise.} \end{cases}$$

Then $\xi_n \to \xi$ (even uniformly). It follows that we need only prove (4.28) for ξ with the following form: there exist $m \geq 1$, $0 \leq t_1 < \cdots t_m$ and $1 < x_1 < \cdots < x_m < \infty$ such that $\xi(t) = 1$ for $0 \leq t < t_1$ (if $t_1 > 0$), $\xi(t) = x_i^A$ for $t_i \leq t < t_{i+1}$, $i = 1, 2, m-1$, and $\xi(t) = x_m^A$ for $t \geq x_m$. By Lemma 4.4, it suffices to show that for every J_1-neighborhood N of ξ and every $\eta > 0$,

$$\liminf P^{\alpha}[(I_1(x, \gamma, \cdot) \vee 1)^{\alpha} \in N] \geq \exp(-R(\xi)) - \eta. \tag{4.29}$$

for sufficiently small $x > 1$.

For $\varepsilon > 0$ and $T > 0$, let $N_{\varepsilon, T}$ be the set of functions $\xi' \in D$ such that for some $s_i \in (t_i, t_i + \varepsilon)$ and $y_i \in (x_i, x_i + 2\varepsilon)$, $i = 1, 2, \ldots, m$, $\xi'(t) = 1$ if $0 \leq t < s_1$, $\xi'(t) = y_i$ if $s_i \leq t < s_{i+1}, i = 1, 2, \ldots, m-1$, and $\xi'(t) = y_m$ if $s_m \leq t \leq T$. If ε is sufficiently small and T is sufficiently large, $N_{\varepsilon, T} \subset N$, so it suffices to prove (4.29) with N replaced by $N_{\varepsilon, T}$. Assuming in particular that $x \in (1, x_i)$ and $T > t_m + 1 + \varepsilon$, we find that $(I_1(x, \gamma, \cdot) \vee 1)^{\alpha} \in N_{\varepsilon, T}$ for large γ if Φ satisfies the following two conditions:

$$\Phi((\gamma t_i, \gamma(t_i + \varepsilon)) \times (\gamma x_i^{\beta}, \gamma(x_i + \varepsilon)^{\beta})) > 0, \qquad i = 1, \ldots, m, \tag{4.30}$$

$$\Phi([0, \gamma T] \times [\gamma x^{\beta}, \infty]) < m + 1. \tag{4.31}$$

Thus, it suffices to show that for all sufficiently small $\varepsilon > 0$,

$$\liminf P^{\alpha}[(4.30) \text{ and } (4.31) \text{ hold}] \geq \exp(-R(\xi)) - \eta.$$

This follows from Theorem 3.4.

5. Proof of Theorem 1.5. Once again we concentrate on the partial sums case. The steps of the proof follow a similar pattern to the steps in Section 4 and we use the same notation. Also, in the following, if we have events $B(\gamma)$, we write "$B(\gamma)$ i.a.u.s." to indicate the event that $B(\gamma)$ occurs for all γ *in an unbounded set*. If λ takes values only in the set $\{1, e, e^2, \ldots\}$, we use the more usual expression "$B(\lambda)$ infinitely often (i.o.)." We begin with an analogue of (4.7).

LEMMA 5.1. *Assume the hypotheses of Theorem* 1.5(a) *hold. Then for* $T > 0$ *and* $v > 1$,

$$P\Big[\sup_{0 \leq t \leq T} |I_2(x, \gamma, t)|^{\alpha} > v^A \ i.a.u.s.\Big] = 0, \tag{5.1}$$

for sufficiently small $x > 1$.

PROOF. In order to apply the Borel–Cantelli (B–C) lemma, we reduce the problem to one where γ takes values in a suitable discrete set. For $\gamma \geq 1$, define $\lambda := \lambda(\gamma) := \exp(\lfloor \log \gamma \rfloor)$, so $\lambda \leq \gamma < e\lambda$. Next, define $s := s(\gamma, t) := \lambda^{-1}\gamma t$ and $z := z(\gamma, x) := (\lambda^{-1}\gamma x^{\beta(\gamma)})^{\alpha(\lambda)}$. A change of variable gives us

$$\begin{aligned} I_2(x, \gamma, t) &= \gamma^{-A}\lambda^A I_2(z, \lambda, s) \\ &= \gamma^{-A}\lambda^A I_2(x, \lambda, s) + \int_{\lambda x^{\beta(\lambda)}}^{\lambda z^{\beta(\lambda)}} \gamma^{-A} y^A [\Phi([0, \lambda s], dy) - \lfloor \lambda s \rfloor \nu(dy)]. \end{aligned}$$

Since β is nondecreasing and $C := 2 \limsup \alpha(\lambda)\beta(\gamma) < \infty$ by (1.20), we have $x \leq z(\gamma, x) \leq x^C$ for large γ. Also, $(\lambda\gamma^{-1})^{\alpha(\lambda)} \to 1$ and $s \leq et$. Therefore, it suffices to show that for all $T > 0$ and $v > 1$, the following hold for sufficiently small $x > 1$,

$$P\Big[\sup_{0 \leq t \leq eT} |I_2(x, \lambda, s)|^{\alpha(\lambda)} > v^A \text{ i.o.}\Big] = 0, \tag{5.2}$$

$$P\bigg[\bigg(\int_{\lambda x^{\beta(\lambda)}}^{\lambda z^{\beta(\lambda)}} \lambda^{-A} y^A \Phi([0, \lambda eT], dy)\bigg)^{\alpha(\lambda)} > v^A \text{ i.o.}\bigg] = 0, \tag{5.3}$$

$$\bigg(\int_{\lambda x^{\beta(\lambda)}}^{\lambda z^{\beta(\lambda)}} \lambda^{1-A} y^A eT\nu(dy)\bigg)^{\alpha(\lambda)} \leq v^A \quad \text{eventually.} \tag{5.4}$$

Now choose $\delta > 0$ such that (1.21) holds. By (4.7),

$$P\Big[\sup_{0 \leq t \leq T} |I_2(x, \lambda, t)|^{\alpha(\lambda)} > v^A\Big] < \delta^{\beta(\lambda)}$$

for large λ. By (1.21) and the B–C lemma, we have (5.2). Similarly, with $x^C < v$, (5.3) follows from

$$\begin{aligned} &P\bigg[\int_{\lambda x^{\beta(\lambda)}}^{\lambda z^{\beta(\lambda)}} \lambda^{-A} y^A \Phi([0, \lambda eT], dy) > v^{A\beta(\lambda)}\bigg] \\ &\quad \leq P\big[x^{CA\beta(\lambda)}\Phi([0, \lambda e^T] \times [\lambda x^{\beta(\lambda)}, \infty]) \geq v^{A\beta(\lambda)}\big] \\ &\quad < \delta^{\beta(\lambda)} \end{aligned}$$

for large λ, by (3.6). Also, (1.7) implies (5.4) for $x^C < v$,

$$\begin{aligned} &\bigg(\int_{\lambda x^{\beta(\lambda)}}^{\lambda z^{\beta(\lambda)}} \lambda^{1-A} y^A eT\nu(dy)\bigg)^{\alpha(\lambda)} \\ &\quad \leq \bigg(\int_{\lambda x^{\beta(\lambda)}}^{\infty} \lambda^{1-A} (\lambda z^{\beta(\lambda)})^A eT\nu(dy)\bigg)^{\alpha(\lambda)} \\ &\quad = (\lambda z^{A\beta(\lambda)} eT\nu(\lambda x^{\beta(\lambda)}, \infty))^{\alpha(\lambda)} \\ &\quad \to z^A x^{-1} < x^{CA-1} < v^A. \end{aligned}$$

□

LEMMA 5.2. *For all* $\rho > 0$,

$$P[d'((\hat{S}_\gamma \vee 1)^\alpha, (I_1(x, \gamma, \cdot) \vee 1)^\alpha) > \rho \ i.a.u.s.] = 0$$

for sufficiently small $x > 1$.

The proof follows from Lemma 5.1 and (4.5) and (4.6) by an argument like the proof of Lemma 4.4.

Since $(I_1(x, \gamma, \cdot) \vee 1)^\alpha$ is always in the compact set $D^\uparrow$, Lemma 5.2 implies Theorem 1.5(a). We now prove Theorem 1.5(b). Let

$$K^* := \{\xi \in D^\uparrow \colon R(\xi) \le 1\} = \left\{\xi \in D^\uparrow \colon \sum_{x \in \mathrm{Range}(\xi)} \log x \le A\right\}.$$

Then K^* is closed in $D^\uparrow$ and hence is compact. For our Strassen upper bound, we prove that for any open $G \supset K^*$, $(\hat{S}_\gamma \vee 1)^\alpha$ is wp1 eventually in G. By Theorem 1.5(a) and the compactness of $D^\uparrow$, it suffices to prove that for each $\xi \in D^\uparrow \backslash K^*$, there is a neighborhood N of ξ such that

$$P[(\hat{S}_\gamma \vee 1)^\alpha \in N \text{ i.a.u.s.}] = 0 \tag{5.5}$$

Assume that $\xi \in D^\uparrow \backslash K^*$, and for now assume also that ξ is everywhere finite. By Lemma 5.2, it suffices to prove that for some $T > 0$ and $\rho > 0$,

$$P[E_0(\gamma) \text{ i.a.u.s.}] = 0 \tag{5.6}$$

for sufficiently small $x > 1$, where $E_0(\gamma)$ is the event defined in (4.15). Following the discussion below (4.15), there exist $m > 0, t_1, \dots, t_m, x_1, \dots, x_m, r, \delta$ and so on, satisfying the conditions given there, with the extra condition,

$$\sum_{i=1}^m \log(x_i - r\varepsilon) > A. \tag{5.7}$$

By (4.22) and (3.6), with $\lambda = e^k$ for some $k \in \{0, 1, \dots\}$ and any $\zeta > 0$ we have

$$\begin{aligned} &P[E_0(\gamma) \cap E_2(\gamma) \text{ for some } \gamma \in [\lambda, e\lambda)] \\ &\quad \le \sum_{i=1}^m P[\Phi([0, \gamma(t_i + \rho)] \times [\gamma x^{\beta(\gamma)}, \infty]) > p_i^{\beta(\gamma)} \text{ for some } \gamma \in [\lambda, e\lambda)] \\ &\quad \le \sum_{i=1}^m P[\Phi([0, e\lambda(t_i + \rho)] \times [\lambda x^{\beta(\lambda)}, \infty]) > p_i^{\beta(\lambda)}] \\ &\quad < m\zeta^{\beta(\lambda)} \end{aligned}$$

eventually. By (1.21) and the B–C lemma, $P[E_0(\gamma) \cap E_2(\gamma) \text{ i.a.u.s.}] = 0$. Therefore,

$$P[E_0(\gamma) \text{ i.a.u.s.}] \le P[E_3(\gamma) \text{ i.a.u.s.}].$$

By (5.7) and Theorem 3.5, $P[E_3(\gamma) \text{ i.a.u.s.}] = 0$. The proof of (5.5) for the case where $\xi(t) = \infty$ for some $t \in [0, \infty)$ is similar and is omitted [but see near (4.25)]. This proves the Strassen upper bound.

We now prove the Strassen lower bound. By analogy with the proof of the lower bound in Section 4, it suffices to prove that for $\sum_{i=1}^{m} \log x_i < A$, P[(4.30) and (4.31) hold for γ i.a.u.s.] = 1. This follows from Theorem 3.5. □

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