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THE SECOND LOWEST EXTREMAL INVARIANT MEASURE OF THE CONTACT PROCESS II

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We continue the investigation of the behavior of the contact process on infinite connected graphs of bounded degree. Some questions left open by Salzano and Schonmann (1997) concerning the notions of complete convergence, partial convergence and the criterion $r = s$ are answered.

The continuity properties of the survival probability and the recurrence probability are studied. These order parameters are found to have a richer behavior than expected, with the possibility of the survival probability being discontinuous at or above the threshold for survival. A condition which guarantees the continuity of the survival probability above the survival point is introduced and exploited. The recurrence probability is shown to always be left-continuous above the recurrence point, and a necessary and sufficient condition for its right-continuity is introduced and exploited. It is shown that for homogeneous graphs the survival probability can only be discontinuous at the survival point, and the recurrence probability can only be discontinuous at the recurrence point.

For graphs which are obtained by joining a finite number of severed homogeneous trees by means of a finite number of vertices and edges, the survival point, the recurrence point and the discontinuity points of the survival and recurrence probabilities are located.

1. Introduction.

1.1. *Preliminaries.* This paper is a continuation of Salzano and Schonmann (1997), to which we refer from this point on as Part I. In the next paragraphs we summarize the main content of this paper, using terminology from Part I (to be reviewed in the next subsection).

In Section 2 we will answer some questions left open in Part I. We will see that above the recurrence point the contact process may still fail to satisfy partial convergence. We will see that the criterion $r = s$, which implies that there are at most two extremal invariant distributions, may hold in situations in which complete convergence fails. And we will also find the survival and recurrence points of the graph called “basic example” in Part I (and do the same for a natural class of graphs which generalize that one).

In Section 3 we will study the continuity properties of the order parameters of the contact process on a graph (by “order parameters” we mean the

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probabilities of survival and of recurrence). We will show that the survival probability may be discontinuous at or above the survival point. This contradicts Conjecture 3 of Pemantle (1992). [Conjecture 1 of the same paper has already been contradicted by results in Part I, and Conjecture 2 there has been contradicted by Pemantle and Stacey (1999).] We will also present positive results, including a useful condition which guarantees the continuity of the survival probability above the survival point and which can be used to show that the basic example of Part I has a continuous survival probability everywhere. The recurrence probability will be shown to always be left-continuous above the recurrence point, and a necessary and sufficient condition for its right-continuity will also be presented. When applied to the basic example of Part I, this condition will give us the location of the discontinuity points of the recurrence probability.

As in Part I, our examples in this paper will be contact processes on trees, but our positive results will refer to very general graphs.

For a detailed motivation to the problems treated in Part I and here, the reader is referred to the introduction of Part I. Nevertheless, for the readers benefit, we present in the next subsection a summary of the basic definitions, notation, background and results from Part I, along with some introductory material not contained in Part I.

1.2. *Notation and background.* This section can be seen as a summary. Readers who need more details are invited to read the introduction of Part I and consult Liggett (1985) and Durrett (1988).

The graphs. We will denote by $\mathscr{G}$ the class of infinite connected graphs of bounded degree. For a graph $G \in \mathscr{G}$ we denote by $\mathscr{V}_G$ its set of vertices, also called sites, and we denote by $\mathscr{E}_G$ its set of edges; note that $\mathscr{V}_G$ and $\mathscr{E}_G$ are both countable sets. One of the sites of G will be distinguished from the others and called its root, denoted by 0. The distance between two sites x and y in $\mathscr{V}_G$ is the length of the shortest path along neighboring sites which join x to y and will be denoted by $\operatorname{dist}(x, y)$. The ball of center $x \in \mathscr{V}_G$ and radius N will be denoted by $B(x, N)$. The notation $A \Subset \mathscr{V}_G$ will mean that A is a finite subset of $\mathscr{V}_G$.

A graph is said to be homogeneous if for each pair x and y of its vertices there is an automorphism of the graph which maps x into y (most authors use the term "transitive graphs"). The class of homogeneous graphs in $\mathscr{G}$ will be denoted by $\mathscr{H}$. Everything that we say about homogeneous graphs in this paper applies also (with essentially the same proofs) to the larger class of quasi-transitive graphs, defined as those graphs in $\mathscr{G}$ for which there is a finite set of vertices, V_0, with the property that each vertex of the graph can be mapped into one of the vertices of V_0 by an automorphism.

Trees will play an important role in this paper. The sites which are at distance n from the root are said to be in generation n. A site y is said to be a descendent of a site $x \neq 0$ of a tree G, if the only path along neighboring sites which joins 0 to y passes through x. A tree is said to be spherically symmetric, with branching numbers $\{d_n\}_{n=0,1,\dots}$, if for each $n \geq 0$, each site in generation n has $d_n + 1$ neighbors.

The configurations. We think of configurations either as elements of $\{0, 1\}^{\mathscr{V}_G}$ or as subsets of $\mathscr{V}_G$. These two sets are identified in the usual way and endowed with metric, a σ-field and partial order also in the usual way.

Probability distributions on the configuration space are determined by their finite dimensional distributions, and weak convergence, denoted by $\Rightarrow$, is equivalent to the convergence of these finite-dimensional distributions. The probability measure which puts all mass on the configuration η will be denoted by δ_η.

The contact process. We will denote by $(\xi^\mu_{G,\lambda;t}\colon t \geq 0)$ the contact process on the graph $G \in \mathscr{G}$, with infection parameter $\lambda > 0$, started from a configuration which is randomly chosen according to the law μ. Usually G and λ will be omitted from the notation. When μ is concentrated on the configuration η we write simply $(\xi^\eta_t\colon t \geq 0)$. We also write $(\xi^x_t\colon t \geq 0)$ for the contact process started from a single particle at $x \in \mathscr{V}_G$. Similar notational conventions will be used systematically without further notice.

We suppose that the contact process on the graph $G \in \mathscr{G}$, with infection parameter $\lambda > 0$ is constructed in the usual graphical additive fashion, by means of Poisson death marks (at rate 1 for each site in $\mathscr{V}_G$) and Poisson arrows [at rate λ for each oriented edge (x, y) such that $\{x, y\} \in \mathscr{E}_G$]. In order to use ergodicity we think of the graphical construction as being made on $G \times \mathbb{R}$, rather than just on $G \times \mathbb{R}_+$. We will use $\mathbb{P}_{G,\lambda} = \mathbb{P}$ to denote the probability measure corresponding to the graphical construction. Given two space–time points, $(x, s), (y, u) \in \mathscr{V}_G \times \mathbb{R}$, with $s < u$, we say that there is a path from (x, s) to (y, u) if there is a sequence of times $s = t_0 < t_1 < \cdots < t_n < t_{n+1} = u$ and spatial locations $x = x_0, x_1, \ldots, x_n = y$ so that for $i = 1, 2, \ldots, n$ there is an arrow from x_{i-1} to x_i at time t_i and the vertical segments $\{x_i\} \times (t_i, t_{i+1})$ for $i = 0, 1, \ldots, n$ do not contain any death mark. Given $A \subset \mathscr{V}_G$ and $s \leq u$, we set

$$\xi^{A,s}_u = \{y \in \mathscr{V}_G\colon \text{there is a path from } (x, s) \text{ to } (y, u) \text{ for some } x \in A\}.$$

Thus $\{\xi^{A,s}_{s+t}\colon t \geq 0\}$ is a version of the contact process started from A. We abbreviate $\xi^{A,0}_t = \xi^A_t$ and note that this is consistent with the notation introduced before. We will also need to use dual processes, and for this purpose, given $A \subset \mathscr{V}_G$ and $s \leq u$, we set

$$\hat{\xi}^{A,u}_s = \{x \in \mathscr{V}_G\colon \text{there is a path from } (x, s) \text{ to } (y, u) \text{ for some } y \in A\}.$$

Note that $\{\hat{\xi}^{A,u}_{u-t}\colon t \geq 0\}$ has the same law as $\{\xi^{A,u}_{u+t}\colon t \geq 0\}$, and therefore it is also a version of the contact process started from A.

For any $A \subset \mathscr{V}_G$, define

$$\Omega^A_\infty = \{\xi^A_t \neq \varnothing \text{ for all } t \geq 0\},$$
$$\Omega^A_r = \{\xi^A_t(0) = 1, \text{ for a unbounded set of values of } t\}.$$

The two corresponding order parameters are:

$$\rho(A, \lambda) = \rho_G(A, \lambda) = \mathbb{P}(\Omega^A_\infty),$$

and

$$\beta(A, \lambda) = \beta_G(A, \lambda) = \mathbb{P}(\Omega_r^A).$$

When the argument A is omitted in the functions β and ρ, it should be understood that $A = \{0\}$.

Next we define the corresponding critical points,

$$\lambda_s = \lambda_s(G) = \inf\{\lambda\colon \rho(\lambda) > 0\}$$

and

$$\lambda_r = \lambda_r(G) = \inf\{\lambda\colon \beta(\lambda) > 0\}.$$

These are, respectively, called the survival point and the recurrence point of the graph G (they clearly do not depend on the choice of the root of G).

The ergodic behavior of the contact process. For fixed $t \geq 0$, the law of ξ_t^μ will be denoted by $\mu_t = \mu S(t) = \mu S_{G,\lambda}(t)$. The set of invariant probability measures will be denoted by $\mathscr{I} = \{\mu\colon \mu_t = \mu \text{ for all } t \geq 0\}$. This is a convex set, and the set of its extremal points will be denoted by $\mathscr{I}_e$.

It is obvious that $\delta_\varnothing \in \mathscr{I}_e$, regardless of the value of λ. As $t \to \infty$, $\delta_{\mathscr{V}_G} S(t) \Rightarrow \bar{\nu} \in \mathscr{I}_e$. Having $\delta_\varnothing = \bar{\nu}$ is equivalent to having $\mu S(t) \Rightarrow \delta_\varnothing$ for all laws μ; the process is in this case said to be ergodic. If this happens, in particular, $\mathscr{I} = \{\delta_\varnothing\}$.

The self-duality of the contact process implies that

$$\bar{\nu}(\zeta\colon \zeta \cap A \neq \varnothing) = \mathbb{P}(\Omega_\infty^A) = \rho(A, \lambda).$$

Motivated by this, we introduced in Part I the probability distribution ν_r, defined by

$$\nu_r(\zeta\colon \zeta \cap A \neq \varnothing) = \mathbb{P}(\Omega_r^A) = \beta(A, \lambda).$$

While $\delta_\varnothing$ and $\bar{\nu}$ are, respectively, the smallest and the largest invariant distributions in the usual sense of stochastic order, Theorem 1(b), (c) in Part I gives that ν_r is in a sense the second lowest extremal invariant distribution. More precisely, $\nu_r \in \mathscr{I}_e$, and for every $\mu \in \mathscr{I}$ such that $\mu \perp \delta_\varnothing$ the following order relation holds:

$$\text{for every } A \Subset \mathscr{V}_G,\ \ \nu_r(\zeta\colon \zeta \cap A \neq \varnothing) \leq \mu(\zeta\colon \zeta \cap A \neq \varnothing);$$

in particular this is the case for all $\mu \in \mathscr{I}_e \setminus \{\delta_\varnothing\}$.

When $\beta(A, \lambda) = 0$, we have $\nu_r = \delta_\varnothing$, but otherwise ν_r can be seen as the lowest nontrivial extremal invariant measure of the contact process. Clearly $\beta(A, \lambda) \leq \rho(A, \lambda)$, so that the opposite extremal possibility is that the following equivalent statements hold.

CRITERION $r = s$.

$$\nu_r = \bar{\nu}.$$

Or equivalently,

$$\text{for any } A \Subset \mathscr{V}_G,\ \ \beta(A) = \rho(A).$$

Or still equivalently,

$$\text{for some nonempty } A \Subset \mathcal{V}_G,\ \beta(A) = \rho(A).$$

Theorem 1(d) in Part I states that if the criterion $r = s$ is satisfied, then $\mathcal{I}_e = \{\delta_\varnothing, \overline{\nu}\}$.

A classical notion in the study of the contact process is the following.

COMPLETE CONVERGENCE (*cc*).

$$\text{For any } A \Subset \mathcal{V}_G,\ \xi_t^A \Rightarrow (1 - \rho(A))\delta_\varnothing + \rho(A)\overline{\nu} \quad \text{as } t \to \infty.$$

Or equivalently,

$$\text{for any } A, B \Subset \mathcal{V}_G,\ \mathbb{P}(\xi_t^A \cap B \neq \varnothing) \to \rho(A)\rho(B) \quad \text{as } t \to \infty.$$

We introduce also the notation *s*&*cc* (for "survival with complete convergence") to denote the statement that not only *cc* holds, but also $\rho(\lambda) > 0$.

Motivated by the definition of *cc*, we introduced in Part I the following similar notion.

PARTIAL CONVERGENCE (*pc*).

$$\text{For any } A \Subset \mathcal{V}_G,\ \xi_t^A \Rightarrow (1 - \beta(A))\delta_\varnothing + \beta(A)\nu_r \quad \text{as } t \to \infty.$$

Or equivalently,

$$\text{for any } A, B \Subset \mathcal{V}_G,\ \mathbb{P}(\xi_t^A \cap B \neq \varnothing) \to \beta(A)\beta(B) \quad \text{as } t \to \infty.$$

In analogy with *s*&*cc*, we define *r*&*pc* (for "recurrence with partial convergence") as the property that *pc* holds and $\beta(\lambda) > 0$.

Theorem 2 in Part I contains a number of results concerning $r = s$, *cc*, *s*&*cc*, *pc* and *r*&*pc*, including properties of these notions, their consequences regarding the ergodic behavior of the contact process and tools to check whether they hold or not for a given graph $G \in \mathcal{G}$ at a given value of λ. Before we can summarize some of these results, we need to introduce the following terminology.

MONOTONE INCREASING PROPERTY. A property of the contact process is said to be monotone increasing when both of the following hold.

(a) If the property holds for the contact process on a graph $G \in \mathcal{G}$ at some λ, then it also holds for the same graph for all $\lambda' > \lambda$.

(b) If the property holds for the contact process on some subgraph $G_0 \in \mathcal{G}$ of some graph $G \in \mathcal{G}$ at some value of λ, then it also holds for G at the same λ.

We will also say that a property is $\mathcal{F}$-monotone increasing, for some family of graphs $\mathcal{F} \subset \mathcal{G}$, in case the statements in the definition above are true when $\mathcal{G}$ is replaced by $\mathcal{F}$ in each place where it appears in the definition.

The following results were proven in Part I: Statements (2)–(6) are parts of Theorem 2 and Statement (7) is Theorem 5.

1. The property $s\&cc$ is not monotone increasing.
2. The property $r\&pc$ is monotone increasing.
3. If cc holds, then

$$\text{for any } \eta \subset \mathscr{V}_G,\ \xi_t^\eta \Rightarrow (1-\beta(\eta))\delta_\varnothing + \beta(\eta)\bar{\nu} \quad \text{as } t \to \infty.$$

   Or equivalently,

   for any $\eta \subset \mathscr{V}_G$ and any finite $B \subset \mathscr{V}_G$, $\mathbb{P}(\xi_t^\eta \cap B \neq \varnothing) \to \beta(\eta)\rho(B)$ as $t \to \infty$.

4. Property cc is equivalent to having simultaneously pc and $r = s$. In particular, if $s\&cc$ holds, then $r\&pc$ also holds.
5. If $r = s$ fails, then there is a configuration η with infinitely many particles for which ξ_t^η does not converge weakly as $t \to \infty$. By the previous statement, this happens in particular if pc holds but cc fails.
6. If $G \in \mathscr{H}$, then whenever $\beta(\lambda) > 0$, the criterion $r = s$ is satisfied. In particular, we have the following for homogeneous graphs: (a) $r\&pc$ is equivalent to $s\&cc$, (b) $s\&cc$ is a $\mathscr{H}$-monotone increasing property, (c) ν_r coincides with $\delta_\varnothing$ when $\beta(\lambda) = 0$ and with $\bar{\nu}$ when $\beta(\lambda) > 0$, (d) if $\beta(\lambda) > 0$, then $\mathscr{I}_e = \{\delta_\varnothing, \bar{\nu}\}$.
7. For every graph $G \in \mathscr{G}$, $s\&cc$ holds for $\lambda > \lambda_c(\mathbb{Z})$, where $\lambda_c(\mathbb{Z})$ denotes the common value of $\lambda_s(\mathbb{Z})$ and $\lambda_r(\mathbb{Z})$.

The contact process on homogeneous trees. We denote by $\mathbb{T}_d$ the homogeneous tree of degree $d+1$. The case $d = 1$ corresponds to the linear chain $\mathbb{Z}$. We immerse the graph $\mathbb{Z}^+$, which has vertices $\{0, 1, 2, \ldots\}$ and edges connecting points which differ by 1 unit, into $\mathbb{T}_d$, in an arbitrary fashion. This allows us to refer to the sites $0, 1, 2, \ldots$ of $\mathbb{T}_d$. As said before, the site 0 of $\mathbb{T}_d$ is called its root.

An important subgraph of $\mathbb{T}_d$ is obtained from this tree by removing one of the neighbors of the root and defining it as the remaining connected component which contains the root. We will suppose that the removed vertex is not the vertex 1, so that the set of sites $\{0, 1, 2, \ldots\}$ is contained in the set of vertices of the new graph. This new graph is called the severed homogeneous tree of degree $d+1$ and will be denoted $\mathbb{T}_d^+$; its root will be chosen as the vertex 0. Note that the root of $\mathbb{T}_d^+$ is the only vertex of this graph which has d neighbors; the others all have $d+1$ neighbors.

The contact process on $\mathbb{T}_d$ and also on $\mathbb{T}_d^+$ is an object of great current interest [see Pemantle (1992), Madras and Schinazi (1992), Morrow, Schinazi and Zhang (1994), Durrett and Schinazi (1995), Wu (1995), Zhang (1996), Liggett (1996a), Stacey (1996), Liggett (1996b), Lalley and Sellke (1998), Salzano and Schonmann (1998), Lalley (1999) and Schonmann (1998)]. Supposing that $d > 1$, to exclude the case of the linear chain, it is known that $0 < \lambda_s(\mathbb{T}_d) = \lambda_s(\mathbb{T}_d^+) < \lambda_r(\mathbb{T}_d) = \lambda_r(\mathbb{T}_d^+) < \infty$. These two critical points give rise to three distinct phases with the following features (what we say holds on $\mathbb{T}_d$ and on

$\mathbb{T}_d^+$): for $\lambda \le \lambda_s$ the process is ergodic; for $\lambda_s < \lambda \le \lambda_r$ there are infinitely many measures in $\mathscr{I}_e$, but $\beta(\lambda) = 0$, so that if the process is started from a finite set $A \subset \mathscr{V}_G$ then $\xi_t^A \Rightarrow \delta_\varnothing$; finally for $\lambda > \lambda_r$ there are exactly two extremal invariant measures, $\delta_\varnothing$ and $\bar{\nu}$, and *cc* holds (in particular $r = s$ holds).

The objects that we discuss next will play a major role in this paper, as they did in Liggett (1996a), Lalley and Sellke (1998), Salzano and Schonmann (1998) and Lalley (1999). First set

$$u_{d,n}(\lambda) = u_n = \mathbb{P}(\xi^0_{\mathbb{T}_d;t}(n) = 1 \text{ for some } t \ge 0).$$

From the inequality $u_{n+m} \ge u_n u_m$, it follows that

$$\lim_{n\to\infty} (u_n)^{1/n} = \alpha = \alpha_d = \alpha_d(\lambda) = \sup\{(u_n)^{1/n}\colon n \ge 1\}. \tag{1.1}$$

[Our α was called ρ in Liggett (1996a), but this conflicts with the standard use of ρ for the survival probability; it was called β in Lalley and Sellke (1998) and in Lalley (1999), but we used β for the recurrence probability in Part I and therefore prefer to use a different notation here, as we did in Salzano and Schonmann (1998).]

From Lalley and Sellke (1998) [see also Salzano and Schonmann (1998)], we know that when $\lambda \le \lambda_r(\mathbb{T}_d)$, then $\alpha_d(\lambda) \le 1/\sqrt{d}$. From Lalley (1999), we know that $\alpha_d(\cdot)$ is strictly increasing on $(0, \lambda_r(\mathbb{T}_d)]$. Therefore,

$$\alpha_d(\lambda) < 1/\sqrt{d} \quad \text{for } \lambda < \lambda_r(\mathbb{T}_d), \tag{1.2}$$

a result which will be very useful in this paper.

2. Answer to some questions from Part I.

2.1. *Collage of graphs.* Recall that in Part I the following operation with graphs was defined. If G_1 and G_2 are two disjoint graphs, then $G_1 \vee G_2$ denotes the graph obtained by connecting their roots, or more precisely, the graph in which the set of vertices is the union of the sets of vertices of G_1 and G_2 and the set of edges is the union of the set of edges of these two graphs plus an edge connecting their roots. The "basic example" in Part I, is a graph of the type $\mathbb{T}_j \vee \mathbb{T}_k$ with $j \ge 2$ and k sufficiently larger than j, so that we have

$$\lambda_s(\mathbb{T}_k) < \lambda_r(\mathbb{T}_k) < \lambda_s(\mathbb{T}_j) < \lambda_r(\mathbb{T}_j). \tag{2.1}$$

In Part I we proved that the contact process on $\mathbb{T}_j \vee \mathbb{T}_k$ has the following features:

1. In the interval $(\lambda_s(\mathbb{T}_j), \lambda_r(\mathbb{T}_j)]$ *r&pc* holds, but *cc* fails.
2. In the intervals $(\lambda_r(\mathbb{T}_k), \lambda_s(\mathbb{T}_j)]$ and $(\lambda_r(\mathbb{T}_j), \infty)$ *s&cc* holds.

The main result in this section, Theorem 2.1.2, implies in particular that $\lambda_s(\mathbb{T}_j \vee \mathbb{T}_k) = \lambda_s(\mathbb{T}_k)$ and $\lambda_r(\mathbb{T}_j \vee \mathbb{T}_k) = \lambda_r(\mathbb{T}_k)$, solving a problem left open in Part I. Note that as a consequence, we learn that *pc* holds on $\mathbb{T}_j \vee \mathbb{T}_k$, immediately above its recurrence point.

The concept that we introduce next is a natural generalization of the operation "$\vee$". Suppose that $G_1, \ldots, G_n \in \mathscr{G}$ are disjoint graphs. We say that $G \in \mathscr{G}$ is a collage of $G_1, \ldots, G_n$ if the following conditions are satisfied: (1) the set of vertices of G is $\mathscr{V}_G = (\bigcup_{i=1,\ldots,n} \mathscr{V}_{G_i}) \cup V_0$, where V_0 is a finite set (disjoint from $\bigcup_{i=1,\ldots,n} \mathscr{V}_{G_i}$). (2) The set of edges of G is $(\bigcup_{i=1,\ldots,n} \mathscr{E}_{G_i}) \cup E_0$, where E_0 is a finite set (disjoint from $\bigcup_{i=1,\ldots,n} \mathscr{E}_{G_i}$). Note that since we suppose that $G \in \mathscr{G}$, G must be connected. In particular, if $n \geq 2$, then E_0 is not empty. We will use the notation V_{glue} for the set of vertices which are endpoints of edges in E_0.

The following theorem is a simple generalization of Theorem 6 in Part I, and it can be proved in the same way that that theorem was proved.

THEOREM 2.1.1. *If G is a collage of $G_1, \ldots, G_n$, then for each value of $\lambda > 0$ the condition $r = s$ holds for G if and only if it holds for each one of the graphs G_i, $i = 1, \ldots, n$.*

To avoid repeating what is already in Part I, we omit the formal proof of Theorem 2.1.1, but nevertheless mention the intuitive reason behind this theorem: if $r = s$ fails for one of the G_i, then the contact process on G can survive inside of G_i, without recurring, so that $r = s$ should also fail for G. On the other hand, if $r = s$ holds for all G_i, $i = 1, \ldots, n$, then we cannot have survival in one or more of the G_i without the infection recurring to each fixed site of this G_i, so that $r = s$ should also hold for G.

To explain why the next theorem indeed applies to the basic example of Part I, $\mathbb{T}_j \vee \mathbb{T}_k$, as we claimed above, note that $\mathbb{T}_d$ is a collage of two copies of $\mathbb{T}_d^+$, so that a collage of copies of $\mathbb{T}_{d_1}, \mathbb{T}_{d_2}, \ldots, \mathbb{T}_{d_m}$ fits the hypothesis of the theorem.

THEOREM 2.1.2. *Suppose that G is a collage of copies of $\mathbb{T}_{d_1}^+, \mathbb{T}_{d_2}^+, \ldots, \mathbb{T}_{d_n}^+$, and set $D = \max\{d_i : i = 1, \ldots, n\}$. Then $\lambda_s(G) = \lambda_s(\mathbb{T}_D)$ and $\lambda_r(G) = \lambda_r(\mathbb{T}_D)$. If $D \geq 2$, then also $\rho_G(\lambda_s(G)) = 0$.*

(Corollary 3.4.6 and Corollary 3.5.2 contain further information about such graphs.) The following lemma will be used in the proof of Theorem 2.1.2.

LEMMA 2.1.1. *Suppose that G is a collage of n copies of $\mathbb{T}_d^+$ and that $\lambda < \lambda_r(\mathbb{T}_d)$. Then $\beta_G(\lambda) = 0$.*

PROOF. To prove this lemma we introduce a notion of "cutting time intervals," and then use ergodicity to show that such "cutting time intervals" will almost surely be present.

We will use the notation in the definition of "collage" of graphs, so that, in particular, G_i, $i = 1, \ldots, n$ are disjoint subgraphs of G and each one is isomorphic to T_d^+. Let 0_i, $i = 1, \ldots, n$, be the roots of these graphs. With no loss in generality, we will suppose that $n \geq 1$ (to assure that $V_{\text{glue}} \neq \varnothing$), that $V_{\text{glue}} \cap \mathscr{V}_{G_i} = 0_i$, $i = 1, \ldots, n$ (this can always be obtained by enlarging V_0 and n), and that $0 \in V_{\text{glue}}$.

For each vertex x which belongs to G_i for some i, we define for $j \in \{0, 1, \ldots\}$,

$$E_x^{\uparrow j} = \{\xi_t^{j+1/2, x}(0_i) = 1 \text{ for some } t \geq j + \tfrac{1}{2}\},$$

$$E_x^{\downarrow j} = \{\hat{\xi}_t^{j+1/2, x}(0_i) = 1 \text{ for some } t \leq j + \tfrac{1}{2}\},$$

$$E_x^j = E_x^{\uparrow j} \cap E_x^{\downarrow j}.$$

For $j \in \{0, 1, \ldots\}$, the time interval $[j, j+1]$ is called a cutting time interval in case during this time interval, in the graphical construction, there is a death mark at each site in V_0, no arrows have endpoints at these sites and the event $\bigcap_{i=1}^n \bigcap_{x \in G_i} (E_x^j)^c$ happens.

It is clear that if $[j, j+1]$ is a cutting time interval, then $\xi_t^0(0) = 0$ for all $t \geq j+1$. Therefore, if there is any cutting time interval then $(\Omega_r^0)^c$ happens. By ergodicity of the Poisson processes in the graphical construction, the proof that $\beta_G(\lambda) = 0$ is reduced to showing that

$$\mathbb{P}([0, 1] \text{ is a cutting time interval}) > 0. \tag{2.2}$$

This can be done as follows. Note that if $x \in G_i$,

$$\mathbb{P}(E_x^0) = \mathbb{P}(E_x^{\uparrow 0})\mathbb{P}(E_x^{\downarrow 0}) = (u(\mathrm{dist}(x, 0_i)))^2 \leq (\alpha_d(\lambda))^{2\,\mathrm{dist}(x, 0_i)}.$$

Since $\alpha_d(\lambda) < d^{-1/2}$, by (1.2), we have now $\sum_{i=1}^n \sum_{x \in G_i} \mathbb{P}(E_x^0) < \infty$. But the events $(E_x^0)^c$, as well as the finitely many events involving the sites in V_0 which must happen for $[0, 1]$ to be a cutting time interval are all decreasing events. Therefore (2.2) follows from Lemma 3.1(a) in Madras, Schinazi and Schonmann (1994). $\square$

Proof of Theorem 2.1.2. Obviously $\lambda_s(G) \leq \lambda_s(\mathbb{T}_D)$ and $\lambda_r(G) \leq \lambda_r(\mathbb{T}_D)$.

Since G is a subgraph of a collage of n copies of $\mathbb{T}_D^+$, Lemma 2.1.1 implies that

$$\beta_G(\lambda) = 0 \quad \text{for } \lambda < \lambda_r(\mathbb{T}_D). \tag{2.3}$$

The proof that $\lambda_r(G) = \lambda_r(\mathbb{T}_D)$ is therefore complete.

If $\lambda \leq \lambda_s(\mathbb{T}_D)$, then the contact process on each $\mathbb{T}_{d_i}^+$, $i = 1, \ldots, n$ dies out, and hence satisfies $r = s$. Theorem 2.1.1 then gives that $r = s$ also holds for the contact process on G. But since $\lambda_s(\mathbb{T}_D) \leq \lambda_r(\mathbb{T}_D)$ [resp. $\lambda_s(\mathbb{T}_D) < \lambda_r(\mathbb{T}_D)$ in case $D \geq 2$], (2.3) implies now that for $\lambda < \lambda_s(\mathbb{T}_D)$ [resp. $\lambda \leq \lambda_s(\mathbb{T}_D)$ in case $D \geq 2$] we have $\rho_G(\lambda) = \beta_G(\lambda) = 0$. This completes the proof. $\square$

The condition that $D \geq 2$, needed for the final conclusion in Theorem 2.1.2, may be just technical. The case which has been excluded is that in which G is a collage of copies of $\mathbb{T}_1^+ = \mathbb{Z}_+$. From the part of Theorem 2.1.2 which can still be applied in this case, we know that $\lambda_s(G) = \lambda_r(G) = \lambda_c(\mathbb{Z})$. The open question is then whether $\rho_G(\lambda_s(G)) = 0$ or not. Surprisingly enough, even the following apparently much simpler question seems also to be open. Consider the graph G' obtained by starting with $\mathbb{Z}$ and adding to it a unique extra site and a unique extra edge connecting this extra site to the origin of $\mathbb{Z}$. Clearly $\lambda_s(G') = \lambda_r(G') = \lambda_c(\mathbb{Z})$ (since $\mathbb{Z}$ is a subgraph of G', which is a subgraph of a collage of three copies of $\mathbb{Z}_+$). But is it the case that $\rho_{G'}(\lambda_s(G')) = 0$ or not? This question is akin to Open Problem 1 in Madras, Schinazi and Schonmann (1994).

2.2. *Recurrence without pc, and $r = s$ without cc.* In this subsection we will answer Questions 5 and 6 in Part I. First we recall what these questions are and why they are relevant. From Theorem 2(f) of Part I, we know that *cc* is equivalent to having simultaneously *pc* and $r = s$. The basic example of Part I provides us with a case in which, for some values of λ, we have *pc*, while *cc* fails (see the beginning of the last subsection). It is natural to ask then the following question.

(Q5) Are there examples in which $r = s$ is satisfied, but *cc* is not?

From Theorem 2(c) of Part I, we know that $r\&pc$ is a monotone increasing property. It is natural then to define

$$\lambda_{r\&pc} = \inf\{\lambda\colon r\&pc \text{ holds}\}.$$

It is equally natural to ask the following:

(Q6) Is it always the case that $\lambda_r = \lambda_{r\&pc}$?

As we proved in the previous subsection, this equality of critical points holds for the basic example of Part I.

Here we will prove the following theorem.

THEOREM 2.2.1. *There exists a spherically symmetric tree for which there is a non-degenerate interval of values of λ on which*:

(a) $\liminf_{t\to\infty} \mathbb{P}(\xi_t^0(0) = 1) = 0$.

(b) *The contact process survives and $r = s$ holds. In other words*, $\rho(\lambda) = \beta(\lambda) > 0$.

(c) $\limsup_{t\to\infty} \mathbb{P}(\xi_t^0(0) = 1) > 0$.

Note that under the conditions of this theorem, *pc* fails, since under *pc* the limit in (a) would be $\big(\beta(\lambda)\big)^2$, which is positive by (b). Therefore the answer to (Q6) is negative. Since *pc* fails, also *cc* fails. Therefore the answer to (Q5) is positive.

Note that from (a) and (c) we know that the law of ξ_t^0 does not converge as $t \to \infty$. Nevertheless the set of invariant distributions for this contact process is very simple for the values of λ being considered. Since we know that $r = s$ holds, Theorem 1(d) in Part I tells us that there are exactly two extremal invariant distributions: $\mathscr{I}_e = \{\delta_\varnothing, \bar{\nu}\}$.

PROOF. To construct the required spherically symmetric tree, G, we take two numbers $d < D$, so that $\lambda_r(\mathbb{T}_D) < \lambda_s(\mathbb{T}_d)$. As in Part I, we observe that this is possible, because $\lambda_r(\mathbb{T}_D) \to 0$ as $D \to \infty$, by the estimates in Pemantle (1992). Fix a sequence $0 = N_0 < N_1 < N_2 < \cdots$, to be specified later. Our spherically symmetric tree is defined by the branching numbers $\{d_i\}_{i=0,1,\ldots}$, given by

$$d_i = \begin{cases} D, & \text{if } N_{2j} \le i < N_{2j+1} \text{ for some } j \in \{0, 1, \ldots\}, \\ d, & \text{otherwise.} \end{cases}$$

The interval of values of λ in the statement of the proposition is $(\lambda_s(\mathbb{T}_d), \lambda_r(\mathbb{T}_d))$.

Two sequences of spherically symmetric trees, which may be thought of as approximating our G from opposite sides will be introduced next.

For $j = 0, 1, \ldots$, we define $G^{(2j)}$ as the spherically symmetric tree which has the same branching numbers d_i as G has, for $i < N_{2j}$, and for $i \ge N_{2j}$ has $d_i = d$.

For $j = 0, 1, \ldots$, we define $G^{(2j+1)}$ as the spherically symmetric tree which has the same branching numbers d_i as G has, for $i < N_{2j+1}$ and for $i \ge N_{2j+1}$ has $d_i = D$.

For $\lambda \in (\lambda_s(\mathbb{T}_d), \lambda_r(\mathbb{T}_d))$, the contact process on each graph $G^{(2j+1)}$, $j = 0, 1, \ldots$ satisfies *cc*. To see this, first recall that *cc* is equivalent to having *pc* and $r = s$ [Theorem 2(f) of Part I]. Recall also that since $\lambda > \lambda_r(\mathbb{T}_D)$, *cc* holds on $\mathbb{T}_D^+$ (Theorem 4 of Part I), so both $r\&pc$ and $r = s$ hold on $\mathbb{T}_D^+$. Observe now that the graphs $G^{(2j+1)}$ are collages of copies of $\mathbb{T}_D^+$. Therefore $r = s$ for the contact process on each such graph by Theorem 2.1.1. On the other hand, since $r\&pc$ is a monotone increasing property [Theorem 2(c) of Part I], it must also be satisfied by the contact process on the graph $G^{(2j+1)}$, which has $\mathbb{T}_D^+$ as a subgraph.

We will explain below how the sequence $0 = N_0 < N_1 < N_2 < \cdots$, and a sequence of times, $t_1 < t_2 < \cdots$ can be chosen so that

$$\lim_{j\to\infty} \mathbb{P}(\xi^0_{t_{2j}}(0) = 1) = 0 \tag{2.4}$$

and

$$\inf_j \inf_{x:\, \mathrm{dist}(0,x) < N_{2j}} \mathbb{P}(\xi^x_{t_{2j+1}}(0) = 1) \ge \tfrac{1}{2}\big(\rho_{\mathbb{T}_d^+}\big)^2 > 0, \tag{2.5}$$

in particular

$$\liminf_{j\to\infty} \mathbb{P}(\xi^0_{t_{2j+1}}(0) = 1) > 0. \tag{2.6}$$

Before we explain how to choose the sequences $0 = N_0 < N_1 < N_2 < \cdots$ and $t_1 < t_2 < \cdots$, we explain why (2.4), (2.5) and (2.6) solve our problem. Clearly (2.4) implies (a), and (2.6) implies (c). The statement in (b) that the contact process survives is trivial, since $\mathbb{T}_d^+$ is a subgraph of G. The statement that $r = s$ holds is a consequence of (2.5) and Theorem 3 in Part I, since for any site x,

$$\mathbb{P}(\xi^x_t(0) = 1 \text{ for some } t > 0) \ge \inf_j \inf_{x:\, \mathrm{dist}(0,x) < N_{2j}} \mathbb{P}(\xi^x_{t_{2j+1}}(0) = 1).$$

Now we return to the choice of the sequences $0 = N_0 < N_1 < N_2 < \cdots$ and $t_1 < t_2 < \cdots$. Choose t_1 and N_1 arbitrarily and proceed recursively as follows. Given $t_1, \dots, t_{2j-1}$, and $N_1, \dots, N_{2j-1}$ take t_{2j} such that

$$\mathbb{P}\big(\xi^0_{G^{(2j)};\, t_{2j}}(0) = 1\big) \le 1/j,$$

which is possible by Lemma 2.1.1, since $G^{(2j)}$ is a collage of copies of $\mathbb{T}_d^+$. Next take N_{2j} such that

$$\mathbb{P}\big(\xi^0_{G;\, t_{2j}}(0) = 1\big) \le \mathbb{P}\big(\xi^0_{G^{(2j)};\, t_{2j}}(0) = 1\big) + 1/j,$$

which is clearly possible, since t_{2j} is held fixed and G and $G^{(2j)}$ are identical up to generation N_{2j}. Therefore (2.4) is assured.

Next take t_{2j+1} such that

$$\inf_{x:\ \mathrm{dist}(0,x) < N_{2j}} \mathbb{P}\big(\xi^x_{G^{(2j+1)};\, t_{2j+1}}(0) = 1\big) \ge \tfrac{3}{4}\big(\rho_{\mathbb{T}_d^+}\big)^2,$$

which is possible because the contact process on $G^{(2j+1)}$ satisfies *cc* and the probability of survival on this graph, starting from any single occupied site is bounded below by $\rho_{\mathbb{T}_d^+}$.

Finally, take next N_{2j+1} such that

$$\inf_{x:\ \mathrm{dist}(0,x) < N_{2j}} \mathbb{P}\big(\xi^x_{G;\, t_{2j+1}}(0) = 1\big) \ge \mathbb{P}\big(\xi^x_{G^{(2j+1)};t_{2j+1}}(0) = 1\big) - \tfrac{1}{4}\big(\rho_{\mathbb{T}_d^+}\big)^2,$$

which is clearly possible, since t_{2j+1} is held fixed and G and $G^{(2j+1)}$ are identical up to generation N_{2j+1}. Thus (2.5) is also assured. $\square$

3. Continuity properties of $\rho(\cdot)$ and $\beta(\cdot)$.

3.1. *Preliminaries.* In statistical mechanics there are two distinct ways to search for "transition points." One of these relies on finding values of the parameters of the model at which the set of equilibrium distributions changes qualitatively. The other one is based on finding values of these parameters at which quantities of relevance (called "order parameters") have some sort of nonsmooth behavior, for example, a discontinuity, or a discontinuity of a derivative of some order, or just a lack of analyticity. When studying an interacting particle system like the contact process, the first of these ideas can be expanded to encompass any sort of qualitative modification in the ergodic behavior (even when the set of invariant distributions itself does not present any qualitative modification). For instance a value of the parameter λ immediately below which, say, *cc* fails, and immediately above which *cc* holds would be considered a "transition point." Regarding "order parameters" for the contact process, it is very natural to consider the behavior of the functions $\rho(\cdot)$ and $\beta(\cdot)$ and to try to locate any value of λ at which they are not "smooth." Of course, λ_s is a transition point in the sense that $\rho(\cdot)$ changes there from being 0 to being positive, and it is also a transition point in the sense that it separates the region in which there is a unique invariant distribution from the one where this is no longer the case. Similarly, λ_r is a transition point in the sense that $\beta(\cdot)$ changes there from being 0 to being positive. For the contact process on a homogeneous tree (with $d \geq 2$) this point also separates a region where there are infinitely many extremal invariant distributions from one where there are exactly two of these.

An interesting direction, in which we will nevertheless not go in this paper, would be to relate in a precise way, if possible, the two notions of phase transition discussed above. It should be clear, though, that the presence of discontinuities and other types of nonanalytic behavior of the functions $\rho(\cdot)$ and $\beta(\cdot)$ is a matter of intrinsic interest. Here we will only address issues of continuity of these functions.

In the most studied cases, the complete answer is known. First, on $\mathbb{Z}^d$, the contact process has a continuous $\rho(\lambda) = \beta(\lambda)$ for all $\lambda > 0$ [this was one of the major open problems about the contact process until it was solved by Bezuidenhout and Grimmett (1990)]. Second, on $\mathbb{T}_d$, with $d \geq 2$, the contact process has a continuous $\rho(\lambda)$ for all $\lambda > 0$ [see Morrow, Schinazi and Zhang (1994)], while $\beta(\lambda) = 0$ on $(0, \lambda_r(\mathbb{T}_d)]$ and $\beta(\lambda) = \rho(\lambda)$ on $(\lambda_r(\mathbb{T}_d), \infty)$ (recall that $r = s$ holds there), so that $\beta(\cdot)$ is also continuous on this open interval, but it is discontinuous at $\lambda_r(\mathbb{T}_d)$.

The theorems which we state and prove in the next two subsections, 3.2 and 3.3, address the continuity properties of $\rho(\cdot)$ and $\beta(\cdot)$, respectively, on general graphs in $\mathscr{G}$. In subsection 3.4 we present a sufficient condition for continuity of $\rho(\cdot)$ above λ_s and apply it. In subsection 3.5 we present a necessary and sufficient condition for right-continuity of $\beta(\cdot)$ and apply it.

The continuity properties of $\rho(\cdot)$ and $\beta(\cdot)$ on $\mathbb{Z}^d$ and on $\mathbb{T}_d$ will be seen to be special, as compared to what can happen on general graphs, but will also be seen to be typical of what happens on homogeneous graphs (see Corollary 3.4.1).

We introduce next some technical tools. We will sometimes need to couple versions of the contact process with values ranging in an interval of the type $(0, \lambda]$. This will be done in a standard fashion, by enlarging the probability space of Poisson processes on which $\mathbb{P}_{G,\lambda}$ is defined (the graphical construction), introducing independent random variables uniformly distributed on $(0, 1)$ associated to each one of the arrows in the graphical construction with infection rate λ. The corresponding probability measure will be denoted by $\bar{\mathbb{P}}_{G,\lambda}$, and the contact process with infection parameter $\lambda' < \lambda$ can be obtained by only keeping the arrows associated to random variables which take value at most λ'/λ. We will say in this situation that we are keeping the arrows only up to level λ'.

It is a standard matter to use the coupling above to prove the continuity in λ of various probabilities which concern the contact process started from some finite set and depend only on what happens up to a fixed time. For instance, this is the case of $\mathbb{P}_{G,\lambda}(\xi_t^0 \neq \varnothing)$, seen as a function of λ, with t fixed. We will use the term "finite-time-continuity," when referring to results of this nature.

Suppose that G' is a subgraph of G, $\lambda' \leq \lambda$, $x, y \in \mathscr{V}_{G'}$ and $s < u$. We will use the notation $\{A \to_{G',\lambda'} B\}$ for the event that there is a path from A to B

in the graphical construction when we keep the arrows only up to level λ', and moreover this path only passes through sites and edges of G'. In this notation A and B can be subsets or elements of $\mathcal{V}_G \times \mathbb{R}$. The event that there is such a path, starting from A and reaching space–time locations at arbitrarily large times will be denoted by $\{A \rightarrow_{G',\lambda'} \infty\}$. In this notation, if G' is omited, it is understood that $G' = G$, and if λ' is omited, it is understood that $\lambda' = \lambda$. If G' is the largest subgraph of G which has $\mathcal{V}_{G'} = C \subset \mathcal{V}_G$, then in the notation above we can replace G' with C.

A minor issue which nevertheless requires our attention is whether the choice of the root of G plays any role in the continuity properties which we will be discussing. The following simple proposition settles the question, as expected, in the negative and also relates continuity properties of $\rho(\cdot)$ to those of the distribution $\bar{\nu}$ and continuity properties of $\beta(\cdot)$ to those of the distribution ν_r.

PROPOSITION 3.1.1. *Suppose $G \in \mathcal{G}$ and $\lambda > 0$. Then*:

(a) *Either $\rho(A,\cdot)$ is left-continuous at λ for all nonempty $A \Subset \mathcal{V}_G$ or for no such A.*

(b) *Either $\rho(A,\cdot)$ is right-continuous at λ for all nonempty $A \Subset \mathcal{V}_G$ or for no such A.*

(c) *Either $\beta(A,\cdot)$ is left-continuous at λ for all nonempty $A \Subset \mathcal{V}_G$ or for no such A.*

(d) *Either $\beta(A,\cdot)$ is right-continuous at λ for all nonempty $A \Subset \mathcal{V}_G$ or for no such A.*

PROOF. We will only prove (a), the other claims having analogous proofs. Set

$$\Delta(A,\lambda) = \rho(A,\lambda) - \lim_{\lambda' \nearrow \lambda} \rho(A,\lambda').$$

By monotonicity in λ, it is clear that $\Delta(A,\lambda) \geq 0$, so that left-continuity of $\rho(A,\cdot)$ at λ is equivalent to the statement that $\Delta(A,\lambda) \leq 0$.

A standard application of the Markov property, of finite-time-continuity and of the dominated convergence theorem gives, for $A \Subset \mathcal{V}_G$,

$$\Delta(A,\lambda) = \sum_{B \Subset \mathcal{V}_G} \mathbb{P}_{G,\lambda}(\xi_1^A = B)\,\Delta(B,\lambda). \tag{3.1}$$

That (3.1) suffices for our purpose should be clear, since if $\Delta(B,\lambda) > 0$ for some $B \Subset \mathcal{V}_G$, this equation gives us that $\Delta(A,\lambda) > 0$ for all nonempty $A \Subset \mathcal{V}_G$ [the fact that G is connected is being used to assure us that all the probabilities which appear in (3.1) are strictly positive when $A \neq \varnothing$]. $\square$

3.2. *General continuity properties of $\rho(\cdot)$.*

THEOREM 3.2.1. *For every $G \in \mathcal{G}$, the function $\rho(\cdot)$ is right-continuous. On the other hand, there are trees in $\mathcal{G}$ for which $\rho(\cdot)$ is discontinuous at λ_s, and there are trees in $\mathcal{G}$ for which $\rho(\cdot)$ is discontinuous at some $\lambda > \lambda_s$.*

PROOF. The right-continuity of $\rho(\cdot)$ is a well-known fact and easy to prove. Set $\rho^{(n)}(\lambda) = \mathbb{P}_{G,\lambda}(\xi_n^0 \neq \varnothing)$. For each n this is a continuous function of λ, by finite-time-continuity. But $\rho(\lambda) = \inf_n \rho^{(n)}(\lambda)$, so that $\rho(\cdot)$ is upper-semicontinuous. Since $\rho(\cdot)$ is a nondecreasing function also, it is right-continuous.

We describe next a tree for which we will show that there is positive probability of survival at the survival point, so that $\rho(\cdot)$ is discontinuous there. This tree is obtained from a tree $\mathbb{T}_d^+$, with some $d \geq 2$, by removing certain sites from it (and the edges which have at least one endpoint at a removed site). To explain how we remove sites from $\mathbb{T}_d^+$ to obtain the new tree, it is convenient to label the sites of $\mathbb{T}_d^+$ in a certain standard way. The origin will be labeled (0). The sites in generation 1 will be labeled $(0,1)$, $(0,2), \dots, (0,d) \dots$. The sites in generation n which are descendents of the site of generation $n-1$ labeled $(0, g_1, g_2, \dots, g_{n-1})$ will be labeled $(0, g_1, g_2, \dots, g_{n-1}, 1)$, $(0, g_1, g_2, \dots, g_{n-1}, 2), \dots, (0, g_1, g_2, \dots, g_{n-1}, d)$. We think of the sites in each generation of the tree $\mathbb{T}_d^+$ as being ordered lexicographically, with basis on the labels (which then run from $(0,0,0,\dots,0)$, to $(0,d,d,\dots,d)$).

Suppose we are given two sequences of strictly positive integer numbers: $\mathbf{l} = (l_i)_{i=1,2,\dots}$ and $\mathbf{k} = (k_i)_{i=1,2,\dots}$ which satisfy $k_i \leq d^{l_i}$. We will denote by $[\mathbb{T}_d^+; \mathbf{l}, \mathbf{k}]$ the tree which is obtained from $\mathbb{T}_d^+$ by deleting sites and edges from this tree in the following recursive fashion. In the first step, select the first k_1 sites in generation l_1 and call them (1)-head sites; now delete all the descendents of the sites in generation l_1 which are not (1)-head sites. In the nth step, $n = 2, 3, \dots$ take the tree which was obtained in the $(n-1)$th step and for each $(n-1)$-head site select the first k_n of its descendents which are in generation $l_1 + \cdots + l_n$ and call them (n)-head sites; now delete all the descendents of the sites in generation $l_1 + \cdots + l_n$ which are not (n)-head sites. The graph $[\mathbb{T}_d^+; \mathbf{l}, \mathbf{k}]$ is the remaining graph after the procedure just described is applied indefinitely.

We will show that for a certain choice of the sequences $\mathbf{l}$ and $\mathbf{k}$ we have

$$\rho_{[\mathbb{T}_d^+; \mathbf{l}, \mathbf{k}]}(\lambda_s([\mathbb{T}_d^+; \mathbf{l}, \mathbf{k}])) > 0 \tag{3.2}$$

and

$$\beta_{[\mathbb{T}_d^+; \mathbf{l}, \mathbf{k}]}(\lambda_s([\mathbb{T}_d^+; \mathbf{l}, \mathbf{k}])) = 0. \tag{3.3}$$

The relevance of this second claim will become clear later, when we use this graph as a building block to obtain another one for which $\rho(\cdot)$ is discontinuous at some $\lambda > \lambda_s$.

Let λ^* be an arbitrary point in the interval $(\lambda_s(\mathbb{T}_d^+), \lambda_r(\mathbb{T}_d^+)]$. We will make the choices of the sequences $\mathbf{l} = (l_i)_{i=1,2,\dots}$ and $\mathbf{k} = (k_i)_{i=1,2,\dots}$ in such a way that

$$\lambda_s([\mathbb{T}_d^+; \mathbf{l}, \mathbf{k}]) = \lambda^*. \tag{3.4}$$

For this purpose we introduce first a new quantity related to $u_{d,n}(\lambda)$. Recall that $B(0,n)$ denotes the ball of center 0 and radius n, and define $\tilde{u}_{d,n}(\lambda)$ and

$s(n)$ by

$$\tilde{u}_{d,n}(\lambda) = \max_{s>0} \mathbb{P}_{\mathbb{T}_d^+,\lambda}\Big((0,0) \xrightarrow{B(0,n)} (n,s)\Big) = \mathbb{P}_{\mathbb{T}_d^+,\lambda}\Big((0,0) \xrightarrow{B(0,n)} (n,s(n))\Big).$$

Clearly such a maximizing $s(n)$ exists; if it is not unique, we can define $s(n)$ as the minimal one. Obviously $\tilde{u}_{d,n}(\lambda) \le u_{d,n}(\lambda)$, but nevertheless

$$\lim_{n\to\infty} (\tilde{u}_{d,n}(\lambda))^{1/n} = \alpha_d(\lambda). \tag{3.5}$$

This result is a marginal strengthening of Lemma 1 in Salzano and Schonmann (1998) and can be proved in essentially the same way as that lemma. We know from Lalley and Sellke (1998) and Lalley (1999) that for any $\lambda_s(\mathbb{T}_d) < \lambda' < \lambda^*$, we have

$$1/d < \alpha_d(\lambda') < \alpha_d(\lambda^*) \le 1/\sqrt{d}. \tag{3.6}$$

[The first inequality follows from the strict monotonicity of $\alpha_d(\cdot)$ in $(0, \lambda_r(\mathbb{T}_d)]$ and the fact that if $\alpha_d(\lambda) < 1/d$, then the expected number of sites of $\mathbb{T}_d$ ever to be infected in the process $\{\xi^0_{\mathbb{T}_d,\lambda;t}: t \ge 0\}$ is finite, so that $\rho_{\mathbb{T}_d}(\lambda) = 0$.]

The choice of the sequence $\mathbf{l} = (l_i)_{i=1,2,\dots}$ will be made later, but we anticipate that it will satisfy

$$\lim_{i\to\infty} l_i = \infty. \tag{3.7}$$

Supposing $\mathbf{l}$ given, we specify $\mathbf{k}$ via

$$k_i = \lfloor c/\tilde{u}_{d,l_i}(\lambda^*)\rfloor,$$

where $c > 1$ is an arbitrary constant. Note that (3.5) and (3.6) guarantee that $k_i \le d^{l_i}$ (as required, for the construction of $[\mathbb{T}_d^+;\mathbf{l},\mathbf{k}]$ to make sense), provided only that $l_i \ge L_1$, $i = 1, 2, \dots$, where $L_1 < \infty$ is some appropriate constant.

It is not hard to show that the choice above yields

$$\rho_{[\mathbb{T}_d^+;\mathbf{l},\mathbf{k}]}(\lambda') = 0 \quad \text{for all } \lambda' < \lambda^*. \tag{3.8}$$

To this end, observe that in $[\mathbb{T}_d^+;\mathbf{l},\mathbf{k}]$ there are exactly $k_1k_2\cdots k_n$ (n)-head sites and they are at distance $l_1 + \cdots + l_n$ from the root. Let E_n be the event that the process $\{\xi^0_t: t \ge 0\}$ ever infects one of these sites. Then, since $[\mathbb{T}_d^+;\mathbf{l},\mathbf{k}]$ is a subgraph of $\mathbb{T}_d$,

$$\begin{aligned}\rho_{[\mathbb{T}_d^+;\mathbf{l},\mathbf{k}]}(\lambda') &= \lim_{n\to\infty} \mathbb{P}_{[\mathbb{T}_d^+;\mathbf{l},\mathbf{k}],\lambda'}(E_n) \le \limsup_{n\to\infty} k_1k_2\cdots k_n\, u_{d,l_1+\cdots+l_n}(\lambda')\\ &\le \limsup_{n\to\infty} k_1k_2\cdots k_n\,(\alpha_d(\lambda'))^{l_1+\cdots+l_n} \le \limsup_{n\to\infty} \prod_{i=1}^n c\frac{(\alpha_d(\lambda'))^{l_i}}{\tilde{u}_{d,l_i}(\lambda^*)},\end{aligned}$$

where (1.1) was used in the second inequality. Thanks to (3.5), (3.6) and (3.7), the nth term in this product vanishes as $n \to \infty$, and therefore (3.8) follows.

In order to complete the proof of the claim (3.2), it remains now to show that we can make the choice of $\mathbf{l}$, satisfying (3.7), in such a way that

$$\rho_{[\mathbb{T}_d^+;\mathbf{l},\mathbf{k}]}(\lambda^*) > 0. \tag{3.9}$$

For this purpose, we consider now the following modification of the contact process on $[\mathbb{T}_d^+; \mathbf{l}, \mathbf{k}]$ started from a single particle at the origin. Until time $s(l_1)$ we run the usual contact process on this graph. At time $s(l_1)$ we remove all particles except for those which are at (1)-head sites; from this time on we keep the set $B(0, l_1 - 1)$ free of particles, and until time $s(l_1) + s(l_2)$ we let the system evolve in the remaining sites with the usual contact process rules. Recursively, at time $s(l_1) + \cdots + s(l_n)$ we remove all particles except for those which are at (n)-head sites; from this time on we keep the set $B(0, l_1 + \cdots + l_n - 1)$ free of particles, and until time $s(l_1) + \cdots + s(l_{n+1})$ we let the system evolve in the remaining sites with the usual contact process rules.

Let Z_n be the number of particles in the process described above at time $s(l_1) + \cdots + s(l_n)$. Obviously

$$\rho_{[\mathbb{T}_d^+; \mathbf{l}, \mathbf{k}]}(\lambda^*) \geq \mathbb{P}(Z_n > 0 \quad \text{for all } n \geq 1). \tag{3.10}$$

The process $(Z_n)_{n=1,\ldots}$ is a time-dependent branching process. Each particle counted in Z_{n-1} gives rise, independently and with the same distribution, to a random number of offspring, which are particles counted in Z_n. The average number of offspring of each particle of Z_{n-1} is $\mu_n \geq k_n \tilde{u}_{d, l_n}(\lambda^*)$. Note that if we take $c' \in (1, c)$, then there is $L_2 \in [L_1, \infty)$ so that if $l_n \geq L_2$, we have $\mu_n \geq c'$. If $(Z_n)_{n=1,\ldots}$ were a branching process, this would be enough to conclude that the right-hand side of (3.10) is positive. In order to use branching process theory to handle the process $(Z_n)_{n=1,\ldots}$, we will take the sequence $\mathbf{l}$ increasing by steps, with long stretches in which it is constant. More precisely, we will take

$$1 \leq n_1 < n_2 < n_3 < \cdots$$

and

$$\begin{aligned} l_1 = l_2 = \cdots = l_{n_1} &< l_{n_1+1} = l_{n_1+2} = \cdots = l_{n_1+n_2} \\ &< l_{n_1+n_2+1} = l_{n_1+n_2+2} = \cdots = l_{n_1+n_2+n_3} < \cdots. \end{aligned}$$

Define $\bar{l}_j = l_{n_1+\cdots+n_j}$ and $\bar{k}_j = k_{n_1+\cdots+n_j} = \lfloor c/u_{d, \bar{l}_j}(\lambda^*) \rfloor$. Let also $\mathbf{l}^{(j)} = (\bar{l}_j, \bar{l}_j, \ldots)$ and $\mathbf{k}^{(j)} = (\bar{k}_j, \bar{k}_j, \ldots)$ be constant sequences. Let $(Z_n^{(j)})_{n=1,\ldots}$ be defined as the process $(Z_n)_{n=1,\ldots}$, but for the tree $[\mathbb{T}_d^+; \mathbf{l}^{(j)}, \mathbf{k}^{(j)}]$ instead of the tree $[\mathbb{T}_d^+; \mathbf{l}, \mathbf{k}]$. For each j, the process $(Z_n^{(j)})_{n=1,\ldots}$ is a branching process with offspring distribution having mean $\mu_j = \bar{k}_j \tilde{u}_{d, \bar{l}_j}(\lambda^*) \geq c' > 1$, provided that $l_j \geq L_2$. Since the offspring distribution has a finite support (namely $\{0, 1, \ldots, \bar{k}_j\}$), and in particular a finite second moment, it follows from standard branching-process theory [see, e.g., Example 4.3 in Section 4.4, page 254 of Durrett (1996)] that for some random variable X_j with mean $\mathbb{E}(X_j) = 1$,

$$\mathbb{P}\left(\lim_{n \to \infty} \frac{Z_n^{(j)}}{(\mu_j)^n} = X_j \right) = 1.$$

In particular, there is $\varepsilon_j > 0$ and $N_j < \infty$ such that

$$\mathbb{P}\left(Z_n^{(j)} \geq \frac{(\mu_j)^n}{2}\right) \geq \varepsilon_j, \tag{3.11}$$

for all $n \geq N_j$.

Back to the process $(Z_n)_{n=1,\ldots}$, it is easy to see that from (3.11) we obtain

$$\begin{aligned}
&\mathbb{P}(Z_n > 0 \text{ for all } n \geq 1)\\
&\quad \geq \mathbb{P}\left(Z_{n_1} \geq \frac{(\mu_1)^{n_1}}{2}\right)\\
&\qquad \times \prod_{j=1}^{\infty} \mathbb{P}\left(Z_{n_1+\cdots+n_{j+1}} \geq \frac{(\mu_{j+1})^{n_{j+1}}}{2} \,\middle|\, Z_{n_1} \geq \frac{(\mu_1)^{n_1}}{2}, \ldots, Z_{n_1+\cdots+n_j} \geq \frac{(\mu_j)^{n_j}}{2}\right)\\
&\quad \geq \mathbb{P}\left(Z_{n_1}^{(1)} \geq \frac{(\mu_1)^{n_1}}{2}\right) \prod_{j=1}^{\infty} \left\{1 - \left(1 - \mathbb{P}\left(Z_{n_{j+1}}^{(j+1)} \geq \frac{(\mu_{j+1})^{n_{j+1}}}{2}\right)\right)^{(\mu_j)^{n_j}/2}\right\}\\
&\quad \geq \varepsilon_1 \prod_{j=1}^{\infty} \left\{1 - (1 - \varepsilon_{j+1})^{(c')^{n_j}/2}\right\},
\end{aligned}$$

provided that $n_j \geq N_j$, for $j \geq 1$. This infinite product can be assured to be positive if we make our choices, for instance, in the following fashion. We will take $\bar{l}_j = L_2 + j$; this gives us values for ε_j and N_j, $j \geq 1$. The infinite product above is clearly positive if n_j (besides satisfying $n_j \geq N_j$ for each j) is chosen to grow fast enough. This establishes the choice of $\mathbf{l}$ and shows that, thanks to the comparison (3.10), (3.9) holds.

From (3.9) and (3.8) we have (3.4) and (3.2). This completes the proof that there is a graph on which the contact process survives at the survival point.

The claim (3.3) is an immediate consequence of the fact that $[\mathbb{T}_d^+; \mathbf{l}, \mathbf{k}]$ is a subgraph of $\mathbb{T}_d^+$, and $\lambda^* \leq \lambda_r(\mathbb{T}_d^+)$. Suppose now that $d \leq D$ and set $G = [\mathbb{T}_d^+; \mathbf{l}, \mathbf{k}] \vee \mathbb{T}_D$. Clearly $\lambda_s(G) \leq \lambda_s(\mathbb{T}_D) < \lambda_s([\mathbb{T}_d^+; \mathbf{l}, \mathbf{k}])$ and from Proposition 3.2.1 below, (3.2) and (3.3), we obtain that $\rho_G(\cdot)$ is discontinuous at $\lambda_s([\mathbb{T}_d^+; \mathbf{l}, \mathbf{k}])$. This completes the proof of Theorem 3.2.1. □

PROPOSITION 3.2.1. *Suppose that $G = G_1 \vee G_2$ and the contact process on G_1 does not satisfy $r = s$ when the infection parameter takes the value $\lambda_s(G_1)$. Then $\rho_G(\cdot)$ is discontinuous at $\lambda_s(G_1)$.*

PROOF. For convenience, we will take for the origin of G the origin of G_1.

Consider the coupling $\bar{\mathbb{P}}_{G, \lambda_s(G_1)}$ of versions of the contact process with values ranging in the interval $(0, \lambda_s(G_1)]$, as reviewed in subsection 3.1. Let E be the event that when we keep all the arrows in this construction, the contact process started from a single particle at the origin survives, but no site of G_2 is ever infected. The hypothesis on G_1 assures us that $\bar{\mathbb{P}}_{G, \lambda_s(G_1)}(E) > 0$.

For $\lambda < \lambda_s(G_1)$ we have

$$\begin{aligned}\rho_G(\lambda_s(G_1)) - \rho_G(\lambda) &= \bar{\mathbb{P}}_{G,\lambda_s(G_1)}\Big((0,0) \xrightarrow{\lambda_s(G_1)} \infty,\ \Big\{(0,0) \xrightarrow{\lambda} \infty\Big\}^c\Big)\\ &\geq \bar{\mathbb{P}}_{G,\lambda_s(G_1)}(E) - \bar{\mathbb{P}}_{G,\lambda_s(G_1)}\Big((0,0) \xrightarrow{G_1,\lambda} \infty\Big)\\ &= \bar{\mathbb{P}}_{G,\lambda_s(G_1)}(E) - \rho_{G_1}(\lambda) = \bar{\mathbb{P}}_{G,\lambda_s(G_1)}(E).\end{aligned}$$

Hence

$$\lim_{\lambda \nearrow \lambda_s(G_1)} \rho_G(\lambda) \leq \rho_G(\lambda_s(G_1)) - \bar{\mathbb{P}}_{G,\lambda_s(G_1)}(E) < \rho_G(\lambda_s(G_1)). \qquad \square$$

We will now present another tree, which has the property that $\rho(\cdot)$ is discontinuous at λ_s. This tree is somewhat simpler than $[\mathbb{T}_d^+; \mathbf{l}, \mathbf{k}]$, but satisfies $r = s$, at λ_s, so that we cannot use it, as we did with $[\mathbb{T}_d^+; \mathbf{l}, \mathbf{k}]$, to construct another tree for which $\rho(\cdot)$ is discontinuous at some $\lambda > \lambda_s$. Moreover, the complete proof that $\rho(\lambda_s) > 0$ for our new tree is rather long and involved and will only be sketched here. The example is adapted from Example B, Proposition 1.3 of Madras, Schinazi and Schonmann (1994), and we refer the reader to that paper for the details of the proof. Later in this paper we will refer to the example which we introduce below as the "desert-oasis example."

Suppose we are given two sequences of strictly positive integer numbers: $\mathbf{d} = (d_i)_{i=1,2,\dots}$ and $\mathbf{o} = (o_i)_{i=1,2,\dots}$. We will denote by $[\mathbf{d}, \mathbf{o}]$ the tree obtained from $\mathbb{Z}^+$ by adding sites and edges to this tree in the following fashion. First partition the sites of $\mathbb{Z}^+$ into two sets: desert-sites and oasis-sites. The first d_1 sites of $\mathbb{Z}^+$ are declared to be desert-sites, then the next o_1 sites of $\mathbb{Z}^+$ are declared to be oasis-sites, then the next d_2 sites of $\mathbb{Z}^+$ are declared to be desert-sites, then the next o_2 sites of $\mathbb{Z}^+$ are declared to be oasis-sites and so on. A new site is added to the tree in association with each oasis-site and connected to this oasis-site by means of a new edge. This completes the construction. The added sites are leaves of the tree and will be called palm-sites.

For our purpose, we will choose $d_i = i^3$ and $o_i = i^2$. The tree $[\mathbf{d}, \mathbf{o}]$ turns out to have then the following features:

$$\lambda_s([\mathbf{d}, \mathbf{o}]) = \lambda_r([\mathbf{d}, \mathbf{o}]) = \lambda_s(\mathbb{Z}) \tag{3.12}$$

and

$$\rho_{[\mathbf{d}, \mathbf{o}]}(\lambda_s([\mathbf{d}, \mathbf{o}])) > 0. \tag{3.13}$$

Moreover $r = s$ holds for the contact process on $[\mathbf{d}, \mathbf{o}]$ for all values of λ.

We will explain below why the claims above hold. But before we can do it we need to introduce another graph, which may be seen as a doubly infinite oasis. This graph will be denoted by O, and is obtained from $\mathbb{Z}$ in the same fashion in which $[\mathbf{d}, \mathbf{o}]$ is obtained from $\mathbb{Z}^+$, but with all sites of $\mathbb{Z}$ being declared to be oasis-sites. More precisely, to each site of $\mathbb{Z}$ we associate a new site and connect it to that site of $\mathbb{Z}$. The resulting graph is O.

From Aizenman and Grimmett (1991) (see the end of Section 2 there), we know that

$$\lambda_s(O) < \lambda_s(\mathbb{Z}).$$

Moreover O can be studied by means of the dynamic rescaling approach of Bezuidenhout and Grimmett (1990), which gives, for instance $\lambda_s(O) = \lambda_r(O)$, and the validity of *cc* for the contact process on O above $\lambda_s(O)$.

The idea of the proof of (3.12) and (3.13) is as follows. Think of $[\mathbf{d}, \mathbf{o}]$ as a sequence of desert stretches and oasis stretches. If $\lambda < \lambda_s(\mathbb{Z})$, then the time needed for infection to cross a desert stretch of length i^3 grows with i as $\exp(C_1 i^3)$ [this follows from self-duality and exponential estimates on the survival time of of the subcritical one-dimensional contact process; see, for instance, Chapter 6 of Liggett (1985)], while clearly infection can typically only persist in an oasis stretch of length i^2 for at most a time of order $\exp(C_2 i^2)$. Therefore the infection should eventually disappear, that is,

$$\rho_{[\mathbf{d}, \mathbf{o}]}(\lambda) = 0 \quad \text{for } \lambda < \lambda_s(\mathbb{Z}). \tag{3.14}$$

On the other hand, when $\lambda = \lambda_s(\mathbb{Z})$, then because $\lambda_s(O) < \lambda_s(\mathbb{Z})$, infection should persist in an oasis stretch of length i^2 for at least a time of order $\exp(Ci^2)$ [a rigorous version of this claim can be proved in a standard fashion using the dynamic rescaling scheme of Bezuidenhout and Grimmett (1990)]. But the time needed to cross a desert stretch of length i^3 grows then only as a power of this length, that is, as i^{C_3} for some C_3 [see (3.4) and the proof of Lemma 3.2 in Madras, Schinazi and Schonmann (1994), which are based on estimates in Durrett, Schonmann and Tanaka (1989)]. This allows the system to survive with positive probability, that is,

$$\rho_{[\mathbf{d}, \mathbf{o}]}(\lambda_s(\mathbb{Z})) > 0, \tag{3.15}$$

with infection crossing from oasis to oasis, through desert, to the right.

The claims (3.12) and (3.13) are equivalent to (3.14) and (3.15). Next we sketch the proof that $r = s$ holds for the contact process on $[\mathbf{d}, \mathbf{o}]$ at $\lambda_s([\mathbf{d}, \mathbf{o}])$ [that it holds above $\lambda_s([\mathbf{d}, \mathbf{o}])$ is clear, for instance from the fact that *cc* holds then, by Theorem 5 of Part I].

The same argument from Madras, Schinazi and Schonmann (1994) which gives (3.15) also implies that there is $\delta > 0$ such that for all oasis-site x,

$$\mathbb{P}_{[\mathbf{d}, \mathbf{o}], \lambda_s([\mathbf{d}, \mathbf{o}])}\big(0 \in \xi_t^x \text{ for some } t > 0\big) \geq \delta. \tag{3.16}$$

Intuitively, in the same way that the infection can cross from oasis to oasis, through desert, to the right, it can also move in a similar fashion in the opposite direction. Now, if (3.16) were true for all site x in the tree, then Theorem 3 in Part I would give the validity of $r = s$ at $\lambda_s([\mathbf{d}, \mathbf{o}])$. Nevertheless, in spite of our only having (3.16) for oasis-sites x, this is enough to obtain the desired conclusion, because the set of all oasis-sites has the the property that if the process survives, then sites in this set will be visited at arbitrarily large times almost surely. It is easy to adapt the proof of Theorem 3 in Part I, to obtain the conclusion that $r = s$ holds, from the knowledge that (3.16) is satisfied for all site x in a set with the property just described.

3.3. *General continuity properties of* $\beta(\cdot)$.

THEOREM 3.3.1. *For every* $G \in \mathscr{G}$ *the function* $\beta(\cdot)$ *is left-continuous on* (λ_r, ∞). *On the other hand, there are trees in* $\mathscr{G}$ *for which* $\beta(\cdot)$ *is not right-continuous at* λ_r, *there are trees in* $\mathscr{G}$ *for which* $\beta(\cdot)$ *is not left-continuous at* λ_r *and there are trees in* $\mathscr{G}$ *for which* $\beta(\cdot)$ *is discontinuous at some* $\lambda > \lambda_r$.

PROOF. To prove the left-continuity of $\beta(\cdot)$ on (λ_r, ∞), we recall a definition introduced in the beginning of Section 2 of Part I. For $A \subset \mathscr{V}_G$, and $s, R > 0$, set $\Omega^\eta(t, R) = \{\xi_s^\eta \supset B(0, R)$ for some $s < t\}$. By Lemma 1 of Part I,

$$\lim_{R\to\infty} \lim_{t\to\infty} \mathbb{P}(\Omega^A(t, R)) = \mathbb{P}(\Omega_r^A). \tag{3.17}$$

We will also use the fact that by Theorem 1(a) of Part I we have that if $\beta(\lambda) > 0$, then

$$\lim_{R\to\infty} \mathbb{P}(\Omega_r^{B(0, R)}) = 1. \tag{3.18}$$

Suppose $\lambda > \lambda_r$. If $\tilde{\lambda} < \lambda' < \lambda$, then for any R and t, the strong Markov property, attractiveness and monotonicity in λ imply that

$$\begin{aligned} \beta_G(\lambda') &= \mathbb{P}_{G,\lambda'}(\Omega_r^0) \\ &\geq \mathbb{P}_{G,\lambda'}(\Omega^0(t, R))\mathbb{P}_{G,\lambda'}(\Omega_r^{B(0,R)}) \geq \mathbb{P}_{G,\lambda'}(\Omega^0(t, R))\mathbb{P}_{G,\tilde{\lambda}}(\Omega_r^{B(0,R)}). \end{aligned}$$

By finite-time-continuity,

$$\lim_{\lambda'\nearrow\lambda} \mathbb{P}_{G,\lambda'}(\Omega^0(t, R)) = \mathbb{P}_{G,\lambda}(\Omega^0(t, R)).$$

Therefore

$$\liminf_{\lambda'\nearrow\lambda} \beta_G(\lambda') \geq \mathbb{P}_{G,\lambda}(\Omega^0(t, R))\mathbb{P}_{G,\tilde{\lambda}}(\Omega_\infty^{B(0,R)}).$$

Now let first $t \to \infty$ and then $R \to \infty$ and use (3.17) and (3.18) to obtain

$$\liminf_{\lambda'\nearrow\lambda} \beta_G(\lambda') \geq \beta_G(\lambda).$$

Since $\beta_G(\lambda') \leq \beta_G(\lambda)$, for all $\lambda' < \lambda$, the proof of left-continuity of $\beta_G(\cdot)$ at λ is complete.

The homogeneous trees $\mathbb{T}_d$, with $d \geq 2$, provide well-known examples for which $\beta(\cdot)$ is not right-continuous at λ_r.

The desert-oasis example of subsection 3.2 provides an example in which $\beta(\cdot)$ is not left-continuous at λ_r. Indeed, we know that for this example $\rho(\cdot)$ is discontinuous from the left at $\lambda_s = \lambda_r$ and that $r = s$ holds for every λ.

Our final task is to provide now an example for which $\beta(\cdot)$ is not right-continuous at some $\lambda > \lambda_r$. The basic example of Part I works for this purpose.

So, set $G = \mathbb{T}_j \vee \mathbb{T}_k$ with $j \geq 2$ and k sufficiently larger than j, so that we have

$$\lambda_s(\mathbb{T}_k) < \lambda_r(\mathbb{T}_k) < \lambda_s(\mathbb{T}_j) < \lambda_r(\mathbb{T}_j). \tag{3.19}$$

From Theorem 2.1.2, we know that $\lambda_r(G) = \lambda_r(\mathbb{T}_k)$ [actually, we only need to know that $\lambda_r(G) \leq \lambda_r(\mathbb{T}_k)$, which is trivially true]. We will show that for some $x \in \mathscr{V}_{\mathbb{T}_j} \subset \mathscr{V}_G$, the function $\beta(\{x\}, \cdot)$ is not right-continuous at $\lambda_r(\mathbb{T}_j)$. Thanks to Proposition 3.1.1, this is all we have to show.

Under recurrence, each site of G will eventually be infected. Therefore, if n denotes the distance between x and the root of $\mathbb{T}_j$, we have from (1.1),

$$\begin{aligned} \beta_G(\{x\}, \lambda_r(\mathbb{T}_j)) &\leq \mathbb{P}_{G, \lambda_r(\mathbb{T}_j)}(0 \in \xi_t^x \text{ for some } t > 0) \\ &= \mathbb{P}_{\mathbb{T}_j, \lambda_r(\mathbb{T}_j)}(0 \in \xi_t^x \text{ for some } t > 0) \\ &= u_{j,n}(\lambda) \leq (\alpha_j(\lambda_r(\mathbb{T}_j))^n. \end{aligned} \tag{3.20}$$

Since from Lalley and Sellke (1998) we know that $\alpha_j(\lambda_r(\mathbb{T}_j)) \leq 1/\sqrt{d}$, the upper bound in (3.20) vanishes as $n \to \infty$.

Suppose now that $\lambda > \lambda_r(\mathbb{T}_j)$. If the origin of $\mathbb{T}_j$ is ever infected and before recovering it infects the origin of $\mathbb{T}_k$, then the conditional probability of recurrence is at least $\beta_{\mathbb{T}_k}(\lambda)$. Therefore,

$$\begin{aligned} \beta_G(\{x\}, \lambda) &\geq \mathbb{P}_{\mathbb{T}_j, \lambda}(0 \in \xi_t^x \text{ for some } t > 0)\frac{\lambda}{\lambda+1}\beta_{\mathbb{T}_k}(\lambda) \\ &\geq \beta_{\mathbb{T}_j}(\lambda)\frac{\lambda}{\lambda+1}\beta_{\mathbb{T}_k}(\lambda) \\ &= \rho_{\mathbb{T}_j}(\lambda)\frac{\lambda}{\lambda+1}\rho_{\mathbb{T}_k}(\lambda). \end{aligned}$$

Hence

$$\liminf_{\lambda \searrow \lambda_r(\mathbb{T}_j)} \beta_G(\{x\}, \lambda) \geq \rho_{\mathbb{T}_j}(\lambda_r(\mathbb{T}_j))\frac{\lambda_r(\mathbb{T}_j)}{\lambda_r(\mathbb{T}_j)+1}\rho_{\mathbb{T}_k}(\lambda_r(\mathbb{T}_j)) > 0. \tag{3.21}$$

If n is large enough, the claimed discontinuity of $\beta_G(\{x\}, \cdot)$ at $\lambda_r(\mathbb{T}_j^+)$ follows from the comparison between (3.20) and (3.21). □

3.4. *A sufficient condition for continuity of* $\rho(\cdot)$. We will say that the contact process on G *survives uniformly* at λ on a set $A \subset \mathscr{V}_G$ if

$$\lim_{R\to\infty} \inf_{x \in A} \mathbb{P}_{G,\lambda}(\Omega_\infty^{B(x,R)}) = 1.$$

In case $A = \mathscr{V}_G$ and the condition above is satisfied, then we just say that the contact process on G survives uniformly at λ.

If $\rho(\lambda) > 0$, then the contact process on G survives uniformly at λ on every finite set $A \subset \mathscr{V}_G$. To see this, observe that there is no loss in considering

$A = \{x\}$ to be a singleton, and note that

$$\lim_{R\to\infty} \mathbb{P}_{G,\lambda}\big(\Omega_\infty^{B(x,R)}\big) = \lim_{R\to\infty} \overline{\nu}\{\eta\colon \eta \cap B(x,R) \neq \varnothing\} = 1, \tag{3.22}$$

where we used the fact that $\overline{\nu}(\varnothing) = 0$.

If $\rho(\lambda) > 0$, a set $A \subset \mathscr{V}_G$ will be called a *recurrence set* for the contact process on G at λ if

$$\mathbb{P}_{G,\lambda}\big(\Omega_\infty^0,\ \{t\colon \xi_t^0 \cap A \neq \varnothing\} \text{ is bounded}\big) = 0.$$

Note that if $\rho_G(\lambda) > 0$, then having $r = s$ for the contact process on G at λ is equivalent to saying that $\{0\}$ is a recurrence set for the contact process on G at λ.

THEOREM 3.4.1. *If there exists $A \subset \mathscr{V}_G$ which is a recurrence set for the contact process on G at λ and on which the contact process on G survives uniformly at some $\tilde{\lambda} < \lambda$, then $\rho_G(\cdot)$ is continuous at λ.*

PROOF. Thanks to Theorem 3.2.1, only left-continuity has to be shown. Set

$$E_{R,t} = \{\xi_s^0 \supset B(x,R) \text{ for some } x \in A \text{ and } 0 \le s \le t\}.$$

If $\tilde{\lambda} < \lambda' < \lambda$, then for any R and t, the Strong Markov property, attractiveness and monotonicity in λ imply that

$$\begin{aligned}\rho_G(\lambda') &= \mathbb{P}_{G,\lambda'}\big(\Omega_\infty^0\big)\\ &\ge \mathbb{P}_{G,\lambda'}(E_{R,t}) \inf_{x\in A} \mathbb{P}_{G,\lambda'}\big(\Omega_\infty^{B(x,R)}\big) \ge \mathbb{P}_{G,\lambda'}(E_{R,t}) \inf_{x\in A} \mathbb{P}_{G,\tilde{\lambda}}\big(\Omega_\infty^{B(x,R)}\big).\end{aligned}$$

By finite-time-continuity,

$$\lim_{\lambda'\nearrow\lambda} \mathbb{P}_{G,\lambda'}(E_{R,t}) = \mathbb{P}_{G,\lambda}(E_{R,t}).$$

Therefore

$$\liminf_{\lambda'\nearrow\lambda} \rho_G(\lambda') \ge \mathbb{P}_{G,\lambda}(E_{R,t}) \inf_{x\in A} \mathbb{P}_{G,\tilde{\lambda}}\big(\Omega_\infty^{B(x,R)}\big). \tag{3.23}$$

Since A is a recurrence set for the contact process on G at λ, for any R we have

$$\liminf_{t\to\infty} \mathbb{P}_{G,\lambda}(E_{R,t}) \ge \rho_G(\lambda).$$

So, in (3.23) let first $t \to \infty$ and then $R \to \infty$ to obtain

$$\liminf_{\lambda'\nearrow\lambda} \rho_G(\lambda') \ge \rho_G(\lambda),$$

where the hypothesis that the contact process on G survives uniformly at $\tilde{\lambda}$ on A was used.

Since $\rho_G(\lambda') \le \rho_G(\lambda)$, for all $\lambda' < \lambda$, the proof is complete. □

It is interesting to compare the similarities between this proof and that of the left-continuity of $\beta(\cdot)$ above λ_r (Theorem 3.3.1).

In the rest of this subsection we will apply Theorem 3.4.1 to various examples and collect several of its consequences.

Some of the simplest applications of Theorem 3.4.1 concern cases in which we can take $A = \mathscr{V}_G$. Regardless of what G is, if $\rho_G(\lambda) > 0$, the set $\mathscr{V}_G$ is, tautologically, a recurrence set. Therefore, to apply the theorem at some point λ we only have to check whether the contact process on G survives uniformly at some $\tilde{\lambda} < \lambda$. The following is a case in which this is immediate.

COROLLARY 3.4.1. *If $G \in \mathscr{H}$, then $\rho_G(\cdot)$ can only be discontinuous at $\lambda_s(G)$, and $\beta_G(\cdot)$ can only be discontinuous at $\lambda_r(G)$.*

PROOF. If $\lambda > \lambda_s$, the homogeneity of G reduces the statement of uniform survival of the contact process on G at some $\tilde{\lambda} \in (\lambda_s, \lambda)$, to the statement that

$$\lim_{R\to\infty} \mathbb{P}_{G,\tilde{\lambda}}\big(\Omega_\infty^{B(0,R)}\big) = 1,$$

which is a particular case of (3.22). So the claim concerning the function $\rho_G(\cdot)$ is a consequence of Theorem 3.4.1.

Regarding the function $\beta_G(\cdot)$, just note that below $\lambda_r(G)$ this function is identically 0, and above this point it coincides with the function $\rho_G(\cdot)$, by Theorem 2(i) of Part I. However, since $\lambda_s(G) \leq \lambda_r(G)$, we have just shown that $\rho_G(\cdot)$ is continuous above $\lambda_r(G)$. $\square$

In the two basic examples of graphs in $\mathscr{H}$, $\mathbb{Z}^d$ and $\mathbb{T}_d$, it is also known that $\rho(\cdot)$ is continuous at λ_s. It is natural to ask whether this is the case for all homogeneous graphs. Unfortunately we are unable to answer this question.

It may be worthwhile to point out that the proof given above that for the contact process on $\mathbb{T}_d$, $\rho(\cdot)$ is continuous above λ_s is different from the proof contained in the papers by Pemantle (1992) and Morrow, Schinazi and Zhang (1994).

We can apply Theorem 3.4.1 with $A = \mathscr{V}_G$ not only for homogeneous graphs. Another example is that of $\mathbb{T}_d^+$. For these graphs it is indeed easy to see that the contact process survives uniformly at any $\lambda > \lambda_s(\mathbb{T}_d^+)$. For this purpose, first observe that each site $x \in \mathbb{T}_d^+$ can be seen as the root of a subgraph G_x which is isomorphic to $\mathbb{T}_d^+$ itself. Note that this isomorphism maps $B(x,R)\cap G_x$ onto $B(0,R)$ and therefore,

$$\lim_{R\to\infty} \inf_{x\in\mathscr{V}_{\mathbb{T}_d^+}} \mathbb{P}_{\mathbb{T}_d^+,\lambda}\big(\Omega_\infty^{B(x,R)}\big) \geq \lim_{R\to\infty} \mathbb{P}_{\mathbb{T}_d^+,\lambda}\big(\Omega_\infty^{B(0,R)}\big) = 1$$

by (3.22).

Another consequence of Theorem 3.4.1 follows.

COROLLARY 3.4.2. *If the criterion $r = s$ holds for the contact process on G at $\lambda > \lambda_s(G)$, then $\rho_G(\cdot)$ is continuous at λ.*

PROOF. Take $A = \{0\}$ and note that from the remarks made when the notions of uniform survival and of recurrent set were introduced, we have satisfied the hypothesis of Theorem 3.4.1. $\square$

The hypothesis that $\lambda > \lambda_s(G)$ in Corollary 3.4.2 is crucial. Note that the desert-oasis example of subsection 3.2 satisfies $r = s$ at λ_s, but $\rho(\cdot)$ is discontinuous there. Since for that example $r = s$ also holds for all $\lambda > \lambda_s = \lambda_r =: \lambda_c$, we conclude that for it $\rho(\cdot) = \beta(\cdot)$ is discontinuous only at λ_c.

From Theorem 5 of Part I we know that for any graph $G \in \mathscr{G}$, *cc* (and hence $r = s$) holds when λ is above the unique critical point, $\lambda_c(\mathbb{Z})$, for the contact process on $\mathbb{Z}$. Therefore we have also the following.

COROLLARY 3.4.3. *For any graph* $G \in \mathscr{G}$, $\rho_G(\cdot) = \beta_G(\cdot)$ *is continuous on* $[\lambda_c(\mathbb{Z}), \infty)$.

The desert-oasis example shows that Corollary 3.4.3 is optimal in the sense that $\lambda_c(\mathbb{Z})$ cannot be replaced in that proposition with any smaller number.

To state the next corollary to Theorem 3.4.1, we need a new definition. Suppose that Λ is a subset of $(0, \infty)$. We say that the contact process on a graph $G \in \mathscr{G}$ has the *uniform survivability property on* Λ in case there exists $A \subset \mathscr{V}_G$, which is a recurrence set for the contact process on G at all $\lambda \in \Lambda$ and on which the contact process on G survives uniformly at all $\lambda \in \Lambda$. In case $\Lambda = (\lambda_s(G), \infty)$, we say that the contact process on G has the *supercritical uniform survivability property*. In case $\Lambda = \{\lambda\colon \rho_G(\lambda) > 0\}$, we simply say that the contact process on G has the *uniform survivability property*.

Examples of graphs with the uniform survivability property are the homogeneous graphs and the graphs $\mathbb{T}_d^+$. The basic example of Part I is an example of a graph which does not have this property (but is a collage of graphs with the property).

COROLLARY 3.4.4. *If* $G \in \mathscr{G}$ *is a collage of graphs* $G_1, G_2, \dots, G_n$ *which have the supercritical uniform survivability property, then the function* $\rho_G(\cdot)$ *is continuous except possibly at the points* $\lambda_s(G_1), \lambda_s(G_2), \dots, \lambda_s(G_n)$ *and* $\lambda_s(G)$. *If in addition the contact process on each one of the graphs* G_i, $i = 1, 2, \dots, n$ *satisfies* $r = s$ *at its survival point* [*for instance, if* $\rho_{G_i}(\lambda_s(G_i)) = 0$], *then* $\rho_G(\cdot)$ *is continuous, except possibly at the point* $\lambda_s(G)$.

PROOF. We only have to consider $\lambda > \lambda_s(G)$. Suppose that also $\lambda \notin \{\lambda_s(G_1), \lambda_s(G_2), \dots, \lambda_s(G_n)\}$ and set

$$I_S = \{i : \lambda_s(G_i) < \lambda\} \quad \text{and} \quad I_D = \{1, 2, \dots, n\} \setminus I_S.$$

We denote by A_i the subset of $\mathscr{V}_{G_i}$, which is a recurrence set for the contact process on G_i at all $\lambda > \lambda_s(G_i)$, and on which the contact process on G_i survives uniformly at all $\lambda > \lambda_s(G_i)$.

We will first show that the set $V_{\text{glue}} \cup (\bigcup_{i \in I_S} A_i)$ is a recurrence set for the contact process on G at λ. Indeed, on the event Ω^0_∞, the set $V_{\text{glue}} \cup (\bigcup_{i \in I_S} A_i)$

must be infected at arbitrarily large times, $\mathbb{P}_{G,\lambda}$-a.s. This is so because the complementary part of Ω_∞^0 can be covered by a countable union of events of the following type: for one of the $i \in \{1, \dots, n\}$, the contact process on the graph G_i starting at the integer time k from a finite set $B \in \mathscr{V}_{G_i}$ survives, but never reaches any site in the set $V_{\text{glue}} \cup (\bigcup_{i \in I_S} A_i)$. But such events indexed by i, k and B have probability 0. To see this, there are two cases to consider: if $i \in I_D$, then the probability of this event is bounded above by $\rho_{G_i}(B, \lambda) = 0$. If $i \in I_S$, then the probability of the referred event is bounded above by the probability that the contact process on G_i, started from B survives but never reaches the set A_i; this probability is zero by the definition of the supercritical uniform survivability property.

It is clear now from the finiteness of V_{glue} and I_S, (3.22), the definition of I_S and the hypothesis of supercritical uniform survivability on the graphs $G_1, G_2, \dots, G_n$ that the contact process on G survives uniformly at $\tilde{\lambda} < \lambda$ on the set $V_{\text{glue}} \cup (\bigcup_{i \in I_S} A_i)$, provided that $\tilde{\lambda}$ is chosen close enough to λ.

The continuity of $\rho_G(\cdot)$ at λ follows from combining the conclusions of the two paragraphs above, therefore verifying the hypothesis of Theorem 3.4.1. This completes the proof of the first claim in the corollary.

If the contact process on each one of the graphs $G_1, G_2, \dots, G_n$ satisfies $r = s$ at its survival point, then for any $\lambda > \lambda_s(G)$ the argument above with a minor modification still applies to give the continuity of $\rho_G(\cdot)$ at λ. This minor modification is that when $i \in I_d \cap \{i: \lambda_s(G_i) = \lambda\}$ the probability of the event indexed by i, k and B used above should now be bounded above by the probability that the contact process on G_i started from B survives without ever infecting the sites in $V_{\text{glue}} \cap \mathscr{V}_{G_i}$; such probabilities are 0, since $r = s$ is supposed to hold. □

The corollary above should be contrasted with Proposition 3.2.1. Combining the two results, we can state the following.

COROLLARY 3.4.5. *Suppose that $G = G_1 \vee G_2$, with $\lambda_s(G_2) < \lambda_s(G_1)$, and suppose also that G_2 satisfies the supercritical uniform survivability property. Then $\rho_G(\cdot)$ is continuous at $\lambda_s(G_1)$ if and only if the contact process on G_1 satisfies $r = s$ when the infection parameter takes the value $\lambda_s(G_1)$.*

The following is a somewhat surprising application of the corollary above. Suppose that $G_2 \in \mathscr{G}$ has the supercritical uniform survivability property and $\lambda_s(G_2) < \lambda_s(\mathbb{Z})$ (for instance, $G_2 = \mathbb{Z}^2$ or $G_2 = \mathbb{T}_2$) and let G_1 be the desert-oasis example of subsection 3.2. In this case, G_1 satisfies $r = s$ when the infection parameter takes the value $\lambda_s(G_1) = \lambda_s(\mathbb{Z})$, and therefore $\rho_{G_1 \vee G_2}(\cdot)$ is continuous at this point. This may seem surprising, because one could expect the discontinuity in $\rho_{G_1}(\cdot)$ at $\lambda_s(G_1)$ to reflect in a discontinuity of $\rho_{G_1 \vee G_2}(\cdot)$ at the same point, but this turns out not to be the case.

Another interesting particular case in which Corollary 3.4.4 applies is the following.

COROLLARY 3.4.6. *If $G \in \mathscr{G}$ is a collage of copies of $\mathbb{T}^+_{d_1}, \mathbb{T}^+_{d_2}, \dots, \mathbb{T}^+_{d_n}$, then $\rho_G(\cdot)$ is continuous, except possibly at $\lambda_s(G)$. If in addition* $\max\{d_i\colon i = 1, \dots, n\} \geq 2$, *then $\rho_G(\cdot)$ is continuous.*

PROOF. From Corollary 3.4.4 we obtain the first statement, and, combining this conclusion with Theorem 2.1.2 we obtain the second statement. □

Note that the basic example of Part I is covered by Corollary 3.4.6, so that in spite of its various transitions from one type of ergodic behavior to another, it has a continuous $\rho(\cdot)$. It is natural at this point to ask what the continuity properties of $\beta(\cdot)$ are for this example. This problem will be treated in the next subsection.

3.5. *A necessary and sufficient condition for right-continuity of $\beta(\cdot)$.* For $G \in \mathscr{G}$, $A, S \Subset \mathscr{V}_G$ and $\lambda > 0$, we define

$$\gamma_G^S(A, \lambda) = \mathbb{P}_{G,\lambda}(\xi_t^A \cap S \neq \varnothing \text{ for some } t \geq 0) = \mathbb{P}_{G,\lambda}(A \times \{0\} \longrightarrow S \times [0, \infty)).$$

Since $\gamma_G^S(A, \lambda) = \sup_{T \geq 0} \mathbb{P}_{G,\lambda}(\xi_t^A \cap S \neq \varnothing \text{ for some } t \in [0, T])$, the function $\gamma_G^S(A, \cdot)$ is left-continuous everywhere [by the same sort of argument used to prove the right-continuity of $\rho_G(\cdot)$]. Regarding its right-continuity, we have the result stated as the next theorem. [Note that this theorem has no analogue for left-continuity, since there are graphs in $\mathscr{G}$ for which $\beta_G(\cdot)$ is not left-continuous at $\lambda_r(G)$.]

THEOREM 3.5.1. *Suppose that $G \in \mathscr{G}$ and $S \Subset \mathscr{V}_G$, $S \neq \varnothing$. Then $\beta_G(\cdot)$ is right-continuous at λ if and only if for each $A \Subset \mathscr{V}_G$ the function $\gamma_G^S(A, \cdot)$ is right-continuous at λ.*

PROOF. We first prove the "if" part. For arbitrary $\lambda', T, N > 0$, the Markov property gives

$$\begin{aligned} 1 - \beta_G(\lambda') &\geq \mathbb{P}_{G,\lambda'}(\xi_t^A \cap S = \varnothing \text{ for all } t \geq T) \\ &= \sum_{A \Subset \mathscr{V}_G} \mathbb{P}_{G,\lambda'}(\xi_T^0 = A)(1 - \gamma_G^S(A, \lambda')) \\ &\geq \sum_{\substack{A \Subset \mathscr{V}_G \\ A \subset B(0,N)}} \mathbb{P}_{G,\lambda'}(\xi_T^0 = A)(1 - \gamma_G^S(A, \lambda')). \end{aligned}$$

Therefore,

$$1 - \limsup_{\lambda' \searrow \lambda} \beta_G(\lambda') \geq \sum_{\substack{A \Subset \mathscr{V}_G \\ A \subset B(0, N)}} \mathbb{P}_{G,\lambda}(\xi_T^0 = A)(1 - \gamma_G^S(A, \lambda)).$$

Since N is arbitrary, this implies

$$1 - \limsup_{\lambda' \searrow \lambda} \beta_G(\lambda') \geq \sum_{A \in \mathscr{V}_G} \mathbb{P}_{G,\lambda}(\xi_T^0 = A)(1 - \gamma_G^S(A, \lambda))$$
$$= \mathbb{P}_{G,\lambda}(\xi_t^A \cap S = \varnothing \text{ for all } t \geq T).$$

Letting $T \to \infty$ yields now

$$\limsup_{\lambda' \searrow \lambda} \beta_G(\lambda') \leq 1 - (1 - \beta_G(\lambda)) = \beta_G(\lambda).$$

Since $\beta_G(\cdot)$ is a nondecreasing function, this shows that it is right-continuous at λ, finishing the proof of the "if" part.

We turn now to the proof of the "only if" part. We will use the coupling $\bar{\mathbb{P}}_{G,\tilde{\lambda}}$, where $\tilde{\lambda} > \lambda$ is arbitrary. The notation $\Omega_{r,\lambda}^A$ will denote the event that when the arrows are kept only up to level λ, the contact process started from A is recurrent, that is, has each site infected at arbitrarily large times. For $\lambda < \lambda' < \tilde{\lambda}$,

$$\begin{aligned}
0 &\leq \gamma_G^S(A, \lambda') - \gamma_G^S(A, \lambda) \\
&= \mathbb{P}_{G,\tilde{\lambda}}\Big(\Big\{A \times \{0\} \xrightarrow{\lambda'} S \times [0, \infty)\Big\} \cap \Big\{A \times \{0\} \xrightarrow{\lambda} S \times [0, \infty)\Big\}^c\Big) \\
&\leq \mathbb{P}_{G,\tilde{\lambda}}\Big(\Omega_{r,\lambda}^A \cap \Big\{A \times \{0\} \xrightarrow{\lambda'} S \times [0, \infty)\Big\} \cap \Big\{A \times \{0\} \xrightarrow{\lambda} S \times [0, \infty)\Big\}^c\Big) \\
&\quad + \mathbb{P}_{G,\tilde{\lambda}}\Big((\Omega_{r,\tilde{\lambda}}^A)^c \cap \Big\{A \times \{0\} \xrightarrow{\lambda'} S \times [0, \infty)\Big\} \cap \Big\{A \times \{0\} \xrightarrow{\lambda} S \times [0, \infty)\Big\}^c\Big) \\
&\quad + \mathbb{P}_{G,\tilde{\lambda}}\big((\Omega_{r,\lambda}^A)^c \cap \Omega_{r,\tilde{\lambda}}^A\big).
\end{aligned}$$

The first term in the right-hand side is clearly null. We introduce an arbitrary $T > 0$ and break down the second term into two parts, to write

$$\begin{aligned}
0 &\leq \gamma_G^S(A, \lambda') - \gamma_G^S(A, \lambda) \\
&\leq \mathbb{P}_{G,\tilde{\lambda}}\Big(\Big\{A \times \{0\} \xrightarrow{\lambda'} S \times [0, T)\Big\} \cap \Big\{A \times \{0\} \xrightarrow{\lambda} S \times [0, \infty)\Big\}^c\Big) \\
&\quad + \mathbb{P}_{G,\tilde{\lambda}}\Big((\Omega_{r,\tilde{\lambda}}^A)^c \cap \Big\{A \times \{0\} \xrightarrow{\lambda'} S \times [T, \infty)\Big\}\Big) \\
&\quad + \mathbb{P}_{G,\tilde{\lambda}}\big((\Omega_{r,\lambda}^A)^c \cap \Omega_{r,\tilde{\lambda}}^A\big) \\
&\leq \mathbb{P}_{G,\tilde{\lambda}}\Big(\Big\{A \times \{0\} \xrightarrow{\lambda'} S \times [0, T)\Big\} \cap \Big\{A \times \{0\} \xrightarrow{\lambda} S \times [0, T)\Big\}^c\Big) \\
&\quad + \mathbb{P}_{G,\tilde{\lambda}}\Big((\Omega_{r,\tilde{\lambda}}^A)^c \cap \Big\{A \times \{0\} \xrightarrow{\tilde{\lambda}} S \times [T, \infty)\Big\}\Big) \\
&\quad + \mathbb{P}_{G,\tilde{\lambda}}\big((\Omega_{r,\lambda}^A)^c \cap \Omega_{r,\tilde{\lambda}}^A\big).
\end{aligned}$$

As we let $\lambda' \searrow \lambda$, the first term in the right-hand side vanishes, by finite-time-continuity. As we then let $T \to \infty$, the second term in the right-hand side vanishes, since the corresponding event decreases to the empty set. Finally, as we then let $\tilde{\lambda} \searrow \lambda$, the third term in the right-hand side vanishes, since it equals $\beta_G(A, \tilde{\lambda}) - \beta_G(A, \lambda)$, which vanishes by hypothesis. The conclusion, as desired, is that

$$\lim_{\lambda' \searrow \lambda} \gamma_G^S(A, \lambda') = \gamma_G^S(A, \lambda). \qquad \square$$

The theorem above may at first sight seem to be difficult to apply in order to prove the right-continuity of $\beta_G(\cdot)$ for any specific G. The corollaries below show otherwise. The point is that since the theorem provides an equivalence, it can be used to extend the known right-continuity of $\beta_G(\cdot)$ for certain graphs, in certain intervals of values of λ, to other graphs built using those.

COROLLARY 3.5.1. *Suppose that G is a collage of $G_1, \dots, G_n$, then $\beta_G(\cdot)$ is right-continuous at λ if and only if $\beta_{G_i}(\cdot)$ is right-continuous at λ for each one of the graphs G_i, $i = 1, \dots, n$.*

PROOF. It is easy to see that for any $A \Subset \mathscr{V}_G$,

$$\gamma_G^{V_{\text{glue}}}(A, \cdot) = 1 - \prod_{i=1}^{n} \left(1 - \gamma_{G_i}^{V_{\text{glue}} \cap \mathscr{V}_{G_i}}(A \cap \mathscr{V}_{G_i}, \cdot)\right). \tag{3.24}$$

If for all $i \in \{1, \dots, n\}$ and $A_i \Subset \mathscr{V}_{G_i}$ the function $\gamma_{G_i}^{V_{\text{glue}} \cap \mathscr{V}_{G_i}}(A_i, \cdot)$ is right-continuous at λ, then, from (3.24), the same is true for all the functions $\gamma_G^{V_{\text{glue}}}(A, \cdot)$, $A \Subset \mathscr{V}_G$.

On the other hand, if for some $i \in \{1, \dots, n\}$ and some $A_i \Subset \mathscr{V}_{G_i}$ the function $\gamma_{G_i}^{V_{\text{glue}} \cap \mathscr{V}_{G_i}}(A_i, \cdot)$ is not right-continuous at λ, then take $A = A_i \Subset \mathscr{V}_G$ and note that (3.24) gives then

$$\gamma_G^{V_{\text{glue}}}(A, \cdot) = \gamma_{G_i}^{V_{\text{glue}} \cap \mathscr{V}_{G_i}}(A_i, \cdot).$$

In particular $\gamma_G^{V_{\text{glue}}}(A, \cdot)$ is not right-continuous at λ.

Our proof is complete by referring to the equivalence provided by Theorem 3.5.1. $\square$

As with Theorem 3.5.1, there is no analogue of Corollary 3.5.1 for left-continuity, as the following example shows. Suppose that $G = \mathbb{Z}^2 \vee [\mathbf{d}, \mathbf{o}]$, where $[\mathbf{d}, \mathbf{o}]$ is the desert-oasis example of subsection 3.2. We know that $\beta_{\mathbb{Z}^2}(\cdot)$ is continuous everywhere and that $\beta_{[\mathbf{d},\mathbf{o}]}(\cdot)$ is continuous except at the point $\lambda_s([\mathbf{d}, \mathbf{o}]) = \lambda_r([\mathbf{d}, \mathbf{o}]) = \lambda_s(\mathbb{Z}) > \lambda_s(\mathbb{Z}^2)$, where it is right-continuous but not left-continuous. From Corollary 3.4.5 and the discussion which followed it, we know that $\rho_G(\cdot)$ is continuous at $\lambda_s(\mathbb{Z})$. We know also that the contact processes on $\mathbb{Z}^2$ and on $[\mathbf{d}, \mathbf{o}]$ satisfy $r = s$ for all λ. Therefore, by Theorem 2.1.1, so does the contact process on G. Hence $\beta_G(\cdot)$ is continuous at $\lambda_s(\mathbb{Z})$.

An interesting application of Theorem 3.5.1 follows.

COROLLARY 3.5.2. *If $G \in \mathscr{G}$ is a collage of copies of $\mathbb{T}_{d_1}^+, \mathbb{T}_{d_2}^+, \dots, \mathbb{T}_{d_n}^+$, with $d_i \geq 2$, then $\beta_G(\cdot)$ is continuous, except at the points $\lambda_r(G_i)$, $i = 1, \dots, n$, where* *it is not right-continuous* [*it is left-continuous at these points*, *except possibly at the smallest of them*, *which coincides with* $\lambda_r(G)$].

PROOF. For the contact process on $\mathbb{T}_d^+$, $d \geq 2$, we know (Theorem 4 of Part I) that *cc*, and hence $r = s$, holds above $\lambda_r(\mathbb{T}_d^+)$. From Corollary 3.4.2 we learn then that $\beta_{\mathbb{T}_d^+}(\cdot) = \rho_{\mathbb{T}_d^+}(\cdot)$ is continuous above this point. But since $\lambda_s(\mathbb{T}_d^+) < \lambda_r(\mathbb{T}_d^+)$,

$$\lim_{\lambda \searrow \lambda_r(\mathbb{T}_d^+)} \beta_{\mathbb{T}_d^+}(\lambda) = \rho_{\mathbb{T}_d^+}(\lambda_r(\mathbb{T}_d^+)) > 0 = \beta_{\mathbb{T}_d^+}(\lambda_r(\mathbb{T}_d^+)),$$

so that $\beta_{\mathbb{T}_d^+}(\cdot)$ is not right-continuous at $\lambda_r(\mathbb{T}_d^+)$.

From Corollary 3.5.1 we obtain now the claim in the corollary, except for the part in parenthesis. That part has already been proved in Theorem 2.1.2 and Theorem 3.3.1. □

From Theorem 2.1.2, Corollary 3.4.6 and Corollary 3.5.2, a great deal of information has been learned about the contact process on graphs which are collages of copies of severed homogeneous trees. When we apply these results, for instance, to the basic example of Part I, $\mathbb{T}_j \vee \mathbb{T}_k$, with (2.1) satisfied, we learn that its survival point is $\lambda_s(\mathbb{T}_j)$, its recurrence point is $\lambda_r(\mathbb{T}_j)$, its survival probability is a continuous function of λ and its recurrence probability is discontinuous precisely at the points $\lambda_r(\mathbb{T}_j)$ and $\lambda_r(\mathbb{T}_k)$.

Note that the arguments presented in this subsection provided a second proof of the claim in Theorem 3.3.1, that there are trees in $\mathscr{G}$ for which $\beta(\cdot)$ is discontinuous at some $\lambda > \lambda_r$.

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