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EXISTENCE OF HYDRODYNAMICS FOR THE TOTALLY ASYMMETRIC SIMPLE K-EXCLUSION PROCESS

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In a totally asymmetric simple K-exclusion process, particles take nearest-neighbor steps to the right on the lattice $\mathbf{Z}$, under the constraint that each site contain at most K particles. We prove that such processes satisfy hydrodynamic limits under Euler scaling, and that the limit of the empirical particle profile is the entropy solution of a scalar conservation law with a concave flux function. Our technique requires no knowledge of the invariant measures of the process, which is essential because the equilibria of asymmetric K-exclusion are unknown. But we cannot calculate the flux function precisely. The proof proceeds via a coupling with a growth model on the two-dimensional lattice. In addition to the basic K-exclusion with constant exponential jump rates, we treat the site-disordered case where each site has its own jump rate, randomly chosen but frozen for all time. The hydrodynamic limit under site disorder is new even for the simple exclusion process (the case $K = 1$). Our proof makes no use of the Markov property, so at the end of the paper we indicate how to treat the case with arbitrary waiting times.

1. Introduction. The family of simple K-exclusion processes naturally interpolates between the simple exclusion process (SEP) and the zero-range process (ZRP). In K-exclusion the particles interact through the K-exclusion rule that stipulates that each site contain at most K particles. The case $K = 1$ is the SEP. The ZRP has no such restriction, so it can be thought of as the case $K = \infty$.

The SEP and ZRP are among the most fruitful models for studies of hydrodynamic behavior of interacting particle systems. By contrast, the K-exclusion processes have been harder to work with. The difficulty posed by the symmetric K-exclusion is that it violates the gradient condition [see II.2.4 in Spohn (1991)]. On the positive side, there are reversible product measures that can be easily identified. With Varadhan's (1993) nongradient techniques, the diffusive hydrodynamic limit for symmetric K-exclusion can be proved, as was demonstrated by Kipnis, Landim and Olla (1994).

In this paper we address the hydrodynamics of totally asymmetric K-exclusion. In the asymmetric case, invariant measures are unknown, and even the belief that there is a spatially ergodic equilibrium for each density remains unproved. We prove the existence of a hydrodynamic limit for the empirical particle density and a qualitative characterization of the limiting

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macroscopic density. As expected, it is the unique entropy solution of a scalar conservation law with a concave flux function. However, without knowledge of the equilibria, or without some other new idea, we cannot explicitly express the flux $f_K(\rho)$ as a function of density ρ. We can bound f_K with the explicitly known flux functions of the cases $K = 1$ and $K = \infty$.

We also prove the hydrodynamic limit for the site-disordered totally asymmetric K-exclusion. Now the jump rates of the sites are i.i.d. random variables, picked once and fixed for the duration of the dynamics. This situation goes by the term "quenched disorder." This result is new even for the case $K = 1$, the totally asymmetric simple exclusion process (TASEP). The case $K = \infty$, a ZRP with site disorder, is more accessible because it has product-form equilibria that can be written down explicitly. The hydrodynamics of the site-disordered asymmetric ZRP, or equivalently, the particle-disordered SEP, have been studied by Benjamini, Ferrari and Landim (1996), and Seppäläinen and Krug (1998).

The theorems are stated and proved in the continuous-time setting. Without essential changes, the same proofs work for the discrete-time K-exclusion process where the entire configuration is updated simultaneously at each time step. One just replaces exponential waiting times with geometric ones and Poisson point processes of jump times with i.i.d. processes of points on the positive integers. The approach also extends to non-Markovian processes, but here new proofs are needed part of the way. The non-Markovian K-exclusion operates with i.i.d. waiting times that have an arbitrary common distribution on $(0, \infty)$.

The key to our proof is a coupling of a process with an arbitrary initial configuration with a family of processes with simple initial configurations. This coupling gives a variational equation that relates the general process to the simple ones. The hydrodynamics of the simple processes can be handled by Kingman's (1968) subadditive ergodic theorem. To achieve this, the simple process is recast as a growth model on the planar lattice. This growth model in turn is formulated in terms of "vertex greedy lattice paths" [Cox, Gandolfi, Griffin and Kesten (1993)] that satisfy certain constraints on the admissible steps of the path.

Along the way we obtain a result of interest for the growth model, namely the Legendre duality of the macroscopic shape and the macroscopic velocity. The macroscopic shape is a function defined on a reference line, which in our model is the y-axis, and the macroscopic velocity is a function of the local slope. See Section 2 in Krug and Spohn (1991).

Another process closely related to the K-exclusion is the marching soldiers model. (The increments of the marching soldiers model can be transformed into a K-exclusion). As a corollary of our proofs, we obtain a hydrodynamic limit for totally asymmetric marching soldiers as well.

The strength of our approach is that it gives sharp results with minimal assumptions. For example, for the hydrodynamic limits we need no extra assumptions on the initial distributions of the processes, only the minimal assumption that a macroscopic profile exists. Furthermore, we get not only weak laws of large numbers, but also strong laws. The disadvantage is that the technique relies on the existence of a special structure, namely, the coupling with simple processes mentioned above. Other applications of related ideas to hydrodynamic limits, asymptotic shapes, and large deviations appear in Aldous and Diaconis (1995), Seppäläinen (1996, 1997a, b, 1998a, b, c) and Seppäläinen and Krug (1998).

Further relevant literature. For general accounts of particle systems and their construction with Poisson point processes of jump times we refer the reader to Durrett (1988), Griffeath (1979) and Liggett (1985). The equilibria of the zero-range process were characterized by Andjel (1982).

For hydrodynamical limits there are lectures by De Masi and Presutti (1991), the monograph of Spohn (1991) and review papers by Ferrari (1994, 1996), together with their references. For ZRP and SEP, the results that correspond to our theorems were proved by Rezakhanlou (1991). The approach of his paper was extended to some inhomogeneous models by Landim (1996) and Covert and Rezakhanlou (1997).

Organization of the paper. The reader who wants a quick look at the central ideas of the paper should set $\alpha(j) = 1$ everywhere, read Section 2 up to Theorem 1, Section 4, Section 5 up to Corollary 5.1, and Section 6. The key to the proof of the hydrodynamic limit is equation (6.5) in Section 6, to which one applies assumption (2.4) [in the form (6.3)] and Corollary 5.1.

The results for K-exclusion and the marching soldiers model, without disorder, are stated and discussed in Section 2. In this setting we prove both a weak law and a strong law for the empirical particle density. A weak law for disordered K-exclusion is stated in Section 3.

The coupling that is central to the proof is developed in Section 4. The growth model is defined and its scaling limit proved in Section 5. The law of large numbers for the interface of the growth model drives the hydrodynamic limits, through the coupling explained in Section 4.

In Section 6 the weak laws stated in Sections 2 and 3 are proved. The passage from weak law to strong law goes through a Borel–Cantelli argument. Section 7 is devoted to a summable upper tail probability estimate for the passage times of the growth model. Armed with this estimate, Section 8 proves the strong laws stated in Section 2. Section 9 develops some bounds for the limiting shape of the growth model and for the flux function of the conservation law of the particle density.

Section 10 treats the non-Markovian case. We outline the proof of the weak law of the hydrodynamic limit. With additional assumptions on the waiting times, the strong law proof generalizes as well.

Notational remarks. $\mathbf{N}$ is the set $\{1, 2, 3, \dots\}$ of natural numbers, while $\mathbf{Z}_+ = \{0, 1, 2, 3, \dots\}$. A nonnegative random variable Y is Exp(β)-distributed if $P(Y > t) = e^{-\beta t}$ for all $t > 0$, $[x] = \max\{n \in \mathbf{Z}: n \le x\}$ for $x \in \mathbf{R}$. I_A and $I\{A\}$ denote the indicator random variable of the event A.

2. Totally asymmetric simple K-exclusion. The process we study has the following informal description. Sites on the lattice $\mathbf{Z}$ are occupied by indistinguishable particles. Each site contains at most K particles. K is a positive integer constant whose value is arbitrary, but fixed. Each site $i \in \mathbf{Z}$ has an independent rate 1 Poisson point process $\mathscr{D}_i$ on the time line $0 \le t < \infty$. The epochs of the Poisson processes are potential jump times. At each epoch of $\mathscr{D}_i$, this event takes place: if there is at least one particle at i and at most $K - 1$ particles at $i + 1$, one particle from site i jumps to site $i + 1$. In other words, jumps are executed as long as there is a particle to jump and as long as the K-exclusion rule is not violated. All jumps proceed to the right, hence the modifier "totally asymmetric" in the name of the process. The modifier "simple" refers to the restriction that only nearest-neighbor jumps are allowed.

The state of the process is a configuration $\eta = (\eta(i)\colon i \in \mathbf{Z})$ of occupation numbers, where $\eta(i) \in \{0, 1, \dots, K\}$ specifies the number of particles at site i. The compact state space of the process is $X = \{0, 1, \dots, K\}^{\mathbf{Z}}$. The dynamics can be conveniently represented by the generator L of the process that acts on cylinder functions f on X:

$$Lf(\eta) = \sum_{i \in \mathbf{Z}} I_{\{\eta(i) \ge 1,\, \eta(i+1) \le K-1\}} \big[f(\eta^{i,i+1}) - f(\eta) \big], \tag{2.1}$$

where $\eta^{i,i+1}$ is the configuration that results from the jump of a single particle from site i to site $i + 1$,

$$\eta^{i,i+1}(j) = \begin{cases} \eta(i) - 1, & j = 1, \\ \eta(i+1) + 1, & j = i + 1, \\ \eta(j), & j \ne i, i+1. \end{cases}$$

We shall construct the process through a graphical representation, so the generator is of interest only as a precise statement of the dynamics. The process is denoted by $\eta(t) = (\eta(i,t)\colon i \in \mathbf{Z})$ where $t \ge 0$ is the time variable.

We are interested in the scaling behavior of the particle density of the process. By suitably scaling space and time we expect to find a deterministic evolution around which the actual particle density fluctuates. Think of an asymmetric random walk. Its position has a nonrandom limit if space shrinks by a factor n, time speeds up by the same factor n and then n goes to infinity. This suggests the scaling for nontrivial behavior in the totally asymmetric K-exclusion. As we shall prove, under this scaling the particle density converges to a nonrandom density function $u(x,t)$ that is determined by the initial density, by a conservation law, and by the entropy criterion for picking the relevant solution from the many possible weak solutions.

To make this precise, suppose we have a sequence of K-exclusion processes $\eta_n(\cdot)$, $n = 1, 2, 3, \dots$. The empirical particle density of the process is the random Radon measure on $\mathbf{R}$ defined by

$$\pi_n(t) = \frac{1}{n} \sum_{i \in \mathbf{Z}} \eta_n(i,t)\, \delta_{i/n}, \tag{2.2}$$

where δ_x is a unit mass at the point $x \in \mathbf{R}$. To clarify, the integral of an element $\phi \in C_0(\mathbf{R})$ (the space of continuous, compactly supported functions on $\mathbf{R}$) against the measure $\pi_n(t)$ is given by

$$\pi_n(t, \phi) = \frac{1}{n} \sum_{i \in \mathbf{Z}} \eta_n(i, t) \phi(i/n).$$

Radon measures are topologized weakly by such functions: $\mu_n \to \mu$ in the space of Radon measures on $\mathbf{R}$ if $\mu_n(\phi) \to \mu(\phi)$ for all $\phi \in C_0(\mathbf{R})$.

Notice that π_n already incorporates the space scaling that shrinks the lattice distance to $1/n$. In the theorem below, this space scaling is complemented by the time scaling that requires us to look at $\pi_n(nt)$ instead of $\pi_n(t)$. The final ingredient is the assumption that the initial distribution of the process has a well-defined macroscopic density. Let u_0 be a measurable function on $\mathbf{R}$ such that

$$0 \le u_0(x) \le K. \tag{2.3}$$

Write P_n for the probability measure on the probability space of the process $\eta_n(\cdot)$.

(2.4) *For all* $\phi \in C_0(\mathbf{R})$ *and* $\varepsilon > 0$,
$$\lim_{n \to \infty} P_n\left(\left| \pi_n(0, \phi) - \int_{\mathbf{R}} \phi(x) u_0(x)\, dx \right| \ge \varepsilon \right) = 0.$$

This assumption says that the random measure π_n satisfies a weak law of large numbers at time $t = 0$, and the limit is a deterministic measure $u_0(x)\,dx$. The theorem is that such a law of large numbers continues to hold at later times. In addition, we prove a strong law under this assumption:

(2.5) The random initial configurations $(\eta_n(i,0)\colon i \in \mathbf{Z})$, $n \in \mathbf{N}$, are all defined on a common probability space, and for all $\phi \in C_0(\mathbf{R})$, $\lim_{n\to\infty} \pi_n(0, \phi) = \int_{\mathbf{R}} \phi(x) u_0(x)\, dx$ almost surely.

We need a brief discussion about entropy solutions before stating the theorem. Let $g_K(x)$ be a nonincreasing, nonnegative convex function on $\mathbf{R}$ that satisfies

$$g_K(x) = 0 \text{ for } x \ge 1 \quad \text{and} \quad g_K(x) = -Kx \text{ for } x \le -1. \tag{2.6}$$

Given u_0 as above, pick a function U_0 that satisfies

$$U_0(b) - U_0(a) = \int_{(a,b]} u_0(x)\, dx \quad \text{for all } a < b. \tag{2.7}$$

Then for $x \in \mathbf{R}$ and $t > 0$ define

$$U(x,t) = \sup_{y \in \mathbf{R}} \left\{ U_0(y) - t g_K\left(\frac{x-y}{t} \right) \right\}. \tag{2.8}$$

The supremum is in fact attained at some $y \in [x - t, x + t]$. For each fixed $t > 0$, $U(\cdot, t)$ is nondecreasing and Lipschitz continuous with Lipschitz constant K. All this follows from (2.3), 2.6) and (2.7), because by convexity g_K has slope in $[-K, 0]$. Let

$$u(x,t) = \frac{\partial}{\partial x} U(x,t). \tag{2.9}$$

Define the flux function f_K as the Legendre conjugate of (the negative of) g_K:

$$f_K(\rho) = \inf_x \{\rho x + g_K(x)\}, \qquad 0 \le \rho \le K. \tag{2.10}$$

Then $U(x,t)$ is the unique viscosity solution of the Hamilton–Jacobi equation

$$U_t + f_K(U_x) = 0, \qquad U(x,0) = U_0(x), \tag{2.11}$$

while $u(x,t)$ is the unique entropy solution of the scalar conservation law

$$u_t + f_K(u)_x = 0, \qquad u(x,0) = u_0(x). \tag{2.12}$$

Equation (2.8) is well-known in the literature on partial differential equations. The names of Hopf, Lax and Oleinik are attached to it in various combinations by different authors. Here are some of the relevant references: Bardi and Evans (1984), Evans (1984), Hopf (1965), Lax (1957), Lions (1982), and Lions, Souganidis and Vásquez (1987).

In our proof we use only formula (2.8) and nothing else about the partial differential equations. In fact, the centerpiece of our proof is a microscopic version of (2.8) [(4.9) in Section 4] that emerges from the graphical construction of the K-exclusion.

THEOREM 1. *For each positive integer K, there exists a nonnegative convex function g_K that satisfies* (2.6), *and such that this holds: Let $u(x,t)$ be defined by* (2.7)–(2.9). *Under assumption* (2.4), *for each $t > 0$ the random measure $\pi_n(nt)$ converges in probability to $u(x,t)\,dx$. Precisely, for all $\phi \in C_0(\mathbf{R})$ and $\varepsilon > 0$,*

$$\lim_{n\to\infty} P_n\left(\left| \pi_n(nt, \phi) - \int_{\mathbf{R}} \phi(x) u(x,t)\, dx \right| \ge \varepsilon \right) = 0. \tag{2.13}$$

Under assumption (2.5) *we can construct the processes $\eta_n(\cdot)$ on a common probability space so that the strong law holds: $\pi_n(nt) \to u(x,t)\,dx$ almost surely as $n \to \infty$, for all $t > 0$.*

Properties of f_K. Only in the case $K = 1$, the totally asymmetric simple exclusion process (TASEP), has f_K been computed: $f_1(\rho) = \rho(1-\rho)$. This was first done by Rost (1981), for a special class of initial profiles.

Presently we can prove that, if the initial distribution of the process is spatially ergodic with expectation $E[\eta(i,0)] = \rho$, then

$$f_K(\rho) = \lim_{t\to\infty} \frac{1}{t}\int_0^t P(\eta(i,s) \ge 1, \eta(i+1,s) \le K-1)\, ds. \tag{2.14}$$

(See Section 9.) The scenario one expects is that for each density $\rho \in [0, K]$ there is a unique spatially ergodic equilibrium measure ν_ρ with expectation

$\nu_\rho[\eta(0)] = \rho$. From this and (2.14), it would follow that

$$f_K(\rho) = \nu_\rho(\eta(i) \ge 1, \eta(i+1) \le K-1). \tag{2.15}$$

Partial results on the existence and uniqueness of equilibria appear in the unpublished manuscript of Ekhaus and Gray (1994).

Here are some general properties of f_K. Convexity of g_K and definition (2.10) imply that f_K is concave. On $[0, K/2]$ f_K is nondecreasing and on $[K/2, K]$ nonincreasing. We have symmetry

$$f_K(\rho) = f_K(K-\rho) \quad \text{for } \rho \in [0, K] \tag{2.16}$$

and monotonic dependence on K:

$$f_K(\rho) \le f_{K+1}(\rho) \quad \text{for } \rho \in [0, K]. \tag{2.17}$$

Monotonicity, symmetry and the explicitly known f_1 give a lower bound,

$$f_K(\rho) \ge \begin{cases} \rho(1-\rho), & 0 \le \rho \le 1/2, \\ 1/4, & 1/2 \le \rho \le K - 1/2, \\ (K-\rho)(1-(K-\rho)), & K - 1/2 \le \rho \le K. \end{cases} \tag{2.18}$$

An upper bound is obtained by comparison with the zero-range process,

$$f_K(\rho) \le \begin{cases} \rho/(1+\rho), & 0 \le \rho \le K/2, \\ (K-\rho)/(1+K-\rho), & K/2 \le \rho \le K. \end{cases} \tag{2.19}$$

The function g_K is more fundamental to our approach than the flux f_K, which is derived from g_K by (2.10) after the proof is complete. In Section 5 we define g_K as the asymptotic shape of a growth model that is closely associated with the K-exclusion process. Properties of both g_K and f_K are derived in Section 9.

The marching soldiers model. As a corollary of our proof, we also get a hydrodynamic limit for a totally asymmetric marching soldiers model. The rules for this process are these: For each integer $i \in \mathbf{Z}$ there is a soldier $\sigma(i)$ whose location at time t is $\sigma(i,t) \in \mathbf{Z}$. Neighboring soldiers are not allowed to be too far apart, so there are constants $L_1, L_2 \in \mathbf{Z}_+$ such that

$$-L_1 \le \sigma(i+1,t) - \sigma(i,t) \le L_2 \quad \text{for all } i \in \mathbf{Z} \text{ and } t \ge 0. \tag{2.20}$$

Subject to condition (2.20), the soldiers march forward at exponential rate 1, independently of each other. We state and prove a theorem that corresponds to the strong law part of Theorem 1. Suppose V_0 is a function on $\mathbf{R}$ that satisfies

$$-L_1(x-y) \le V_0(x) - V_0(y) \le L_2(x-y) \quad \text{for all } y < x. \tag{2.21}$$

We make this assumption:

(2.22) A sequence of random initial configurations $(\sigma_n(i,0): i \in \mathbf{Z})$, $n \in \mathbf{N}$, is defined on a probability space, and for all $y \in \mathbf{R}$, $\lim_{n\to\infty}(1/n)\sigma_n([ny],0) = V_0(y)$ almost surely.

THEOREM 2. *Under assumption* (2.22) *we can construct the processes* $\sigma_n(\cdot)$ *on a common probability space so that the strong law continues to hold*:

$$\lim_{n\to\infty} \frac{1}{n}\sigma_n([nx], nt) = V(x,t) \quad a.s.$$

for all $t > 0$ *and* $x \in \mathbf{R}$. *The limit* $V(x,t)$ *satisfies*

$$V(x,t) = \inf_{y\in\mathbf{R}} \left\{V_0(y) - L_1(x-y) + tg_{L_1+L_2}((y-x)/t)\right\}. \tag{2.23}$$

The function $g_{L_1+L_2}$ is the same as the one appearing in Theorem 1. By the remarks preceding Theorem 1, Theorem 2 implies that the motion of the soldiers' front is governed by the differential equation $\partial V/\partial t = f_{L_1+L_2}(L_1 + \partial V/\partial x)$, where $f_{L_1+L_2}$ is the function defined by (2.10).

If we consider just a single process [$\sigma_n(\cdot) = \sigma(\cdot)$ for all n], then (2.22) implies that $V_0(y) = \rho y$ for some $-L_1 \le \rho \le L_2$. Equation (2.23) gives $V(x,t) = \rho x + tf_{L_1+L_2}(L_1 + \rho)$. We see that from an initial slope ρ, each soldier has the same well-defined asymptotic speed,

$$\lim_{t\to\infty} \frac{1}{t}\sigma(i,t) = V(0,1) = f_{L_1+L_2}(L_1 + \rho) \quad \text{a.s.} \tag{2.24}$$

for any fixed i. By (2.18), the asymptotic speed is nonzero if $-L_1 < \rho < L_2$.

The $M/M/1/K$ queueing picture. There is a natural queueing interpretation for K-exclusion. The sites represent a sequence of servers, and the particles are customers moving through the system. Servers serve at rate 1, and the queues at the servers have capacity K and FIFO discipline. The ordering of customers is preserved by this convention: The queue of $\eta(i)$ customers at server i is represented by an ordered stack, with the oldest customer at the bottom and currently in service and with the most recently arrived customer at the top. When a jump from site i occurs, the bottom customer of queue i leaves (his service was completed at server i), and he becomes the top customer of queue $i+1$. In Section 4 we introduce another particle system where the customers are fixed in space and the servers jump past the customers. To keep the two processes separate we shall call the K-exclusion $\eta(\cdot)$ the *customer process*.

3. Disordered *K*-exclusion. The disordered process operates as the one described in Section 2, except for this difference: the rate of jumping from site i is now a number $\alpha(i)$, instead of a uniform rate 1 for all sites. Let $\boldsymbol{\alpha} = (\alpha(i): i \in \mathbf{Z})$ denote the sequence of jump rates. Once $\boldsymbol{\alpha}$ is picked, the generator (2.1) becomes

$$L^{\boldsymbol{\alpha}} f(\eta) = \sum_{i\in\mathbf{Z}} \alpha(i) I_{\{\eta(i)\ge 1,\, \eta(i+1)\le K-1\}}\left[f(\eta^{i,i+1}) - f(\eta)\right]. \tag{3.1}$$

We assume that $a_0 \le \alpha(i) \le 1$ for all $i \in \mathbf{Z}$, where $a_0 > 0$ is a fixed constant. Thus $\boldsymbol{\alpha}$ is an element of the space $\mathscr{A} = [a_0, 1]^{\mathbf{Z}}$. We want random jump rates, so we assume that we have a probability measure Q on $\mathscr{A}$ under which the $\alpha(i)$'s are i.i.d. random variables. The marginal distribution of $\alpha(0)$ on $[a_0, 1]$ is arbitrary.

The idea of *quenched disorder* is that $\boldsymbol{\alpha}$, though random, is picked once and fixed for the duration of the dynamics. So, in the setting of a sequence of processes $\eta_n(\cdot)$ described in Section 2, we assume that a choice of $\boldsymbol{\alpha} \in \mathscr{A}$ is made at the outset, and then each process $\eta_n(\cdot)$ operates under these same rates, so that jumps from site i always occur at rate $\alpha(i)$. Write $P_n^{\boldsymbol{\alpha}}$ for the probability measure of the nth process $\eta_n(\cdot)$ with rates $\boldsymbol{\alpha}$. We only prove a weak law here, and the assumption is the same as for Theorem 1.

(3.2) *For all $\phi \in C_0(\mathbf{R})$ and $\varepsilon > 0$,*

$$\lim_{n\to\infty} P_n^{\boldsymbol{\alpha}}\left(\left|\pi_n(0,\phi) - \int_{\mathbf{R}} \phi(x) u_0(x)\, dx\right| \geq \varepsilon\right) = 0.$$

Let $a_1 = (E^Q[\alpha(0)^{-1}])^{-1} \in [a_0, 1]$. Property (2.6) of the basic setting now becomes

$$g_K(y) = 0 \text{ for } y \geq a_1 \quad \text{and} \quad g_K(y) = -Ky \text{ for } y \leq -a_1. \tag{3.3}$$

THEOREM 3. *Fix a positive integer K and an i.i.d. distribution Q for $\boldsymbol{\alpha} = (\alpha(i))$. Then there exists a nonnegative convex function g_K that depends on Q and has property* (3.3), *and a subset $\mathscr{A}_0 \subseteq \mathscr{A}$ such that $Q(\mathscr{A}_0) = 1$ and such that the following statement holds for g_K and all $\boldsymbol{\alpha} \in \mathscr{A}_0$: if assumption* (3.2) *is satisfied and $u(x,t)$ is defined by* (2.7)–(2.9), *then for each $t > 0$, $\phi \in C_0(\mathbf{R})$, and $\varepsilon > 0$:*

$$\lim_{n\to\infty} P_n^{\boldsymbol{\alpha}}\left(\left|\pi_n(nt,\phi) - \int_{\mathbf{R}} \phi(x) u(x,t)\, dx\right| \geq \varepsilon\right) = 0. \tag{3.4}$$

We emphasize that the good set $\mathscr{A}_0$ of disorder variables $\boldsymbol{\alpha}$ does not depend on the initial distributions of the processes $\eta_n(\cdot)$. The way this set is constructed can be seen from Proposition 5.2 below.

Notice also this: Even though assumption (3.2) is stated in terms of the measures $P_n^{\boldsymbol{\alpha}}$, the disorder variable $\boldsymbol{\alpha}$ really has nothing to do with the initial distribution of the process. The rates $\boldsymbol{\alpha}$ are fixed, and then the process is constructed on the product probability space of the initial distribution and of the Poisson processes $\{\mathscr{D}_i\}$ of the graphical construction (see Section 4).

It is of course possible to look at the process simultaneously under many or all $\boldsymbol{\alpha}$'s and to pick initial distributions that depend on $\boldsymbol{\alpha}$. Then Theorem 3 says that (3.4) holds for each $\boldsymbol{\alpha}$ in $\mathscr{A}_0$ for which assumption (3.2) is valid.

The assumption that $(\alpha(i))$ are i.i.d. as opposed to merely ergodic is used only in the proof of Lemma 5.4, to obtain a probability estimate good enough for the Borel–Cantelli lemma. Consequently, any mixing assumption on $(\alpha(i))$ strong enough to give a summable bound in Lemma 5.4 is good enough for our proof of Theorem 3. The existence of g_K, and thereby the existence of f_K through (2.10), requires only ergodicity for $(\alpha(i))$. (See Proposition 5.1.) Ergodicity of $(\alpha(i))$ is sufficient for K-monotonicity (2.17) of f_K. We prove symmetry (2.16) under reversibility of $(\alpha(i))$ (Lemma 9.1), which we have in the i.i.d case. A recent result of Goldstein and Speer (1998) suggests that symmetry may hold much more generally.

If the initial distribution of the process is spatially ergodic with expectation $E[\eta(i,0)] = \rho$, then (2.14) takes the form

$$(3.5) \qquad f_K(\rho) = \alpha(0) \lim_{t\to\infty} \frac{1}{t}\int_0^t P^{\alpha}(\eta(0,s) \ge 1, \eta(1,s) \le K-1)\, ds,$$

for any $\boldsymbol{\alpha} \in \mathscr{A}_0$. The reader may find it somewhat surprising that the right-hand side of (3.5) should converge to something independent of $\boldsymbol{\alpha}$. We can check the reasonableness of this contention by comparing with a site-disordered zero-range process for which explicit calculations are possible. If we take $K = \infty$ in the generator (3.1), we get a zero-range process with product-form equilibria $\nu_\rho^{\boldsymbol{\alpha}}$. Under $\nu_\rho^{\boldsymbol{\alpha}}$ the variables $(\eta(i)\colon i \in \mathbf{Z})$ are mutually independent with geometric marginals

$$(3.6) \quad \nu_\rho^{\boldsymbol{\alpha}}(\eta(i) = k) = (1 - v/\alpha(i))(v/\alpha(i))^k, \qquad k = 0, 1, 2, 3, \dots.$$

Here $v \in (0, a_0)$ is the average jump rate, common to all sites, and determined by the density ρ through the equation $\rho = E^Q[v/(\alpha(0) - v)]$. [See Benjamini, Ferrari and Landim (1996) or Seppäläinen and Krug (1998).] If the zero-range process is started with $\nu_\rho^{\boldsymbol{\alpha}}$, then the right-hand side of (3.5) becomes $\alpha(0)\nu_\rho^{\boldsymbol{\alpha}}(\eta(0) \ge 1) = v$, as expected.

In case the reader wonders about the necessity to assume $a_0 > 0$: if we allowed rates arbitrarily close to zero, the particle profile would not move on the hydrodynamic scale. See the last paragraph of Section 6.

4. The coupling. We begin by constructing the disordered K-exclusion process. Pick and fix the rates $\boldsymbol{\alpha} = (\alpha(i))$. The reader only interested in the basic case without disorder should set $\alpha(i) \equiv 1$ throughout this section. As hinted in the remark about the queueing interpretation in Section 2, we do the construction [and prove the theorems] through the corresponding *server process* $z(\cdot)$.

This process $z(t) = (z(i,t)\colon i \in \mathbf{Z})$ satisfies the following rules: (1) Its state is an ordered, labeled particle configuration $z = (z(i)\colon i \in \mathbf{Z}) \in \mathbf{Z}^{\mathbf{Z}}$ with $z(i) \le z(i+1) \le z(i) + K$ for all $i \in \mathbf{Z}$. (2) The dynamics is determined by a collection $\{\mathscr{D}_i\}$ of mutually independent Poisson processes on $[0,\infty)$, where $\mathscr{D}_i$ has rate $\alpha(i)$. At the epochs of $\mathscr{D}_i$, $z(i)$ attempts to jump one step to the *left*, and the jump is executed unless it threatens to violate the inequalities $z(i) \ge z(i-1)$ and $z(i) \ge z(i+1) - K$.

It is clear that if we construct a process $z(\cdot)$ with these two properties, then the disordered K-exclusion of Section 3 can be constructed by defining

$$(4.1) \qquad \eta(i,t) = z(i,t) - z(i-1,t).$$

In other words, particle (server) $z(i)$ jumps precisely when a K-exclusion particle (customer) jumps from site i to $i+1$. All our remaining work will be in terms of the server process $z(\cdot)$, and the results for the customer process follow from (4.1).

Construction of the server process $z(\cdot)$. This goes by standard arguments. Assume given an initial configuration $(z(i,0)\colon i \in \mathbf{Z})$ that satisfies rule (1) above. The bound $\alpha(i) \le 1$ on the rates assures that the following assumptions are valid for almost every realization of the Poisson point processes:

(4.2a) The $\{\mathscr{D}_i\}$ are such that there are no simultaneous jump attempts.
(4.2b) Each $\mathscr{D}_i$ has only finitely many epochs in any bounded time interval.
(4.2c) Given any $t_1 > 0$, there are arbitrarily faraway indices $i_0 \ll 0 \ll i_1$ such that neither $\mathscr{D}_{i_0}$ nor $\mathscr{D}_{i_1}$ has any epochs in the time interval $[0, t_1]$.

Then $z(i_0)$ and $z(i_1)$ do not attempt to jump during $[0, t_1]$, and the evolution of the servers $\{z(i_0+1), \dots, z(i_1-1)\}$ is isolated from the rest of the process up to time t_1. There are only finitely many jump attempts in $\bigcup_{i_0 \le i \le i_1} \mathscr{D}_i \cap [0, t_1]$, so they can be ordered, and then the evolution of $z(i,t)$, for $i_0 \le i \le i_1$ and $0 \le t \le t_1$, can be computed by considering the jump attempts in their temporal order and applying rule (2). This completes the construction of $z(\cdot)$. With $\boldsymbol{\alpha} = (\alpha(i))$ and the initial $(z(i,0))$ fixed, the process is defined on the probability space of the $\{\mathscr{D}_i\}$.

This construction can be used in a standard way to produce couplings that preserve order. Proof of the following lemma is left to the reader.

LEMMA 4.1. *Pick a realization* $\{\mathscr{D}_i\}$ *of the Poisson processes that satisfies properties* (4.2a)–(4.2c). *Suppose* $w(t) = (w(i,t)\colon i \in \mathbf{Z})$ *and* $y(t) = (y(i,t)\colon i \in \mathbf{Z})$ *are server processes constructed exactly as described above, from initial configurations* $(w(i,0)\colon i \in \mathbf{Z})$ *and* $(y(i,0)\colon i \in \mathbf{Z})$, *and so that for each* i, *server particles* $w(i)$ *and* $y(i)$ *read their jump attempts from* $\mathscr{D}_i$. *Then if initially* $w(i,0) \le y(i,0)$ *for all* $i \in \mathbf{Z}$, *the same ordering* $w(i,t) \le y(i,t)$ *holds for all later times* $t \ge 0$ *and for all* $i \in \mathbf{Z}$.

Write $w^l(\cdot)$ *for the server process with this special initial configuration*:

$$w^l(j,0) = \begin{cases} l, & j \ge 0, \\ l + Kj, & j < 0. \end{cases} \tag{4.3}$$

These processes occupy a special role in our development, hence the distinct notation. From the full collection $\{w^l(\cdot)\colon l \in \mathbf{Z}\}$, we actually construct only the subcollection $\{w^{z(i,0)}(\cdot)\colon i \in \mathbf{Z}\}$ for a given initial configuration $(z(i,0)\colon i \in \mathbf{Z})$ of the $z(\cdot)$-process. Once the rates $\boldsymbol{\alpha} = (\alpha(i))$ and a realization $\{\mathscr{D}_i\}$ of the Poisson event times have been chosen, we construct the processes $w^{z(i,0)}(\cdot)$ so that particle $w^{z(i,0)}(j)$ reads its jump attempts from the Poisson process $\mathscr{D}_{i+j}$. In particular,

$$\text{the jump rate of particle } w^{z(i,0)}(j) \text{ is } \alpha(i+j). \tag{4.4}$$

Now $z(\cdot)$ and $\{w^{z(i,0)}(\cdot)\}$ are coupled, that is, constructed on the same probability space. We have the following crucial lemma.

LEMMA 4.2. *Assume that* $\{\mathscr{D}_i\}$ *satisfies assumptions* (4.2a)–(4.2c). *Then the following equality holds for all* $k \in \mathbf{Z}$ *and* $t \geq 0$:

$$z(k,t) = \sup_{i \in \mathbf{Z}} w^{z(i,0)}(k-i,t). \tag{4.5}$$

PROOF. Choosing i_0 and i_1 for t_1 as in (4.2c), it suffices to prove inductively that (4.5) holds after each epoch of $\bigcup_{i_0<k<i_1} \mathscr{D}_k \cap [0,t_1]$, for $i_0 < k < i_1$. Notice that all the particles involved in (4.5) for a particular k read their jump commands from $\mathscr{D}_k$, so the validity of (4.5) for k cannot change at epochs of $\mathscr{D}_j$ for $j \neq k$.

Equation (4.5) certainly holds at time $t = 0$; the supremum is realized by $i = k$, by any $i > k$ such that $z(i,0) = z(k,0) + (i-k)K$ and by any $i < k$ such that $z(i,0) = z(k,0)$. Also, (4.5) holds for $k = i_0$ and $k = i_1$ for all $t \in [0,t_1]$ because none of the particles involved even attempts to jump during this time interval.

Now suppose τ is an epoch of the Poisson process $\mathscr{D}_k$ for some $i_0 < k < i_1$, and assume by induction that (4.5) holds for all $i_0 \leq k \leq i_1$ and $t \in [0,\tau)$. Three cases need to be considered.

CASE 1 [$z(k)$ jumps at time τ]. We must show that if $w^{z(i,0)}(k-i,\tau-) = z(k,\tau-)$, then $w^{z(i,0)}(k-i)$ also jumps at time τ. Since $z(k)$ can execute its jump, we must have $z(k-1,\tau-) \leq z(k,\tau-) - 1$ and $z(k+1,\tau-) \leq z(k,\tau-) + K - 1$. By the induction assumption,

$$\begin{aligned} w^{z(i,0)}(k-i-1,\tau-) &\leq z(k-1,\tau-) \leq z(k,\tau-) - 1 \\ &= w^{z(i,0)}(k-i,\tau-) - 1 \end{aligned}$$

and

$$\begin{aligned} w^{z(i,0)}(k-i+1,\tau-) &\leq z(k+1,\tau-) \leq z(k,\tau-) + K - 1 \\ &= w^{z(i,0)}(k-i,\tau-) + K - 1, \end{aligned}$$

so that $w^{z(i,0)}(k-i)$ can also jump at time τ. Since this argument applies to any i such that $w^{z(i,0)}(k-i,\tau-) = z(k,\tau-)$, (4.5) continues to hold for k at time τ.

CASE 2 [$z(k)$ does not jump at time τ because $z(k-1,\tau-) = z(k,\tau-)$]. By induction there exists an i such that $z(k-1,\tau-) = w^{z(i,0)}(k-i-1,\tau-)$, and consequently, again by induction,

$$\begin{aligned} z(k,\tau-) = z(k-1,\tau-) &= w^{z(i,0)}(k-i-1,\tau-) \\ &\leq w^{z(i,0)}(k-i,\tau-) \leq z(k,\tau-), \end{aligned}$$

from which we conclude that $w^{z(i,0)}(k-i,\tau-) = z(k,\tau-)$ and that $w^{z(i,0)}(k-i)$ cannot jump at time τ because it is blocked by $w^{z(i,0)}(k-i-1)$. Thus $z(k,\tau) = w^{z(i,0)}(k-i,\tau)$, and (4.5) continues to hold for k at time τ.

CASE 3 [$z(k)$ does not jump at time τ because $z(k+1, \tau-) = z(k, \tau-) + K$]. This case is in principle like the previous one, and we leave the details to the reader. $\square$

An application of Lemma 4.1 to $w(k,t) = w^{z(j,0)}(k-j,t)$ and $y(k,t) = w^{z(i,0)}(k-i,t) + z(j,0) - z(i,0)$ gives the next lemma.

LEMMA 4.3. *For $i < j$, all k, and all $t \ge 0$,*

$$w^{z(i,0)}(k-i,t) \ge w^{z(j,0)}(k-j,t) - [z(j,0) - z(i,0)].$$

The evolution of the processes $w^{z(i,0)}(\cdot)$ comes from two sources: the initial configuration $(z(i,0)\colon\ i \in \mathbf{Z})$ that determines the initial configurations $(w^{z(i,0)}(j,0)\colon\ j \in \mathbf{Z})$, and the increments $w^{z(i,0)}(j,t) - w^{z(i,0)}(j,0)$ that come from the Poisson processes. To explicitly separate these, define the nonnegative, increasing processes $\xi^{z(i,0)}(\cdot)$ by

$$\xi^{z(i,0)}(j,t) = z(i,0) - w^{z(i,0)}(j,t), \qquad j \in \mathbf{Z},\ t \ge 0. \tag{4.6}$$

We regard $\xi^{z(i,0)}$ as the position of an interface that moves to the right on the plane. At time t, $\xi^{z(i,0)}(j,t)$ is the distance at level j from the y-axis to the interface. Once the servers $w^{z(i,0)}(j)$ have been assigned rates as in (4.4), the dynamics of $\xi^{z(i,0)}$ is determined by the rules of the server process $w^{z(i,0)}$. Initially

$$\xi^{z(i,0)}(j,0) = 0 \text{ for } j \ge 0 \quad \text{and} \quad \xi^{z(i,0)}(j,0) = -Kj \text{ for } j < 0. \tag{4.7}$$

Then $\xi^{z(i,0)}(j)$ jumps one step to the right at rate $\alpha(i+j)$, under the restrictions

$$\xi^{z(i,0)}(j,t) \le \xi^{z(i,0)}(j-1,t) \le \xi^{z(i,0)}(j,t) + K. \tag{4.8}$$

By (4.6) the process $\xi^{z(i,0)}(\cdot)$ registers only increments of the $w^{z(i,0)}(\cdot)$-process and not absolute location. In particular, the distribution of the process $\xi^{z(i,0)}(\cdot)$ does not depend on the actual location $z(i,0)$ in the initial z-configuration, and this distribution depends on the index i only by a translation of the rates $\boldsymbol{\alpha}$. Consequently, in the basic case without disorder [$\alpha(j) \equiv 1$], the processes $\{\xi^{z(i,0)}(\cdot)\colon\ i \in \mathbf{Z}\}$ are identically distributed.

With these interface processes, we rewrite (4.5) as

$$z(k,t) = \sup_{i \in \mathbf{Z}} \{z(i,0) - \xi^{z(i,0)}(k-i,t)\}. \tag{4.9}$$

This equation is the basis for all our results. In the next section we develop an alternative definition for the process ξ to prove its scaling limit. As the final point of this section we state one lemma about (4.9) for future use.

LEMMA 4.4. *Suppose there are $j_1 < k < j_2$ such that*

$$\xi^{z(j_1,0)}(k-j_1,t) = 0 \quad \text{and} \quad \xi^{z(j_2,0)}(k-j_2,t) = -K(k-j_2).$$

Then

$$z(k,t) = \max_{j_1 \le i \le j_2} \{z(i,0) - \xi^{z(i,0)}(k-i,\ t)\}.$$

PROOF. For $i < j_1$,

$$\begin{aligned} z(i,0) - \xi^{z(i,0)}(k-i,t) &\le z(i,0) \\ &\le z(j_1,0) \\ &= z(j_1,0) - \xi^{z(j_1,0)}(k-j_1,t), \end{aligned}$$

and consequently

$$z(k,t) = \sup_{i:\, i \ge j_1} \{z(i,0) - \xi^{z(i,0)}(k-i,t)\}.$$

We leave the other half to the reader. □

5. The growth model. In this section we study the planar growth model encountered at the end of the previous section. We develop a "last-passage" formulation for it (as opposed to "first-passage" percolation) to apply the subadditive ergodic theorem. After Proposition 5.1 we define the function g_K that appears in Theorems 1–3. The reader who wishes to consider only the basic case of Section 2, without disorder, should set $\alpha(j) \equiv 1$ throughout this section.

For each K there is a notion of a *K-admissible path*. A K-admissible path on the lattice $\mathbf{Z}^2$ is a finite sequence $r = \{\kappa_1, \kappa_2, \dots, \kappa_m\}$ of sites or points of $\mathbf{Z}^2$ that satisfy this constraint: There are three admissible kinds of steps in the path. For each $i = 1, \dots, m-1$,

$$\kappa_{i+1} - \kappa_i = (1,0), (0,1), \text{ or } (K, -1). \tag{5.1}$$

Geometrically speaking, a K-admissible path can take steps up, steps right, and chess knight-type steps K lattice increments to the right and one down. (In the case $K = 2$ this is exactly a knight's move in chess. For $K = 1$ it is actually a bishop's move diagonally southeast, while for $K \ge 3$ it is a "stretched" knight's move.) The empty path with $m = 0$ is also deemed K-admissible.

Let $\mathscr{L}_K$ denote the set of integer sites that can be reached from the point $(1,0)$ along K-admissible paths. Quite obviously

$$\mathscr{L}_K = \{(k,l) \in \mathbf{N} \times \mathbf{Z}\colon l \ge -[(k-1)/K]\}. \tag{5.2}$$

This set is a wedge bounded by the positive y-axis and a lattice equivalent of the line with slope $-1/K$. All K-admissible paths started inside $\mathscr{L}_K$ remain inside $\mathscr{L}_K$.

Next we introduce the passage times of sites and paths. Let $\boldsymbol{\lambda} = (\lambda(i,j)\colon (i,j) \in \mathbf{Z}^2)$ be an assignment of nonnegative numbers to the sites of $\mathbf{Z}^2$. To the points of the lattice y-axis, attach the quenched rate variables $\boldsymbol{\alpha} = (\alpha(j)\colon j \in \mathbf{Z})$. The passage time of site (i,j) is then $\tau(i,j) = \alpha(j)^{-1}\lambda(i,j)$. The configuration $\boldsymbol{\lambda}$ is an element of the space $\mathscr{S} = [0,\infty)^{\mathbf{Z}^2}$, while the sequence $\boldsymbol{\alpha}$ is an element of the space $\mathscr{A} = [a_0, 1]^{\mathbf{Z}}$ as defined in Section 3. Elements of the product space $\Sigma = \mathscr{S} \times \mathscr{A}$ are indexed by $\mathbf{Z}^2$ in the obvious way: $(\boldsymbol{\lambda}, \boldsymbol{\alpha})(i,j) = (\lambda(i,j), \alpha(j))$. Define translation operators $\Theta(k,l), (k,l) \in \mathbf{Z}^2$, on Σ by

$$[\Theta(k,l)(\boldsymbol{\lambda}, \boldsymbol{\alpha})](i,j) = (\lambda(i+k, j+l), \alpha(j+l)). \tag{5.3}$$

Let μ be the probability measure on $\mathscr{S}$ under which the $(\lambda(i,j))$ are i.i.d. Exp(1)-distributed, and let Q be the i.i.d. measure on $\mathscr{A}$ as denoted in Section 3. The probability measure we put on Σ is $\mathbb{P} = \mu \otimes Q$, the product measure under which $\boldsymbol{\lambda}$ and $\boldsymbol{\alpha}$ are mutually independent. Expectation under $\mathbb{P}$ is denoted by $\mathbb{E}$. When $\boldsymbol{\alpha}$ is fixed, $\mathbb{P}^{\boldsymbol{\alpha}} = \mu \otimes \delta_{\boldsymbol{\alpha}}$ is the conditional distribution on Σ. Under $\mathbb{P}^{\boldsymbol{\alpha}}$, the passage time $\tau(i,j)$ is Exp$(\alpha(j))$-distributed. This is quenched disorder, where the $\alpha(j)$'s are fixed but the $\lambda(i,j)$'s remain random. However, at this stage we regard both $\boldsymbol{\lambda}$ and $\boldsymbol{\alpha}$ random.

The passage time of a path $r = \{\kappa_1, \kappa_2, \dots, \kappa_m\}$ is simply the sum of the passage times of the sites along the path

$$T(r) = \sum_{i=1}^{m} \tau(\kappa_i). \tag{5.4}$$

For any two integer points (u_1, v_1) and (u_2, v_2) on the plane such that $(u_2, v_2) \in (u_1, v_1) + \mathscr{L}_K$, let $\mathscr{R}_K((u_1, v_1), (u_2, v_2))$ denote the collection of K-admissible paths $r = \{\kappa_1, \kappa_2, \dots, \kappa_m\}$ such that $\kappa_1 = (u_1 + 1, v_1)$ and $\kappa_m = (u_2, v_2)$, and m of course may vary. Our convention is $\kappa_1 = (u_1 + 1, v_1)$ instead of $\kappa_1 = (u_1, v_1)$ so that elements of $\mathscr{R}_K((u_1, v_1), (u_2, v_2))$ and $\mathscr{R}_K((u_2, v_2), (u_3, v_3))$ can be joined to form an element of $\mathscr{R}_K((u_1, v_1), (u_3, v_3))$ without repeating the point (u_2, v_2) The basic case is $(u_1, v_1) = (0,0)$, in which case paths start at $(1,0)$. The passage time from (u_1, v_1) to (u_2, v_2) is defined by

$$T_K\big((u_1, v_1), (u_2, v_2)\big) = \max\big\{T(r): r \in \mathscr{R}_K\big((u_1, v_1), (u_2, v_2)\big)\big\}. \tag{5.5}$$

Abbreviate $T_K(u,v) = T_K((0,0),(u,v))$.

The growth model is a randomly growing subset of $\mathscr{L}_K$, defined by proclaiming that point (i,j) joins the growing cluster at time $T_K(i,j)$. So, if $B(t)$ denotes the cluster at time t, the rule is

$$B(t) = \big\{(i,j) \in \mathscr{L}_K : T_K(i,j) \le t\big\}. \tag{5.6}$$

A random process ξ marks the location of the interface of $B(t)$, in the sense that

$$B(t) = \big\{(i,j) \in \mathscr{L}_K : i \le \xi(j,t)\big\}. \tag{5.7}$$

So the interface moves to the right, in the increasing x-direction. For $j \in \mathbf{Z}$ and $t \ge 0$, $\xi(j,t)$ denotes the distance from point j on the y-axis to the interface on the right-hand side of this axis. In terms of the passage times,

$$\xi(j,t) = \min\big\{i: (i+1,j) \in \mathscr{L}_K, T_K(i+1,j) > t\big\}. \tag{5.8}$$

Initially $\xi(j,0) = 0$ for $j \ge 0$ and $\xi(j,0) = -Kj$ for $j < 0$.

At this point the reader should observe Lemma 5.1.

LEMMA 5.1. *Fix K and the rates $\boldsymbol{\alpha}$. The distribution of the process $\xi(\cdot)$ under $\mathbb{P}^{\boldsymbol{\alpha}}$ is equal to the distribution of the process $\xi^{z(0,0)}(\cdot)$ defined by* (4.6), *for any random or deterministic initial configuration* $(z(i,0): i \in \mathbf{Z})$.

Indeed, the whole point of the definition of K-admissible paths is that the evolution of the surface $\xi(\cdot)$ replicates the evolution of $\xi^{z(0,0)}(\cdot)$ that comes from the server process. We prove Lemma 5.1 in a more general setting in Section 10 [see the proof following (10.36)], so we will not waste space on it here. But this should convince the reader: according to (5.8) and the definition of K-admissible paths, $\xi(j,t)$ increases from i to $i+1$ at rate $\alpha(j)$ provided

$$\xi(j,t) = i,$$
$$\xi(j-1,t) \geq i+1$$

and

$$\xi(j+1,t) \geq i-K+1.$$

On the other hand, by (4.6) and the definition of the server dynamics, $\xi^{z(0,0)}(j,t)$ increases from i to $i+1$ precisely when $w^{z(0,0)}(j,t)$ jumps from $z(0,0)-i$ to $z(0,0)-i-1$, and this happens at rate $\alpha(j)$ provided

$$w^{z(0,0)}(j,t) = z(0,0)-i,$$
$$w^{z(0,0)}(j-1,t) \leq z(0,0)-i-1$$

and

$$w^{z(0,0)}(j+1,t) \leq z(0,0)+K-i-1,$$

or, equivalently, provided

$$\xi^{z(0,0)}(j,t) = i,$$
$$\xi^{z(0,0)}(j-1,t) \geq i+1$$

and

$$\xi^{z(0,0)}(j+1,t) \geq i-K+1.$$

Thus both the initial condition and the infinitesimal rates for $\xi(\cdot)$ and $\xi^{z(0,0)}(\cdot)$ are the same.

Let

$$\mathscr{U}_K = \{(x,y) \in \mathbf{R}^2 : x > 0,\ y > -x/K\}$$

be the continuum analogue of the lattice $\mathscr{L}_K$.

PROPOSITION 5.1. *There exists a finite, concave function $\gamma_K(x,y)$ defined on $\mathscr{U}_K$ and a subset $\Sigma_0 \subseteq \Sigma$ such that $\mathbb{P}(\Sigma_0) = 1$, and*

$$\lim_{n\to\infty} \frac{1}{n} T_K([nx],[ny]) = \gamma_K(x,y) \tag{5.9}$$

holds for all $(x,y) \in \mathscr{U}_K$ and all $(\boldsymbol{\lambda},\boldsymbol{\alpha}) \in \Sigma_0$. Furthermore, γ_K can be extended to a continuous function on the closure $\overline{\mathscr{U}_K}$.

PROOF. Fix first a lattice point $(k,l) \in \mathscr{L}_K$. The limit (5.9) for $(x,y) = (k,l)$ will come from the subadditive ergodic theorem, so the first task is the moment bound

$$\mathbb{E}[T_K(nk,nl)] \leq Cn \tag{5.10}$$

for some constant C, for all n.

Begin by observing that

$$T_K(i,j) \le T_1(i,j) \tag{5.11}$$

for any $(i,j) \in \mathscr{L}_K$. This is because a K-admissible path can be turned into a 1-admissible path by replacing each $(K,-1)$-step by $K-1$ $(1,0)$-steps and one $(1,-1)$-step. This transformation cannot decrease the passage time of the path because no sites are removed.

To bound $T_1(i,j)$ for $(i,j) \in \mathscr{L}_1$, define a bijection p: $\mathscr{L}_1 \to \mathbf{N}^2$ by $p(i,j) = (i, i+j)$. A 1-admissible path from $(1,0)$ to (i,j) in $\mathscr{L}_1$ becomes under p a path in $\mathbf{N}^2$ that connects $(1,1)$ to $(i,i+j)$ and takes steps up, to the right and diagonally northeast. A northeast step can be replaced by a step up followed by a step right, so this image path is contained in a path from $(1,1)$ to $(i,i+j)$ that takes steps only up and to the right. Define i.i.d. Exp(1) passage times $v(i,j)$ for sites $(i,j) \in \mathbf{N}^2$ by $v(i,j) = \lambda(p^{-1}(i,j))$. ($p^{-1}$ is the inverse mapping.) For $(i,j) \in \mathbf{N}^2$, let

$$V(i,j) = \max_{r'} \sum_{(s,t)\in r'} v(s,t), \tag{5.12}$$

where the maximum is over up-right paths r' from $(1,1)$ to (i,j). In other words, if $r' = \{(s_1,t_1),\dots,(s_m,t_m)\}$, then $(s_1,t_1) = (1,1)$, $(s_m,t_m) = (i,j)$ and for $l = 1,\dots,m-1$, $(s_{l+1},t_{l+1}) - (s_l,t_l) = (1,0)$ or $(0,1)$. The steps we took to construct the passage times $V(i,j)$, and the bound $\tau(i,j) = \alpha(j)^{-1}\lambda(i,j) \le a_0^{-1}\lambda(i,j)$ imply that

$$T_1(i,j) \le a_0^{-1} V(i, i+j). \tag{5.13}$$

The advantage gained is that $V(i,j)$ is easy to bound. We insert a lemma.

LEMMA 5.2. *We have the stochastic dominance $V(i,j) \le S^{1/2}(i+j)$, where $S^{1/2}(m)$ denotes a sum of m i.i.d.* Exp(1/2) *random variables.*

PROOF. We appeal to a well-known $M/M/1$ queueing interpretation of $V(i,j)$ [see Section 2 in Glynn and Whitt (1991)]. Imagine j $M/M/1$ servers in series, each with service rate 1, unlimited waiting space, and FIFO queueing discipline. At time 0, i customers are in queue at server 1, while servers $2,\dots,j$ have empty queues. Each customer moves through the entire series of j servers, joining the queue at server $l+1$ as soon as service with server l is completed. Let $v(k,l)$ denote the service time of customer $k = 1,\dots,i$ at server $l = 1,\dots,j$. Then one can show inductively that $V(i,j)$ defined by (5.12) is the time when customer i leaves server j.

Next assume that, instead of queueing up at server 1 at time 0, customers $1,\dots,i$ arrive at server 1 in a Poisson(1/2) process, and that furthermore, at time 0 the queues at servers $1,\dots,j$ are not empty, but are i.i.d. Geom(1/2)-distributed. The system starts in equilibrium and consequently stays in equilibrium. [See Kelly (1979) for more on this.] The waiting times an individual customer experiences at the successive queues are i.i.d. Exp(1/2)-distributed. Thus the time for customer i to leave server j is $S^{1/2}(i+j)$, because $S^{1/2}(i)$ is the time when customer i arrives in queue 1 in the Poisson(1/2) arrival process, and $S^{1/2}(j)$ is the time it takes customer i to get through j queues.

Compared to the original system, customer i is slowed down. This is made precise by a coupling argument. The upshot is that $V(i, j)$ is stochastically dominated by $S^{1/2}(i+j)$. $\square$

REMARK 5.1. The process $V(i, j)$ has been studied by several authors. Glynn and Whitt (1991) derive a moment bound for $V(i, j)$ with an elegant associativity argument. The limit

$$\lim_{n\to\infty} \frac{1}{n} V([nx],[ny]) = \left(\sqrt{x} + \sqrt{y}\right)^2 \quad \text{a.s. for } (x, y) \in \mathbf{R}_+^2 \tag{5.14}$$

was first obtained by Rost (1981).

We return to the proof of Proposition 5.1. Lemma 5.2 implies that

$$\mathbb{E}[V(i, j)] \le 2(i+j) \tag{5.15}$$

which, through (5.11) and (5.13), gives (5.10).

Now define $X_{m,n} = -T_K((mk, ml),(nk, nl))$ for $0 \le m < n$. We leave it to the reader to verify that $(X_{m,n})$ satisfies the assumptions of a suitable subadditive ergodic theorem, such as that in Chapter 6 of Durrett (1991). The conclusion is that for all $(k, l) \in \mathscr{L}_K$, there is a finite number $\gamma_K(k, l)$ such that (5.9) holds $\mathbb{P}$-a.s. for $(x, y) = (k, l)$.

Next we extend the limit and the definition of γ_K, first to rational (x, y), and then to all (x, y). First check that we have homogeneity: $\gamma_K(mk, ml) = m\gamma_K(k, l)$ for $m \in \mathbf{N}$. This implies that for $(x, y) \in \mathscr{U}_K \cap \mathbf{Q}^2$, we can unambiguously define

$$\gamma_K(x, y) = \frac{1}{m}\gamma_K(mx, my) \tag{5.16}$$

for any $m \in \mathbf{N}$ that satisfies $(mx, my) \in \mathscr{L}_K$. Keep such an m fixed, and for arbitrary n, let k be such that $km \le n < (k+1)m$. Then

$$T_K(kmx, kmy) \le T_K([nx],[ny]) \le T_K((k+1)mx,(k+1)my),$$

and by letting $n, k \to \infty$, we get the limit (5.9) a.s. for rational $(x, y) \in \mathscr{U}_K$. Check that we have superadditivity

$$\gamma_K(x_1 + x_2, y_1 + y_2) \ge \gamma_K(x_1, y_1) + \gamma_K(x_2, y_2), \tag{5.17}$$

homogeneity

$$\gamma_K(sx_1, sy_1) = s\gamma_K(x_1, y_1) \tag{5.18}$$

and, consequently, concavity

$$\begin{aligned}\gamma_K(\theta x_1 + (1-\theta)x_2, \theta y_1 + (1-\theta)y_2)\\ \ge \theta\gamma_K(x_1, y_1) + (1-\theta)\gamma_K(x_2, y_2),\end{aligned} \tag{5.19}$$

for rational $(x_1, y_1), (x_2, y_2) \in \mathscr{U}_K$, $s > 0$ and $\theta \in (0,1)$. Quite obviously also, if $(x, y), (x', y') \in \mathscr{U}_K \cap \mathbf{Q}^2$ are such that $(x', y') \neq (x, y)$ and $(x', y') \in (x, y) + \overline{\mathscr{U}_K}$, then

$$\gamma_K(x, y) < \gamma_K(x', y'). \tag{5.20}$$

We need a continuity result which will be given in Lemma 5.3.

LEMMA 5.3. *For* $(x, y) \in \mathscr{U}_K \cap \mathbf{Q}^2$,

$$\lim_{\mathbf{Q} \ni x' \searrow x} \gamma_K(x', y) = \gamma_K(x, y). \tag{5.21}$$

PROOF. If $y = 0$, the statement follows from homogeneity (5.18). Suppose $y > 0$ and $x' > x$. Then because $(x, y) \in (x, xy/x') + \overline{\mathscr{U}_K}$, we have by (5.20) and (5.18) that

$$\gamma_K(x', y) \geq \gamma_K(x, y) \geq \gamma_K(x, yx/x') = (x/x')\gamma_K(x', y).$$

The result follows by letting $x' \searrow x$.

For $y < 0$, the result follows by the same principle. Take $x'' = x'(y + x/K)/(y + x'/K)$ and $y'' = yx''/x'$. Then $(x, y) \in (x'', y'') + \overline{\mathscr{U}_K}$, so that

$$\gamma_K(x, y) \geq \gamma_K(x'', y'') = (x''/x')\gamma_K(x', y),$$

and again we may let $x' \searrow x$ which also sends $x''/x' \to 1$. This proves (5.21). □

We return to the proof of Proposition 5.1. Define the set Σ_0 in the statement of the proposition as that subset of Σ on which the limit (5.9) holds for all rational $(x, y) \in \mathscr{U}_K$. Extend $\gamma_K(x, y)$ to all $(x, y) \in \overline{\mathscr{U}_K}$ by the formula

$$\gamma_K(x, y) = \inf\{\gamma_K(x', y') \colon (x', y') \in [(x, y) + \mathscr{U}_K] \cap \mathbf{Q}^2\}. \tag{5.22}$$

On the right-hand side are the values $\gamma_K(x', y')$ for rational (x', y') already defined by (5.9). By (5.20) and (5.21), for $(x, y) \in \mathscr{U}_K \cap \mathbf{Q}^2$, (5.22) is a valid identity, so it is a sensible extension of γ_K. Superadditivity (5.17) and homogeneity (5.18) can be proved again, this time for all $(x, y) \in \overline{\mathscr{U}_K}$. From this follows concavity. A finite, concave function is continuous on an open, convex set [Rockafellar (1970), Theorem 10.1]. Consequently, $\gamma_K(x, y)$ is continuous on $\mathscr{U}_K$. Continuity at boundary points $(x, y) \in \overline{\mathscr{U}_K} \setminus \mathscr{U}_K$ requires separate arguments in the style of Lemma 5.3, and we leave them to the reader.

The final point is the validity of the limit (5.9) for all $(x, y) \in \mathscr{U}_K$ and all $(\boldsymbol{\lambda}, \boldsymbol{\alpha}) \in \Sigma_0$. This follows from the continuity of γ_K, by approximating a general lattice point $([nx], [ny])$ by $([nx'], [ny'])$ with rational (x', y'). The proof of Proposition 5.1 is complete. □

Since $\gamma_K(x, y)$ is strictly increasing in x and y, the following equation defines uniquely a finite, convex, continuous, nonincreasing function g_K on $\mathbf{R}$:

$$g_K(y) = \inf\{x > 0 \colon (x, y) \in \mathscr{U}_K, \gamma_K(x, y) \geq 1\}, \qquad y \in \mathbf{R}.$$

By applying the strong law to the passage times $\tau(1, j)$, $j \geq 0$, and $\tau(-Kj, j)$, $j \leq -1$, we get

$$\gamma_K(0, y) \geq y/a_1 \quad \text{and} \quad \gamma_K(Ky, -y) \geq y/a_1 \text{ for } y \geq 0. \tag{5.23}$$

This implies (2.6) and (3.3).

REMARK 5.2. In the basic case without disorder $[\alpha(j) \equiv 1]$, we can obtain the exact boundary values

$$\gamma_K(0, y) = \gamma_K(Ky, -y) = y \quad \text{for } y \geq 0. \tag{5.24}$$

This is proved in Section 9 by comparison with the $K = 1$ case; see the remarks after Lemma 9.1.

Proposition 5.1 has the following consequence.

COROLLARY 5.1. *The interface process $\xi(\cdot)$ defined by* (5.8) *satisfies a strong law of large numbers*: *for any $y \in \mathbf{R}$ and $t > 0$,*

$$\lim_{n\to\infty} \frac{1}{n}\xi([ny], nt) = tg_K(y/t), \quad \mathbb{P}\text{-}a.s. \tag{5.25}$$

PROOF. The proof has two parts. We do the limsup part and leave the liminf part to the reader. Let $\varepsilon > 0$. By strict monotonicity we can pick $\delta > 0$ so that

$$\gamma_K\big(tg_K(y/t) + \varepsilon, y\big) \geq \gamma_K\big(tg_K(y/t), y\big) + \delta \geq t + \delta.$$

The last inequality is actually an equality if $(tg_K(y/t), y)$ is not a boundary point of $\mathscr{U}_K$. Let $x = tg_K(y/t) + \varepsilon$. Almost surely the inequality

$$T_K([nx],[ny]) \geq n\gamma_K(x, y) - n\delta/2 \geq n(t + \delta/2)$$

holds eventually, and this implies by (5.8) that

$$\xi([ny], nt) \leq [nx] \leq ntg_K(y/t) + n\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary,

$$\limsup_{n\to\infty} \frac{1}{n}\xi([ny], nt) \leq tg_K(y/t), \quad \mathbb{P}\text{-a.s.} \qquad \square$$

Next we deduce a one-sided exponential bound valid for both the basic setting and the disordered setting. It is needed for Theorem 3 and for the strong law part of Theorem 1. Up to now, this section has required only ergodicity of $(\alpha(j))$, but this lemma requires a mixing assumption. So for convenience we take $\alpha(j)$ i.i.d. as assumed in Section 3.

LEMMA 5.4. *Fix $(x, y) \in \mathscr{U}_K$ and $\varepsilon > 0$. For some constant $C > 0$,*

$$\mathbb{P}\big(T_K([nx],[ny]) \leq n\gamma_K(x, y) - n\varepsilon\big) \leq e^{-Cn} \tag{5.26}$$

for large enough n.

PROOF. Suppose first that $y > 0$. Since $n^{-1}\mathbb{E}[T_K([nx],[ny])] \to \gamma_K(x,y)$, we may pick a number M such that

$$\mathbb{E}[T_K([Mx],[My])] \geq M(\gamma_K(x,y) - \varepsilon/2). \tag{5.27}$$

Let

$$\tau_l = T_K((l[Mx], l[My]), ((l+1)[Mx], (l+1)[My])), \qquad l = 0, 1, 2, \dots. \tag{5.28}$$

Consider the mutual dependencies of the passage times $\{\tau_l\}$ under the measure $\mathbb{P}$. By definition (5.5) and the definition of K-admissibility, τ_l is computed from

$$\{\tau(i,j)\colon l[Mx] + 1 \leq i \leq (l+1)[Mx],$$
$$l[My] - [Mx]/K \leq j \leq (l+1)[My] + [Mx]/K\}.$$

The variables $(\lambda(i,j))$ induce no dependencies between the τ_l's, and the $\alpha(j)$'s force only a finite range dependence on the τ_l's. Thus there is a number R such that $\{\tau_{l+jR}\colon j \geq 0\}$ are i.i.d. random variables, for any $l \geq 0$.

Let $L = [n/MR]$. Then $\sum_{l=0}^{LR-1} \tau_l \leq T_K([nx],[ny])$. Abbreviate $\gamma = \gamma_K(x,y)$ and let $\beta > 0$. We apply standard large deviations reasoning,

$$\begin{aligned}
&\frac{1}{n}\log \mathbb{P}(T_K([nx],[ny]) \leq n(\gamma - \varepsilon)) \\
&\quad \leq \frac{1}{n}\log \mathbb{P}\left(\sum_{l=0}^{LR-1} \tau_l \leq n(\gamma-\varepsilon)\right) \\
&\quad \leq \frac{1}{n}\log\left[\exp(\beta n(\gamma - \varepsilon))\mathbb{E}\left(\exp\left\{-\beta\sum_{l=0}^{LR-1}\tau_l\right\}\right)\right].
\end{aligned} \tag{5.29}$$

To the expectation, apply first Hölder's inequality and then independence,

$$\begin{aligned}
&\mathbb{E}\left(\exp\left\{-\beta\sum_{l=0}^{LR-1}\tau_l\right\}\right) \\
&\quad = \mathbb{E}\left(\prod_{l=0}^{R-1}\exp\left\{-\beta\sum_{j=0}^{L-1}\tau_{l+jR}\right\}\right) \\
&\quad \leq \mathbb{E}\left(\exp\left\{-\beta R\sum_{j=0}^{L-1}\tau_{jR}\right\}\right) \\
&\quad = \mathbb{E}(\exp\{-\beta R\tau_0\})^L.
\end{aligned}$$

[Note to the reader: The above step is where more than ergodicity of $(\alpha(i))$ is needed.] Substitute this back into the last line of (5.29):

$$\begin{aligned}
&\frac{1}{n}\log \mathbb{P}\big(T_K([nx],[ny]) \le n(\gamma-\varepsilon)\big)\\
&\quad \le \beta(\gamma-\varepsilon) + \frac{L}{n}\log \mathbb{E}\big(\exp(-\beta R\tau_0)\big)\\
&\quad \le \beta(\gamma-\varepsilon) + \frac{1}{MR}\left(1-\frac{\varepsilon}{4\gamma}\right)\log \mathbb{E}\big(\exp(-\beta R\tau_0)\big)\\
&\quad \equiv H(\beta).
\end{aligned}$$

The last inequality is valid for large enough n. It remains to show that $H(\beta) < 0$ for some $\beta > 0$. Since $H(0) = 0$, it suffices to show that $H'(0) < 0$.

$$H'(\beta) = \gamma - \varepsilon - \frac{1}{M}\left(1-\frac{\varepsilon}{4\gamma}\right)\frac{\mathbb{E}\big(\tau_0 \exp(-\beta R\tau_0)\big)}{\mathbb{E}\big(\exp(-\beta R\tau_0)\big)},$$

from which

$$H'(0) = \gamma - \varepsilon - \frac{1}{M}\left(1-\frac{\varepsilon}{4\gamma}\right)\mathbb{E}(\tau_0).$$

This is less than 0, for by (5.27) and (5.28),

$$\mathbb{E}(\tau_0) = \mathbb{E}\big[T_K([Mx],[My])\big] \ge M(\gamma - \varepsilon/2).$$

This proves (5.26) for $y > 0$. The argument for $y < 0$ is the same.

For $y = 0$, choose by continuity $x' < x$ and $y < 0$ such that $\gamma_K(x', y) \ge \gamma_K(x, 0) - \varepsilon/2$. Since $([nx], 0) \in ([nx'], [ny]) + \mathscr{L}_K$ for large n, we have $T_K([nx'],[ny]) \le T_K([nx], 0)$. By the case already proved,

$$\begin{aligned}
&\mathbb{P}\big(T_K([nx],0) \le n(\gamma_K(x,0)-\varepsilon)\big)\\
&\quad \le \mathbb{P}\big(T_K([nx'],[ny]) \le n(\gamma_K(x',y)-\varepsilon/2)\big)\\
&\quad \le e^{-Cn}.
\end{aligned}$$

□

The reader not interested in the disordered setting can move on to Section 6. For the disordered model, we need to think about the quenched setting. Proposition 5.1 implies that the limit (5.9) holds in $\mathbb{P}^{\boldsymbol{\alpha}}$-probability, for Q-almost every $\boldsymbol{\alpha}$. However, our proof of the hydrodynamic limit needs a stronger version where the quenched variable $\boldsymbol{\alpha}$ is translated simultaneously as the limit is taken. Write $\theta(k)$ for translations on the space $\mathscr{A}$: for $\boldsymbol{\alpha} = (\alpha(i)), [\theta(k)\boldsymbol{\alpha}](i) = \alpha(i+k)$.

PROPOSITION 5.2. *There exists a subset* $\mathscr{A}_0 \subseteq \mathscr{A}$ *such that this holds*: $Q(\mathscr{A}_0) = 1$, *and for all* $\boldsymbol{\alpha} \in \mathscr{A}_0$, $b \in \mathbf{R}$, $(x, y) \in \mathscr{U}_K$ *and* $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}^{\theta([nb])\boldsymbol{\alpha}}\big(\big|n^{-1}T_K([nx],[ny]) - \gamma_K(x,y)\big| \ge \varepsilon\big) = 0. \tag{5.30}$$

PROOF. Start by observing this:

$$
\begin{aligned}
&\text{the } \mathbb{P}^{\theta([nb])\boldsymbol{\alpha}}\text{-distribution of } T_K([nx],[ny]) \\
&\qquad = \text{the } \mathbb{P}^{\boldsymbol{\alpha}}\text{-distribution of } T_K([nx],[ny]) \circ \Theta(0,[nb]) \\
&\qquad = \text{the } \mathbb{P}^{\boldsymbol{\alpha}}\text{-distribution of } T_K([nx],[ny]) \circ \Theta([na],[nb])
\end{aligned}
\tag{5.31}
$$

for any $a \in \mathbf{R}$. The second equality follows because the variables $(\lambda(i,j))$ are i.i.d. Exp(1) under any $\mathbb{P}^{\boldsymbol{\alpha}}$, so they can be translated arbitrarily with no effect on the distribution. Recall from (5.3) that only the second index l of the joint translation $\Theta(k,l)$ affects $\boldsymbol{\alpha}$.

Fix $b \in \mathbf{R}$ and $(x, y) \in \mathscr{U}_K$. Since $T_K([nx],[ny]) \circ \Theta(0,[nb]) =_d T_K([nx],[ny])$ under $\mathbb{P}$ [$=_d$ denotes equality in distribution], Lemma 5.4 implies that for all $\varepsilon > 0$,

$$
\sum_{n=1}^{\infty} \mathbb{P}\big(n^{-1}T_K([nx],[ny]) \circ \Theta(0,[nb]) \le \gamma_K(x,y) - \varepsilon\big) < \infty. \tag{5.32}
$$

Consequently, by the Borel–Cantelli lemma,

$$
\liminf_{n\to\infty} \frac{1}{n} T_K([nx],[ny]) \circ \Theta(0,[nb]) \ge \gamma_K(x,y) \quad \mathbb{P}\text{-a.s.},
$$

and hence also $\mathbb{P}^{\boldsymbol{\alpha}}$-a.s., for Q-a.e. $\boldsymbol{\alpha}$. By (5.31), for Q-a.e. $\boldsymbol{\alpha}$,

$$
\lim_{n\to\infty} \mathbb{P}^{\theta([nb])\boldsymbol{\alpha}}\big(n^{-1}T_K([nx],[ny]) \le \gamma_K(x,y) - \varepsilon\big) = 0. \tag{5.33}
$$

At this point we have an exceptional Q-null set of $\boldsymbol{\alpha}$'s that depends on b, (x, y) and ε. To get a single null set simultaneously valid for all b, (x, y) and ε requires an approximation step and use of the continuity of γ_K as in the proof of Proposition 5.1. We leave this to the reader.

Equation (5.33) is one half of the goal (5.30). To get the remaining half, define the event $\Sigma_1 \subseteq \Sigma$ as the set of $(\boldsymbol{\lambda}, \boldsymbol{\alpha})$ for which both of these statements hold:

$$
\lim_{n\to\infty} \frac{1}{n} T_K((0,0),([nx],[ny])) = \gamma_K(x,y) \quad \text{for all } (x,y) \in \mathscr{U}_K \tag{5.34}
$$

and

$$
\lim_{n\to\infty} \frac{1}{n} T_K(([nu],[nv]),(0,0)) = \gamma_K(-u,-v) \quad \text{for all } (u,v) \in -\mathscr{U}_K. \tag{5.35}
$$

Statement (5.34) is of course just Proposition 5.1, and (5.35) is the same thing with a change in lattice direction. So $\mathbb{P}(\Sigma_1) = 1$. Define the "good" subset $\mathscr{B} \subseteq \mathscr{A}$ as the set of $\boldsymbol{\alpha}$ such that $\mathbb{P}^{\boldsymbol{\alpha}}(\Sigma_1) = 1$. Then $Q(\mathscr{B}) = 1$. We shall demonstrate that there always is a number a, depending on b, (x, y) and ε, such that

$$\lim_{n\to\infty} \mathbb{P}^{\boldsymbol{\alpha}}\big(n^{-1}T_K([nx],[ny])\circ\Theta([na],[nb]) \geq \gamma_K(x,y)+\varepsilon\big)=0 \tag{5.36}$$

for $\boldsymbol{\alpha}\in\mathscr{B}$. This step is carried out in several cases according to the values of b and (x, y). Then we are done, for another application of (5.31) shows that (5.36) and (5.33) together give (5.30).

CASE I ($b, y > 0$ or $b, y < 0$). Set $a = bx/y > 0$. By superadditivity,

$$\begin{aligned} T_K((0,0),([nx],[ny]))&\circ\Theta([na],[nb]) \\ &= T_K(([na],[nb]),([na]+[nx],[nb]+[ny])) \\ &\leq T_K((0,0),([na]+[nx],[nb]+[ny])) \\ &\quad - T_K((0,0),([na],[nb])), \end{aligned} \tag{5.37}$$

from which

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n}&T_K((0,0),([nx],[ny]))\circ\Theta([na],[nb]) \\ &\leq \gamma_K(a+x,b+y)-\gamma_K(a,b) \\ &= (1+b/y)\gamma_K(x,y)-(b/y)\gamma_K(x,y) \\ &= \gamma_K(x,y) \end{aligned} \tag{5.38}$$

on the event Σ_1. The second-last step used homogeneity (5.18). It follows that (5.36) holds for all $\boldsymbol{\alpha}\in\mathscr{B}$ in Case I.

For the next case we need an intermediate lemma.

LEMMA 5.5. *For $x > 0$ and $b \in \mathbf{R}$,*

$$\lim_{a\to\infty}\big[\gamma_K(a+x,b)-\gamma_K(a,b)\big]=\gamma_K(x,0). \tag{5.39}$$

PROOF. Superadditivity says that the liminf of the left-hand side of (5.39) is at least as large as the right-hand side. The nontrivial direction is the opposite inequality. The case $b = 0$ follows from homogeneity. Suppose $b > 0$. Set

$$x' = \frac{x(a+x)}{a+x+Kb} \in (0,x).$$

Check from a picture that $(x,0) \in (x', bx'/(a+x)) + \overline{\mathscr{U}_K}$. From this we get the first inequality below, and then we use homogeneity.

$$\begin{aligned}
\gamma_K(x,0) &\geq \gamma\left(x', \frac{bx'}{a+x}\right)\\
&= \frac{x'}{a+x}\gamma_K(a+x,b)\\
&= \gamma_K(a+x,b) - \left(1 + \frac{x-x'}{a}\right)\gamma_K\left(a, \frac{ab}{a+x}\right)\\
&\geq \gamma_K(a+x,b) - \gamma_K(a,b) - \left(\frac{x-x'}{a}\right)\gamma_K(a,b)\\
&\qquad\qquad \left[\text{because } \gamma_K(a, ab/(a+x)) \leq \gamma_K(a,b) \text{ for } a,b,x>0\right]\\
&\geq \gamma_K(a+x,b) - \gamma_K(a,b) - O(a^{-1}),
\end{aligned}$$

because $x - x' = O(a^{-1})$ and because the moment bound (5.10) implies that $\gamma_K(a,b) = O(a)$ for a fixed b. This proves (5.39) for $b > 0$. We leave the similar argument for $b < 0$ to the reader. □

We return to the proof of Proposition 5.2.

Case II ($y = 0$). Pick a large enough so that

$$\gamma_K(a+x,b) - \gamma_K(a,b) \leq \gamma_K(x,0) + \varepsilon/2. \tag{5.40}$$

Repeat the reasoning of Case I down to the second line of (5.38) with $y = 0$. Then (5.40) gives

$$\limsup_{n\to\infty} \frac{1}{n} T_K((0,0),([nx],0)) \circ \Theta([na],[nb]) \leq \gamma_K(x,0) + \varepsilon/2$$

everywhere on the event Σ_1. Again (5.36) holds for all $\boldsymbol{\alpha} \in \mathscr{B}$.

Case III ($b > 0 > y \geq -b$ or $b < 0 < y \leq -b$). If $y = -b$ we can directly apply (5.35) to

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} T_K((0,0),([nx],[ny])) \circ \Theta([-nx],[nb])\\
&\quad= \lim_{n\to\infty} \frac{1}{n} T_K(([-nx],[-ny]),([nx]+[-nx],[ny]+[-ny]))\\
&\quad= \lim_{n\to\infty} \frac{1}{n} T_K(([-nx],[-ny]),(0,0)).
\end{aligned}$$

The point of the last equality is simply that, due to the continuity of γ_K, the difference between $([nx]+[-nx],[ny]+[-ny])$ and $(0,0)$ is not felt in the limit.

Otherwise set $a = bx/y < 0$, and we have $(a, b), (a + x, b + y) \in -\mathscr{U}_K$. By superadditivity,

$$\begin{aligned} &T_K((0,0),([nx],[ny])) \circ \Theta([na],[nb]) \\ &\qquad = T_K(([na],[nb]),([na]+[nx],[nb]+[ny])) \\ &\qquad \le T_K(([na],[nb]),(0,0)) \\ &\qquad\quad - T_K(([na]+[nx],[nb]+[ny]),(0,0)), \end{aligned} \tag{5.41}$$

from which by (5.35),

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{n} T_K((0,0),([nx],[ny])) \circ \Theta([na],[nb]) \\ &\qquad \le \gamma_K(-a,-b) - \gamma_K(-a-x,-b-y) \\ &\qquad = \left(-\frac{b}{y}\right)\gamma_K(x,y) - \left(-\frac{b}{y} - 1\right)\gamma_K(x,y) \\ &\qquad = \gamma_K(x,y) \end{aligned} \tag{5.42}$$

on the event Σ_1. The assumption $|y| < |b|$ was needed for the homogeneity step, so that $-b/y - 1 > 0$. (5.36) holds for all $\boldsymbol{\alpha} \in \mathscr{B}$ also in this case.

Case IV ($b > 0$ and $-b > y$). Pick a finite partition $\{x_0, x_1, \ldots, x_m\}$ of the interval $(0, x)$ such that

$$x_0 < Kb < x_1 < \cdots < x_{m-1} < x + K(b+y) < x_m.$$

The point here is that any K-admissible path from $(1, [nb])$ to $([nx], [nb] + [ny])$ must cross the x-axis at one of the sites $(i, 0)$ for $[nx_0] \le i \le [nx_m]$. By uniform continuity of γ_K on the compact set $[Kb, x] \times \{-b\}$, we may further choose the above partition fine enough to have

$$\left|\gamma_K(x_j, -b) - \gamma_K(x_{j+1}, -b)\right| \le \varepsilon/2 \quad \text{for all } j. \tag{5.43}$$

Here and below, interpret $\gamma_K(x_0, -b)$ as $\gamma_K(Kb, -b)$ because $\gamma_K(x_0, -b)$ is not defined for $x_0 < Kb$. Now we can estimate

$$\begin{aligned} &T_K((0,0),([nx],[ny])) \circ \Theta(0,[nb]) \\ &\qquad = T_K((0,[nb]),([nx],[nb]+[ny])) \\ &\qquad \le \max_{0\le j\le m-1} \Big\{ T_K\big((0,[nb]),([nx_{j+1}],0)\big) \\ &\qquad\qquad\qquad + T_K\big(([nx_j],0),([nx],[nb]+[ny])\big)\Big\}. \end{aligned} \tag{5.44}$$

By superadditivity of γ_K and (5.43),

$$\begin{aligned} \gamma_K(x,y) + \varepsilon &\ge \gamma_K(x_j, -b) + \gamma_K(x - x_j, b + y) + \varepsilon \\ &\ge \gamma_K(x_{j+1}, -b) + \gamma_K(x - x_j, b + y) + \varepsilon/2. \end{aligned} \tag{5.45}$$

Combining (5.44) and (5.45) allows us to write

$$\begin{aligned}
&\mathbb{P}^{\boldsymbol{\alpha}}\{T_K([nx],[ny])\circ\Theta(0,[nb]) \ge n\gamma_K(x,y)+n\varepsilon\}\\
&\quad\le \sum_{j=0}^{m-1}\Big[\mathbb{P}^{\boldsymbol{\alpha}}\Big\{T_K\big((0,[nb]),([nx_{j+1}],0)\big) \ge n\gamma_K(x_{j+1},-b)+n\varepsilon/4\Big\}\\
&\qquad\qquad +\mathbb{P}^{\boldsymbol{\alpha}}\Big\{T_K\big(([nx_j],0),([nx],[nb]+[ny])\big)\\
&\qquad\qquad\qquad \ge n\gamma_K(x-x_j,b+y)+n\varepsilon/4\Big\}\Big]\\
&\quad= \sum_{j=0}^{m-1}\Big[\mathbb{P}^{\boldsymbol{\alpha}}\Big\{T_K\big((0,0),([nx_{j+1}],-[nb])\big)\circ\Theta(0,[nb])\\
&\qquad\qquad\qquad \ge n\gamma_K(x_{j+1},-b)+n\varepsilon/4\Big\}\\
&\qquad\qquad +\mathbb{P}^{\boldsymbol{\alpha}}\Big\{T_K\big((0,0),([nx]-[nx_j],[nb]+[ny])\big)\\
&\qquad\qquad\qquad \ge n\gamma_K(x-x_j,b+y)+n\varepsilon/4\Big\}\Big].
\end{aligned}$$

In the last step we do a trivial translation in the first probability inside the brackets, and a translation that utilizes the horizontal translation-invariance of $(\lambda(i,j))$ in the second probability. The last sum tends to 0 as $n\to\infty$, because Case III takes care of the first probability, and assumption (5.34) of the second.

CASE V ($b<0$ and $-b<y$). This is the only remaining case. Perform a partition argument similar to the one for Case IV and appeal to earlier cases. □

6. Proof of the weak laws. In this section we prove Theorem 3. Simultaneously we obtain statement (2.13) of Theorem 1 under assumption (2.4). [This is just the special case $\alpha(i)\equiv 1$ of Theorem 3.] Fix a configuration $\boldsymbol{\alpha}\in\mathscr{A}_0$, the set defined by Proposition 5.2. Let $(\eta_n(i,0): i\in\mathbf{Z})$ be the given random initial configurations that satisfy (3.2). To define initial configurations for the corresponding server processes, set

$$\begin{aligned}
&z_n(0,0)=0,\\
&z_n(i,0)=\sum_{j=1}^{i}\eta_n(j,0)\quad\text{for } i>0,\text{ and}\\
&z_n(i,0)=-\sum_{j=i+1}^{0}\eta_n(j,0)\quad\text{for } i<0.
\end{aligned}\tag{6.1}$$

Let $z_n(\cdot)$ denote the server process constructed from initial configuration $(z_n(i,0): i\in\mathbf{Z})$ according to the description of Section 4, and denote its probability measure by $P_n^{\boldsymbol{\alpha}}$. Then define the K-exclusion $\eta_n(\cdot)$ by (4.1).

The empirical measure of an interval $(a, b]$ for $a < b$ is given by

$$\begin{aligned}
\pi_n(t,(a,b]) &= \frac{1}{n}\sum_{j\in\mathbf{Z}} \eta_n(j,t) I_{(a,b]}\left(\frac{j}{n}\right) \\
&= \frac{1}{n}\sum_{j=[na]+1}^{[nb]} \eta_n(j,t) \\
&= \frac{1}{n} z_n([nb],t) - \frac{1}{n} z_n([na],t).
\end{aligned} \tag{6.2}$$

Pick U_0 that satisfies (2.7) and $U_0(0) = 0$. Then assumption (3.2) together with (6.1) and (6.2) implies that, for all $y \in \mathbf{R}$ and $\varepsilon > 0$,

$$\lim_{n\to\infty} P_n^{\boldsymbol{\alpha}}\left(\left|n^{-1} z_n([ny],0) - U_0(y)\right| \geq \varepsilon\right) = 0. \tag{6.3}$$

Note that behind this claim is the fact that the limit measure $u_0(x)\,dx$ of $\pi_n(0, dx)$ has no atoms, so that $\pi_n(0,(a, b])$ converges to $U_0(b) - U_0(a)$ for all intervals $(a, b]$.

Define $U(x,t)$ by (2.8). By (6.2) and the definition (2.9) of $u(x,t)$, we shall have proved Theorem 3 if we show that, for all $x \in \mathbf{R}$, $t > 0$, and $\varepsilon > 0$,

$$\lim_{n\to\infty} P_n^{\boldsymbol{\alpha}}\left(\left|n^{-1} z_n([nx],nt) - U(x,t)\right| \geq \varepsilon\right) = 0. \tag{6.4}$$

Fix (x,t). The starting point is the coupling (4.5) and (4.9) that now reads

$$\begin{aligned}
z_n([nx],nt) &= \sup_{i\in\mathbf{Z}} w^{z_n(i,0)}([nx]-i,nt) \\
&= \sup_{i\in\mathbf{Z}} \left\{ z_n(i,0) - \xi^{z_n(i,0)}([nx]-i,nt)\right\}.
\end{aligned} \tag{6.5}$$

An important point to keep straight is that, by the same reasoning that justified Lemma 5.1, we have this equality in distribution:

$$\left[\xi^{z_n([nb],0)}(\cdot) \text{ under } P_n^{\boldsymbol{\alpha}}\right] \stackrel{d}{=} \left[\xi(\cdot) \text{ under } \mathbb{P}^{\theta([nb])\boldsymbol{\alpha}}\right]. \tag{6.6}$$

The first task is to restrict the range of indices i that need to be considered in (6.5). For $r_1 < r_2$, define

$$\zeta_n(r_1,r_2) = \max_{[nr_1]\leq i\leq [nr_2]} \left\{ z_n(i,0) - \xi^{z_n(i,0)}([nx]-i,nt)\right\}. \tag{6.7}$$

LEMMA 6.1. *For any $\boldsymbol{\alpha} \in \mathscr{A}$ and $r_1 < x - t < x + t < r_2$, there exists a constant $C > 0$ such that*

$$P_n^{\boldsymbol{\alpha}}\left(z_n([nx],nt) \neq \zeta_n(r_1,r_2)\right) \leq e^{-Cn} \tag{6.8}$$

for all n.

PROOF. By Lemma 4.4 it suffices to show that

$$P_n^{\boldsymbol{\alpha}}\left(\xi^{z_n([nr_1],0)}([nx]-[nr_1],nt) > 0\right) \leq e^{-Cn}$$

and

$$P_n^{\boldsymbol{\alpha}}\big(\xi^{z_n([nr_2],0)}([nx]-[nr_2],nt)>K([nr_2]-[nx])\big)\le e^{-Cn}.$$

By (6.6), this will follow from showing that, for any $b\in\mathbf{R}$ and $y>t$,

$$\mathbb{P}^{\theta([nb])\boldsymbol{\alpha}}\big(\xi([ny],nt)>0\big)\le e^{-Cn} \tag{6.9}$$

and

$$\mathbb{P}^{\theta([nb])\boldsymbol{\alpha}}\big(\xi(-[ny],nt)>K[ny]\big)\le e^{-Cn}. \tag{6.10}$$

We show (6.9) and leave the argument for (6.10) to the reader. By definitions (5.5) and (5.8), and because $\alpha(j)\le 1$ always,

$$\begin{aligned}\xi([ny],nt)\circ\Theta(0,[nb])>0 &\Rightarrow T_K(1,[ny])\circ\Theta(0,[nb])\le nt\\ &\Rightarrow \sum_{j=[nb]}^{[nb]+[ny]}\alpha(j)^{-1}\lambda(1,j)\le nt\\ &\Rightarrow \sum_{j=[nb]}^{[nb]+[ny]}\lambda(1,j)\le nt.\end{aligned}$$

This last event has exponentially small probability because the $(\lambda(1,j))$ are i.i.d. with expectation 1 and $y>t$. □

The second point is to note the weak law of the interface processes in (6.5). This is a corollary of Proposition 5.2.

LEMMA 6.2. *For any $\boldsymbol{\alpha}\in\mathscr{A}_0$, $b,y\in\mathbf{R}$, $t>0$ and $\varepsilon>0$,*

$$\lim_{n\to\infty}P_n^{\boldsymbol{\alpha}}\big(\big|n^{-1}\xi^{z_n([nb],0)}([ny],nt)-tg_K(y/t)\big|\ge\varepsilon\big)=0. \tag{6.11}$$

PROOF. The lemma follows from (6.6) and Proposition 5.2, by the same sort of reasoning that Corollary 5.1 followed from Proposition 5.1. □

One half of (6.4), namely,

$$\lim_{n\to\infty}P_n^{\boldsymbol{\alpha}}\big(n^{-1}z_n([nx],nt)\le U(x,t)-\varepsilon\big)=0, \tag{6.12}$$

is now immediate because setting $i=[ny]$ inside the braces in (6.5) gives a lower bound for $z_n([nx],nt)$, and by (2.8), (6.3) and (6.11), the limit of the quantity in braces can be taken arbitrarily close to $U(x,t)$ by choice of y.

For the other half, fix $r_1<r_2$ so that (6.8) holds, and pick a partition

$$r_1=b_0<b_1<\cdots<b_{m-1}<b_m=r_2$$

fine enough so that

$$\big|tg_K((x-b_{l+1})/t)-tg_K((x-b_l)/t)\big|\le\varepsilon/2\quad\text{for all } l. \tag{6.13}$$

By Lemma 4.3 we may reason as follows:

$$
\begin{aligned}
\zeta_n(r_1, r_2) &= \max_{[nr_1] \le i \le [nr_2]} w^{z_n(i,0)}([nx] - i, nt) \\
&= \max_{0 \le l < m} \max_{[nb_l] \le i \le [nb_{l+1}]} w^{z_n(i,0)}([nx] - i, nt) \\
&\le \max_{0 \le l < m} \big\{ w^{z_n([nb_l],0)}([nx] - [nb_l], nt) \\
&\qquad\qquad + z_n([nb_{l+1}], 0) - z_n([nb_l], 0) \big\} \\
&= \max_{0 \le l < m} \big\{ z_n([nb_{l+1}], 0) - \xi^{z_n([nb_l],0)}([nx] - [nb_l], nt) \big\}.
\end{aligned}
\tag{6.14}
$$

By (6.3) and (6.11), the limit in P_n^{α}-probability of $(1/n) \cdot$[the last line of (6.14)] is

$$
\max_{0 \le l < m} \big\{ U_0(b_{l+1}) - t g_K((x - b_l)/t) \big\},
$$

which by (6.13), and by the definition (2.8) of $U(x, t)$, is bounded above by $U(x, t) + \varepsilon/2$. We have shown that

$$
\lim_{n \to \infty} P_n^{\alpha}\big(n^{-1} \zeta_n(r_1, r_2) \ge U(x, t) + \varepsilon \big) = 0, \tag{6.15}
$$

which together with (6.8) and (6.12) implies (6.4), the goal of the argument. Theorem 3 and the weak law part of Theorem 1 are thereby proved.

As a final matter in this section, we can now see why we have to bound the rates $\alpha(j)$ away from zero in order to get nontrivial hydrodynamics. Suppose $Q(\alpha(0) < \varepsilon) > 0$ for all $\varepsilon > 0$. Then the limit in Proposition 5.1 would be $\gamma_K(x, y) \equiv \infty$. For $y > 0$ this can be seen from

$$
\frac{1}{n} T_K([nx], [ny]) \ge \Big\{ \min_{0 \le j \le [ny]} \alpha(j) \Big\}^{-1} \frac{1}{n} \sum_{i=1}^{[nx]} \lambda(i, j_n),
$$

where the random j_n satisfies $\alpha(j_n) = \min_{0 \le j \le [ny]} \alpha(j)$. For $y \le 0$, similar reasoning works. From $\gamma_K \equiv \infty$ follows that $g_K(y) = 0$ for $y \ge 0$ and $g_K(y) = -Ky$ for $y \le 0$. This in turn makes $U(x, t) = U_0(x)$ in (2.8).

7. An upper tail estimate for the last-passage times. To obtain the strong laws in Theorems 1 and 2, we need estimates that allow us to apply the Borel–Cantelli lemma. In Lemma 5.4 we already have a lower tail estimate. This section is devoted to the next lemma that gives the complementary upper tail estimate. Its proof is reminiscent of a proof of Grimmett and Kesten (1984) for a lower tail estimate on the passage times of first-passage percolation, although the main challenge here is different. The geometry of our paths is simpler than in first-passage percolation. What makes our upper tail estimate problematic is that passage times of individual sites are unbounded, so an excessively high value of $T_K([nx], [ny])$ can be produced in a large number of ways. In contrast, a lower tail estimate can use the fact that passage times are bounded below by zero. In the end we do not quite get an exponential bound, although we certainly expect one, by analogy with other similar models [Seppäläinen (1998b, c)].

LEMMA 7.1. *Suppose $a_0 = \alpha(j) = 1$ for all $j \in \mathbf{Z}$. Fix $(\bar{x}, \bar{y}) \in \mathscr{U}_K$ and $\varepsilon > 0$. Then there is a constant $C > 0$ such that*

$$\mathbb{P}\big(T_K([n\bar{x}], [n\bar{y}]) \geq n\gamma_K(\bar{x}, \bar{y}) + n\varepsilon\big) \leq \exp\Big[-Cn(\log n)^{-1}\Big] \tag{7.1}$$

for large enough n.

The proof of this lemma takes several steps. Regard $(\bar{x}, \bar{y})$ fixed for the duration of this section. We start with an exponential upper tail bound on large enough deviations of order n.

LEMMA 7.2. *Suppose $a_0 = \alpha(j) = 1$ for all $j \in \mathbf{Z}$. For $(x, y) \in \mathscr{U}_K$, there exist positive constants b_0, n_0 and c_0 depending on (x, y) such that*

$$\mathbb{P}\big(T_K([nx], [ny]) \geq bn\big) \leq \exp(-c_0 bn) \tag{7.2}$$

for all $b \geq b_0$ and $n \geq n_0$.

PROOF. By (5.11), (5.13) and Lemma 5.2, it suffices to prove the bound for the sum $S^{1/2}(2[nx] + [ny])$ of $2[nx] + [ny]$ i.i.d. Exp(1/2)-distributed random variables. The conclusion then follows from a standard large deviation bound,

$$\begin{aligned}\text{Prob}\big(S^{1/2}(2[nx] + [ny]) \geq bn\big) \\ \leq \exp\Big\{-(2[nx] + [ny])I\big(bn(2[nx] + [ny])^{-1}\big)\Big\},\end{aligned} \tag{7.3}$$

where

$$I(x) = x/2 - 1 - \log(x/2) \tag{7.4}$$

is the Cramér rate function for the Exp(1/2) distribution. The reader can prove (7.3) easily with the exponential Chebyshev argument of Lemma 5.4 above. Alternatively, see Section 2.2 in Dembo and Zeitouni (1993), Section 1.2 in Deuschel and Stroock (1989), Section 1.9 in Durrett (1991) or Section 3 in Varadhan (1984). The inequality (7.3) is true already for finite n due to superadditivity. By (7.4),

$$I\big(bn(2[nx] + [ny])^{-1}\big) \geq \tfrac{1}{4}bn(2[nx] + [ny])^{-1}$$

for large enough b, so the bound $\exp[-c_0 bn]$ follows. □

Consider the level curve $\{(x, y) \in \overline{\mathscr{U}_K}\colon \gamma_K(x, y) = \gamma_K(\bar{x}, \bar{y})\}$. Since γ_K is concave and strictly increasing in both variables, this level curve is the graph of a convex function $y = h(x)$. Pick a tangent line of the curve at $(\bar{x}, \bar{y})$. This line has slope $-\rho \in (-\infty, -1/K)$. (The slope cannot be equal to $-\infty$ or $-1/K$ because the level curve connects a point on the positive y-axis to a point on the positive part of the line $y = -x/K$.) The tangent line itself intersects the y-axis and the line $y = -x/K$ at the points $(0, y_1)$ and $(Ky_2, -y_2)$ for $y_1 = \bar{y} + \rho\bar{x}$ and $y_2 = (\bar{y} + \rho\bar{x})/(K\rho - 1)$. Set $y_0 = (y_1 + y_2)/2$.

On the lattice $\mathbf{Z}^2$ define, for large $M \in \mathbf{N}$,

$$\mathscr{B}_M = \{(i,j) \in \mathbf{Z}^2 \colon -[\rho i] \le j \le [My_1] - [\rho i] - 1, \textit{ and for some } 0 \le k \le [My_0/\rho], \textit{ there is a K-admissible path from } (k, -[\rho k]) \textit{ to } (i,j)\}. \tag{7.5}$$

The set $\mathscr{B}_M$ is the lattice analogue, in the scale $1/M$, of the trapezoid between the lines $y = y_1 - \rho x$ and $y = -\rho x$, with vertices (clockwise) $(0,0)$, $(0, y_1)$, $(3y_0/\rho, -y_0 - y_2)$ and $(y_0/\rho, -y_0)$. Let F be the closure of this trapezoid on the plane. See Figure 1.

Recall the definition (5.4) of the passage time $T(r)$ of a K-admissible path. Let

$$Y(\mathscr{B}_M) = \max\{T(r)\colon r \text{ is a } K\text{-admissible path} \subseteq \mathscr{B}_M\}. \tag{7.6}$$

LEMMA 7.3. *The limit*

$$\lim_{M\to\infty} \frac{1}{M} Y(\mathscr{B}_M) = \gamma_K(\bar{x}, \bar{y}) \tag{7.7}$$

holds $\mathbb{P}$*-a.s. There exist positive constants* b_0, M_0 *and* c_0 *such that*

$$\mathbb{P}(Y(\mathscr{B}_M) \ge Mb) \le \exp[-c_0 Mb] \tag{7.8}$$

for all $b \ge b_0$ *and* $M \ge M_0$.

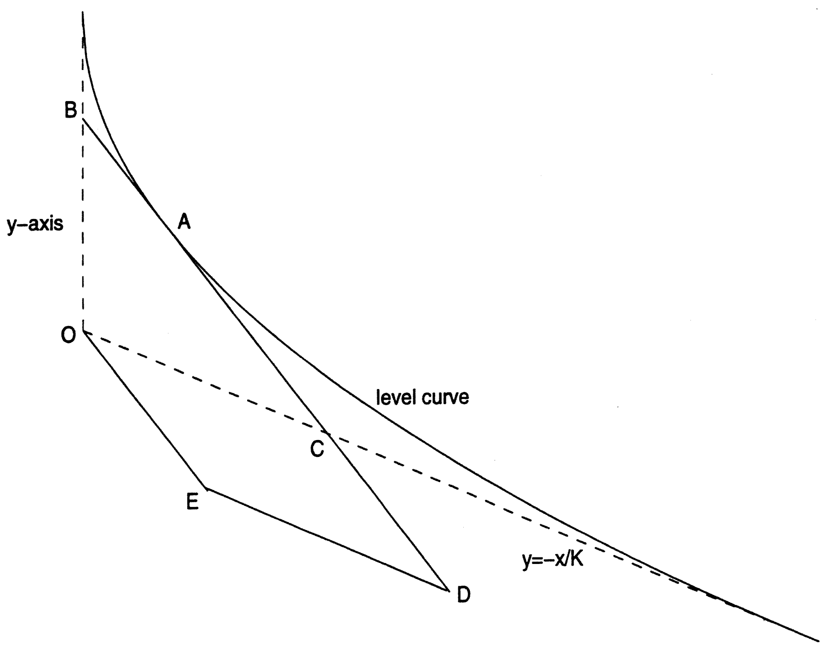

FIG. 1. *The region $\mathscr{U}_K$ is bounded by the y-axis and the line $y = -x/K$ (dashed lines) that meet at the origin O. $A = (\bar{x}, \bar{y})$ on the level curve $\{\gamma_K = \gamma_K(\bar{x}, \bar{y})\}$. BD is the tangent line at $A = (\bar{x}, \bar{y})$ of slope $-\rho$. It intersects the boundary of $\mathscr{U}_K$ at the points $B = (0, y_1)$ and $C = (Ky_2, -y_2)$. OE is a line of slope $-\rho$ through the origin, ED a line of slope $-1/K$. The vertices of the trapezoid F are OBDE.*

PROOF. Since

$$[M\bar{y}] = [My_1 - M\rho\bar{x}] \leq [My_1] - [M\rho\bar{x}], \tag{7.9}$$

every K-admissible path from $(0, 1)$ to $([M\bar{x}], [M\bar{y}] - 1)$ lies inside $\mathscr{B}_M$, and consequently by the continuity of γ_K,

$$\liminf_{M\to\infty} \frac{1}{M} Y(\mathscr{B}_M) \geq \gamma_K(\bar{x}, \bar{y}), \quad \mathbb{P}\text{-a.s.}$$

For the converse, suppose a small number $\delta_1 > 0$ is given. Then by the compactness of F it is possible to pick two even smaller numbers $\delta, \varepsilon > 0$ and two finite sequences

$$x'_0 < x'_1 < \cdots < x'_I \quad \text{and} \quad x''_0 < x''_1 < \cdots < x''_J$$

such that this holds: with $y'_k = -\varepsilon - \rho x'_k$ and $y''_l = y_1 + \varepsilon - \rho x''_l$, we have the inclusions

$$F^{(\delta)} \subseteq \bigcup_{k=0}^{I} [(x'_k, y'_k) + \mathscr{U}_K] \quad \text{and} \quad F^{(\delta)} \subseteq \bigcup_{l=0}^{J} [(x''_l, y''_l) - \mathscr{U}_K], \tag{7.10}$$

where $F^{(\delta)}$ is the δ-neighborhood of F, and moreover, all (x'_k, y'_k) and (x''_l, y''_l) lie in $F^{(\delta_1)}$. See Figure 2.

Let $\mathscr{I}$ be the set of pairs (k, l) such that $(x''_l, y''_l) \in (x'_k, y'_k) + \mathscr{U}_K$. It then follows, for large enough M, that if r is any K-admissible path in $\mathscr{B}_M$, there is a pair $(k, l) \in \mathscr{I}$ and a K-admissible path $r' \supseteq r$ from $([Mx'_k] + 1, [My'_k])$ to $([Mx''_l], [My''_l])$. Consequently,

$$Y(\mathscr{B}_M) \leq \max_{(k,l)\in\mathscr{I}} T_K(([Mx'_k], [My'_k]), ([Mx''_l], [My''_l])), \tag{7.11}$$

and then

$$\limsup_{M\to\infty} \frac{1}{M} Y(\mathscr{B}_M) \leq \max_{(k,l)\in\mathscr{I}} \gamma_K(x''_2 - x'_k, y''_2 - y'_k), \quad \mathbb{P}\text{-a.s.}$$

To establish (7.7) it remains to argue that, given ε_0, we can make

$$\max_{(k,l)\in\mathscr{I}} \gamma_K(x''_l - x'_k, y''_l - y'_k) \leq \gamma_K(\bar{x}, \bar{y}) + \varepsilon_0$$

by choosing δ_1 small enough. Consider any point $(x', -\rho x')$, $x' \in [0, y_0/\rho]$, on the southwest edge of F (edge OE in Figure 1). By convexity and the choice of the slope ρ, the translated level curve

$$\{(x, y) \in (x', -\rho x') + \overline{\mathscr{U}_K}: \gamma_K(x - x', y + \rho x') = \gamma_K(\bar{x}, \bar{y})\}$$

is on or above the line $y = y_1 - \rho x$ (line BD in Figure 1), and consequently $\gamma_K(x - x', y + \rho x') \leq \gamma_K(\bar{x}, \bar{y})$ for all $(x, y) \in [(x', -\rho x') + \overline{\mathscr{U}_K}] \cap F$. This works for all points on the southwest edge of F. Therefore

$$\sup\{\gamma_K(x - x', y - y'): (x, y), (x', y') \in F, (x, y) \in (x', y') + \overline{\mathscr{U}_K}\} = \gamma_K(\bar{x}, \bar{y}).$$

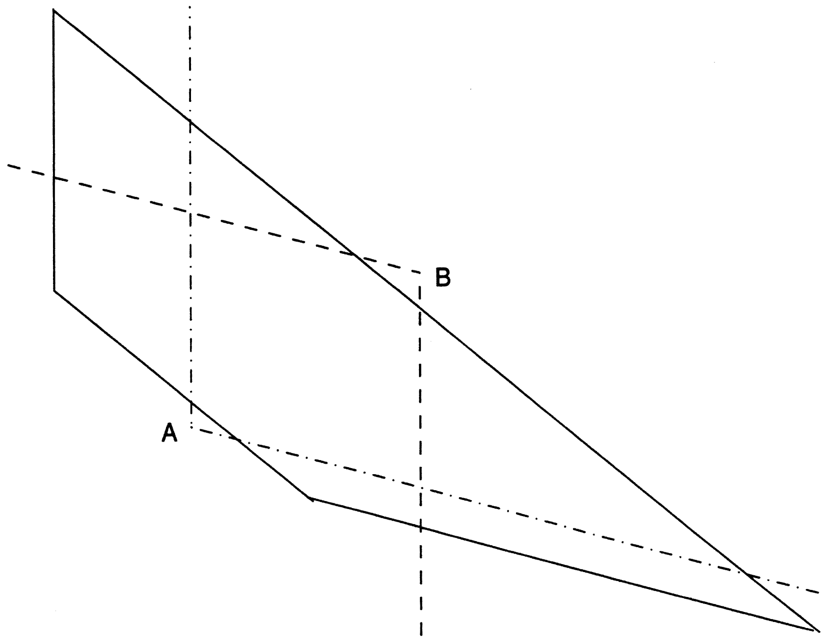

FIG. 2. [*To justify* (7.10)]. *The trapezoid represents the neighborhood* $F^{(\delta)}$, $A = (x'_k, y'_k)$, *and* $B = (x''_l, y''_l)$. *The dash-dot line bounds the set* $(x'_k, y'_k) + \mathscr{U}_K$. *The reader sees that finitely many such sets can cover* $F^{(\delta)}$, *however close to* $F^{(\delta)}$ *we require the basepoints A to be. The same reasoning applies to* $(x''_l, y''_l) - \mathscr{U}_K$ *bounded by the dashed line.*

By continuity of γ_K and compactness of F, for small enough $\delta_1 > 0$, we have

$$\sup\{\gamma_K(x - x', y - y') : (x, y), (x', y') \in F^{(\delta_1)}, (x, y) \in (x', y') + \overline{\mathscr{U}_K}\} \\ \leq \gamma_K(\bar{x}, \bar{y}) + \varepsilon_0.$$

This completes the proof of (7.7).

The deviation bound (7.8) follows by applying Lemma 7.2 to the finitely many random variation on the right-hand side of (7.11). □

The preliminaries are done and we can attack Lemma 7.1

PROOF OF LEMMA 7.1. Abbreviate $\bar{\gamma} = \gamma_K(\bar{x}, \bar{y})$. Suppose M is large enough so that (7.8) is valid, and let $n \gg M$. Recall the definition of the number y_1 in the paragraph preceding (7.5) (point B in Figure 1). Let R_0 be the smallest integer that satisfies

$$R_0[My_1] \geq [ny_1] + 1. \tag{7.12}$$

Given $\varepsilon_1 > 0$,

$$(1 - \varepsilon_1)n/M \leq R_0 \leq (1 + \varepsilon_1)n/M \tag{7.13}$$

for large enough n. Divide the lattice $\mathscr{L}_K$ into parallel strips of slope $-\rho$ and height $[My_1]$, so that the kth strip, $1 \le k \le R_0$, is given by

$$\mathscr{T}_k = \{(i,j) \in \mathscr{L}_K : (k-1)[My_1] - [\rho i] \le j \le k[My_1] - [\rho i] - 1\}. \tag{7.14}$$

Equations (7.9) and (7.12) guarantee that $([n\bar{x}],[n\bar{y}])$ lies in the union $\mathscr{T}_1 \cup \cdots \cup \mathscr{T}_{R_0}$.

The definition (7.5) of $\mathscr{B}_M$ was tailored so that the height of $\mathscr{B}_M$ matches the height of a strip $\mathscr{T}_k$. We can cover each strip with finitely many translates of the set $\mathscr{B}_M$ so that these translates do not intersect on the bottom boundary line of the strip. That is, if $\mathscr{B}'$, $\mathscr{B}''$ are two such translates inside $\mathscr{T}_k$ and $(i,(k-1)[My_1] - [\rho i]) \in \mathscr{B}'$, then $(i,(k-1)[My_1] - [\rho i]) \notin \mathscr{B}''$.

Fix such a covering of each strip $\mathscr{T}_k$ by translates of $\mathscr{B}_M$, henceforth called *basic regions*. For $\mathscr{T}_1$, only one basic region is needed. If M is large enough, any K-admissible path that starts in a basic region $\mathscr{B} \subseteq \mathscr{T}_k$ can enter only five possible basic regions inside $\mathscr{T}_{k+1}$. (Figure 3.) Thus there are at most 5^{R_0}

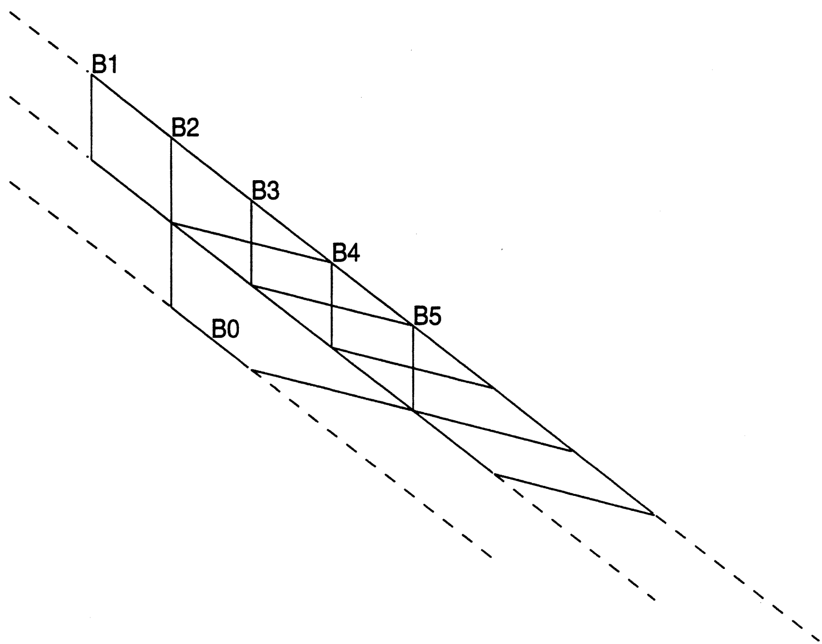

FIG. 3. *The dashed lines bound the strips $\mathscr{T}_k$ and $\mathscr{T}_{k+1}$. Here $\mathscr{B}_0$ is a basic region in $\mathscr{T}_k$, and $\mathscr{B}_1,\dots,\mathscr{B}_5$ are basic regions in $\mathscr{T}_{k+1}$. The figure gives a macroscopic representation. For $i = 1,\dots,5$, the label Bi is placed at the top vertex of the trapezoid that represents $\mathscr{B}_i$. $\mathscr{B}_1,\dots,\mathscr{B}_5$ overlap in the interior of $\mathscr{T}_{k+1}$ but not on the dashed boundary between the two strips. $\mathscr{B}_0$ can touch at most five basic regions in $\mathscr{T}_{k+1}$ due to our choice of $y_0 = (y_1 + y_2)/2$. This makes the length of the northeast edge* (*BD in Figure* 1) *of the trapezoid three times the length of the southwest edge* (*OE in Figure* 1).

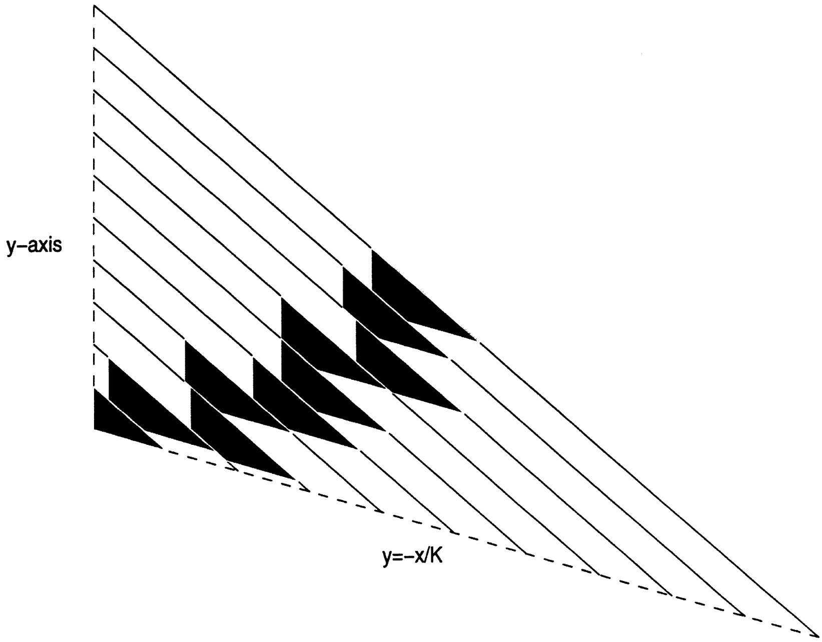

FIG. 4. *An example of a sequence $\mathscr{B}(1), \dots, \mathscr{B}(R_0)$ of basic regions whose union contains K-admissible paths from $(1,0)$ to the northeast edge of $\mathscr{B}(R_0)$. The solid lines mark the boundaries of the strips $\mathscr{T}_k$. The figure gives a macroscopic view so the basic regions are represented by shaded trapezoids. Here $\mathscr{B}(1)$ in $\mathscr{T}_1$ is reduced to a triangle because the trapezoid is larger than $\mathscr{T}_1$.*

sequences $\mathscr{B}(1), \dots, \mathscr{B}(R_0)$ of basic regions such that $\mathscr{B}(j) \subseteq \mathscr{T}_j$ and such that the union of the $\mathscr{B}(j)$'s can contain a K-admissible path from $(1,0)$ to $([n\bar{x}], [n\bar{y}])$. (Figure 4). If $\mathscr{B}(1), \dots, \mathscr{B}(R_0)$ is such a sequence that contains the K-admissible path r, then $T(r) \geq n\bar{\gamma} + n\varepsilon$ implies $Y(\mathscr{B}(1)) + \cdots + Y(\mathscr{B}(R_0)) \geq n\bar{\gamma} + n\varepsilon$, where the random variable $Y(\mathscr{B}(j))$ is defined as in (7.6).

Now fix a sequence $\mathscr{B}(1), \dots, \mathscr{B}(R_0)$. We shall estimate the probability

$$\mathbb{P}\left(\sum_{j=1}^{R_0} Y(\mathscr{B}(j)) \geq n\bar{\gamma} + n\varepsilon \right).$$

Incidentally, an ordinary exponential large deviation estimate for the i.i.d. random variables $Y(\mathscr{B}(j))$ of the type $\mathbb{P}(\sum_1^{R_0} Y(\mathscr{B}(j)) \geq R_0 \beta) \leq \exp[-R_0 Q(\beta)]$ for some unknown rate function Q would not be of help now because, in the final step, our estimate has to be multiplied by the factor 5^{R_0}, to account for all the possible ways of choosing the sequence $\mathscr{B}(1), \dots, \mathscr{B}(R_0)$.

Pick and fix a number $\beta_0 > 0$. In the first step we decompose like this:

$$\begin{aligned} &\mathbb{P}\left(\sum_{j=1}^{R_0} Y(\mathscr{B}(j)) \geq n\bar{\gamma} + n\varepsilon \right) \\ &\quad \leq \sum_{j=1}^{R_0} \mathbb{P}(Y(\mathscr{B}(j)) \geq \beta_0 n) \\ &\qquad + \mathbb{P}\left(\sum_{j=1}^{R_0} Y(\mathscr{B}(j)) \geq n\bar{\gamma} + n\varepsilon, Y(\mathscr{B}(j)) < \beta_0 n \text{ for } 1 \leq j \leq R_0 \right). \end{aligned} \tag{7.15}$$

Next we partition $(0, \beta_0 n)$ and count the number of $Y(\mathscr{B}(j))$'s that fall into each partition interval. First, pick small $\delta > 0$ and $\varepsilon_2 \in (0, \varepsilon)$. Let

$$L_0 = [b_0/\delta] + 1, \tag{7.16}$$

where b_0 is the constant that appears in Lemma 7.3, and

$$L_1 = [\beta_0 n / M\delta] + 1. \tag{7.17}$$

Set

$$\zeta_0 = \sum_{j=1}^{R_0} I\{Y(\mathscr{B}(j)) < M(\bar{\gamma} + \varepsilon_2)\},$$

$$\zeta_{L_0} = \sum_{j=1}^{R_0} I\{M(\bar{\gamma} + \varepsilon_2) \leq Y(\mathscr{B}(j)) < ML_0\delta\}$$

and

$$\zeta_l = \sum_{j=1}^{R_0} I\{M(l-1)\delta \leq Y(\mathscr{B}(j)) < Ml\delta\} \quad \text{for } L_0 < l \leq L_1.$$

In the next step, apply (7.8) to the probabilities $\mathbb{P}(Y(\mathscr{B}(j)) \geq \beta_0 n)$ in (7.15). This is valid for n/M large enough. In the last probability in (7.15), pick $\varepsilon_3 > 0$ and separate cases according to the value of ζ_{L_0}. Then (7.15) becomes

$$\begin{aligned} &\mathbb{P}\left(\sum_{j=1}^{R_0} Y(\mathscr{B}(j)) \geq n\bar{\gamma} + n\varepsilon \right) \\ &\quad \leq R_0 \exp[-c_0 \beta_0 n] + \mathbb{P}(\zeta_{L_0} > \varepsilon_3 R_0) \\ &\qquad + \mathbb{P}\Bigg(\sum_{j=1}^{R_0} Y(\mathscr{B}(j)) \geq n\bar{\gamma} + n\varepsilon, \\ &\qquad\qquad Y(\mathscr{B}(j)) < \beta_0 n \text{ for } 1 \leq j \leq R_0, \zeta_{L_0} \leq \varepsilon_3 R_0 \Bigg). \end{aligned} \tag{7.18}$$

In the last probability, the possible realizations of $(\zeta_0, \zeta_{L_0}, \dots, \zeta_{L_1})$ are sequences $\mathbf{w} = (w_0, w_{L_0}, \dots, w_{L_1}) \in \mathbf{Z}_+^{L_1 - L_0 + 2}$ of nonnegative integers that satisfy these properties:

$$w_0 + \sum_{l=L_0}^{L_1} w_l = R_0, \tag{7.19}$$

$$w_0 M(\bar{\gamma} + \varepsilon_2) + \sum_{l=L_0}^{L_1} w_l M l \delta \geq n(\bar{\gamma} + \varepsilon) \tag{7.20}$$

and

$$w_{L_0} \leq \varepsilon_3 R_0. \tag{7.21}$$

Let

$$\mathscr{G} = \left\{ \mathbf{w} \in \mathbf{Z}_+^{L_1 - L_0 + 2} : \mathbf{w} \text{ satisfies } (7.19), (7.20) \text{ and } (7.21) \right\}.$$

The last probability in (7.18) is bounded above by $\mathbb{P}((\zeta_0, \zeta_{L_0}, \dots, \zeta_{L_1}) \in \mathscr{G})$. Let us estimate the probability of a single element $\mathbf{w} \in \mathscr{G}$. Abbreviate

$$\binom{R_0}{\mathbf{w}} = \binom{R_0}{w_0, w_{L_0}, \dots, w_{L_1}}$$

for the number of ways of arranging the $Y(\mathscr{B}(j))$'s in the partition intervals, with counts $(w_0, w_{L_0}, \dots, w_{L_1})$. Then, since the $Y(\mathscr{B}(j))$'s are i.i.d. and distributed like $Y(\mathscr{B}_M)$,

$$\begin{aligned}
&\mathbb{P}\big((\zeta_0, \zeta_{L_0}, \dots, \zeta_{L_1}) = (w_0, w_{L_0}, \dots, w_{L_1})\big) \\
&\quad \leq \binom{R_0}{\mathbf{w}} \prod_{l=L_0+1}^{L_1} \mathbb{P}\big(M(l-1)\delta \leq Y(\mathscr{B}_M) < Ml\delta\big)^{w_l} \\
&\quad \leq \binom{R_0}{\mathbf{w}} \exp\left\{ -c_0 \sum_{l=L_0+1}^{L_1} w_l M(l-1)\delta \right\} && [\text{by } (7.8)] \\
&\quad \leq \binom{R_0}{\mathbf{w}} \exp\left\{ -c_0 \left[n(\bar{\gamma} + \varepsilon) - w_0 M(\bar{\gamma} + \varepsilon_2) - w_{L_0} M L_0 \delta \right.\right. \\
&\qquad\qquad \left.\left. - \sum_{l=L_0+1}^{L_1} w_l M \delta \right] \right\} [\text{by } (7.20)] \\
&\quad \leq \binom{R_0}{\mathbf{w}} \exp\big\{ -c_0 \big[n(\bar{\gamma} + \varepsilon) - R_0 M \delta - w_0 M(\bar{\gamma} + \varepsilon_2 - \delta) \\
&\qquad\qquad - w_{L_0} M(L_0 - 1)\delta \big] \big\} [\text{by } (7.19)]
\end{aligned}$$

$$\leq \binom{R_0}{\mathbf{w}} \exp\{-c_0 n[\bar{\gamma} + \varepsilon - (1+\varepsilon_1)(\bar{\gamma} + \varepsilon_2) - \varepsilon_3(1+\varepsilon_1) b_0]\}$$

$$\text{[by (7.13), (7.16) and (7.21) and by choosing } \delta \text{ small enough]}$$

$$\leq \binom{R_0}{\mathbf{w}} \exp[-c_1 n],$$

for a constant $c_1 > 0$, by choosing ε_1, ε_2 and ε_3 small enough. [Note to the reader: The first step above would not work for the disordered case, because the $Y(\mathscr{B}(j))$'s might not be independent. Indeed, for some sequence $\mathscr{B}(1), \dots, \mathscr{B}(R_0)$, the $Y(\mathscr{B}(j))$'s might all depend on some particular $\alpha(j)$. This can be fixed by conditioning on $\boldsymbol{\alpha}$, but then the $Y(\mathscr{B}(j))$'s are no longer identically distributed. On account of these complications we state and prove Lemma 7.1 only for the basic case without disorder.]

Let $\delta(M) = \mathbb{P}(Y(\mathscr{B}_M) \geq M(\bar{\gamma} + \varepsilon_2))$. By (7.7) we may suppose M is large enough to have $\delta(M)^{\varepsilon_3} < 1/2$, for a fixed $\varepsilon_3 > 0$. Returning to (7.18), we have

$$\begin{aligned}
&\mathbb{P}\left(\sum_{j=1}^{R_0} Y(\mathscr{B}(j)) \geq n\bar{\gamma} + n\varepsilon\right) \\
&\quad \leq R_0 \exp[-c_0 \beta_0 n] + \mathbb{P}(\zeta_{L_0} > \varepsilon_3 R_0) + \sum_{\mathbf{w} \in \mathscr{G}} \binom{R_0}{\mathbf{w}} \exp[-c_1 n] \\
&\quad \leq R_0 \exp[-c_0 \beta_0 n] + 2^{R_0} \delta(M)^{\varepsilon_3 R_0} + (L_1 - L_0 + 2)^{R_0} \exp[-c_1 n] \qquad (7.22)\\
&\quad \leq \frac{n}{M}(1+\varepsilon_1) \exp[-c_0 \beta_0 n] + \left[2\delta(M)^{\varepsilon_3}\right]^{(1-\varepsilon_1)n/M} \\
&\qquad + \exp\left[-c_1 n + \left(\frac{n}{M}\right)(1+\varepsilon_1) \log\left(\frac{\beta_0 n}{M\delta}\right)\right].
\end{aligned}$$

To recapitulate, we have come thus far by assuming that M, n and n/M are large enough. By taking $M \geq C \log n$ for a large enough constant C, we can bound the last three terms above by $\exp[-c_2 n (\log n)^{-1}]$ for a constant $c_2 > 0$. Finally,

$$\begin{aligned}
&\mathbb{P}(T_K([n\bar{x}], [n\bar{y}]) \geq n\gamma_K(\bar{x}, \bar{y}) + n\varepsilon) \\
&\quad \leq \sum_{\mathscr{B}(1), \dots, \mathscr{B}(R_0)} \mathbb{P}\left(\sum_{j=1}^{R_0} Y(\mathscr{B}(j)) \geq n\bar{\gamma} + n\varepsilon\right) \\
&\quad \leq 5^{R_0} \exp\left[-c_2 n (\log n)^{-1}\right] \\
&\quad \leq \exp\left[n (\log n)^{-1}((1+\varepsilon_1)(\log 5)/C - c_2)\right] \\
&\quad \leq \exp\left[-c_3 n (\log n)^{-1}\right]
\end{aligned}$$

for yet another constant $c_3 > 0$, by increasing C further if necessary. This completes the proof of Lemma 7.1.

One last word about the stage of the proof where we gave up the genuine exponential bound $\exp[-Cn]$: the culprit is the factor $\sum_{\mathbf{w} \in \mathscr{G}} \binom{R_0}{\mathbf{w}}$ in (7.22), which we bounded above by $(L_1 - L_0 + 2)^{R_0} \le (\beta_0 n/M\delta)^{(1+\varepsilon_1)n/M}$. To control this term, we need M to grow sufficiently fast with n. Then we can no longer assert that $[2\delta(M)^{\varepsilon_3}]^{(1-\varepsilon_1)n/M}$ decays exponentially, because we do not know the true decay rate of $\delta(M)$. □

8. Proofs of the strong laws. In this section we prove the strong law part of Theorem 1 for K-exclusion, and Theorem 2 for the marching soldiers model. In this section the rates are constant $\alpha(j) \equiv 1$.

8.1. *Proof of the strong law of Theorem* 1. According to assumption (2.5), the initial configurations $(\eta_n(i,0))$ are now defined on some common probability space $(\Omega, \mathscr{F}, P)$. Augment this probability space to include the Poisson processes required for the graphical construction. Define the initial server configurations $(z_n(i,0))$ by (6.1). Then construct the server processes $z_n(\cdot)$ and $w^{z_n(i,0)}(\cdot)$ and the interface processes $\xi^{z_n(i,0)}(\cdot)$ on $(\Omega, \mathscr{F}, P)$ according to the recipes of Section 4. As a corollary of our estimates we get the following lemma.

LEMMA 8.1. *Suppose $a_0 = \alpha(j) = 1$ for all $j \in \mathbf{Z}$. For any $b, y \in \mathbf{R}$ and $t > 0$,*

$$\lim_{n\to\infty} \frac{1}{n} \xi^{z_n([nb],0)}([ny], nt) = t g_K(y/t) \quad P\text{-}a.s. \tag{8.1}$$

PROOF. By an argument similar to the one of Corollary 5.1, Lemmas 5.4 and 7.1 apply to deviations of ξ as well as of T_K and imply that

$$\sum_{n=1}^{\infty} \mathbb{P}\big(|\xi([ny], nt) - t g_K(y/t)| \ge \varepsilon\big) < \infty \tag{8.2}$$

for any $\varepsilon > 0$. Next note that the P-distribution of $\xi^{z_n([nb],0)}([ny], nt)$ and the $\mathbb{P}$-distribution of $\xi([ny], nt)$ are equal. Apply the Borel–Cantelli lemma. □

By the reasoning that led to (6.3) under assumption (3.2), under the stronger assumption (2.5) we get that

$$\lim_{n\to\infty} \frac{1}{n} z_n([ny], 0) = U_0(y), \quad P\text{-a.s. for all } y \in \mathbf{R}. \tag{8.3}$$

Fix (x, t). As argued in conjunction with (6.12) above, equations (2.8), (6.5), (8.1) and (8.3) imply that

$$\liminf_{n\to\infty} \frac{1}{n} z_n([nx], nt) \ge U(x, t), \quad P\text{-a.s.} \tag{8.4}$$

On the other hand, by Lemma 6.1 and Borel–Cantelli,

$$\limsup_{n\to\infty} \frac{1}{n} z_n([nx], nt) = \limsup_{n\to\infty} \frac{1}{n} \zeta_n(r_1, r_2), \quad P\text{-a.s.},$$

while by the development in (6.14),

$$\limsup_{n\to\infty} \frac{1}{n} \zeta_n(r_1, r_2) \le U(x,t) + \varepsilon/2, \quad P\text{-a.s.}$$

for any $\varepsilon > 0$. In conclusion, we get

$$\lim_{n\to\infty} \frac{1}{n} z_n([nx], nt) = U(x,t), \quad P\text{-a.s.} \tag{8.5}$$

and this suffices for the strong law of Theorem 1.

8.2. *Proof of Theorem* 2. Given the initial soldier locations $\sigma_n(i,0)$, define initial server configurations by

$$z_n(i,0) = \sigma_n(0,0) - \sigma_n(-i,0) + iL_1, \qquad i \in \mathbf{Z}. \tag{8.6}$$

This defines admissible server configurations with $K = L_1 + L_2$. Next, construct the server processes $z_n(\cdot)$ as in Section 4. Define the soldiers process $\sigma_n(\cdot)$ by

$$\sigma_n(i,t) = \sigma_n(0,0) - z_n(-i,t) - iL_1, \qquad i \in \mathbf{Z}. \tag{8.7}$$

The reader can check that this produces a process that satisfies the rules laid down for the marching soldiers in Section 2. The jump rule of $z_n(\cdot)$ with $K = L_1 + L_2$ implies that $\sigma_n(i,t)$ jumps *right* with rate 1 if $\sigma_n(i,t) \le \sigma_n(i-1,t) + L_2 - 1$ and $\sigma_n(i,t) \le \sigma_n(i+1,t) + L_1 - 1$.

Assumption (2.22) and definition (8.6) imply that

$$\lim_{n\to\infty} \frac{1}{n} z_n([ny], 0) = U_0(y) \equiv V_0(0) - V_0(-y) + L_1 y \quad \text{a.s.} \tag{8.8}$$

for all $y \in \mathbf{R}$. From this assumption, we have just proved in (8.5) that

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n} z_n([nx], nt) \\ &\quad = U(x,t) \equiv \sup_y \{U_0(y) - t g_{L_1+L_2}((x-y)/t)\} \quad \text{a.s.} \end{aligned} \tag{8.9}$$

for all $x \in \mathbf{R}$ and $t > 0$. The limit (8.9) and definition (8.7) then imply the limit of $n^{-1}\sigma_n([nx], nt)$. Theorem 2 is thereby proved. Statement (2.24) follows from (2.20) and Theorem 2 applied to $x = 0$.

9. Properties of g_K and f_K. Suppose that the distribution of $(z(i,0): i \in \mathbf{Z})$ is such that $z(0,0) = 0$ with probability 1 and that $(\eta(i,0) = z(i,0) - z(i-1,0): i \in \mathbf{Z})$ is an ergodic process with expectation $E[\eta(i,0)] = \rho$. Then, with $z_n(i,0) = z(i,0)$ for all n, (6.3) is satisfied with $U_0(y) = \rho y$. By (6.4), in

P^{α}-probability,

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}z(0,t) &= U(0,1)\\ &= \sup_{y\in\mathbf{R}}\{\rho y - g_K(-y)\}\\ &= -f_K(\rho),\end{aligned} \tag{9.1}$$

where we used (2.8) and (2.10). On the other hand, recall that $z(0,t)$ jumps leftward with rate $\alpha(0)$ as long as $\eta(0,t) = z(0,t) - z(-1,t) \geq 1$ and $\eta(1,t) = z(1,t) - z(0,t) \leq K-1$. Then standard generator considerations permit us to write

$$z(0,t) = -\alpha(0)\int_0^t I\{\eta(0,s) \geq 1, \eta(1,s) \leq K-1\}\,ds + M_t, \tag{9.2}$$

where M_t is a martingale. Taking expectations gives

$$E^{\alpha}[z(0,t)] = -\alpha(0)\int_0^t P^{\alpha}(\eta(0,s) \geq 1, \eta(1,s) \leq K-1)\,ds. \tag{9.3}$$

Clearly $|z(0,t)|$ is bounded above by the total number of jump attempts in time $[0,t]$ which is Poisson($\alpha(0)t$)-distributed. This gives uniform integrability, and, consequently, the limit in (9.1) holds also in expectation. So (2.14) and (3.5) are proved.

Next we have a lemma about g_K that will give us the properties of f_K announced after Theorem 1. Recall that the limits of Proposition 5.1 and Corollary 5.1 required only ergodicity of $(\alpha(j))$.

LEMMA 9.1. *Assume $(\alpha(j))$ is ergodic. We have these properties*:

(i) *For $y \geq 0$, $g_K(y) \geq g_{K-1}(y)$ and for $y < 0$, $g_K(y) \geq g_{K-1}(y) - y$.*

(ii) *Suppose in addition that $(\alpha(j)\colon j \in \mathbf{Z})$ is reversible, so that $(\alpha(j)\colon j \in \mathbf{Z}) =_d (\alpha(-j)\colon j \in \mathbf{Z})$. Then for all $y \in \mathbf{R}$, $g_K(-y) = g_K(y) + Ky$.*

PROOF. For (i), use the bijection $p\colon \mathscr{L}_K \to \mathscr{L}_{K-1}$ defined by

$$p(i,j) = \begin{cases}(i,j), & j \geq 0,\\ (i+j,j), & j < 0.\end{cases} \tag{9.4}$$

The image $p(r)$ of a K-admissible path r may fail to be $(K-1)$-admissible, but it is contained in a $(K-1)$-admissible path. This is because, under p, $(1,0)$-steps are preserved; a $(K,-1)$-step becomes either a single $(K-1,-1)$-step or a $(1,0)$-step followed by a $(K-1,-1)$-step, and a $(0,1)$-step becomes either a single $(0,1)$-step or a $(1,0)$-step followed by a $(0,1)$-step. Since passage times are monotone under inclusion of paths, we get the inequality $T_K(i,j) \leq T_{K-1}(p(i,j))$. Passing to the limit (5.9) then gives

$$\gamma_K(x,y) \leq \begin{cases}\gamma_{K-1}(x,y), & y \geq 0,\\ \gamma_{K-1}(x+y,y), & y < 0.\end{cases} \tag{9.5}$$

This implies statement (i).

For (ii) we use the bijection $p\colon \mathscr{L}_K \to \mathscr{L}_K$ defined by $p(i,j) = (i + Kj, -j)$. Check that p is its own inverse, and that p preserves K-admissibility of lattice paths, by interchanging $(K, -1)$-steps with $(0,1)$-steps, and by leaving $(1,0)$-steps intact. The reversibility assumption implies that the distribution of the sitewise passage times is invariant under p: $(\tau(i,j)\colon (i,j) \in \mathscr{L}_K) =_d (\tau(p(i,j))\colon (i,j) \in \mathscr{L}_K)$ where $\tau(i,j) = \alpha(j)^{-1}\lambda(i,j)$ as in Section 5. From all this follows that $T_K(i,j) =_d T_K(p(i,j))$, and consequently $\gamma_K(x,y) = \gamma_K(x + Ky, -y)$. This implies statement (ii). $\square$

Lemma 9.1(ii) implies that any tangent to g_K at $x > 0$ must have slope in $[-K/2, 0]$. This and (2.10) imply that f_K is nondecreasing on $[0, K/2]$. The symmetry (2.16) of f_K follows from (2.10) and Lemma 9.1(ii), while the K-monotonicity (2.17) comes from (2.10) and Lemma 9.1(i).

To get explicitly computable bounds, we restrict ourselves to the basic case without disorder. First we explain Remark 5.2. Definition (2.10), the convexity and continuity of g_K and basic convex duality [Rockafellar (1970)] imply that

$$g_K(y) = \sup_{0 \le \rho \le K} \{ f_K(\rho) - x\rho \}. \tag{9.6}$$

For $K = 1$ we deduce, from (2.15) and Bernoulli equilibria ν_ρ, the well-known flux $f_1(\rho) = \rho(1 - \rho)$, and then (9.6) gives

$$g_1(y) = \begin{cases} -y, & y \le -1, \\ (1/4)(1-y)^2, & -1 \le y \le 1, \\ 0, & y \ge 1. \end{cases}$$

From homogeneity (5.18) and the identity $\gamma_1(g_1(y), y) = 1$ for $y \in [-1, 1]$, one deduces $\gamma_1(x,y) = (\sqrt{x} + \sqrt{x+y})^2$. Equations (5.23) (with $a_1 = 1$) and (9.5) then imply the boundary values (5.24).

Lemma 9.1(i) gives a lower bound for g_K in terms of g_1. To get an upper bound, we utilize Rost's (1981) calculation (5.14). Define a convex function h_K by

$$h_K(y) = \begin{cases} -Ky, & y \le -1, \\ \left(1 - \sqrt{-y}\right)^2 - Ky, & -1 \le y \le -4/(K+2)^2, \\ K/(K+2) - Ky/2, & -4/(K+2)^2 \le y \le 4/(K+2)^2, \\ \left(1 - \sqrt{y}\right)^2, & 4/(K+2)^2 \le y \le 1, \\ 0, & y \ge 1. \end{cases} \tag{9.7}$$

LEMMA 9.2. *Suppose $\alpha(j) = 1$ for all j. Then $g_K(y) \le h_K(y)$ for all $y \in \mathbf{R}$.*

PROOF. Let $(i, j) \in \mathbf{N}^2$. In the definition (5.12) of the last-passage time $V(i, j)$, set $v(i, j) = \lambda(i, j)$. Since the up-right paths r' in (5.12) are K-admissible, $T_K(i, j) \geq V(i, j)$. By passing to the limits (5.9) and (5.14), we get $\gamma_K(x, y) \geq (\sqrt{x} + \sqrt{y})^2$ for $x, y > 0$, and consequently $g_K(y) \leq (1 - \sqrt{y})^2$ for $0 \leq y \leq 1$. An application of Lemma 9.1(ii) then gives

$$g_K(y) \leq \tilde{h}_K(y) = \begin{cases} -Ky, & y \leq -1, \\ \left(1 - \sqrt{-y}\right)^2 - Ky, & -1 \leq y \leq 0, \\ \left(1 - \sqrt{y}\right)^2, & 0 \leq y \leq 1, \\ 0, & 1 \leq y. \end{cases} \tag{9.8}$$

The function $\tilde{h}_K$ is not convex while g_K is convex, so we can replace $\tilde{h}_K$ in the inequality by its convex hull h_K, the greatest convex function majorized by $\tilde{h}_K$. □

Substituting the upper bound h_K for g_K in (2.10) gives the upper bound (2.19) for f_K. The function on the right-hand side of (2.19) has a corner at $\rho = K/2$. This corresponds to the linear segment of slope $-K/2$ that h_K has on an interval around 0.

10. A non-Markovian *K*-exclusion. Let $\beta(l, i), (l, i) \in \mathbf{Z}^2$, be $(0, \infty)$-valued i.i.d. random variables with common cumulative distribution function F. In this section we study a non-Markovian totally asymmetric K-exclusion that uses the process $\mathscr{D}_i = \{\beta(l, i): l \in \mathbf{Z}\}$ as the waiting times for jumps from site i to $i + 1$.

These are the informally stated rules: Let t_0 be the last time a particle jumped from site i to $i + 1$, or let $t_0 = 0$ if we are just starting the process. Let t_1 be the first time in $[t_0, \infty)$ such that

$$\eta(i, t_1) \geq 1 \quad \text{and} \quad \eta(i + 1, t_1) \leq K - 1. \tag{10.1}$$

Let $\beta(l, i)$ be the next unused waiting time from $\mathscr{D}_i$. Then at time $t_2 = t_1 + \beta(l, i)$ one particle jumps from site i to $i + 1$; in other words, $\eta(i, t_2) = \eta(i, t_2 -) - 1$ and $\eta(i + 1, t_2) = \eta(i + 1, t_2 -) + 1$.

Notice that the only events involving sites i and $i + 1$ that can happen in the time interval $[t_1, t_2)$ are particles jumping from $i + 1$ to $i + 2$, or from $i - 1$ to i. Neither event violates the precondition (10.1) for the jump from i to $i + 1$, so this jump can be legitimately executed at time t_2. The process $\mathscr{D}_i$ need not be bi-infinite for this description, but later this property will be useful. The jumping rule can be summarized like this: as soon as (10.1) holds, pick an F-distributed waiting time β independently of everything else, and let one particle jump from i to $i + 1$ after time β.

The possibility of simultaneous jumps is not a problem, so we make no continuity assumptions on F. We shall not attempt a more technical description of the process at this point. In Section 10.1 below we define the process through a rigorous construction of the corresponding server process $z(\cdot)$, as was done in Section 4 for the Markovian K-exclusion.

Otherwise the basic set-up is the same as in Section 2. We assume a sequence $\eta_n(\cdot)$ of processes that satisfies the weak law (2.4) at time zero. Let us normalize the waiting times by assuming $E[\beta(l,i)] = 1$ so that (2.6) becomes part of the conclusion again. (The case $E[\beta(l,i)] = \infty$ gives trivial hydrodynamic behavior where nothing moves, for the reason outlined in the last paragraph of Section 6.)

THEOREM 4. *There exists a nonnegative convex function g_K that depends on F and has property* (2.6), *and such that the following statement holds: if assumption* (2.4) *is satisfied and $u(x,t)$ is defined by* (2.7)–(2.9), *then for each $t > 0$, $\phi \in C_0(\mathbf{R})$, and $\varepsilon > 0$,*

$$\lim_{n\to\infty} P_n\left(\left|\pi_n(nt,\phi) - \int_{\mathbf{R}} \phi(x)u(x,t)\,dx\right| \geq \varepsilon\right) = 0. \tag{10.2}$$

If we wanted a strong law, we would need some boundedness assumptions on F to make Lemma 7.1 valid. To prove Theorem 4 we first construct a process $\eta(\cdot)$ that operates according to the rules laid down above, and then we observe how the arguments of Sections 5 and 6 work in this more general setting.

10.1. *Construction of the process.* As in Section 4, we construct the server process $z(\cdot)$ whose particles jump to the left and then define the customer process $\eta(\cdot)$ by (4.1). Assume we are given an initial configuration $(z(i,0)$: $i \in \mathbf{Z})$ such that (4.1) holds for $t = 0$, where $(\eta(i,0)$: $i \in \mathbf{Z})$ is the desired initial K-exclusion configuration. In particular, the inequalities

$$0 \leq z(i+1,t) - z(i,t) \leq K \tag{10.3}$$

hold at $t = 0$.

The informal rule for the evolution of $z(\cdot)$ follows from the earlier description for $\eta(\cdot)$, and now we specify that $\beta(l,i)$ is the waiting time for server $z(i)$ to execute the jump from site $l+1$ to l. If $z(i,0) > l$, let t_1 be the earliest moment at which these three conditions hold:

$$z(i-1,t_1) \leq l,\ z(i,t_1) = l+1 \quad \text{and} \quad z(i+1,t_1) \leq l+K. \tag{10.4}$$

Then at time $t_2 = t_1 + \beta(l,i)$ server $z(i)$ jumps to site l.

We shall construct the trajectories $z(i,t)$ by rigorously defining the jump times. Let $\tau(l,i)$ denote the earliest time t such that $z(i,t) \leq l$ holds. There is one technical twist; a particular $z(i)$ may never reach certain sites l because $z(i)$ comes to a permanent halt during the dynamics. Then the corresponding $\tau(l,i)$'s must be infinite. This happens if in the initial K-exclusion configuration either $\eta(i,0) = 0$ for all sites i far enough to the left or $\eta(i,0) = K$ for all sites i far enough to the right. For the initial $z(i,0)$'s, the corresponding conditions are these:

$$l_* = \lim_{i\to-\infty} z(i,0) > -\infty \tag{10.5}$$

or

$$\text{(10.6)} \quad \text{there exists } i^* < \infty \text{ such that } z(i+1,0) = z(i,0) + K \text{ for } i \ge i^*.$$

In case (10.5), no $z(i)$ ever makes it past l_*, while in case (10.6), $z(i)$ never makes it beyond $z(i^*,0) + K(i - i^*)$, for all i. This leads us to define the domain $\mathscr{I}$ of $\tau(l,i)$ as follows:

$$\text{(10.7)} \quad \begin{aligned} &\mathscr{I} = \mathbf{Z}^2 \quad \text{if neither (10.5) nor (10.6) holds,} \\ &\mathscr{I} = \{(l,i) \in \mathbf{Z}^2 \colon l \ge l_*\} \quad \text{if (10.5) holds but not (10.6),} \\ &\mathscr{I} = \{(l,i) \in \mathbf{Z}^2 \colon l \ge z(i^*,0) + K(i-i^*)\} \\ &\qquad \text{if (10.6) holds but not (10.5) and} \\ &\mathscr{I} = \{(l,i) \in \mathbf{Z}^2 \colon l \ge \max[l_*, z(i^*,0) + K(i-i^*)]\} \\ &\qquad \text{if both (10.5) and (10.6) hold.} \end{aligned}$$

Outside of this domain we set

$$\text{(10.8)} \qquad \tau(l,i) = \infty \text{ for } (l,i) \notin \mathscr{I}.$$

Let $\mathscr{I}_0 = \{(l,i)\colon l \ge z(i,0)\} \subseteq \mathscr{I}$. The initial configuration $(z(i,0)\colon i \in \mathbf{Z})$ dictates that

$$\text{(10.9)} \qquad \tau(l,i) = 0 \quad \text{for } (l,i) \in \mathscr{I}_0.$$

The remaining values are determined by the equation

$$\text{(10.10)} \quad \begin{aligned} &\tau(l,i) = \beta(l,i) + \max\{\tau(l,i-1), \tau(l+1,i), \tau(l+K,i+1)\}, \\ &(l,i) \in \mathscr{I} \setminus \mathscr{I}_0. \end{aligned}$$

Equation (10.10) incorporates the dynamics, starting with the "initial values" (10.9). It is the formalization of the description around (10.4) above. Observe that "time increases leftward" on the lattice (l,i) in the sense that $\tau(l,i)$ increases as l decreases, because $z(i)$ jumps leftward.

We need to prove that (10.8)–(10.10) define all $\tau(l,i)$'s, and thereby the dynamics $z(\cdot)$. Notice that (10.8) and (10.10) are consistent in this sense: it follows from the definition (10.7) of $\mathscr{I}$ that if $(l, i-1)$, $(l+1, i)$ and $(l+K, i+1)$ all lie in $\mathscr{I}$, then so does (l,i). In other words, equation (10.10) cannot define τ beyond the domain $\mathscr{I}$. Furthermore, since $\beta(l,i) > 0$, (10.9) and (10.10) together imply that $\tau(l,i) = 0$ iff $(l,i) \in \mathscr{I}_0$.

To prevent the jump times from accumulating at a finite value, assume that

$$\text{(10.11)} \qquad \sum_{m=-\infty}^{l} \beta(m,i) = \infty \quad \text{for all } l, i.$$

By the strong law of large numbers for i.i.d. random variables, a.e. realization $\{\beta(l,i)\}$ satisfies (10.11).

LEMMA 10.1. *For given* $\{\beta(l,i): (l,i) \in \mathbf{Z}^2\}$ *that satisfy* (10.11), *there is a unique function* τ *on* $\mathbf{Z}^2$ *that satisfies equations* (10.8)–(10.10). *Furthermore, this function* τ *satisfies* $\lim_{l \to -\infty} \tau(l,i) = \infty$ *for all* i.

PROOF. We first show that (10.9) and $\{\beta(l,i)\}$ determine $\tau(l,i)$ for each $(l,i) \in \mathscr{I} \setminus \mathscr{I}_0$ through (10.10). To do this by induction, let $\mathscr{I}_n$ be the set of $(l,i) \in \mathscr{I}$ such that

$$(l,i) + (0, -n_1) + (n_2, 0) + (Kn_3, n_3) \in \mathscr{I}_0$$

for all choices of $n_1, n_2, n_3 \geq 0$ such that $n_1 + n_2 + n_3 = n$. First note that

$$\mathscr{I}_0 \subseteq \mathscr{I}_1 \subseteq \mathscr{I}_2 \subseteq \cdots$$

because by (10.3), $\mathscr{I}_0$ is closed under the maps

$$(l,i) \mapsto (l,i) + (0, -n_1) + (n_2, 0) + (Kn_3, n_3) \tag{10.12}$$

for all $n_1, n_2, n_3 \geq 0$. Second, (10.10) defines $\tau(l,i)$ inductively over successive $\mathscr{I}_n$'s. Assume that $\tau(l,i)$ is defined for all $(l,i) \in \mathscr{I}_n$. If $(l,i) \in \mathscr{I}_{n+1}$, then $(l, i-1)$, $(l+1, i)$ and $(l+K, i+1)$ all lie in $\mathscr{I}_n$, and equation (10.10) defines $\tau(l,i)$.

Next we argue that $\mathscr{I} = \bigcup_{n \geq 0} \mathscr{I}_n$. Pick and fix $(l,i) \in \mathscr{I}$. By checking the different cases in (10.7), one sees that it is possible to choose m large enough so that $(l, i-m), (l+m, i), (l+Km, i+m)$ all lie in $\mathscr{I}_0$. It follows that $(l,i) \in \mathscr{I}_{3m}$, because $n_1 + n_2 + n_3 = 3m$ forces some $n_i \geq m$, and because $\mathscr{I}_0$ itself is closed under the maps (10.12). We have showed that τ is defined on all of $\mathbf{Z}^2$.

Uniqueness is essentially contained in the previous paragraphs. Suppose $\tilde{\tau}$ is a different function that satisfies (10.8)–(10.10). Then $\tau(l,i) \neq \tilde{\tau}(l,i)$ for some $(l,i) \in \mathscr{I} \setminus \mathscr{I}_0$. Then by (10.10), τ and $\tilde{\tau}$ must differ at one of $(l, i-1)$, $(l+1, i)$, or $(l+K, i+1)$. Proceeding inductively, we eventually conclude that τ and $\tilde{\tau}$ must differ at some $(l,i) \in \mathscr{I}_0$, contradicting (10.9).

Fix i. If $(l,i) \notin \mathscr{I}$ for some l, then $\lim_{l \to -\infty} \tau(l,i) = \infty$ is immediate. Suppose $(l,i) \in \mathscr{I}$ for all l. Set $l_0 = z(i,0) - 1$. Since $(l_0, i) \in \mathscr{I} \setminus \mathscr{I}_0$, we can apply (10.10) repeatedly to get, for any $l < l_0$,

$$\begin{aligned} \tau(l,i) &\geq \beta(l,i) + \tau(l+1,i) \\ &\geq \beta(l,i) + \beta(l+1,i) + \tau(l+2,i) \\ &\geq \cdots \geq \sum_{m=l}^{l_0} \beta(m,i). \end{aligned}$$

Now $\lim_{l \to -\infty} \tau(l,i) = \infty$ follows from (10.11). □

Having defined the jump times $\tau(l,i)$, we define the dynamics by

$$z(i,t) = l \text{ for } \tau(l,i) \leq t < \tau(l-1,i) \quad \text{for all } l, i \in \mathbf{Z}. \tag{10.13}$$

Lemma 10.1 guarantees that $z(i,t)$ is defined for all time $0 \leq t < \infty$. Equation (10.10) implies that (10.3) holds for all $t > 0$.

10.2. *The auxiliary processes.* As in Section 4, given an initial configuration $(z(i,0)\colon i \in \mathbf{Z})$ denote by $w^{z(i,0)}(\cdot)$ the server process with the special initial configuration

$$w^{z(i,0)}(j,0) = \begin{cases} z(i,0), & j \geq 0, \\ z(i,0) + Kj, & j < 0. \end{cases} \tag{10.14}$$

Our goal is to prove a result that corresponds to the central Lemma 4.2. First we define further requirements satisfied by a "good" realization of $\{\beta(l,i)\}$.

$$\sum_{j=-\infty}^{k} \beta(l,j) = \infty \quad \text{for all } l, k \tag{10.15}$$

and

$$\sum_{j=k}^{\infty} \beta(l + (j-k)K, j) = \infty \quad \text{for all } l, k. \tag{10.16}$$

A.e. realization $\{\beta(l,i)\}$ satisfies (10.11), (10.15) and (10.16). Corresponding to (4.4), we stipulate that

(10.17) Particle $w^{z(i,0)}(j)$ uses waiting time $\beta(l, i+j)$ for its jump from site $l+1$ to l.

The point of (10.17) is that both $z(k)$ and $w^{z(i,0)}(k-i)$ use the same waiting time $\beta(l,k)$ to jump from site $l+1$ to l.

LEMMA 10.2. *Assume we are given an initial configuration $(z(i,0)\colon i \in \mathbf{Z})$ for the $z(\cdot)$-process and a realization $\{\beta(l,i)\}$ of the waiting times that satisfies* (10.11), (10.15) *and* (10.16). *Construct the processes $w^{z(i,0)}(\cdot)$ according to the description of Section* 10.1 *and rule* (10.17). *Then the following equality holds for all $k \in \mathbf{Z}$ and $t \geq 0$*:

$$z(k,t) = \sup_{i \in \mathbf{Z}} w^{z(i,0)}(k-i,t). \tag{10.18}$$

PROOF. Let $\sigma^i(l,j)$ be the earliest time t when $w^{z(i,0)}(j,t) \leq l$. By the construction (10.8)–(10.10) and the initial condition (10.14), the domain of σ^i is $\mathbf{Z}^2$ for each i,

$$\sigma^i(l,j) = 0 \quad \text{for } l \geq z(i,0) + K \cdot \min\{j,0\} \tag{10.19}$$

and

$$\begin{aligned} \sigma^i(l,j) = \beta(l, i+j) \\ + \max\{\sigma^i(l,j-1), \sigma^i(l+1,j), \sigma^i(l+K,j+1)\} \\ \text{for } l < z(i,0) + K \cdot \min\{j,0\}. \end{aligned} \tag{10.20}$$

Let $\mathscr{I}$ and $\mathscr{I}_0$ be as defined by (10.7) and by the sentence after (10.8), on the basis of the given $(z(i,0)\colon i \in \mathbf{Z})$. Set

$$\tilde{\tau}(l,k) = \sup_{i \in \mathbf{Z}} \sigma^i(l, k-i). \tag{10.21}$$

Our first goal is to show that $\tilde{\tau}$ satisfies (10.8)–(10.10), and therefore by Lemma 10.1, $\tilde{\tau} = \tau$.

Condition (10.9) for $\tilde{\tau}$ follows immediately because for all i, $\sigma^i(l, k-i) = 0$ for $l \geq z(k,0)$ by (10.19) and (10.3).

To deduce (10.10) for $\tilde{\tau}$, note first that

$$\tilde{\tau}(l,k) = \max\Big\{0, \sup_{i \in J_{l,k}} \sigma^i(l, k-i)\Big\}, \tag{10.22}$$

where

$$J_{l,k} = \{i \in \mathbf{Z}\colon z(i,0) + K \cdot \min\{k-i, 0\} > l\}. \tag{10.23}$$

Equation (10.22) is true because $\sigma^i(l, k-i) = 0$ for $i \notin J_{l,k}$, by (10.19). By (10.3), $J_{l,k} = \emptyset$ when $l \geq z(k,0)$, so we need the comparison with zero on the right-hand side of (10.22). (By convention, the supremum of an empty set is $-\infty$). If $l < z(k,0)$ [equivalently, $(l,k) \notin \mathscr{I}_0$], then $k \in J_{l,k}$, so $J_{l,k} \neq \emptyset$. Then the comparison with zero is not needed in (10.22) because $\sigma^i \geq 0$ always. Now we can deduce (10.10) for $(l,k) \in \mathscr{I} \setminus \mathscr{I}_0$. First,

$$\begin{aligned}
\tilde{\tau}(l,k) &= \sup_{i \in J_{l,k}} \sigma^i(l, k-i) \\
&= \sup_{i \in J_{l,k}} \big[\beta(l,k) + \max\{\sigma^i(l, k-i-1), \sigma^i(l+1, k-i), \\
&\qquad\qquad \sigma^i(l+K, k-i+1)\}\big] \\
&\qquad\qquad\qquad [\text{by } (10.20)] \\
&= \beta(l,k) + \max\Big\{\sup_{i \in J_{l,k}} \sigma^i(l, k-i-1), \sup_{i \in J_{l,k}} \sigma^i(l+1, k-i), \\
&\qquad\qquad \sup_{i \in J_{l,k}} \sigma^i(l+K, k-i+1)\Big\} \\
&\leq \beta(l,k) + \max\{\tilde{\tau}(l, k-1), \tilde{\tau}(l+1, k), \tilde{\tau}(l+K, k+1)\},
\end{aligned} \tag{10.24}$$

where the last inequality follows from (10.21). On the other hand, one can check that $J_{l,k} \supseteq J_{l,k-1} \cup J_{l+1,k} \cup J_{l+K,k+1}$. Then we can continue from the next-to-last line in (10.24) by

$$\begin{aligned}
&\tilde{\tau}(l,k) \\
&\quad \geq \beta(l,k) + \max\Big\{0, \sup_{i \in J_{l,k-1}} \sigma^i(l, k-i-1), \sup_{i \in J_{l+1,k}} \sigma^i(l+1, k-i), \\
&\qquad\qquad \sup_{i \in J_{l+K,k+1}} \sigma^i(l+K, k-i+1)\Big\} \\
&\quad = \beta(l,k) + \max\{\tilde{\tau}(l, k-1), \tilde{\tau}(l+1, k), \tilde{\tau}(l+K, k+1)\},
\end{aligned} \tag{10.25}$$

where the last step now comes from (10.22). We have verified that $\tilde{\tau}$ satisfies (10.10).

It remains to check (10.8), namely, that $\tilde{\tau}(l,k)=\infty$ for $(l,k)\notin\mathscr{I}$. Two cases, according to (10.5) and (10.6), need to be considered.

CASE I. Suppose $l_*>-\infty$ and $l<l_*$. Let i be a large negative integer, large enough so that $z(i,0)=l_*$ and $k-i\gg 0$. By repeated application of (10.20),

$$
\begin{aligned}
\sigma^i(l,k-i) &\geq \beta(l,k)+\sigma^i(l,k-i-1)\\
&\geq \beta(l,k)+\beta(l,k-1)+\sigma^i(l,k-i-2)\\
&\geq \cdots \geq \sum_{j=k-i_0+1}^{k}\beta(l,j)+\sigma^i(l,k-i-i_0)
\end{aligned}
\tag{10.26}
$$

as long as $k-i-i_0\geq 0$ because then (10.20) is valid. Take $i_0=k-i$ and combine with (10.21) to write

$$
\tilde{\tau}(l,k)\geq\sigma^i(l,k-i)\geq\sum_{j=i+1}^{k}\beta(l,j).
$$

Now $\tilde{\tau}(l,k)=\infty$ follows from (10.15) by letting $i\searrow-\infty$.

CASE II. Suppose $i^*<\infty$ and $l<z(i^*,0)+K(k-i^*)$. Let $i\gg\max\{k,i^*\}$, and $i_0\leq i-k$. The assumption on (l,k) and $z(i,0)=z(i^*,0)+K(i-i^*)$ imply that $l+i_0K<z(i,0)+K(k-i+i_0)$, and we can again apply (10.20) inductively to write

$$
\begin{aligned}
\sigma^i(l,k-i) &\geq \beta(l,k)+\sigma^i(l+K,k-i+1)\\
&\geq \beta(l,k)+\beta(l+K,k+1)+\sigma^i(l+2K,k-i+2)\\
&\geq \cdots \geq \sum_{j=k}^{k+i_0-1}\beta(l+(j-k)K,j)+\sigma^i(l+i_0K,k-i+i_0).
\end{aligned}
\tag{10.27}
$$

Take $i_0=i-k$, use (10.21), let $i\nearrow\infty$ and apply (10.16) to conclude that $\tilde{\tau}(l,k)=\infty$.

This concludes the first step of the proof of Lemma 10.2. Since $\tilde{\tau}$ satisfies (10.8)–(10.10), Lemma 10.1 implies that

$$
\tau(l,k)=\sup_{i\in\mathbf{Z}}\sigma^i(l,k-i).
\tag{10.28}
$$

This implies that "$\geq$" holds in (10.18), because no $w^{z(i,0)}(k-i)$ arrives in l any later than $z(k)$. It remains to argue that some $w^{z(i,0)}(k-i)$ remains at site $l+1$ with $z(k)$ until time $\tau(l,k)$.

For the dynamics of $z(\cdot)$, we only concern ourselves with $(l,k)\in\mathscr{I}$ in (10.28). Fix such a pair (l,k). The next step is to argue that the supremum in (10.28) is always realized at some finite i. To this end we show that $\sigma^i(l,k-i)=0$ for all large enough positive and negative i. By (10.19), this follows from showing that

$$
l\geq z(i,0)+K\cdot\min\{k-i,0\}.
\tag{10.29}
$$

First for large negative i, simply pick $i_0 < k$ so that $z(i_0, 0) \le l$, and then (10.29) holds for $i \le i_0$.

For large positive i we need two cases. Suppose first that (10.6) does not hold. Let u_i denote the number of indices $j \in \{k+1, \dots, i\}$ such that $z(i,0) - z(i-1,0) \le K-1$. Since (10.6) does not hold, $u_i \nearrow \infty$ as $i \nearrow \infty$. Pick $i_1 > k$ large enough so that $u_{i_1} \ge z(k,0) - l$. Then for $i \ge i_1$,

$$\begin{aligned} z(i,0) &\le z(k,0) + (i-k-u_i)K + u_i(K-1) \\ &= z(k,0) - u_i + K(i-k) \\ &\le l + K(i-k), \end{aligned}$$

and (10.29) follows.

On the other hand, if (10.6) holds, then $(l,k) \in \mathscr{I}$ forces $l \ge z(i^*, 0) + K(k - i^*)$. This, together with $z(i,0) = z(i^*,0) + K(i-i^*)$ for $i \ge i^*$, implies (10.29).

To summarize, for each $(l,k) \in \mathscr{I}$, there exists an index i_0 such that

$$\tau(l,k) = \sup_{i \in \mathbf{Z}} \sigma^i(l, k-i) = \sigma^{i_0}(l, k-i_0). \tag{10.30}$$

We claim that for $(l,k) \in \mathscr{I}$,

$$w^{z(i_0,0)}(k-i_0,t) = z(k,t) = l+1 \qquad \text{for } \tau(l+1,k) \le t < \tau(l,k). \tag{10.31}$$

As already observed,

$$w^{z(i_0,0)}(k-i_0,t) \le z(k,t) = l+1$$

by virtue of (10.28). On the other hand, $\sigma^{i_0}(l, k-i_0)$ is the earliest time for $w^{z(i_0,0)}(k-i_0,t) \le l$ to hold, so $w^{z(i_0,0)}(k-i_0,t) \ge l+1$ for $t < \tau(l,k)$. This proves (10.31). Consequently, (10.18) holds for $t < \tau(l,k)$ whenever $\tau(l,k) < \infty$.

It remains to consider the case $\tau(l+1,k) \le t < \infty = \tau(l,k)$. Now $z(k,t)$ has come to a halt at site $l+1$, and there are two possible causes for this. The first possibility is that $z(k,t) = l+1 = l_* > -\infty$. By the calculation in (10.26), we can choose $i \ll k$ so that $z(i,0) = l_*$ and $\sigma^i(l,k-i) > t$. Then $w^{z(i,0)}(k-i,t) = l+1 = z(k,t)$, and (10.18) is valid again. The other possibility is that (10.6) holds and $z(k,t) = z(i^*,0) - K(i^* - k)$. Then we choose $i \gg \max\{i^*, k\}$ large enough so that $\sigma^i(l,k-i) > t$, as can be done by the calculation in (10.27). Then $w^{z(i,0)}(k-i,t) = l+1 = z(k,t)$.

This concludes the proof of (10.18). □

10.3. *The growth model.* The growth model is formulated exactly as in Section 5. Now $\alpha(j) = 1$ for all j, and the $(\lambda(i,j)\colon (i,j) \in \mathbf{Z}^2)$ are i.i.d. F-distributed instead of exponentially distributed. The passage times $T_K(i,j)$ are defined in terms of K-admissible paths as in (5.5), and the interface process $\xi(j,t)$ is defined by (5.8).

PROPOSITION 10.1. *Let the common distribution F of the i.i.d. $\{\beta(l,i)\}$ be any distribution on $(0,\infty)$ that satisfies $E[\beta(l,i)] = 1$. Then there is a concave function γ_K defined on $\mathscr{U}_K$ such that*

$$\lim_{n\to\infty} \frac{1}{n} T_K([nx],[ny]) = \gamma_K(x,y) \tag{10.32}$$

holds almost surely, for all $(x,y) \in \mathscr{U}_K$. Either $\gamma_K = \infty$ on all of $\mathscr{U}_K$ or $\gamma_K < \infty$ on all of $\mathscr{U}_K$.

The equation

$$g_K(y) = \inf\{x > 0: (x,y) \in \mathscr{U}_K, \gamma_K(x,y) \ge 1\}$$

defines a finite, convex, continuous, nonincreasing function on $\mathbf{R}$ with property (2.6). *For any $y \in \mathbf{R}$ and $t > 0$,*

$$\lim_{n\to\infty} \frac{1}{n} \xi([ny], nt) = t g_K(y/t) \quad a.s. \tag{10.33}$$

PROOF. The proof of Proposition 5.1 applies with some adjustments. The point to note is that a nonnegative superadditive process converges almost surely with or without the moment bound, as long as the other assumptions are met. This can be seen by applying a truncation argument to the basic subadditive ergodic theorem.

The first step of the proof of Proposition 5.1 now defines a $[0,\infty]$-valued function $\gamma_K(k,l)$ on $\mathscr{L}_K$ that gives the limit (10.32) for $(x,y) = (k,l)$. Suppose that $\gamma_K(i,j) = \infty$ for some $(i,j) \in \mathscr{L}_K$. For any $(k,l) \in \mathscr{L}_K$, find m large enough such that $(mk, ml) \in (i,j) + \mathscr{L}_K$. Then

$$m\gamma_K(k,l) = \gamma_K(mk, ml) \ge \gamma_K(i,j).$$

This shows that either $\gamma_K(k,l) < \infty$ for all $(k,l) \in \mathscr{L}_K$ or $\gamma_K(k,l) = \infty$ for all $(k,l) \in \mathscr{L}_K$. In the former case, follow the proof of Proposition 5.1, in the latter case modify that proof to show that (10.32) holds with $\gamma_K(x,y) \equiv \infty$.

The limit (10.33) follows as Corollary 5.1 did. Note that $\gamma_K = \infty$ poses no problem for the definition of g_K. In this case, $g_K(y) = 0$ for $y \ge 0$ and $g_K(y) = -Ky$ for $y \le 0$. $\square$

Another collection of interface processes is defined, as in Section 4, by

$$\xi^{z(i,0)}(j,t) = z(i,0) - w^{z(i,0)}(j,t). \tag{10.34}$$

LEMMA 10.3. *For any sequence of possibly random initial configurations $(z_n(i,0): i \in \mathbf{Z})$, any sequence of indices i_n and any $y \in \mathbf{R}$ and $t > 0$,*

$$\lim_{n\to\infty} \frac{1}{n} \xi^{z_n(i_n,0)}([ny], nt) = t g_K\left(\frac{y}{t}\right) \quad \text{in probability.} \tag{10.35}$$

PROOF. Suppose first that we have a deterministic initial configuration $(z(i,0): i \in \mathbf{Z})$. We shall show that for any fixed i, j and t,

$$\xi^{z(i,0)}(j,t) =_d \xi(j,t). \tag{10.36}$$

Then (10.35) follows from (10.33) for a deterministic sequence of initial configurations. Subsequently, (10.35) follows for random $(z_n(i,0)\colon i \in \mathbf{Z})$ by integrating over the distribution of $z_n(i_n, 0)$.

Fix i. By (10.34), the time t when $\xi^{z(i,0)}(j,t) = l$ first holds is the same as $\sigma^i(z(i,0) - l, j)$, the time when server $w^{z(i,0)}(j)$ first reaches site $z(i,0) - l$. By (5.8), the time when $\xi(j,t) = l$ first holds is the past-passage time $T_K(l,j)$. To prove (10.36) and thereby the lemma, we show the equality in distribution of the time processes

$$(10.37)\quad \left\{\sigma^i(z(i,0) - l, j)\colon (l,j) \in \mathscr{L}_K\right\} =_d \left\{T_K(l,j)\colon (l,j) \in \mathscr{L}_K\right\}.$$

To establish this, we choose the sitewise passage times $\tau(l,j)$ from a realization of the waiting times $\beta(l,i)$. Set $\tau(l,j) = \beta(z(i,0) - l, i+j)$ for $(l,j) \in \mathscr{L}_K$. By (5.5), for $(l,j) \in \mathscr{L}_K$,

$$(10.38)\quad \begin{aligned} &T_K(l,j)\\ &\quad = \tau(l,j) + \max\{T_K(l,j-1), T_K(l-1,j), T_K(l-K,j+1)\}\\ &\quad = \beta(z(i,0) - l, i+j)\\ &\qquad + \max\{T_K(l,j-1), T_K(l-1,j), T_K(l-K,j+1)\}, \end{aligned}$$

with initial $T(l,j) = 0$ for $(l,j) \notin \mathscr{L}_K$. A comparison with (10.19) and (10.20) shows that $\sigma^i(z(i,0) - l, j)$ satisfies the same recursion (10.38), with the same initial values $\sigma^i(z(i,0) - l, j) = 0$ for $(l,j) \notin \mathscr{L}_K$. It follows inductively that, with this particular choice of $\tau(l,j)$'s, $T_K(l,j)$ actually equals $\sigma^i(z(i,0) - l, j)$. For arbitrary i.i.d. F-distributed passage times $\tau(l,j)$ we then have the equality in distribution (10.37). □

Now rewrite (10.18) as

$$(10.39)\qquad z(k,t) = \sup_{i \in \mathbf{Z}} \left\{z(i,0) - \xi^{z(i,0)}(k-i,t)\right\}.$$

10.4. *Proof of Theorem* 4. The route from (10.39) to (10.2) is the same as the one taken in Section 6. The goal is again to prove (6.4), with $\boldsymbol{\alpha} \equiv 1$. The easy half (6.12) is easy again and for the same reasons. The harder half required two lemmas: Lemma 6.1, which allows us to exclude the extreme values of i in (10.39), and Lemma 4.3 about preserving the ordering of particles. Both lemmas are valid again, and hence the reasoning (6.14) can be repeated to deduce (6.15), which together with (6.8) completes the proof of (10.2). The Borel–Cantelli estimate (5.26) was required only for the disordered weak law, and we do not need it now, although it would be valid here too. With this we consider Theorem 4 proved.

REFERENCES

Aldous, D. and Diaconis, P. (1995). Hammersley's interacting particle process and longest increasing subsequences. *Probab. Theory Related Fields* **103** 199–213.

ANDJEL, E. (1982). Invariant measures for the zero range process. *Ann. Probab.* **10** 525–547.

BARDI, M. and EVANS, L. C. (1984). On Hopf's formulas for solutions of Hamilton–Jacobi equations. *Nonlinear Anal.* **8** 1373–1381.

BENJAMINI, I., FERRARI, P. and LANDIM, C. (1996). Asymmetric conservative processes with random rates. *Stochastic Process. Appl.* **61** 181–204.

COVERT, P. and REZAKHANLOU, F. (1997). Hydrodynamic limit for particle systems with nonconstant speed parameter. *J. Statist. Phys.* **88** 383–426.

COX, J. T., GANDOLFI, A., GRIFFIN, P.S. and KESTEN, H. (1993). Greedy lattice animals I: upper bounds. *Ann. Appl. Probab.* **3** 1151–1169.

DE MASI, A. and PRESUTTI, E. (1991). *Mathematical Methods for Hydrodynamic Limits. Lecture Notes in Math.* **1501**. Springer, Berlin.

DEMBO, A. and ZEITOUNI, O. (1993). *Large Deviations Techniques and Applications*. Jones and Bartlett.

DEUSCHEL, J.-D. and STROOCK, D. W. (1989). *Large Deviations*. Academic Press, New York.

DURRETT, R. (1988). *Lectures on Particle Systems and Percolation*. Wadsworth and Brooks/Cole, Belmont, CA.

DURRETT, R. (1991). *Probability: Theory and Examples*. Wadsworth and Brooks/Cole, Belmont, CA.

EKHAUS, M. and GRAY, L. (1994). Convergence to equilibrium and a strong law for the motion of restricted interfaces. Unpublished manuscript.

EVANS, L. C. (1984). Some min–max methods for the Hamilton–Jacobi equation. *Indiana Univ. Math. J.* **33** 31–50.

FERRARI, P. A. (1994). Shocks in one-dimensional processes with drift. In *Probability and Phase Transitions* (G. Grimmett, ed.) 35–48. Kluwer, Dordrecht.

FERRARI, P. A. (1996). Limit theorems for tagged particles. *Markov Process. Related Fields* **2** 17–40.

GLYNN, P. W. and WHITT, W. (1991). Departures from many queues in a series. *Ann. Appl. Probab.* **1** 546–572.

GOLDSTEIN, S. and SPEER, E. R. (1998). Reflection invariance of the current in the totally asymmetric simple exclusion process with disorder. Preprint.

GRIFFEATH, D. (1979). *Additive and Cancellative Interacting Particle Systems. Lecture Notes in Math.* **724**. Springer, Berlin.

GRIMMETT, G. and KESTEN, H. (1984). First-passage percolation, network flows and electrical resistances. *Z. Wahrsch. Verw. Gebiete* **66** 335–366.

HOPF, E. (1965). Generalized solutions of nonlinear equations of first order. *J. Math. Mech.* **14** 951–973.

KELLY, F. (1979). *Reversibility and Stochastic Networks*. Wiley, New York.

KINGMAN, J. F. C. (1968). The ergodic theory of subadditive stochastic processes. *J. Roy. Statist. Soc. Ser. B* **30** 499–510.

KIPNIS, C., LANDIM, C. and OLLA, S. (1994). Hydrodynamic limit for a nongradient system: the generalized symmetric exclusion process. *Comm. Pure Appl. Math.* **47** 1475–1545.

KRUG, J. and SPOHN, H. (1991). Kinetic roughening of growing surfaces. In *Solids Far from Equilibrium* (C. Godrèche, ed.) 479–582. Cambridge Univ. Press.

LANDIM, C. (1996). Hydrodynamical limit for space inhomogeneous one-dimensional totally asymmetric zero-range processes. *Ann. Probab.* **24** 599–638.

LAX, P. (1957). Hyperbolic systems of conservation laws II. *Comm. Pure Appl. Math.* **10** 537–566.

LIGGETT, T. M. (1985). *Interacting Particle Systems*. Springer, New York.

LIONS, P. L. (1982). *Generalized Solutions of Hamilton–Jacobi Equations*, Pitman, London.

LIONS, P. L., SOUGANIDIS, P. E. and VÁSQUEZ, J. L. (1987). The relation between the porous medium and the eikonal equations in several space dimensions. *Rev. Mat. Iberoamericana* **3** 275–310.

REZAKHANLOU, F. (1991). Hydrodynamic limit for attractive particle systems on $\mathbf{Z}^d$. *Comm. Math. Phys.* **140** 417–448.

ROCKAFELLAR, R. T. (1970). *Convex Analysis*. Princeton Univ. Press.

ROST, H. (1981). Non-equilibrium behaviour of a many particle process: density profile and local equilibrium. *Z. Wahrsch. Verw. Gebiete* **58** 41–53.

SEPPÄLÄINEN, T. (1996). A microscopic model for the Burgers equation and longest increasing subsequences. *Electron. J. Probab.* **1** Paper 5 1–51.

SEPPÄLÄINEN, T. (1997a). A scaling limit for queues in series. *Ann. Appl. Probab.* **7** 855–872.

SEPPÄLÄINEN, T. (1997b). Increasing sequences of independent points on the planar lattice. *Ann. Appl. Probab.* **7** 886–898.

SEPPÄLÄINEN, T. (1998a). Exact limiting shape for a simplified model of first-passage percolation on the plane. *Ann. Probab.* **26** 1232–1250.

SEPPÄLÄINEN, T. (1998b). Large deviations for increasing sequences on the plane. *Probab. Theory Related Fields*. **112** 221–244.

SEPPÄLÄINEN, T. (1998c). Coupling the totally asymmetric simple exclusion process with a moving interface. *Markov Process. Related Fields* **112** 221–244.

SEPPÄLÄINEN, T. and KRUG, J. (1998). Hydrodynamics and platoon formation for a totally asymmetric exclusion model with particlewise disorder. *J. Statist. Phys.* To appear.

SPOHN, H. (1991). *Large Scale Dynamics of Interacting Particles*. Springer, Berlin.

VARADHAN, S. R. S. (1984). *Large Deviations and Applications*. SIAM, Philadelphia.

VARADHAN, S. R. S. (1993). Nonlinear diffusion limit for a system with nearest neighbor interactions II. In *Asymptotic Problems in Probability Theory: Stochastic Models and Diffusions on Fractals* (K. Elworthy and N. Ikeda, eds.) 75–128. Longman, New York.

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