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RELAXATION TO EQUILIBRIUM OF CONSERVATIVE DYNAMICS I: ZERO-RANGE PROCESSES

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Under mild assumptions we prove that for any local function u the decay rate to equilibrium in the variance sense of zero range dynamics on the d-dimensional integer lattice is $C_u t^{-d/2} + o(t^{-d/2})$. The constant C_u is computed explicitly.

0. Introduction. In this article we present a method to estimate the decay to equilibrium in the variance sense of conservative interacting particle systems in infinite volume. Although such issues are interesting for a wide variety of models, we will concentrate here on a particular class of models in order to present the method in a simple setting. These are the symmetric zero-range models, and the key simplifying feature is that the invariant measures are of product form.

By analogy with the heat equation in $\mathbb{R}^d$, which appears in the diffusive scaling limit of these models, and with the noninteracting case, one expects an algebraic decay to equilibrium for such models on $\mathbb{Z}^d$ at rate $O(t^{-d/2})$. Indeed, by a careful choice of test functions, one can show that the decay could not in general be faster. Upper bounds, on the other hand, have proved more difficult to obtain. We will derive an estimate of the form $C_u t^{-d/2} + o(t^{-d/2})$ and compute the constant C_u explicitly; here $\lim_{t\to\infty} t^{d/2} o(t^{-d/2}) = 0$. This answers not only the upper bound of the form $O(t^{-d/2})$ but also identifies the class of functions decaying as $O(t^{-d/2})$ as the class of functions for which $C_u \neq 0$.

One should note the sharp contrast between the algebraic decay for conservative systems and the well-known exponential decay displayed by nonconservative systems. In the first case, the slow decay is a consequence of the need to transport mass across large distances in order to equilibrate, while in the latter case, distant regions equilibrate more or less independently. This manifests itself in the behavior near zero of the spectrum of the generator for the process in infinite volume. For the conservative system, the

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spectrum is continuous at zero, while the nonconservative system has a gap at the bottom of the spectrum. It appears also in the different decay rates for the process on finite regions. On boxes of linear size l, the decay rate for the conservative system is exponential, but with a rate $O(l^{-2})$, either in terms of the spectral gap or the logarithmic Sobolev inequality [1, 7, 13, 12, 24]. These estimates have been used heavily in the hydrodynamic limit [18, 23]. The nonconservative dynamics on the other hand decays exponentially with a rate independent of the size of the box. In fact, the dependence of the exponential decay rate on the size of the box in the conservative system is a key ingredient of the present proof of algebraic decay in infinite volume.

In systems with conservation law, one studies the density–density correlation functions

$$\langle \eta_t(x); \eta_0(0) \rangle,$$

where $\eta_t(x)$ is the number of particles at x at time t. These can be thought of as representing the response at position x at time t to a small initial disturbance at position 0. Physically, the disturbance should diffuse out, so we expect that at least for large t and x the density–density correlation functions decay as

$$\chi(4\pi t \det D)^{-d/2} \exp\left\{ -\frac{x \cdot D^{-1} x}{4t} \right\},$$

where χ is the compressibility and D is the bulk diffusion coefficient given by the Green–Kubo formula (see [20]). Such a picture can be made rigorous at various levels. The simplest is linear response theory or equilibrium fluctuations, which deal with small perturbations of equilibrium and large space and time scales. More difficult is the hydrodynamics limit, where the space and time scales are still large, but the deviations from equilibrium are no longer small. Finally, in the present paper we consider such models without rescaling and show an algebraic decay with correct prefactor depending on the diffusion coefficient but at the loss of the Gaussian factor.

The traditional approach for algebraic decay for heat equations is via Nash estimates [17, 8, 5]. The Nash inequality on $\mathbb{R}^d$ states that

$$\|f\|_{L^2}^2 \le C D(f)^{d/(d+2)} \|f\|_{L^1}^{4/(d+2)},$$

where the Dirichlet form is given by

$$D(f) = \int_{\mathbb{R}^d} |\nabla f|^2 \, dx.$$

To use this estimate, recall that the energy inequality of the standard heat equation gives for the solution f_t of the heat equation $\partial_t f = \Delta f$,

$$\frac{d}{dt} \|f_t\|_{L^2}^2 \le -C D(f_t).$$

Applying the Nash inequality, we obtain that

$$\frac{d}{dt}\|f_t\|_{L^2}^2 \le -C\|f_t\|_{L^1}^{-4/d}\|f_t\|_{L^2}^{2(d+2)/d}.$$

Since the heat kernel is a contraction in L^1, we can bound $\|f_t\|_{L^1}$ by its initial value at time $t = 0$, $\|f_0\|_{L^1}$. We now integrate the differential inequality to have the $t^{-d/2}$ decay estimate

$$\|f_t\|_{L^2}^2 \le C\|f\|_{L^1}^2\left[t + C\|f\|_{L^1}^{4/d}\|f\|_{L^2}^{-4/d}\right]^{-d/2}.$$

To extend this idea to infinite systems, it may appear that the key ingredient is a generalization of the Nash inequality. However, the contractivity of the heat kernel in the L^1 norm plays a central role. As it stands, the Nash inequality is unlikely to be true in the infinite-dimensional setting since the L^1 (or any L^p) norm on the right-hand side is too weak to control the variance. One can generalize the Nash inequality by replacing the L^1 norm by a suitably chosen norm $|||\cdot|||$. For any mean zero function,

$$\|f\|_{L^2}^2 \le CD(f)^{d/(d+2)}|||f|||^{4/(d+2)}.$$

On the other hand, we do not know of any norm other than the standard L^p norm contracting (or uniformly bounded in time) under the zero-range or lattice gas dynamics. In fact, a Nash inequality with a seemingly natural choice of the norm $|||\cdot|||$ can be proved for the zero-range processes and the Ginzburg–Landau models in a few lines. To see this, suppose that we have a Ginzburg–Landau model with invariant measure μ and Dirichlet form

$$D(f) = \sum_{x,\,y\in\mathbb{Z}^d,\,|x-y|=1} E_\mu\left[\left(\frac{\partial f}{\partial \eta_x} - \frac{\partial f}{\partial \eta_y}\right)^2\right],$$

where $\eta_x \in \mathbb{R}$ is the field variable at the lattice site x and where E_μ stands for expectation with respect to μ. Define

$$\alpha_x = \left\{E\left[\left(\frac{\partial f}{\partial \eta_x}\right)^2\right]\right\}^{1/2}. \tag{0.1}$$

The usual Nash inequality for the lattice Laplacian states that

$$\sum_{x\in\mathbb{Z}^d}\alpha_x^2 \le C\left[\sum_{x,\,y\in\mathbb{Z}^d,\,|x-y|=1}|\alpha_x - \alpha_y|^2\right]^{d/(d+2)}\left[\sum_{x\in\mathbb{Z}^d}\alpha_x\right]^{4/(d+2)}.$$

By the triangle inequality,

$$|\alpha_x - \alpha_y|^2 \le E\left[\left(\frac{\partial f}{\partial \eta_x} - \frac{\partial f}{\partial \eta_y}\right)^2\right]. \tag{0.2}$$

Hence,

$$\sum_{x\in\mathbb{Z}^d}\alpha_x^2 \le CD(f)^{d/(d+2)}\left[\sum_{x\in\mathbb{Z}^d}\alpha_x\right]^{4/(d+2)}.$$

Suppose that there is a positive spectral gap for the corresponding Glauber dynamics, that is,

$$\mathrm{Var}(f) \le \gamma \sum_{x\in\mathbb{Z}^d}\alpha_x^2$$

for some $\gamma > 0$. Then we have

$$\mathrm{Var}(f) \le C\gamma D(f)^{d/(d+2)}\,|||f|||^{4/(d+2)};\ |||f||| = \sum_{x\in\mathbb{Z}^d}\alpha_x.$$

This proves a "Nash inequality." A weaker version of Nash inequality was obtained in [3] for lattice gases where the triple norm was defined as above but with the L^2 norm of α_x in (0.1) replaced by the L^∞ norm. Notice that the only inputs of our proof are a spectral gap for the corresponding Glauber dynamics and a triangle inequality (0.2). For the zero-range or the symmetric simple exclusion processes, since the invariant measures are product, the spectral gaps for the corresponding Glauber dynamics are the same as those for the marginal on a single site. Hence we only have to prove the triangle inequality (0.2) for these models. Again, because the invariant measures are product, we only need to prove (0.2) for two sites, which can be easily checked. Similar ideas work for the lattice gases but require a short argument to prove (0.2), to be presented in the Appendix.

Unfortunately, at the present time a uniform control in time of the norm $|||\cdot|||$ can only be obtained for the symmetric simple exclusion process [3]. However, for this model certain special techniques become available and therefore simple proofs of the decay are already available [4, 6, 15]. In the Appendix we shall give an elementary proof of the $t^{-d/2}$ decay for the symmetric simple exclusion process. We emphasize that the simplification in the case of the symmetric simple exclusion process comes mainly from its very special duality property and not so much from the fact that the invariant measures are Bernoulli. For example, at the present time, Nash's ideas cannot be extended to models with speed change even when the invariant measures are Bernoulli.

Next we comment on the sense in which decay to equilibrium is measured in this article. Of the few monotone functionals available, the most natural in which to study the decay are the L^2 and L^∞ norms. In L^p the expected decay rate is $t^{-d(p-1)/2p}$ and for $1 < p < 2$ this can be obtained from the trivial L^1 bound and the L^2 decay by interpolation.

The case of L^∞ is more interesting. The processes under study have a family of extremal invariant measures ν_ρ parametrized by the density. A natural statement of ergodicity is that

$$P_t f(\eta) - E_{\nu_\rho}[f] \to 0 \quad \text{as } t\to\infty, \tag{0.3}$$

if the configuration η has a density

$$\rho = \lim_{L \to \infty} (2L + 1)^{-d} \sum_{|x| \le L} \eta(x). \tag{0.4}$$

We are not aware of any results in this direction except for the simple exclusion process where a fairly complete picture can be obtained using duality [16]. If we wanted to go further and understand the rate of convergence in (0.3), we would need to make assumptions about the rate of convergence in (0.4). One way to eliminate the dependence on the rate of convergence of (0.4) is to allow the choice of ρ in (0.3) to depend on t. So we choose $\rho(t)$ carefully and study

$$P_t f(\eta) - E_{\nu_{\rho(t)}}[f]. \tag{0.5}$$

Note that $\rho(t)$ should be independent of f, for otherwise there is nothing to prove. We can normalize the choice of $\rho(t)$ by requiring equality in (0.5) with $f = \eta_0$. In the case of symmetric simple exclusions, one can compute $P_t f$ explicitly if $f = \eta_x$ and the answer is

$$P_t f = \sum_y p_t(y - x)\eta_y,$$

where $p_t(y - x)$ is the heat kernel on the lattice. Hence $\rho(t) = \sum_y p_t(y)\eta_y$ and in particular, if $f = \eta_x$,

$$P_t f(\eta) - E_{\nu_{\rho(t)}}[f] = \sum_y [p_t(y - x) - p_t(y)]\eta_y. \tag{0.6}$$

We fix $x \neq 0$ and study the behavior of the right-hand side as $t \to \infty$. Even if we require that the convergence of (0.4) is as good as possible, say $(2L + 1)^{-d} \sum_{|x| \le L} \eta(x) = \rho$ for all L large enough, we can find configurations η so that the right-hand side of (0.6) is as large as $t^{-1/2}$ in any dimension. Of course, for typical η the decay will be faster. Under any equilibrium measure ν_ρ, the right-hand side of (0.6) is of order $t^{-(d+2)/4}$ in root mean square.

On the other hand, it is shown in [9] that for noninteracting random walks, if one starts with one particle at each site, then for any local function there are constants c_1 and c_2 so that

$$c_1 t^{-d/2} \le \left| P_t f(\eta) - E_{\nu_\rho}[f] \right| \le c_2 t^{-d/2}.$$

One can see from all this that any L^∞ estimate would have to depend quite subtly on the initial data.

The main body of the paper is concerned with the $t^{-d/2}$ estimate on the variance for a class of reversible zero-range models. Our method is very general and applies to lattice gases with mixing assumptions, to be presented in detail in a subsequent paper. It shares with the Nash inequality the use of the spectral gap estimate on finite cubes as a key input. Otherwise, the idea is quite different. The main observation of this approach is that the L^2 norm of the difference between $P_t u$ and its translation, $\tau_x P_t u$, can be controlled by an entropy argument widely used in hydrodynamic limits. This allows us to

replace $P_t u$ by $t^{-d/2}\sum_{|x|\le\sqrt{t}}\tau_x P_t u$ when combined with a cutoff estimate which shows that disturbances move at speed less than $O(\sqrt{t}\log t)$. Since P_t is a contraction on L^2, the variance of the averaged term is now of order $t^{-d/2}$. To complete the argument, we use the equilibrium fluctuation argument in the hydrodynamic limits to compute the leading term in $t^{-d/2}$ explicitly.

1. Notation and results. We consider zero-range models described as follows. Particles are distributed on the lattice $\mathbb{Z}^d$ with η_x denoting the number of particles at site x. Configurations will be called η and the state space is the set $\mathbb{N}^{\mathbb{Z}^d}$ of such configurations. We also choose jump rates $g\colon \mathbb{N}\to\mathbb{R}_+$ such that $g(0)=0<g(k)$ for $k\ge 1$. The dynamics of the process is described as follows. If there are η_x particles at site x, then at rate $g(\eta_x)$ one of them jumps to nearest neighbor site y. This takes place independently of all the other particles, and the new configuration $\sigma^{x,y}\eta$ obtained from η in this way is given by

$$(\sigma^{x,y}\eta)_z=\begin{cases}\eta_z, & \text{if } z\neq x,y,\\ \eta_x-1, & \text{if } z=x,\\ \eta_y+1, & \text{if } z=y.\end{cases} \tag{1.1}$$

The dynamics we have described is a Markov process on the state space $\mathbb{N}^{\mathbb{Z}^d}$ whose generator acts on functions that depend only on a finite number of coordinates as

$$(\mathscr{L}f)(\eta)=\sum_{x\sim y} g(\eta_x)\big[f(\sigma^{x,y}\eta)-f(\eta)\big], \tag{1.2}$$

where $x\sim y$ denotes nearest neighbors.

To ensure that the process is well defined, we make the following Lipschitz assumption on the jump rates (cf. [1]):

(L) $G_1^*=\sup_{n\ge 0}|g(n+1)-g(n)|<\infty$.

Denote by $Z\colon \mathbb{R}_+\to\mathbb{R}_+$ the partition function defined by

$$Z(\varphi)=\sum_{k\ge 0}\frac{\varphi^k}{g(1)\cdots g(k)}$$

and by φ^* the radius of convergence of Z. In order to avoid some degeneracies, we will also assume that the partition function Z diverges as one approaches the boundary of its domain of definition,

$$\lim_{\varphi\to\varphi^*} Z(\varphi)=\infty. \tag{1.3}$$

For $0\le\varphi<\varphi^*$, let $\bar{\nu}_\varphi$ denote the product measure on $\mathbb{N}^{\mathbb{Z}^d}$ with marginals given by

$$\bar{\nu}_\varphi(\eta_x=j)=\frac{1}{Z(\varphi)}\frac{\varphi^j}{g(1)\cdots g(j)} \tag{1.4}$$

for $j \in \mathbb{N}$, $x \in \mathbb{Z}^d$. The dynamics we have described conserves the total number of particles and the set of measures $\bar{\nu}_\varphi$ represent a full set of extremal invariant measures. Let $\rho(\varphi) = E_{\bar{\nu}_\varphi}[\eta_0]$ be the density of particles for the measure $\bar{\nu}_\varphi$. From assumption (1.3) it follows that $\rho: [0, \varphi^*) \to [0, \infty)$ is a smooth strictly increasing bijection. Since $\rho(\varphi)$ has a physical meaning as the density of particles, instead of parameterizing the above family of measures by φ, we use the density ρ as parameter and we write

$$\nu_{\rho_0} = \bar{\nu}_{\rho^{-1}(\rho_0)}$$

for $\rho \in [0, \infty)$. With this convention,

$$\varphi(\rho) = E_{\nu_\rho}[g(\eta_0)].$$

Moreover, φ is a smooth function whose derivative is bounded above by G_1^* and below by a strictly positive constant on each compact set of $\mathbb{R}_+$ (cf. [11]). Because each nearest neighbor jump is chosen by the particles with equal probability, the process is reversible with respect to each ν_ρ and the corresponding Dirichlet form is given by

$$D(\nu_\rho, f) := -E_{\nu_\rho}[f\mathscr{L}f] = \tfrac{1}{2}\sum_{x\sim y} E_{\nu_\rho}\Big[g(\eta_x)\big(f(\sigma^{x,y}\eta) - f(\eta)\big)^2\Big]. \tag{1.5}$$

We will also consider the process restricted to a box of side length l. Jumps to sites outside the box are simply excluded. The corresponding generator will be denoted by $\mathscr{L}_l$. We now make the following additional assumption which guarantees that $\mathscr{L}_l$ has a gap of order l^{-2} uniformly in the density [12]:

(H) There exists $\delta > 0$ and $k_0 \geq 1$ such that $g(m) - g(n) \geq \delta$ for all $m - n \geq k_0$.

Note that from (H) and (L) it follows that there exists an $\varepsilon_0 > 0$ such that for all $k \in \mathbb{N}$,

$$\varepsilon_0 k \leq g(k) \leq \varepsilon_0^{-1} k. \tag{1.6}$$

Fix a density $\rho > 0$ and denote by P_t the semigroup associated to the generator $\mathscr{L}$ and by $\mathrm{Var}(\nu, u)$ the variance, with respect to a probability measure ν, of a function u in $L^2(\nu)$. The main theorem of this article states that under assumptions (L) and (H), the process relaxes to equilibrium in $L^2(\nu_\rho)$ at rate $t^{-d/2}$.

THEOREM 1.1. *For every bounded cylinder function u,*

$$\mathrm{Var}\big(\nu_\rho, [P_t u]\big) = \frac{[\tilde{u}'(\rho)]^2 \chi(\rho)}{[8\pi\varphi'(\rho)t]^{d/2}} + o(t^{-d/2}),$$

where $\chi(\rho)$ is the static compressibility, which in our model is given by $\chi(\rho) = \mathrm{Var}(\nu_\rho, \eta(0))$, $\tilde{u}(\rho)$ is the expectation of u with respect to ν_ρ, $\tilde{u}(\rho) = E_{\nu_\rho}[u(\eta)]$ and $\tilde{u}'(\rho)$ is the derivative of $\tilde{u}$ with respect to ρ.

2. Proof of the main result. Let us introduce some notations. Fix $\rho \in (0, \infty)$ and a bounded cylinder function u which is mean zero with respect to ν_ρ, that is, $E_{\nu_\rho}[u] = 0$. We will also use $\langle \cdot \rangle$ to denote expectation with respect to ν_ρ. For a positive integer L, denote by Λ_L the cube of length $2L + 1$ centered at the origin

$$\Lambda_L = \{-L, \dots, L\}^d.$$

Denote by $\{P_t,\ t \geq 0\}$ the semigroup associated to the generator $\mathscr{L}$ defined in (1.2). For $t \geq 0$, let u_t stand for $P_t u$ so that u_t is the solution of the backward equation

$$\begin{aligned} \partial_t u_t &= \mathscr{L} u_t, \\ u_0 &= u. \end{aligned}$$

Fix two constants $t_0 > 0$ and $R_0 > 1$. We will soon impose new lower bounds on t_0. Then R_0 will be made explicit later in Theorem 3.2. It can be in principle any constant greater than 1, but a particular choice will simplify notation. For $n \geq 1$, let $t_n = R_0^n t_0$.

For a positive integer n, denote by $\overline{\eta}_n$ the density of particles in a cube of length $2n + 1$ centered at the origin $\overline{\eta}_n = (2n + 1)^{-d} \sum_{x \in \Lambda_n} \eta_x$ and by $G_n u$ the conditional expectation of a cylinder function u, given $\overline{\eta}_n$,

$$(G_n u)(\overline{\eta}) = E\big[u | \overline{\eta}_n\big].$$

We sometimes denote $G_n u$ by $\tilde{u}_n$.

Fix a smooth function with compact support $J\colon (-1, 1)^d \to \mathbb{R}_+$ such that $\int J(u)\, du = 1$ and $\varepsilon > 0$ small. Let $K, k\colon \mathbb{R}_+ \to \mathbb{N}$ be two increasing integer valued functions defined by $K(t) = [t_n^{(1-\varepsilon)/2}]$, $k(t) = [t_n^{2\varepsilon}]$ in the interval $[t_n, t_{n+1}]$, where $[a]$ stands for the integer part of a. For each $t \geq 0$, $E_{\nu_\rho}[u_t^2]$ is bounded above

$$\begin{aligned} &(1 + A) E_{\nu_\rho}\left[\left(u_t - \frac{1}{|\Lambda_K|} \sum_{y \in \Lambda_K} J\Big(\frac{y}{K}\Big) \tau_y (G_k u)_t\right)^2\right] \\ &\quad + (1 + A^{-1}) E_{\nu_\rho}\left[\left(\frac{1}{|\Lambda_K|} \sum_{y \in \Lambda_K} J\Big(\frac{y}{K}\Big) \tau_y (G_k u)_t\right)^2\right] \end{aligned} \tag{2.1}$$

for every $A > 0$. In Section 5 we prove the following proposition.

PROPOSITION 2.1. *For every bounded cylinder function u and every smooth function J as defined above,*

$$E_{\nu_\rho}\left[\left(\frac{1}{|\Lambda_K|} \sum_{y \in \Lambda_K} J\Big(\frac{y}{K}\Big) \tau_y (G_k u)_t\right)^2\right] = \frac{\chi(\rho)[\tilde{u}'(\rho)]^2}{(8\pi\varphi'(\rho)t)^{d/2}} + o(t^{-d/2}).$$

This statement, together with Proposition 2.2, which will be proved in Section 3, concludes the proof of Theorem 1.1. □

PROPOSITION 2.2. *With the same notation as the previous proposition, for every bounded cylinder function u and every smooth function J as defined above,*

$$\lim_{t\to\infty} t^{d/2} E_{\nu_\rho}\left[\left(u_t - \frac{1}{|\Lambda_K|}\sum_{y\in\Lambda_K} J\Big(\frac{y}{K}\Big)\tau_y(G_k u)_t\right)^2\right] = 0.$$

3. Proof of Proposition 2.2. Proposition 2.2 is proved in several steps. Recall the definition of the sequence t_n. For $t > t_0$, denote by $n(t)$ the largest integer n such that $t_n \le t$. To keep notation simple, we shall convey that $t_{n(t)+1} = t$. Let v_t be defined by

$$v_t = u_t - \frac{1}{|\Lambda_K|}\sum_{y\in\Lambda_K} J\Big(\frac{y}{K}\Big)\tau_y(G_k u)_t.$$

We may rewrite $(1+t)^{(d+2)/2}\langle v_t^2\rangle - (1+t_0)^{(d+2)/2}\langle v_{t_0}^2\rangle$ as

$$\sum_{j=0}^{n(t)} (1+t_{j+1})^{(d+2)/2}\langle v_{t_{j+1}}^2\rangle - (1+t_j)^{(d+2)/2}\langle v_{t_j}^2\rangle.$$

Recall the definition of the Dirichlet form given in (1.5). Since K and k are constants in the intervals $[t_j, t_{j+1}]$, a differentiation gives that the last expression is equal to

$$-2\int_{t_0}^t ds(1+s)^{(d+2)/2} D(\nu_\rho, v_s) + \frac{(d+2)}{2}\int_{t_0}^t ds(1+s)^{d/2} E_{\nu_\rho}\big[v_s^2\big] \tag{3.1}$$

for all $t \ge t_0 \ge 0$.

Consider a trajectory $X\colon \mathbb{R}_+ \to \mathbb{Z}^d$ on the lattice such that $X(s)$ is constant in the intervals $[t_n, t_{n+1}]$ and $|X(t_n)| \le (1/4)\sqrt{t_n}$. Since the dynamics is translation invariant, we may replace v_s by $\tau_{X(s)}v_s$ in the previous formula. After this substitution, it becomes

$$\begin{aligned} &-2\int_{t_0}^t ds(1+s)^{(d+2)/2} D(\nu_\rho, \tau_{X(s)}v_s) \\ &\quad + \frac{(d+2)}{2}\int_{t_0}^t ds(1+s)^{d/2} E_{\nu_\rho}\Big[(\tau_{X(s)}v_s)^2\Big]. \end{aligned} \tag{3.2}$$

STEP 1. *Cutoff.* For a positive integer L, denote by $\mathcal{F}_{\Lambda_L}$ the σ-algebra generated by $\{\eta_x,\ x\in\Lambda_L\}$ and by $A_L h$ the conditional expectation of a $L^2(\nu_\rho)$ function h given $\mathcal{F}_{\Lambda_l}$,

$$A_L h = E_{\nu_\rho}\big[h \mid \mathcal{F}_{\Lambda_L}\big]. \tag{3.3}$$

For every $L \geq 1$, the second term in (3.2) is equal to

$$\begin{aligned} &\frac{(d+2)}{2}\int_{t_0}^{t}(1+s)^{d/2}E_{\nu_\rho}\Big[\big(\tau_{X(s)}v_s - A_L\tau_{X(s)}v_s\big)^2\Big]\,ds \\ &\quad + \frac{(d+2)}{2}\int_{t_0}^{t}(1+s)^{d/2}E_{\nu_\rho}\Big[\big(A_L\tau_{X(s)}v_s\big)^2\Big]\,ds. \end{aligned} \tag{3.4}$$

PROPOSITION 3.1. *Fix $\gamma > 0$ and a cylinder function h. Denote by s_h the smallest integer k such that the support of h is contained in the cube Λ_k. There exists a finite constant $C(\rho)$ depending only on ρ such that for each $s \geq \max\{2, s_h^2\}$ and each $L \geq [\gamma\sqrt{s}\,\log s]$,*

$$\Big\langle \big(A_L h_s - h_s\big)^2\Big\rangle \leq \frac{C(\rho)}{s^\gamma}\langle h^2\rangle.$$

Proposition 3.1 is proved in Section 6. Set $\gamma = (d+2)/2$ and $L(s) = [\gamma\sqrt{t_{n+1}}\,\log t_{n+1}]$ on the interval $[t_n, t_{n+1}]$ so that $L(s) \geq [\gamma\sqrt{s}\,\log s]$ for all $s \geq t_0$. Fix an interval $[t_n, t_{n+1}]$. We apply Proposition 3.1 to the cylinder function $h = \tau_{X(t_n)}\{u - |\Lambda_K|^{-1}\sum_{y\in\Lambda_K} J(y/K)\tau_y(G_k u)\}$. Its support is contained in a cube centered at the origin with length $|X(t_n)| + K + k + s_u$. A simple computation, taking into account the definitions of X, K, k and assuming that $\varepsilon < 1/5$ shows that $s_h^2 \leq t_n$ provided $t_0 > (24)^{\varepsilon^{-1}}$.

It follows from (3.4) and the previous proposition that (3.2) is bounded above by

$$\begin{aligned} &\frac{d+2}{2}\int_{t_0}^{t} ds(1+s)^{d/2}\Big\langle\big(A_L\tau_{X(s)}v_s\big)^2\Big\rangle \\ &\quad - 2\int_{t_0}^{t}(1+s)^{(d+2)/2}D(\nu_\rho, \tau_{X(s)}v_s)\,ds + C(\rho,d)\log\left(\frac{t}{t_0}\right)\langle u^2\rangle \end{aligned} \tag{3.5}$$

for all $t \geq t_0$. Here $C(\rho, d)$ is some finite constant that depends only on the dimension d and the density ρ. To deduce this bound, we estimated the $L^2(\nu_\rho)$ norm of $\tau_{X(t_n)}\{u - |\Lambda_K|^{-1}\sum_{y\in\Lambda_K} J(y/K)\tau_y(G_k u)\}$ by the one of u times a constant. Note that in the formula above $L = [\gamma\sqrt{t_{n+1}}\,\log t_{n+1}]$ is a function of s.

STEP 2. *Spectral gap.* For a finite subset Λ of $\mathbb{Z}^d$, denote by ν_ρ^Λ the product measure on $\mathbb{N}^\Lambda$ with marginals equal to the marginals of ν_ρ. For each $K \geq 0$, let $\nu_{\Lambda,K}$ stand for the canonical measure on $\mathbb{N}^\Lambda$. This is the product measure ν_ρ^Λ conditioned so that the total number of particles on Λ is K

$$\nu_{\Lambda,K}(\cdot) = \nu_\rho^\Lambda\Big(\ \cdot\ \Big|\ \sum_{x\in\Lambda}\eta_x = K\Big).$$

Note that the right-hand side does not depend on the particular choice of the parameter ρ.

For a subset Ω of $\mathbb{Z}^d$, a cube $\Lambda \subset \Omega$, a probability measure ν on $\mathbb{N}^\Omega$ and a function f in $L_2(\nu)$ denote by $D_\Lambda(\nu, f)$ the Dirichlet form of f on the cube Λ,

$$D_\Lambda(\nu, f) = -\int f \mathscr{L}_\Lambda f \, d\nu. \tag{3.6}$$

In the case where $\Lambda = \mathbb{Z}^d$, we denote $D_\Lambda(\nu, f)$ simply by $D(\nu, f)$. From [12] we have the following spectral gap estimate.

THEOREM 3.2. *Under the assumptions* (L) *and* (H) *on the jump rate* $g(\cdot)$, *there exists a universal constant* $R_0 > 1$ *such that for all* $l \geq 2$, $K \geq 0$.

$$E_{\nu_{\Lambda_l, K}}\left[\left(f - E_{\nu_{\Lambda_l, K}}[f]\right)^2\right] \leq R_0 l^2 D_{\Lambda_l}(\nu_{\Lambda_l, K}, f)$$

for all f *in* $L^2(\nu_{\Lambda_l, K})$.

The second step in the proof of Theorem 1.1 consists in applying the spectral gap for the dynamics restricted to finite boxes in order to replace $A_L \tau_{X(s)} v_s$ by a function that depends only on the density of particles on boxes of length $O(\sqrt{s})$.

Let $l = l(s) = [\sqrt{2(1 + t_n)/(d + 2)R_0}]$ on the interval $[t_n, t_{n+1}]$. To guarantee that $l \geq 2$, we shall assume that $t_0 \geq 2(d + 2)R_0$. Let $\mathscr{R} = \{(2l + 1)x, x \in \mathbb{Z}^d\}$ and consider an enumeration of this set: $\mathscr{R} = \{x_1, x_2, \ldots\}$ such that $|x_j| \leq |x_k|$ for $j \leq k$. Let $\Omega_j = x_j + \Lambda_l$ and let $M_j = M_j(\eta)$ be the total number of particles in Ω_j for the configuration η,

$$M_j = \sum_{x \in \Omega_j} \eta_x.$$

Let q denote the total number of cubes with nonvoid intersection with Λ_L. Note that $q = O((L/l)^d)$. For each $j = 1, 2, \ldots,$ denote by $\mathbf{M}_j$ the vector $(M_1, \ldots, M_j)$.

For a function h in $L^2(\nu_\rho)$, denote by $B_{l,L}h$ the conditional expectation of h given $\mathbf{M}_q$:

$$B_{l,L}h = E_{\nu_\rho}[h \mid M_1, \ldots, M_q]. \tag{3.7}$$

If l and L are chosen in such a way that $(2L + 1)/(2l + 1)$ is an odd number,

$$B_{l,L} A_L h = B_{l,L} h.$$

Since we may modify the definition of L, increasing it if necessary, without changing our estimates, we can assume that $(2L + 1)/(2l + 1)$ is odd. In this case,

$$\left\langle (A_L h)^2 \right\rangle = \left\langle (B_{l,L} h)^2 \right\rangle + \left\langle (A_L h - B_{l,L} h)^2 \right\rangle. \tag{3.8}$$

LEMMA 3.3. *For any* $h \in L^2(\nu_\rho)$,

$$E_{\nu_\rho}\left[(A_L h - B_{l,L} A_L h)^2\right] \leq R_0 l^2 D_{\Lambda_L}(\nu_\rho, h).$$

PROOF. Fix a $\mathscr{F}_{\Lambda_L}$-measurable function h. For $1 \le j \le q$, denote by $\mathscr{G}_j$ the decreasing sequence of σ-algebras generated by $\{M_1, \dots, M_j\}$ and $\{\eta_x,\ x \in \Omega_{j+1} \cup \cdots \cup \Omega_q\}$. Let $h_0 = h$ and for $1 \le j \le q$, let

$$h_j = E_{\nu_\rho}\big[h \mid \mathscr{G}_j\big].$$

With this notation we have that $B_{l,L}h = h_q$ and that

$$E_{\nu_\rho}\Big[\big(h - B_{l,L}h\big)^2\Big] = \sum_{j=0}^{q-1} E_{\nu_\rho}\Big[\big(h_{j+1} - h_j\big)^2\Big].$$

Fix $0 \le j \le q-1$ and recall the definition of the canonical measures $\nu_{\Lambda,K}$. Taking conditional expectation with respect to $\mathscr{G}_{j+1}$, we have that

$$E_{\nu_\rho}\Big[\big(h_{j+1} - h_j\big)^2\Big] = E_{\nu_\rho}\Big[E_{\nu_\rho}\Big[\big(h_{j+1} - h_j\big)^2 | \mathscr{G}_{j+1}\Big]\Big] = E_{\nu_\rho}\Big[\mathrm{Var}\big(\nu_{\Omega_{j+1},M_{j+1}}, h_j\big)\Big].$$

By Theorem 3.2

$$E_{\nu_\rho}\Big[\mathrm{Var}\big(\nu_{\Omega_{j+1},M_{j+1}}, h_j\big)\Big] \le R_0 l^2 E_{\nu_\rho}\Big[D_{\Omega_{j+1}}\big(\nu_{\Omega_{j+1},M_{j+1}}, h_j\big)\Big].$$

By the convexity of the Dirichlet form, this expression is bounded above by $R_0 l^2 D_{\Omega_{j+1}}(\nu_\rho, h)$. To conclude the proof of the lemma it only remains to sum over j. □

It follows from this lemma, the decomposition (3.8), the convexity of the Dirichlet form and the choice of l that

$$\begin{aligned}
&\frac{d+2}{2}\int_{t_0}^t ds(1+s)^{d/2}\Big\langle\big(A_L\tau_{X(s)}v_s\big)^2\Big\rangle \\
&\qquad \le \frac{d+2}{2}\int_{t_0}^t ds(1+s)^{d/2}\Big\langle\big(B_{l,L}\tau_{X(s)}v_s\big)^2\Big\rangle \\
&\qquad\quad + \int_{t_0}^t (1+s)^{(d+2)/2} D(\nu_\rho, \tau_{X(s)}v_s)\, ds
\end{aligned} \tag{3.9}$$

provided $t_0 \ge 2(d+2)R_0$. In view of (3.5) and (3.9), up to this point we proved that for any cylinder function u,

$$\begin{aligned}
&(1+t)^{(d+2)/2}E_{\nu_\rho}\big[v_t^2\big] - (1+t_0)^{(d+2)/2}E_{\nu_\rho}\big[v_{t_0}^2\big] \\
&\qquad \le \frac{d+2}{2}\int_{t_0}^t ds(1+s)^{d/2}\langle\big(B_{l,L}\tau_{X(s)}v_s\big)^2\rangle + C(\rho,d)\log\left(\frac{t}{t_0}\right)\langle u^2\rangle
\end{aligned} \tag{3.10}$$

for all $t \ge t_0$.

STEP 3. *Space averages.* Since the previous formula holds for all trajectories $X\colon \mathbb{R}_+ \to \mathbb{Z}^d$ that are constant in the time interval $[t_n, t_{n+1}]$ and such

that $|X(t_n)| \le (1/4)\sqrt{t_n}$, we may average in space to obtain that the left-hand side of (3.10) is bounded above by

$$\frac{d+2}{2}\int_{t_0}^{t} ds \frac{1}{|\Lambda_{l/4}|}\sum_{x\in\Lambda_{l/4}} (1+s)^{d/2}\left\langle \left(B_{l,L}\tau_x v_s\right)^2\right\rangle + C(\rho,d)\log\left(\frac{t}{t_0}\right)\langle u^2\rangle$$

for all $t \ge t_0$. Recall the definition of the function v_s. By the Schwarz inequality, the first term on the right-hand side of the previous formula is bounded by

$$\int_{t_0}^{t} ds \frac{(d+2)}{|\Lambda_{l/4}|}\sum_{x\in\Lambda_{l/4}} (1+s)^{d/2} \times E_{\nu_\rho}\left[\left(B_{l,L}\tau_x\left\{u_s - \frac{1}{|\Lambda_K|}\sum_{y\in\Lambda_K} J\left(\frac{y}{K}\right)\tau_y u_s\right\}\right)^2\right]$$

(3.11)

$$+\int_{t_0}^{t} ds \frac{(d+2)}{|\Lambda_{l/4}|}\sum_{x\in\Lambda_{l/4}} (1+s)^{d/2} \times E_{\nu_\rho}\left[\left(\frac{1}{|\Lambda_K|}\sum_{y\in\Lambda_K} J\left(\frac{y}{K}\right)B_{l,L}\tau_{x+y}\left[u_s - (G_k u)_s\right]\right)^2\right].$$

To estimate the first term in (3.11) observe that by the Schwarz inequality,

$$E_{\nu_\rho}\left[\left(B_{l,L}\tau_x\left\{u_s - \frac{1}{|\Lambda_K|}\sum_{y\in\Lambda_K} J\left(\frac{y}{K}\right)\tau_y u_s\right\}\right)^2\right] \le \frac{C_J(K)}{|\Lambda_K|}\sum_{y\in\Lambda_K} J\left(\frac{y}{K}\right)E_{\nu_\rho}\left[\left(B_{l,L}\{\tau_x u_s - \tau_{x+y}u_s\}\right)^2\right].$$

In this formula, $C_J(K) = |\Lambda_K|^{-1}\Sigma_y J(y/K)$ that converges to 1 as $K\uparrow\infty$ (i.e., as $s\uparrow\infty$). The next lemma shows that the first term in (3.11) is $o(t)$.

LEMMA 3.4. *Recall the definition of K given above. We have that*

$$\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t} ds \frac{1}{|\Lambda_{l/4}|}\sum_{x\in\Lambda_{l/4}}\frac{1}{|\Lambda_K|}\sum_{y\in\Lambda_K} J\left(\frac{y}{K}\right)(1+s)^{d/2} \times E_{\nu_\rho}\left[\left(B_{l,L}\{\tau_x u_s - \tau_{x+y}u_s\}\right)^2\right] = 0.$$

The second term in (3.11) can be bounded in a similar way.

LEMMA 3.5. *We have that*

$$\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t} ds \frac{1}{|\Lambda_{l/4}|}\sum_{x\in\Lambda_{l/4}}\frac{1}{|\Lambda_K|}\sum_{y\in\Lambda_K} J\Big(\frac{y}{K}\Big)(1+s)^{d/2} \times E_{\nu_\rho}\Big[\big(B_{l,L}\tau_{x+y}\big[u_s-(G_k u)_s\big]\big)^2\Big] = 0.$$

Lemmas 3.4 and 3.5 conclude the proof of Proposition 2.2. Their proofs are given at the end of the next section.

4. Entropy estimates. We prove Lemmas 3.4 and 3.5 in this section. Fix an interval $[t_n, t_{n+1}]$. Recall the definition of the canonical measures $\nu_{\Lambda,K}$ and the product measure ν_ρ^Λ and the decomposition of Λ_L into subcubes $\Omega_1,\dots,\Omega_q$ of side length $2l+1$ with $\mathbf{M}_q=(M_1,\dots,M_q)$ the number of particles in each. Fix a vector $\mathbf{M}_q$ and let $f=f_{l,L,\mathbf{M}_q}$ be the Radon–Nikodym derivative given by

$$f(\eta)=f_{l,L,\mathbf{M}_q}(\eta)=\frac{d\nu_{\Omega_1,M_1}\otimes\cdots\otimes d\nu_{\Omega_q,M_q}}{d\nu_\rho^{\Lambda_L}}. \tag{4.1}$$

Since ν_ρ is translation invariant and reversible, and since the dynamics is translation invariant, we have that

$$B_{l,L}\tau_x h_s = E_{\nu_\rho}\big[\tau_x h_s \mid \mathbf{M}_q\big] = \int \tau_x h_s(\eta) f(\eta)\,\nu_\rho(d\eta) = E_{\nu_\rho}\big[\tau_x h f_s\big]. \tag{4.2}$$

Denote by $H(f)$ the relative entropy of $f d\nu_\rho$ with respect to ν_ρ:

$$H(f)=E_{\nu_\rho}[f\log f].$$

Since ν_ρ is an invariant measure for the dynamics, $H(f_s)$ is decreasing in time. We now obtain a bound for $H(f)$ which therefore immediately bounds $H(f_s)$ for all times $s\ge 0$, as well as the time integral of the Dirichlet form of $\sqrt{f_s}$ because

$$H(f_t)+4\int_0^t D\big(\sqrt{f_s}\big)\,ds \le H(f).$$

Recall that we are assuming that $(2l+1)$ divides $(2L+1)$.

LEMMA 4.1. *Let $f_{l,L,\mathbf{M}_q}$ be defined as in* (4.1). *There exists a constant $C=C(\rho,d,R_0)$ such that*

$$E_{\nu_\rho}\Big[H\big(f_{l,L,\mathbf{M}_q}\big)\Big] \le C(L/l)^d \log l$$

provided l is chosen sufficiently large. Here the expectation is over the random vector $\mathbf{M}_q$.

PROOF. Recall the definition of the product measures ν_ρ^Λ. Let $\rho_j = K_j/(2l+1)^d$ and $m_j = M_j/(2l+1)^d$. The entropy $H(f)$ can be written as

$$H(f) = \int \log \frac{f d\nu_\rho^{\Lambda_L^0}}{d\nu_{\rho_0}^{\Omega_1} \otimes \cdots \otimes d\nu_{\rho_q}^{\Omega_q}} f d\nu_\rho + \int \log \frac{d\nu_{\rho_0}^{\Omega_1} \otimes \cdots \otimes d\nu_{\rho_q}^{\Omega_q}}{d\nu_\rho^{\Lambda_L^0}} f d\nu_\rho.$$

By the definition of f, this expression is equal to

$$\sum_{j=1}^{q} \log \frac{1}{\nu_{\rho_j}^{\Omega_j}[M_j = K_j]} + \sum_{j=1}^{q} \left\{ M_j \log \frac{\varphi(\rho_j)}{\varphi(\rho)} - |\Lambda_l^0| \log \frac{Z(\varphi(\rho_j))}{Z(\varphi(\rho))} \right\}. \tag{4.3}$$

By the uniform local central limit theorem for zero range distributions (cf. [12], Theorem 6.1), $\nu_{\rho_j}^{\Omega_j}[M_j = K_j]$ is bounded below by $C_0/\sqrt{|\Omega_j|\sigma(\rho_j)^2}$ for some universal constant C_0, provided l is large enough. The expected value, with respect to ν_ρ, of the first term of the previous formula is thus bounded below by

$$\tfrac{1}{2} \sum_{j=1}^{q} E_{\nu_\rho}\left[\log C|\Omega_j|\sigma(\rho_j)^2\right].$$

It follows from assumptions (L) and (H) that there exists universal constants C_1 and C_2 such that $0 < C_1 \le \sigma(\rho)^2/\varphi(\rho) \le C_2 < \infty$ (cf. (5.2) in [12]). Since also $0 < C_3 \le \rho/\varphi(\rho) \le C_4 < \infty$ and $\log \alpha \le \alpha$, by definition of $q = O((L/l)^d)$, the previous expression is bounded above by $C(\rho, b, d, R_0)(L/l)^d \log l$ provided l is sufficiently large.

Let $F(\theta) = \theta \log\{\varphi(\theta)/\varphi(\rho)\} - \log\{Z(\varphi(\theta))/Z(\varphi(\rho))\}$. The expected value, with respect to ν_ρ, of the second term of (4.3) is equal to

$$\sum_{j=0}^{q} |\Lambda_l^0| E_{\nu_\rho}\left[F(\rho_j)\right].$$

It is easy to check that F and its first derivative vanish at ρ: $F(\rho) = F'(\rho) = 0$, that $\lim_{\theta \to 0} F(\theta) = C(\rho)$ and that there exists a finite constant C such that $F(\theta) \le C\theta \log \theta$ for θ large. In particular, there exists a finite constant $C_1(\rho)$ depending only on ρ such that $F(\theta) \le C_1(\theta - \rho)^2$ for all $\theta \ge 0$. The previous sum is thus bounded above by

$$C_1(\rho)|\Lambda_l^0| \sum_{j=1}^{q} E_{\nu_\rho}\left[(\rho_j - \rho)^2\right] \le C_2(\rho)(L/l)^d.$$

This concludes the proof of the lemma. □

The proof of the following perturbation result is standard. However, since we were not able to find an explicit reference and since [11] is still in press, the proof is included for completeness.

LEMMA 4.2. *Let $(\Omega, P, \mathscr{F})$ be a probability space and let $\langle f, g \rangle = \int fg\, dP$ denote the standard inner product on $L^2(\Omega, P, \mathscr{F})$. Let A be a nonnegative*

definite symmetric operator on $L^2(\Omega, P, \mathscr{F})$, *which has* 0 *as a simple eigenvalue with corresponding eigenfunction the constant function* 1, *and second smallest eigenvalue* $\delta > 0$ (*the spectral gap*). *Let* V *be a function of mean zero*, $\langle 1, V \rangle = 0$ *and assume that* V *is essentially bounded. Denote by* λ_ε *the principal eigenvalue of* $-A + \varepsilon V$ *given by the variational formula*

$$\lambda_\varepsilon = \sup_{\|f\|_2 = 1} \langle f, (-A + \varepsilon V) f \rangle.$$

Then for $\varepsilon < \delta(2\|V\|_\infty)^{-1}$,

$$0 \le \lambda_\varepsilon \le \frac{\varepsilon^2 \langle V, A^{-1} V \rangle}{1 - 2\|V\|_\infty \varepsilon \delta^{-1}}.$$

PROOF. The lower bound follows immediately by setting $f = 1$ in the variational formula. Let G be any function with $\|G\|_2 = 1$ and $\langle 1, G \rangle \ge 0$. Since $\langle 1, V \rangle = 0$ we have

$$\lambda_\varepsilon = \varepsilon 2\langle V, G \rangle + \varepsilon \big\langle V, [G - 1]^2 \big\rangle - \langle G, AG \rangle + \big\langle G, (\lambda_\varepsilon + A - \varepsilon V) G \big\rangle.$$

By Schwarz's inequality we can control the first two terms on the right-hand side for any $\beta > 0$ by

$$\varepsilon 2\langle V, G \rangle + \varepsilon \big\langle V, [G - 1]^2 \big\rangle \le \frac{\varepsilon^2}{\beta} \langle V, A^{-1} V \rangle + \beta \langle G, AG \rangle + 2\varepsilon \|V\|_\infty (1 - \langle 1, G \rangle).$$

Also by Schwarz's inequality, $\langle 1, G \rangle \le 1$ so by the spectral gap

$$1 - \langle 1, G \rangle \le \langle G, G \rangle - \langle 1, G \rangle^2 \le \delta^{-1} \langle G, AG \rangle.$$

Therefore for any $\beta > 0$,

$$\lambda_\varepsilon \le \frac{\varepsilon^2}{\beta} \langle V, A^{-1} V \rangle + \big(\beta + 2\varepsilon \|V\|_\infty \delta^{-1} - 1 \big) \langle G, AG \rangle + \big\langle G, (\lambda_\varepsilon + A - \varepsilon V) G \big\rangle.$$

Consider an optimizing sequence $\{G_{\varepsilon, n},\ n = 1, 2, \dots\}$ in the variational formula and, without loss of generality, assume that $\langle 1, G_{\varepsilon, n} \rangle \ge 0$ for all n. Choose $\beta = 1 - 2\varepsilon \|V\|_\infty \delta^{-1}$ and $G = G_{\varepsilon, n}$ in the previous bound. Letting $n \to \infty$ we have $\langle G_{\varepsilon, n}, (\lambda_\varepsilon + A - \varepsilon V) G_{\varepsilon, n} \rangle \to 0$ and the upper bound follows. □

LEMMA 4.3. *Let* u *be any local function. Let* Λ_u *be the smallest cube centered at the origin containing the support of* $u - \tau_e u$ *for all unit vectors* e *in* $\mathbb{Z}^d$. *There exists a finite constant* $C = C(u, \rho)$ *depending only on* u *and* ρ *such that for any unit vector* e *in* $\mathbb{Z}^d$,

$$\Big(E_{\nu_\rho} \big[f(u - \tau_e u) \big] \Big)^2 \le C(u, \rho) D_{\Lambda_u} \Big(\nu_\rho, \sqrt{f} \Big).$$

PROOF. Consider the zero-range process corresponding to the Dirichlet form $D_{\Lambda_u}(\nu_\rho, \sqrt{f})$. This process has a spectral gap of magnitude Γ that depends on the size of the support of u. Assume $\beta \le (1/8)(\|u\|_\infty \Gamma)^{-1}$. In this case by the standard perturbation theorem,

$$E_{\nu_\rho}[f(u - \tau_e u)] - \frac{1}{\beta} D_{\Lambda_u}\left(\nu_\rho, \sqrt{f}\right) \le 2\beta \left\langle (u - \tau_e u), \left(-L_{\Lambda_u}\right)^{-1} (u - \tau_e u) \right\rangle,$$

where L_{Λ_u} is the generator restricted to the box Λ_u (with reflecting boundary conditions). On the other hand, for $\beta \ge (1/8)(\|u\|_\infty \Gamma)^{-1}$, $E_{\nu_\rho}[f(u - \tau_e u)]$ is bounded above by $2\|u\|_\infty \le 16\beta\Gamma\|u\|_\infty^2$. Optimizing over β we obtain the lemma. □

LEMMA 4.4. *There exists a finite constant $C = C(d, u, \rho)$ depending only on u, ρ and the dimension d, such that for n sufficiently large,*

$$\frac{1}{|\Lambda_n|^2} \sum_{x,\, y \in \Lambda_u} \left(E_{\nu_\rho}\left[\tau_x f(u - \tau_y u)\right]\right)^2 \le C(d, u, \rho) n^{2-d} D_{\Lambda_{3n}}\left(\nu_\rho, \sqrt{f}\right).$$

PROOF. Define a canonical path $0 = x_0, x_1, \ldots, x_m = y$ where $m = \sum_{j=1}^d |y_j|$ from the origin to y by nearest neighbor steps, that is, $\tilde{e}_i = x_{i+1} - x_i$ are unit vectors, by first moving toward y in the first coordinate direction, then in the second coordinate direction, and so on. Then,

$$u - \tau_y u = \sum_{i=0}^{m-1} \tau_{x_i} u - \tau_{x_{i+1}} u.$$

Therefore for any f, by the previous lemma and Schwarz's inequality,

$$\begin{aligned}\left(E_{\nu_\rho}\left[f(u - \tau_y u)\right]\right)^2 &= \left(\sum_{i=0}^{m-1} E_{\nu_\rho}\left[\tau_{-x_i} f(u - \tau_{\tilde{e}}, u)\right]\right)^2 \\ &\le C(d, u, \rho) m \sum_{i=0}^{m-1} D_{\Lambda_u}\left(\nu_\rho, \sqrt{\tau_{-x_i} f}\right).\end{aligned}$$

Since $m \le 2dn$, as long as n is larger than the side length of Λ_u we have that $D_{\Lambda_u}(\nu_\rho, \sqrt{\tau_x f}) = D_{\tau_{-x}\Lambda_u}(\nu_\rho, \sqrt{f})$ and by explicit counting, that

$$\frac{1}{|\Lambda_n|^2} \sum_{x,\, y \in \Lambda_u} \left(E_{\nu_\rho}\left[\tau_x f(u - \tau_y u)\right]\right)^2 \le C(d, u, \rho) n^{2-d} D_{\Lambda_{3n}}\left(\nu_\rho, \sqrt{f}\right)$$

with a new constant $C(d, u, \rho)$. □

PROOF OF LEMMA 3.4. Fix an interval $[t_n, t_{n+1}]$. In this interval $l(s)$ and $L(s)$ are constant. Recalling the definition (4.1) of f and (4.2), we have

$$B_{l,L} \tau_x (u_s - \tau_y u_s) = E_{\nu_\rho}\left[(u - \tau_y u) \tau_{-x} f_s\right].$$

Using the same method as for the proof of Lemma 4.4, we prove that there exists a constant $C(\rho, u, d) > 0$ depending only on ρ, u and d such that

$$\frac{1}{|\Lambda_{l/4}|} \sum_{x \in \Lambda_{l/4}} \frac{1}{|\Lambda_K|} \sum_{y \in \Lambda_K} J\Big(\frac{y}{K}\Big) E_{\nu_\rho}\Big[\big(B_{l,L}\tau_x[u_s - \tau_y u_s]\big)^2\Big]$$

is bounded above by

$$C(\rho, u, d)\frac{K^2}{l^d} E_{\nu_\rho}\Big[D_{\Lambda_l}\Big(\nu_\rho, \sqrt{f_s}\Big)\Big].$$

Notice that l^{-d} cancels $(1+s)^{d/2}$ and that $K(s)^2 \le C(R_0, d)t^{1-e}$ for $s \le t$. By the entropy estimate proved in Lemma 4.1, we have

$$E_{\nu_\rho}\left[\int_{t_n}^{t_{n+1}} ds\, D\Big(\nu_\rho, \sqrt{f_s}\Big)\right] \le E_{\nu_\rho}[H(f)] \le C_0(\log C_1 t)^{d+1}$$

for some finite constants C_0, C_1 depending only on d, ρ and R_0. To prove the lemma, it remains to sum over n. □

PROOF OF LEMMA 3.5. This proof follows closely the one of Lemma 3.4. Fix an interval $[t_n, t_{n+1}]$. By (4.2) and since $(u - G_k u)$ is $\mathscr{F}_{\Lambda_k}$-measurable,

$$\begin{aligned} B_{l,L}\tau_{x+y}[u_s - (G_k u)_s] &- E_{\nu_\rho}\big[(u - G_k u)\tau_{-(x+y)} f_s\big] \\ &= E_{\nu_\rho}\big[(u - G_k u) G_k \tau_{-(x+y)} f_s\big]. \end{aligned}$$

By the standard perturbation theorem and the spectral gap for zero range dynamics, the previous expression is bounded above by

$$\beta^{-1} E_{\nu_\rho}\Big[D_{\Lambda_k}\Big(\nu_\rho^{\Lambda_k}, \sqrt{\tau_{-(x+y)} f_s}\Big)\Big] + 2\beta E_{\nu_\rho}\big[(u - G_k u)(-\mathscr{L}_k)^{-1}(u - G_k u)\big]$$

for all $\beta \le C/\|u\|_\infty k^2$. Here we used the convexity of the Dirichlet form to bound the expression $D_{\Lambda_k}(\nu_\rho^{\Lambda_k}, G_k f)$ by $D_{\Lambda_k}(\nu_\rho^{\Lambda_k}, f)$.

In the case where $\beta \ge C/\|u\|_\infty k^2$, since $E_{\nu_\rho}[(u - G_k u)\tau_{-(x+y)} f_s]$ is bounded above by $2\|u\|_\infty$, we have that

$$E_{\nu_\rho}\big[(u - G_k u)\tau_{-(x+y)} f_s\big] \le \beta^{-1} E_{\nu_\rho}\Big[D_{\Lambda_k}\Big(\nu_\rho^{\Lambda_k}, \sqrt{\tau_{-(x+y)} f_s}\Big)\Big] + \beta k^2 C(u).$$

In view of the two previous estimates, minimizing over β, we get that

$$\big(B_{l,L}\tau_{x+y}[u_s - (G_k u)_s]\big)^2 \le C(u, \rho) k^2 D_{\Lambda_k}\Big(\nu_\rho, \sqrt{\tau_{-(x+y)} f_s}\Big).$$

Hence, the time integral appearing in the statement of Lemma 3.5 restricted to the time interval $[t_n, t_{n+1}]$ is bounded above by

$$C(u, \rho, d) C_J E_{\nu_\rho}\left[\int_{t_n}^{t_{n+1}} (1+s)^{d/2} \frac{k^2}{l^d} D_{\Lambda_l}\Big(\nu_\rho, \sqrt{f_s}\Big)\, ds\right].$$

where $C_J = \Sigma_y J(y/K)$. In this formula l^{-d} cancels $(1+s)^{d/2}$ and $k(s)^2$ is bounded by $t^{4\varepsilon}$. By the entropy estimate, this expression is bounded above by $C(u, \rho, d)t^{4\varepsilon}(\log t_n)^{d+1}$. To conclude the proof of the lemma it remains to sum over n. □

5. Equilibrium fluctuations. We prove Proposition 2.1 in this section. The expectation appearing in the statement of the proposition is bounded above by

$$
\begin{aligned}
&(1+A)E_{\nu_\rho}\left[\left(P_t\frac{1}{|\Lambda_K|}\sum_{y\in\Lambda_K}J\left(\frac{y}{K}\right)\tau_y\left\{G_k u - E(u,\rho)(\overline{\eta}_k-\rho)\right)^2\right\}\right]\\
&\quad+(1+A^{-1})E(u,\rho)^2E_{\nu_\rho}\left[\left(P_t\frac{1}{|\Lambda_K|}\sum_{y\in\Lambda_K}J\left(\frac{y}{K}\right)\tau_y(\overline{\eta}_k-\rho)\right)^2\right]
\end{aligned}
\tag{5.1}
$$

for every $A>0$. Here $\{P_s,\ s\ge 0\}$ is the semigroup associated to the generator $\mathscr{L}$ and $E(u,\rho)=(d/d\rho)E_{\nu_\rho}[u]=\tilde{u}'(\rho)$.

Denote by $V(\eta)$ the function $G_k u - E(u,\rho)(\overline{\eta}_k-\rho)$ and notice that it is $\mathscr{F}_{\Lambda_k}$-measurable. Since the semigroup is a contraction, since V has mean zero and since ν_ρ is a translation invariant product measure, the first term in (5.1) is bounded above by

$$
\begin{aligned}
&\frac{(1+A)}{|\Lambda_K|^2}\sum_{\substack{x,\,y\in\Lambda_K\\ |x-y|\le 2k}}J\left(\frac{y}{K}\right)J\left(\frac{x}{K}\right)E_{\nu_\rho}\left[V\tau_{x-y}V\right]\\
&\quad\le\frac{(1+A)|\Lambda_{2k}|}{|\Lambda_K|^2}E_{\nu_\rho}\left[V^2\right]\sum_{x\in\Lambda_K}J\left(\frac{x}{K}\right)^2.
\end{aligned}
\tag{5.2}
$$

We claim that $E_{\nu_\rho}[V^2]$ is bounded above by $C(u,\rho)k^{-d}$ for some finite constant $C(u,\rho)$. Indeed, for $\rho_0>\rho$,

$$
E_{\nu_\rho}\left[V^2\right]=E_{\nu_\rho}\left[V^2\mathbf{1}\{\overline{\eta}_k\le\rho_0\}\right]+E_{\nu_\rho}\left[V^2\mathbf{1}\{\overline{\eta}_k\ge\rho_0\}\right].
$$

Since $|V|\le\|u\|_\alpha+C(u,\rho)|\eta_k-\rho|$, by a large deviations argument the second term is exponentially small. On the other hand, the first term is bounded above by

$$
\begin{aligned}
E_{\nu_\rho}\left[V^2\mathbf{1}\{\overline{\eta}_k\le\rho_0\}\right]&\le 2E_{\nu_\rho}\left[\left(G_k u-\tilde{u}(\overline{\eta}_k)\right)^2\mathbf{1}\{\overline{\eta}_k\le\rho_0\}\right]\\
&\quad+E_{\nu_\rho}\left[\left(\tilde{u}(\overline{\eta}_k)-\tilde{u}(\rho)-\tilde{u}'(\rho)(\overline{\eta}_k-\rho)\right)^2\mathbf{1}\{\overline{\eta}_k\le\rho_0\}\right],
\end{aligned}
$$

provided $\tilde{u}(\rho)$ stand for $E_{\nu_\rho}[u]$. Notice that with this notation, $E(u,\rho)=\tilde{u}'(\rho)$. By the equivalence of ensembles (cf. Appendix 2 in [11]) $\sup_{\overline{\eta}_k\le\rho_0}|G_k u - E_{\nu_{\overline{\eta}_k}}[u]|$ is bounded above by $C\|u\|_\infty k^{-d}$ for some finite constant C depending only on ρ_0. In particular, the first term is bounded above by $C(u,\rho_0)k^{-2d}$. By Taylor expansion and since the measure ν_ρ is product, the second term is bounded above by $C(u,\rho_0)k^{-2d}$. In view of (5.2), this shows that the first term of (5.1) is of order $K^{-d}k^{-d}=O(t^{-(1+\varepsilon)d/2})$.

The next lemma shows that the second term of (5.1) is equal to $Ct^{-d/2}+o(t^{-d/2})$ and concludes the proof of the theorem. □

LEMMA 5.1. *Fix two integers $k \ll K \ll \sqrt{t}$. Then,*

$$E_{\nu_\rho}\left[\left(P_t \frac{1}{|\Lambda_K|}\sum_{y\in\Lambda_K} J\Big(\frac{y}{K}\Big)\tau_y(\overline{\eta}_k - \rho)\right)^2\right] = \frac{\chi(\rho)}{[8\pi\varphi'(\rho)t]^{d/2}}(1 + C(k,K,t)),$$

where $C(k,K,t)$ is a positive expression bounded by

$$C(k,K,t) \le \frac{K}{\sqrt{t}} + \left\{\frac{t^{(d+2)/2}}{K^{d+2}}\left[\frac{1}{k^d} + \frac{k^2}{K^2}\Big(1 + \frac{t}{K^2}\Big)\right]\right\}^{1/2}. \tag{5.3}$$

PROOF. We want to estimate

$$E_{\nu_\rho}\left[\left(E_\eta\left[\frac{1}{|\Lambda_K|}\sum_{x\in\Lambda_K} J\Big(\frac{x}{K}\Big)\tau_x(\overline{\eta}_k(t) - \rho)\right]\right)^2\right]. \tag{5.4}$$

Denote by $J(t,x)$ the solution of the linear discrete equation

$$(\partial_t J)(t,x) = \varphi'(\rho)\Delta_d J(t,x) \tag{5.5}$$

on $\mathbb{R}_+ \times \mathbb{Z}^d$ with initial condition $J(0,x) = J(x/K)$. In this equation Δ_d is the discrete Laplacian so that for $h\colon \mathbb{Z}^d \to \mathbb{R}$, $(\Delta_d h)(x) = \sum_{1\le i\le d}[h(x+e_i) + h(x-e_i) - 2h(x)]$.

Define the martingale $\{M_s,\ 0 \le s \le t\}$ by

$$M_s = \frac{1}{|\Lambda_K|}\sum_{x\in\mathbb{Z}^d} J(t-s,x)\tau_x(\overline{\eta}_k(s) - \rho) - \int_0^s dr(\partial_r + \mathscr{L})\frac{1}{|\Lambda_K|}\sum_{x\in\mathbb{Z}^d} J(t-r,x)\tau_x(\overline{\eta}_k(r) - \rho).$$

Since J is the solution of (5.5), the integral part of M_t is equal to

$$\int_0^s dr \frac{1}{|\Lambda_K|}\sum_{x\in\mathbb{Z}^d}(\Delta_d J)(t-r,x) \times \tau_x\left\{\frac{1}{|\Lambda_k|}\sum_{y\in\Lambda_k}\big[g(\eta_y(r)) - \varphi(\rho)\big] - \varphi'(\rho)(\overline{\eta}_k(r) - \rho)\right\}.$$

Here we were allowed to add the term $\varphi(\rho)$ because the summation on $\mathbb{Z}^d$ of $\Delta_d J$ vanishes. Notice that the martingale at time 0 is just $|\Lambda_K|^{-1}\sum_x J(t,x)\tau_x(\overline{\eta}_k - \rho)$. Therefore,

$$E_\eta\left[\frac{1}{|\Lambda_K|}\sum_{x\in\mathbb{Z}^d} J\Big(\frac{x}{K}\Big)\tau_x(\overline{\eta}_k(t) - \rho)\right] = \frac{1}{|\Lambda_K|}\sum_{x\in\mathbb{Z}^d} J(t,x)\tau_x(\overline{\eta}_k - \rho) + E_\nu\left[\int_0^t dr \frac{1}{|\Lambda_K|}\sum_{x\in\mathbb{Z}^d}(\Delta_d J)(t-r,x)\tau_x U_k(\eta(r))\right].$$

where $U_k(\eta) = |\Lambda_k|^{-1}\sum_{y\in\Lambda_k}[g(\eta_y) - \varphi(\rho)] - \varphi'(\rho)(\overline{\eta}_k - \rho)$. In view of this identity, by the Schwarz inequality, (5.4) is bounded above by

$$
\begin{aligned}
&(1 + A^{-1})E_{\nu_\rho}\left[\left(\frac{1}{|\Lambda_K|}\sum_{x\in\mathbb{Z}^d} J(t,x)\tau_x(\overline{\eta}_k - \rho)\right)^2\right] \\
&+(1 + A)E_{\nu_\rho}\left[\left(E_\eta\left[\int_0^t dr\frac{1}{|\Lambda_K|}\sum_{x\in\mathbb{Z}^d}(\Delta_d J)(t-r,x)\tau_x U_k(\eta(r))\right]\right)^2\right]
\end{aligned}
\tag{5.6}
$$

for every $A > 0$.

We now estimate the expectation in the second term of (5.6). By Schwarz's inequality it is bounded above by

$$
E_{\nu_\rho}\left[\left(\int_0^t dr\frac{1}{|\Lambda_K|}\sum_{x\in\mathbb{Z}^d}(\Delta_d J)(t-r,x)\tau_x U_k(\eta(r))\right)^2\right].
$$

Denote by $\tilde{g}_k(\overline{\eta}_k)$ the conditional expectation of $g(\eta_0)$ given $|\Lambda_k|^{-1}\sum_{x\in\Lambda_k}\eta_x$. By the Schwarz inequality this expression is bounded above by

$$
\begin{aligned}
&2E_{\nu_\rho}\left[\left(\int_0^t dr\frac{1}{|\Lambda_K|}\sum_{x\in\mathbb{Z}^d}(\Delta_d J)(t-r,x)\right.\right. \\
&\qquad\left.\left.\times\tau_x\left\{\frac{1}{|\Lambda_k|}\sum_{y\in\Lambda_k} g\big(\eta_y(r)\big) - \tilde{g}_k\big(\overline{\eta}_k(r)\big)\right\}\right)^2\right] \\
&+2E_{\nu_\rho}\left[\left(\int_0^t dr\frac{1}{|\Lambda_K|}\sum_{x\in\mathbb{Z}^d}(\Delta_d J)(t-r,x)\right.\right. \\
&\qquad\left.\left.\times\tau_x\big\{\tilde{g}_k\big(\overline{\eta}_k(r)\big) - \varphi(\rho) - \varphi'(\rho)\big(\overline{\eta}_k(r) - \rho\big)\big\}\right)^2\right].
\end{aligned}
\tag{5.7}
$$

We shall estimate these two terms separately. The second one is simpler. Denote the $\mathscr{F}_{\Lambda_k}$-measurable function $\tilde{g}_k(\overline{\eta}_k) - \varphi(\rho) - \varphi'(\rho)(\overline{\eta}_k - \rho)$ by $V_k(\eta)$. By the Schwarz inequality, since ν_ρ is invariant and translation invariant, the second term in (5.7) is bounded above by

$$
\begin{aligned}
&2t\int_0^t dr\, E_{\nu_\rho}\left[\left(\frac{1}{|\Lambda_K|}\sum_{x\in\mathbb{Z}^d}(\Delta_d J)(t-r,x)\tau_x V_k(\eta(r))\right)^2\right] \\
&\quad= 2t\int_0^t dr\left(\frac{1}{|\Lambda_K|^2}\sum_{|x-y|\le 2k}(\Delta_d J)(t-r,x)(\Delta_d J)(t-r,y)\right. \\
&\qquad\left.\times E_{\nu_\rho}\big[V_k(\eta)\tau_{x-y}V_k(\eta)\big]\right).
\end{aligned}
$$

Applying the elementary inequality $(a+b)^2 \le 2a^2 + 2b^2$, we bound this expression by

$$\frac{C(\rho)t}{k^d K^d} \int_0^t dr \frac{1}{|\Lambda_K|} \sum_{x \in \mathbb{Z}^d} \big[(\Delta_d J)(t-r, x)\big]^2$$

because $E_{\nu_\rho}[V_k(\eta)^2]$ is less than or equal to $C(\rho)k^{-2d}$. This last estimate, which follows in part from the equivalence of ensembles, has been explained in Step 5 of Section 3. A simple computation shows that

$$\int_0^t dr \frac{1}{|\Lambda_K|} \sum_{x \in \mathbb{Z}^d} \big[(\Delta_d J)(r, x)\big]^2 \le \frac{1}{|\Lambda_K|} \sum_{x \in \mathbb{Z}^d} \left\| (\nabla_d J)\left(\frac{x}{K}\right) \right\|^2.$$

where $\nabla_d J$ stands for the discrete gradient of J. Since J is smooth, the right-hand side is of order K^{-2}. It follows from this estimate that the previous expression is bounded above by $C(\rho, J)tk^{-d}K^{-d-2}$.

We now turn to the first term of (5.7). Denote $|\Lambda_k|^{-1}\Sigma_{y \in \Lambda_k} g(\eta_y) - \tilde{g}_k(\overline{\eta}_k)$ by $W_k(\eta)$ and notice that it is measurable with respect to $\mathscr{F}_{\Lambda_k}$. An integration by parts and the Schwarz inequality permits bounding the first term of (5.7) by

$$\begin{aligned} &4E_{\nu_\rho}\left[\left(\frac{1}{|\Lambda_K|} \sum_{x \in \mathbb{Z}^d} (\Delta_d J)\left(\frac{x}{K}\right) \int_0^t dr\, \tau_x W_k(\eta(r))\right)^2\right] \\ &\quad + 4E_{\nu_\rho}\left[\left(\int_0^t dr \frac{1}{|\Lambda_K|} \sum_{x \in \mathbb{Z}^d} (\Delta_d \partial_r J)(t-r, x) \int_0^r ds\, \tau_x W_k(\eta(s))\right)^2\right]. \end{aligned} \tag{5.8}$$

Recall (cf. [11], Appendix 1) that for Markov processes X_t with generator L symmetric with respect to a probability measure ν,

$$E_\nu\left[\left(\frac{1}{\sqrt{t}} \int_0^t V(X_s)\, ds\right)^2\right] \le C_0 \big\langle V, (-L)^{-1} V \big\rangle \tag{5.9}$$

for some universal constant C_0. From this estimate and the variational formula for the H_{-1} norm, we have that the first term in (5.8) is bounded above by

$$Ct \sup_h \left\{ \frac{2}{|\Lambda_K|} \sum_{x \in \mathbb{Z}^d} (\Delta_d J)\left(\frac{x}{K}\right) \int \tau_x W_k(\eta) h\, d\nu_\rho - D(\nu_\rho, h) \right\}.$$

In this formula the supremum is carried over all functions h in $L^2(\nu_\rho)$. Since ν_ρ is translation invariant, $E_{\nu_\rho}[\tau_x W_k h] = E_{\nu_\rho}[W_k \tau_{-x} h]$. Since W_k is $\mathscr{F}_{\Lambda_k}$-measurable, we may replace τ_{-x} by $G_k \tau_{-x} h$. Finally, since the Dirichlet form is convex, $D(\nu_\rho, h)$ is bounded below by $|\Lambda_k|^{-1}\Sigma_x D_{\Lambda_k}(\nu_\rho, G_k \tau_{-x} h)$. In conclu-

sion, the previous expression is bounded above by

$$\frac{Ct|\Lambda_k|}{|\Lambda_K|^2}\sum_{x\in\mathbb{Z}^d}\left[(\Delta_d J)\left(\frac{x}{K}\right)\right]^2\left\langle W_k, (-\mathscr{L}_k)^{-1}W_k\right\rangle,$$

where $\mathscr{L}_k$ stands for the generator $\mathscr{L}$ restricted to the cube Λ_k. By the spectral gap, $\langle W_k,(-\mathscr{L}_k)^{-1}W_k\rangle$ is bounded above by $k^2\langle W_k, W_k\rangle$, which is less than or equal to $C(\rho)k^{2-d}$ because it is a variance term. In conclusion, the first term of (5.8) is bounded above by $C(J,\rho)tk^2K^{-d-4}$.

We now estimate the second term of (5.8). Since $J(t,x)$ is the solution of (5.5) by one Schwarz inequality, the second term of (5.8) is bounded above by

$$4t\int_0^t dr\, E_{\nu_\rho}\left[\left(\frac{1}{|\Lambda_K|}\sum_{x\in\mathbb{Z}^d}(\Delta_d^2 J)(t-r,x)\int_0^r ds\,\tau_x W_k(\eta(s))\right)^2\right].$$

Applying inequality (5.9) and repeating the previous arguments, we obtain that this expression is less than or equal to

$$C(\rho)t^2\int_0^t dr\frac{k^2}{|\Lambda_K|^2}\sum_{x\in\mathbb{Z}^d}\left[(\Delta_d^2 J)(t-r,x)\right]^2.$$

Here we bounded the term r coming from (5.9) by t. Since J is the solution of (5.5), a simple computation shows that

$$\int_0^t dr\sum_{x\in\mathbb{Z}^d}\left[(\Delta_d^2 J)(r,x)\right]^2\le C(\rho)\sum_{x\in\mathbb{Z}^d}\left\|(\nabla_d\Delta_d J)\left(\frac{x}{K}\right)\right\|^2.$$

J being smooth, this expression is bounded by $C(J,\rho)K^{d-6}$. Therefore, the second term of (5.8) is bounded by $C(J,\rho)t^2k^2K^{-d-6}$.

To conclude the proof of the lemma, it remains to compute the contribution of the main term. The expectation in the first line of (5.6) is equal to

$$\frac{1}{|\Lambda_K|^2}\sum_{x,\,y\in\mathbb{Z}^d}J(t,x)J(t,y)E_{\nu_\rho}\left[\tau_x(\overline{\eta}_k-\rho)\tau_y(\overline{\eta}_k-\rho)\right].$$

Since ν_ρ is a product measure, this expectation is easy to compute. It is equal to

$$\left(1+O\left(\frac{k}{K}\right)\right)\frac{\chi(\rho)}{|\Lambda_K|^2}\sum_{x\in\mathbb{Z}^d}J(t,x)^2,$$

where $\chi(\rho)$ is the static compressibility given in our model by $\chi(\rho)=E_{\nu_\rho}[(\eta(0)-\rho)^2]$. Since J is the solution of (5.5), $J(t,x)=\sum_y K_t(x,y)J(y)$, where K_t is the kernel of the discrete heat equation. Therefore,

$$|\Lambda_K|^{-2}\sum_x J(t,x)^2=|\Lambda_K|^{-2}\sum_x\left[\sum_y K_t(x,y)J\left(\frac{y}{K}\right)\right]^2.$$

Since $|\Lambda_K|^{-1}\sum_x J(x/K) = 1$, this expression is bounded above by

$$(1 + A^{-1}) \sum_{x \in \mathbb{Z}^d} K_t(x,0)^2 + \frac{1+A}{|\Lambda_K|^2} \left\{ \sum_y [K_t(x,y) - K_t(x,0)] J\left(\frac{y}{K}\right) \right\}^2$$

for every $A > 0$. Since K_t is the kernel of the discrete heat equation, $|K_t(x,y) - K_t(x,0)|$ is bounded by $C(\rho,d)(|x|/t)|y|K_t(x,0)$ for some finite constant depending only on the dimension and on ρ. Moreover,

$$\sum_{x \in \mathbb{Z}^d} K_t(x,0)^2 = (8\pi\varphi'(\rho)t)^{-d/2}(1 + O(t^{-1})).$$

Therefore, minimizing in A, the previous expression is bounded above by

$$(8\pi\varphi'(\rho)t)^{-d/2}\left(1 + \frac{K}{\sqrt{t}}\right).$$

To conclude the proof of the lemma, it remains to recollect all the previous estimates.

6. Cutoff estimate. In this section we prove the cutoff estimate stated in Proposition 3.1. The proof will be developed in several lemmas which follow. First we need some notation. Recall that for each positive integer j, Λ_j is a cube of side length $2j + 1$ centered at the origin in $\mathbb{Z}^d$, $\mathscr{F}_j$ is the σ-algebra generated by the variables $\{\eta_x,\ x \in \Lambda_j\}$ and $A_j f = E_{\nu_\rho}[f \mid \mathscr{F}_j]$ is the conditional expectation given those variables. Here $\mathscr{L}$ is the generator of our process which can be written as

$$\mathscr{L} = \sum_b \mathscr{L}_b,$$

where the sum is over nearest neighbor bonds $b = (x,y)$ and

$$(\mathscr{L}_b u)(\eta) = g(\eta_x)[u(\sigma^{x,y}\eta) - u(\eta)] + g(\eta_y)[u(\sigma^{y,x}\eta) - u(\eta)].$$

Likewise, we can write the Dirichlet form $D(\nu_\rho, u)$ as the sum $\sum_b D_b(\nu_\rho, u)$ where $D_b(\nu_\rho, u)$ is the piece of the Dirichlet form corresponding to jumps over the bond b,

$$D_b(\nu_\rho, u) = \langle u, (-\mathscr{L}_b)u \rangle.$$

We shall say that a nearest neighbor bond $b = (x,y)$ belongs to a subset Λ of $\mathbb{Z}^d$ if both ends x and y belong to Λ and that it belongs to the boundary $\partial\Lambda$ of Λ if one and only one of the ends belongs to Λ. In this case, we always denote by x the end that belongs to Λ and by y the end that does not belong to Λ.

Note that if $u_t = P_t u$ is evolving by the dynamics, then for each $j \geq 1$,

$$\partial_t A_j u_t = \partial_t E_{\nu_\rho}[u_t \mid \mathscr{F}_j] = E_{\nu_\rho}[\partial_t u_t \mid \mathscr{F}_j] = E_{\nu_\rho}[\mathscr{L} u_t \mid \mathscr{F}_j]$$

and therefore,

$$\frac{d}{dt}\left\langle (A_j u_t)^2 \right\rangle = 2\langle A_j u_t, \mathscr{L} u_t \rangle.$$

The plan of attack is to control an appropriate combination of the $\langle (A_j u)^2 \rangle$. With this in mind, we first provide an estimate for this last term.

LEMMA 6.1. *There exists a finite constant $C(\rho)$ such that for all $\beta > 0$,*

$$\left\langle A_j u, (-\mathscr{L})u \right\rangle \le \sum_{b \in \Lambda_j} D_b(\nu_\rho, u) + \left(1 + \frac{\beta}{2}\right) \sum_{b \in \partial\Lambda_j} D_b(\nu_\rho, u) + \frac{C(\rho)}{\beta} \left[\left\langle \left(A_{j+1} u\right)^2 \right\rangle - \left\langle \left(A_j u\right)^2 \right\rangle \right].$$

PROOF. We can write $\langle A_j u, (-\mathscr{L})u \rangle$ as the sum of interior terms and boundary terms

$$\sum_{b \in \Lambda_j} \left\langle A_j u, (-\mathscr{L}_b)u \right\rangle + \sum_{b \in \partial\Lambda_j} \left\langle A_j u, (-\mathscr{L}_b)u \right\rangle. \tag{6.1}$$

Note that for interior bonds b the conditional expectation A_j commutes with $\mathscr{L}_b$ and so, by convexity, each interior term is controlled by the Dirichlet term,

$$\left\langle A_j u, (-\mathscr{L}_b)u \right\rangle \le D_b(\nu_\rho, u), \qquad b \in \Lambda_j.$$

Next we consider the case $b = (x, y) \in \partial\Lambda_j$, and now we have to face the fact that A_j and $\mathscr{L}_b$ do not commute. It is convenient to write the bilinear form corresponding to $\mathscr{L}_b$ as

$$\left\langle u, (-\mathscr{L}_b)v \right\rangle = \varphi \langle \nabla_b u, \nabla_b v \rangle,$$

where $\varphi = \varphi(\rho)$ and

$$(\nabla_b u)(\eta) = u(\eta + \mathfrak{d}_x) - u(\eta + \mathfrak{d}_y).$$

Here $\mathfrak{d}_x$ represents the configuration with one particle at x and none elsewhere, and addition is componentwise at each site. Since $y \notin \Lambda_j$,

$$(\nabla_b A_j u)(\eta) = A_j u(\eta + \mathfrak{d}_x) - A_j u(\eta).$$

By changing variables we can write $A_j u(\eta + \mathfrak{d}_x)$ as

$$\int \left\{ u(\eta \circ \zeta + \mathfrak{d}_x) - u(\eta \circ \zeta + \mathfrak{d}_y) \right\} \nu_\rho^{\mathbb{Z}^d \setminus \Lambda_j}(d\zeta) + \int u(\eta \circ \zeta) \tilde{V}(\zeta_y) \nu_\rho^{\mathbb{Z}^d \setminus \Lambda_j}(d\zeta),$$

where $(\eta \circ \zeta)_z$ is equal to η_z for $z \in \Lambda_j$ and ζ_z for $z \notin \Lambda_j$. The Jacobian $\tilde{V}$ is given by $\tilde{V}(k) = g(k)/\varphi$ for each $k \ge 0$. Notice that for each site $y \notin \Lambda_j$, $\tilde{V}(\zeta_y)$ is a $\nu_\rho^{\mathbb{Z}^d \setminus \Lambda_j}$ mean one cylinder function. Define $V_y(\eta) = \tilde{V}(\eta_y) - 1$. We obtained that

$$\nabla_b A_j u = A_j \nabla_b u + A_j(V_y u),$$

that is, that the commutator $[\nabla_b, A_j]$ of ∇_b and A_j is equal to $A_j(V_y \cdot)$. In particular, the second term of (6.1) can be written as

$$\varphi \sum_{b \in \partial\Lambda_j} \left\{ \langle \nabla_b u, A_j \nabla_b u \rangle + \left\langle \nabla_b u, A_j(u V_y) \right\rangle \right\}.$$

By the Schwarz inequality and the convexity of the Dirichlet form, this expression is bounded above by

$$\left(1+\frac{\beta}{2}\right)\sum_{b\in\partial\Lambda_j} D_b(\nu_\rho,u)+\frac{\varphi}{2\beta}\sum_{b\in\partial\Lambda_j}\left\langle\left\{A_j(uV_y)\right\}^2\right\rangle \tag{6.2}$$

for each $\beta>0$.

Consider a collection $\{\psi_i,\ 1\le i\le m\}$ of orthogonal vectors in a Hilbert space $\mathscr{H}$ with inner product denoted by $\langle\langle\cdot,\cdot\rangle\rangle$. It is easy to see that for every ψ in $\mathscr{H}$,

$$\sum_{i=1}^m\left(\langle\langle\psi,\psi_i\rangle\rangle\right)^2\le\max_{1\le i\le m}\langle\langle\psi_i,\psi_i\rangle\rangle\langle\langle\psi,\psi\rangle\rangle. \tag{6.3}$$

Since for each bond b in $\partial\Lambda_j$, V_y is $\mathscr{F}_{j+1}$-measurable and has mean zero with respect to $E_{\nu_\rho}[\cdot|\mathscr{F}_j]$, we have that

$$\begin{aligned}A_j(uV_y)&=E_{\nu_\rho}\left[uV_y\mid\mathscr{F}_j\right]=E_{\nu_\rho}\left[(A_{j+1}u)V_y\mid\mathscr{F}_j\right]\\&=E_{\nu_\rho}\left[(A_{j+1}u-A_ju)V_y\mid\mathscr{F}_j\right]=A_j\left(\{A_{j+1}u-A_ju\}V_y\right).\end{aligned}$$

Since the functions $\{V_y,\ b\in\partial\Lambda_j\}$ are orthogonal with respect to $E_{\nu_\rho}[\cdot\mid\mathscr{F}_j]$, by inequality (6.3) the second term of (6.2) is bounded above by

$$\frac{\varphi}{2\beta}\max_{b\in\partial\Lambda_j}\langle V_y^2\rangle\left\langle(A_{j+1}u-A_ju)^2\right\rangle=\frac{C(\rho)}{\beta}\left\{\left\langle(A_{j+1}u)^2\right\rangle-\left\langle(A_ju)^2\right\rangle\right\}$$

for some finite constant $C(\rho)$ depending on ρ only. This completes the proof of the lemma. □

For positive integers $k<K$ and $\beta>0$, define $\mathfrak{A}=\mathfrak{A}_{k,K,\beta}$ on $L^2(\nu_\rho)$ by

$$\mathfrak{A}u=\alpha_{k+1}\left\langle(A_ku)^2\right\rangle+\sum_{j=k}^{K-1}\alpha_{j+1}\left\langle(A_{j+1}u-A_ju)^2\right\rangle+\alpha_{K+1}\left\langle(u-A_Ku)^2\right\rangle,$$

where

$$\alpha_j=\exp\{j/\beta\}.$$

LEMMA 6.2. *There exists a finite constant $C(\rho)$ such that for each k, K and β satisfying $\beta\ge 2$, for each $t\ge 0$,*

$$\mathfrak{A}_{k,K,\beta}u_t\le\exp\left\{\frac{C(\rho)t}{\beta^2}\right\}\mathfrak{A}_{k,K,\beta}u.$$

PROOF. Notice that $\mathfrak{A}u$ may be rewritten as

$$\mathfrak{A}u=\alpha_{K+1}\langle u^2\rangle-\sum_{j=k+1}^K(\alpha_{j+1}-\alpha_j)\left\langle(A_ju)^2\right\rangle.$$

In particular,

$$\frac{d}{dt}\mathfrak{A}u_t=-2\alpha_{K+1}D(\nu_\rho,u_t)-2\sum_{j=k+1}^K(\alpha_{j+1}-\alpha_j)\langle A_ju_t,\mathscr{L}u_t\rangle.$$

By the previous lemma and since $\beta \geq 2$, the right-hand side is bounded above by

$$
\begin{aligned}
&- 2\alpha_{K+1} D(\nu_\rho, u_t) + 2 \sum_{j=k+1}^{K} (\alpha_{j+1} - \alpha_j) \sum_{b \in \Lambda_j} D_b(\nu_\rho, u_t) \\
&+ 2\beta \sum_{j=k+1}^{K} (\alpha_{j+1} - \alpha_j) \sum_{b \in \partial\Lambda_j} D_b(\nu_\rho, u_t) \\
&+ \frac{C(\rho)}{\beta} \sum_{j=k+1}^{K} (\alpha_{j+1} - \alpha_j) \Big\{ \big\langle (A_{j+1} u_t)^2 \big\rangle - \big\langle (A_j u_t)^2 \big\rangle \Big\}.
\end{aligned} \tag{6.4}
$$

From the definition of the sequence α_j, we have that $\alpha_{j+1} \geq \beta(\alpha_{j+1} - \alpha_j)$. It follows from this inequality and a summation by parts that the Dirichlet part of the previous expression is negative. Applying the inequality for the α's again, the third line of (6.4) is bounded above by

$$
\frac{C(\rho)}{\beta^2} \sum_{j=k+1}^{K} \alpha_{j+1} \Big\{ \big\langle (A_{j+1} u_t)^2 \big\rangle - \big\langle (A_j u_t)^2 \big\rangle \Big\} \leq \frac{C(\rho)}{\beta^2} \mathfrak{I} u_t.
$$

The lemma follows by Gronwall's inequality. □

PROOF OF PROPOSITION 3.1. Fix a cylinder function u and $s \geq \max\{4, s_u^2\}$. Let $\beta = \sqrt{s}$, $k = [\sqrt{s}]$ and $K = [\gamma\sqrt{s} \log s]$ in Lemma 6.2. Since $s \geq \max\{4, s_u^2\}$, $\beta \geq 2$ and $\operatorname{supp} u \subset \Lambda_k$. It follows from this last property that $\mathfrak{I} u_0$ is equal to $\alpha_{k+1} \langle u^2 \rangle$. By definition of $\mathfrak{I} u_s$ and Lemma 6.2,

$$
\alpha_{K+1} \big\langle (A_K u_s - u_s)^2 \big\rangle \leq \mathfrak{I} u_s \leq \exp\left\{ \frac{C(\rho) s}{\beta^2} \right\} \mathfrak{I} u_0 = \exp\left\{ \frac{C(\rho) s}{\beta^2} \right\} \alpha_{k+1} \langle u^2 \rangle.
$$

Therefore, by our choice of β,

$$
\big\langle (A_K u_s - u_s)^2 \big\rangle \leq C(\rho) \frac{\alpha_{k+1}}{\alpha_{K+1}} \langle u^2 \rangle.
$$

To conclude the proof of the lemma, all that remains is to use the definition of α_j and recall that $K \leq L$. □

APPENDIX

A. Symmetric simple exclusion process. In this model particles are distributed on $\mathbb{Z}^d$ with at most one particle per site. Each particle performs a continuous time symmetric random walk with jump law $p(\cdot)$. However, jumps to already occupied sites are excluded. The state space is $\{0, 1\}^{\mathbb{Z}^d}$ and the generator is given by

$$
Lf(\eta) = \sum_{x, y} p(y - x) \big[\eta_x (1 - \eta_y) + \eta_y (1 - \eta_x) \big] \big(f(\sigma^{x, y} \eta) - f(\eta) \big),
$$

where $(\sigma^{x,y}\eta)_x = \eta_y$, $(\sigma^{x,y}\eta)_y = \eta_x$ and $(\sigma^{x,y}\eta)_z = \eta_z$ if $z \neq x, y$. Note that the term $[\eta_x(1-\eta_y) + \eta_y(1-\eta_x)]$ can be dropped from this expression without changing the meaning. The invariant measures are product measures μ_ρ, $0 \le \rho \le 1$ with marginals $\mu_\rho(\eta_x = 1) = \rho$ and the process is reversible (generator is symmetric) with respect to this family of measures. Fix $\rho \in [0,1]$ and let P_t denote the semigroup on $L^2(\mu_\rho)$ corresponding to the process. For $0 \le p \le \infty$ and $0 \le q < \infty$, define the following seminorms:

$$|||f|||_{p,q} = \left(\sum_{x \in \mathbb{Z}^d} \left\| \frac{\partial f}{\partial \eta_x} \right\|_p^q \right)^{1/q},$$

where $\|f\|_p$ is the standard L_p norm, $\|f\|_p = (E_{\mu_\rho}[|f|^p])^{1/p}$. In the present context,

$$\frac{\partial f}{\partial \eta_x} = f(\sigma^x \eta) - f(\eta),$$

where σ^x is given by $(\sigma^x\eta)_x = 1 - \eta_x$ and $(\sigma^x\eta)_y = \eta_y$, $y \neq x$.

THEOREM A.1. *For any $\rho \in [0,1]$ and $q \in [1,2)$, there exists a finite positive constant $C_q(d)$ such that for all functions f,*

$$\mathrm{Var}(\mu_\rho, [P_t f]) \le C_q(d)\, t^{-d(1/q - 1/2)}\, |||f|||_{2,q}^2 \quad \text{for all } t > 0. \tag{A.1}$$

REMARK. The theorem was proved previously by [3] using Nash type inequalities with the norm $|||\cdot|||_{\infty,q}$ on the right-hand side. This follows from the above since $|||f|||_{2,q} \le |||f|||_{\infty,q}$ always.

PROOF. Let f be a function on $\{0,1\}^{\mathbb{Z}^d}$, let $x_1, x_2, \ldots$ be an enumeration of $\mathbb{Z}^d$, let $\mathscr{F}_k$ be the σ-field generated by $\eta_{x_k}, \eta_{x_{k+1}}, \ldots$ and define $f_k = E_{\mu_\rho}[f \mid \mathscr{F}_k]$. Then

$$\mathrm{Var}(\mu_\rho, f) = E_{\mu_\rho}\left[\sum_{k=0}^{\infty} E_{\mu_\rho}[f_k^2 \mid \mathscr{F}_{k+1}] - \left(E_{\mu_\rho}[f_k \mid \mathscr{F}_{k+1}] \right)^2 \right],$$

so that for any such function we have

$$\mathrm{Var}(\mu_\rho, f) \le \sum_{x \in \mathbb{Z}^d} \left\| \frac{\partial f}{\partial \eta_x} \right\|_2^2 = |||f|||_{2,2}^2. \tag{A.2}$$

In fact, this is just a statement of the well-known spectral gap for the Glauber dynamics with respect to the product measure μ_ρ.

By definition,

$$\frac{\partial [P_t f]}{\partial \eta_x}(\eta) = E_{\sigma^x \eta}[f(\eta(t))] - E_\eta[f(\eta(t))],$$

where $E_\eta[\cdot]$ denotes the expectation with respect to the process starting at η. Now consider the following coupling (η, η') of the symmetric simple exclusion

process with itself. The η process starts at η and the η' process starts at $\sigma_x\eta$. The generator of the coupled process is

$$\mathbb{L}f(\eta,\eta') = \sum_{x,y} p(y-x)\big(f(\sigma^{x,y}\eta, \sigma^{x,y}\eta') - f(\eta,\eta')\big).$$

Let $\mathbb{E}_{(x,\eta)}$ denote expectation with respect to this coupled process. Note that in this coupling it is true that, for all times, η and η' differ at exactly one site, which we call $x(t)$, the position of the "second class particle." We have

$$E_{\sigma^x\eta}\big[f(\eta(t))\big] = \mathbb{E}_{(x,\eta)}\big[f(\sigma^{x(t)}(\eta(t)))\big],$$

so that for every f we have the following formula:

$$\frac{\partial[P_t f]}{\partial\eta_x} = \mathbb{E}_{(x,\eta)}\left[\frac{\partial f}{\partial\eta_{x(t)}}(\eta(t))\right]. \tag{A.3}$$

Note that we have reparametrized the coupled process as $(x(t),\eta(t))$ where $(\eta(t),\eta'(t)) = (\eta(t),\sigma^{x(t)}\eta(t))$. The generator of the (x,η) process is simply computed as $\tilde{\mathbb{L}} = L_1 + L_2$ where L_1 is the generator of the symmetric simple exclusion process with jumps to x disallowed,

$$L_1 f(x,\eta) = \sum_{y,z\neq x} p(y-z)\big(f(x,\sigma^{y,z}\eta) - f(x,\eta)\big)$$

and L_2 corresponds to jumps involving the second class particle at x,

$$L_2 f(x,\eta) = \sum_y p(y-x)\big(f(y,\sigma^{x,y}\eta) - f(x,\eta)\big).$$

Let

$$a(x,\eta,t) = \frac{\partial[P_t f]}{\partial\eta_x}(\eta).$$

Then, by (7.3) we have $a(x,\eta,t) = \mathbb{E}_{(x,\eta)}[a(x(t),\eta(t),0)]$ and therefore

$$\partial_t a(x,\eta,t) = L_1 a(x,\eta,t) + L_2 a(x,\eta,t).$$

Note that L_1 alone is symmetric with respect to the product measure μ_ρ. Note also that μ_ρ is invariant under the map $\sigma^{x,y}$ for any fixed x and y. Therefore, if we take the expectation over η with respect to the measure μ_ρ, we have

$$\begin{aligned}\partial_t \sum_x E\Big[\big(a(x,\eta)\big)^2\Big] = &-\sum_x \sum_{y,z\neq x} p(y-z)E\Big[\big(a(x,\sigma^{y,z}\eta) - a(x,\eta)\big)^2\Big] \\ &- \sum_{x,y} p(y-x)E\Big[\big(a(y,\sigma^{x,y}\eta) - a(x,\eta)\big)^2\Big].\end{aligned} \tag{A.4}$$

Here and below to keep notation simple we shall sometimes omit the time dependence of $a(x,\eta,t)$. By the triangle inequality,

$$\begin{aligned}\Big\{E\Big[\big(a(y,\sigma^{x,y}\eta) - a(x,\eta)\big)^2\Big]\Big\}^{1/2} \geq &\Big\{E\Big[\big(a(y,\sigma^{x,y}\eta)\big)^2\Big]\Big\}^{1/2} \\ &- \Big\{E\Big[\big(a(x,\eta)\big)^2\Big]\Big\}^{1/2}.\end{aligned}$$

Define

$$g_x(t) = \Big\{E\Big[\big(a(x,\eta,t)\big)^2\Big]\Big\}^{1/2} = \left\|\frac{\partial P_t f}{\partial \eta_x}\right\|_2.$$

Since our product measure is invariant under $\sigma^{x,y}$ we also have $\{E[(a(y,\sigma^{x,y}\eta))^2]\}^{1/2} = g_y$. We have thus shown that

$$\partial_t \sum_x \big(g_x(t)\big)^2 \le -\sum_{x,y} p(y-x)\big(g_y(t)-g_x(t)\big)^2,$$

which we can write as

$$\partial_t |||P_t f|||_{2,2}^2 \le -\mathbb{D}(P_t f), \tag{A.5}$$

where we have introduced the notation $\mathbb{D}(f) = \sum_{x,y} p(y-x)(g_y - g_x)^2$.

We now want to show that our semigroup is a contraction in $|||\cdot|||_{2,q}$ for each $q \in [1,2]$. Recall that $|||P_t f|||_{2,1} = \sum_x \{E[(a(x,\eta,t))^2]\}^{1/2}$. As earlier,

$$\begin{aligned}
&\partial_t \sum_x \Big\{E\Big[\big(a(x,\eta)\big)^2\Big]\Big\}^{1/2} \\
&\quad = -\sum_x \Big\{E\Big[\big(a(x,\eta)\big)^2\Big]\Big\}^{-1/2} \\
&\qquad \times \sum_{y,z\neq x} p(y-z)E\Big[\big(a(x,\sigma^{y,z}\eta)-a(x,\eta)\big)^2\Big] \qquad \text{(A.6)}\\
&\qquad + \Big\{E\Big[\big(a(x,\eta)\big)^2\Big]\Big\}^{-1/2} \\
&\qquad \times \sum_{x,y} p(y-x)E\big[a(x,\eta)\big(a(y,\sigma^{x,y}\eta)-a(x,\eta)\big)\big].
\end{aligned}$$

We drop the first term on the right-hand side, which is negative. By Schwarz's inequality, the last term on the right-hand side of (A.6) is dominated by

$$\sum_{x,y} p(y-x)\left[-\Big\{E\Big[\big(a(x,\eta)\big)^2\Big]\Big\}^{1/2} + \Big\{E\Big[\big(a(y,\sigma^{x,y}\eta)^2\Big]\Big\}^{1/2}\right].$$

After changing variables $\sigma^{x,y}\eta \mapsto \eta$, one can see that this expression vanishes identically. This proves the contraction for $q=1$. For $q=2$, the contraction follows from (A.5). By the standard interpolation theorem, it follows that for each $q \in [1,2]$,

$$|||P_t f|||_{2,q} \le |||f|||_{2,q}. \tag{A.7}$$

The well-known Nash inequality on $\mathbb{Z}^d$ states that there exists a finite, positive constant C_q depending only on q, the jump law $p(\cdot)$ and the dimension d such that for all $g\colon \mathbb{Z}^d \to \mathbb{R}$,

$$\sum_x g_x^2 \le C_q\left(\sum_{x,y} p(y-x)(g_y-g_x)^2\right)^{\gamma}\left(\sum_x g_x^q\right)^{2(1-\gamma)/q},$$

where

$$\gamma = \left(\frac{1}{q}-\frac{1}{2}\right)\left(\frac{1}{d}+\frac{1}{q}-\frac{1}{2}\right)^{-1}.$$

Translating into an expression for f, we obtain

$$\|\!|\!| f |\!|\!|_{2,2}^2 \le C_q \mathbb{D}(f)^{\gamma} \|\!|\!| f |\!|\!|_{q,2}^{2(1-\gamma)}. \tag{A.8}$$

Integrating (A.8) with (A.5), (A.7) and (A.2), we obtain the decay estimate.

REMARK 1. If one is willing to use the norm $\|\!|\!| \cdot |\!|\!|_{\infty,q}$ in the estimate (A.1), there is an even easier proof. Rewrite (A.3) as

$$\frac{\partial [P_t f]}{\partial \eta_x} = \sum_y \mathbb{E}_{(x,\eta)} \left[\frac{\partial f}{\partial \eta_y}(\eta(t)) \,\middle|\, x(t) = y \right] \mathbb{P}_{(x,\eta)}(x(t) = y).$$

For the symmetric simple exclusion the marginal distribution of the second-class particle is a simple random walk, so

$$\mathbb{P}_{(x,\eta)}(x(t) = y) = p_t(y - x),$$

where $p_t(y - x)$ is the solution of the (discrete) heat equation $\partial_t p = \Delta_d p$ with the discrete Laplacian $\Delta_d f(x) = \sum_y p(y - x)(f(y) - f(x))$ and $p_0(y) = 1_{\{y=0\}}$. By the previous formula we have

$$\left\| \frac{\partial [P_t f]}{\partial \eta_x} \right\|_{\infty} \le \sum_y \left\| \frac{\partial f}{\partial \eta_y} \right\|_{\infty} p_t(x, y).$$

The result follows immediately from standard estimates for the heat kernel on $\mathbb{Z}^d$, which state that for $q \in [1, 2)$ there exists a constant $C_q(d)$ so that for any function $a \in l^q(\mathbb{Z}^d)$,

$$\|a * p_t\|_{l^2}^2 \le t^{-d(1/q - 1/2)} \|a\|_{l^q}^2.$$

REMARK 2. The coupling method described above for the symmetric simple exclusion model can also be applied, for example, to the zero-range model, in the case that the model is attractive [$g(k)$ increasing in k in (1.1)]. However, the rates of the resulting second-class particle depend in a nontrivial way on the process and therefore the method described in this section does not seem to apply to this setting.

B. Nash inequality. We prove in this section a Nash inequality for conservative lattice gases. Consider the state space $\{0, 1\}^{\mathbb{Z}^d}$ and denote the configurations by $\eta = \{\eta_x,\ x \in \mathbb{Z}^d\}$.

Let $F(\eta)$ be a local function. Formally, the Hamiltonian H is given by

$$H(\eta) = \sum_{x \in \mathbb{Z}^d} F(\tau_x \eta)$$

and the Gibbs measure is a probability measure with density proportional to $\exp\{-\beta H(\eta)\}$. Here τ_x is the translation by x units: $(\tau_x \eta)_y = \eta_{x+y}$. To make the definition of the Gibbs measure rigorous, we need to introduce finite volume approximations or the DLR equations. Since this is well known and it does not affect our argument, we shall omit it.

Denote by b an unoriented bond $(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d$ with x and y two sites at distance 1. We have limited ourselves to nearest neighbor bonds mainly to simplify notation. As long as bonds with a fixed finite bound on length are used, the proofs will remain the same. Let η^b be the configuration obtained by interchanging the occupation variables η_x and η_y,

$$(\eta^b)_z = (\eta^{x,y})_z = \begin{cases} \eta_y, & \text{if } z = x, \\ \eta_x, & \text{if } z = y, \\ \eta_z, & \text{otherwise,} \end{cases}$$

and define T_b by

$$T_b f(\eta) = f(\eta^b) - f(\eta).$$

Let $\mathscr{L}_b$ be the symmetric generator defined by

$$-\int f \mathscr{L}_b g \, d\mu = \tfrac{1}{2} \int c_b(\eta) [T_b f(\eta)] [T_b g(\eta)] \, d\mu$$

and $\mathscr{L} = \sum_{b \in \Lambda} \mathscr{L}_b$. The rate $c_b(\eta)$ is assumed to be a local function that is translation invariant ($c_{\tau_x b}(\tau_x \eta) = c_b(\eta)$) and bounded away from zero and infinity: $0 < \delta < c_b(\eta) < \delta^{-1}$ for some δ. Explicitly, $\mathscr{L}_b$ is given by

$$\mathscr{L}_b f = A(b, \eta) T_b f(\eta),$$

where

$$A(b, \eta) = (1/2) c_b(\eta) \big(1 + \exp\{(T_b \log c_b)(\eta) - \beta (T_b H)(\eta)\}\big).$$

Recall the proof of the Nash inequality from the introduction. Define

$$\alpha_x = \left\{ E_\mu \left[\left(\frac{\partial f}{\partial \eta_x} \right)^2 \right] \right\}^{1/2}.$$

In this formula, $\partial f / \partial \eta_x = f(\eta^x) - f(\eta)$ where η^x is the configuration η with the occupation variable η_x flipped:

$$(\eta^x)_z = \begin{cases} 1 - \eta_x, & \text{if } z = x, \\ \eta_z, & \text{otherwise.} \end{cases}$$

The usual Nash inequality for the discrete Laplacian states that

$$\sum_{x \in \mathbb{Z}^d} \alpha_x^2 \le C \left\{ \sum_{x, y \in \mathbb{Z}^d, |x-y|=1} |\alpha_x - \alpha_y|^2 \right\}^{d/(d+2)} \left\{ \sum_{x \in \mathbb{Z}^d} \alpha_x \right\}^{4/(d+2)}$$

for some finite constant C. By the triangle inequality,

$$|\alpha_x - \alpha_y|^2 \le E_\mu \left[\left(\frac{\partial f}{\partial \eta_x} - \frac{\partial f}{\partial \eta_y} \right)^2 \right].$$

If the right-hand side could be estimated by the Dirichlet form $E_\mu[c_b (T_b f)^2]$, we would have that

$$\sum_{x \in \mathbb{Z}^d} \alpha_x^2 \le C D(f)^{d/(d+2)} \left[\sum_{x \in \mathbb{Z}^d} \alpha_x \right]^{4/(d+2)} \tag{B.1}$$

and a Nash inequality would follow from a spectral gap for the corresponding Glauber dynamics [1, 22, 14, 13]. The last estimate is, however, incorrect, even in the infinite temperature case, where all computations can be done explicitly. We need to prove instead the following estimate:

$$|\alpha_x - \alpha_y|^2 \le E_\mu\Big[(T_b f)^2\Big]. \tag{B.2}$$

Consider first the infinite temperature case $\beta = 0$. Here the measure μ is product, all one-dimensional marginals are equal and we only have to prove (B.2) for functions f that depend only on η_x and η_y. This is easy and we leave to the reader to check the correctness of the assertion.

For general lattice gases, the Gibbs measure is no longer product and (B.2) may fail. Instead, a simple computation shows that there is a finite constant γ such that

$$\alpha_x^2 \le \gamma\Big\{E_\mu\Big[(T_{x,y} f)^2\Big] + \alpha_y^2\Big\}. \tag{B.3}$$

Though inequality (B.2) may not hold, we shall prove that (B.1) always holds and thus a Nash inequality holds for the lattice gas dynamics. For simplicity, we take the cylinder function F in the definition of the Hamiltonian to be of the form $F(\eta) = \eta_0 \sum_{|x|=1} c_x \eta_x$ and we consider the one-dimensional case. Of course, this argument can be generalized to finite ranged interactions in higher dimension.

Let Γ_o be the odd sites. Conditioning on the odd sites, μ becomes a product measure on the even sites Γ_e with possibly different one-site marginals. Fix a cube Λ_L of length L centered at the origin.

The marginal measures at even sites conditioned on the configuration at the odd sites have only finite choices. For simplicity, assume that we have only two choices, denoted by ν_1 and ν_2. Denote by A, the even sites of Λ_L where the marginal distribution is ν_i, $i = 1, 2$, so that $\Gamma_o \cap \Lambda_L = A_1 \cup A_2$. Assume without loss of generality that

$$|A_1| \ge L^d/4$$

and define

$$a_x = \left\{E_\mu\left[\left(\frac{\partial f}{\partial \eta_x}\right)^2 \,\middle|\, \Gamma_o\right]\right\}^{1/2}.$$

From the spectral gap of the Bernoulli–Laplace model,

$$|A_1|^{-1} \sum_{x \in A_1} a_x^2 - \bar{a}_1^2 \le C|A_1|^{-2} \sum_{x,\, y \in A_1} (a_x - a_y)^2$$

for some finite constant C, provided $\bar{a}_1$ stands for the average of a_x in A_1,

$$\bar{a}_1 = |A_1|^{-1} \sum_{x \in A_1} a_x.$$

Since the marginal at x and y are identical for $x, y \in A_1$, we have from (B.2) for homogeneous product measures that

$$(a_x - a_y)^2 \le E_\mu\Big[(T_{x,y} f)^2 \,\Big|\, \Gamma_o\Big].$$

Therefore, since by assumption $|A_1| \geq L^d/4$,

$$\sum_{x \in A_1} a_x^2 - |A_1| a_1^2 \leq C|A_1|^{-1} \sum_{x,\, y \in A_1} E_\mu\Big[\big(T_{x,\,y} f\big)^2 \,\Big|\, \Gamma_o \Big] \leq 4CL^{-d} \sum_{x,\, y \in \Lambda_L} E_\mu\Big[\big(T_{x,\,y} f\big)^2 \,\Big|\, \Gamma_o \Big].$$

From our choice of A_1, we also have that

$$|A_1| E_\mu(\bar{a}_1)^2 = |A_1|^{-1} \left(E_\mu\left[\sum_{x \in A_1} a_x \right] \right)^2 \leq 4L^{-d} \left(E_\mu\left[\sum_{x \in \Lambda_L} a_x \right] \right)^2.$$

We now use a lemma proved in [21] (Lemma 2) and [23] (Lemma 6.1) stating that

$$L^{-d} \sum_{x,\, y \in \Lambda_L} E_\mu\Big[\big(T_{x,\,y} f\big)^2 \Big] \leq CL^2 \sum_{b \in \Lambda_L} E_\mu\Big[\big(T_b f\big)^2 \Big] \tag{B.4}$$

for some finite constant C. Thus

$$E_\mu\left[\sum_{x \in A_1} a_x^2 \right] \leq CL^2 \sum_{b \in \Lambda_L} E_\mu\Big[\big(T_b f\big)^2 \Big] + L^{-d} \left[E_\mu \sum_{x \in \Lambda_L} a_x \right]^2. \tag{B.5}$$

From (B.3), there is a constant γ such that

$$E_\mu\big[a_x^2\big] \leq \gamma \Big\{ E_\mu\Big[\big(T_{x,\,y} f\big)^2 \Big] + E_\mu\big[a_y^2\big] \Big\}$$

for any two sites $x, y \in \Gamma_e$. Applying this inequality for $x \in A_2$ and $y \in A_1$, summing over $x \in A_2$ and averaging over $y \in A_2$, we have

$$E_\mu\left[\sum_{x \in A_2} a_x^2 \right] \leq C_1 L^2 \sum_{b \in \Lambda} E_\mu\Big[\big(T_b f\big)^2 \Big] + C_2 E_\mu\left[\sum_{y \in A_1} a_y^2 \right]$$

for some finite constants C_1, C_2. Here we used that $|A_1| \geq L^d/4$ and we applied the estimate (B.4).

Multiplying the inequality (B.5) by an appropriate constant and adding that inequality with the one just obtained, we see that the restriction $x \in A_1$ on the left-hand side of (B.5) can be replaced by $x \in \Lambda_L \cap \Gamma_e$. For the same reason, the restriction Γ_e can be dropped. We have thus proved that

$$E_\mu\left[\sum_{x \in \Lambda_L} \left(\frac{\partial f}{\partial \eta_x} \right)^2 \right] \leq CL^2 \sum_{b \in \Lambda_L} E_\mu\Big[\big(T_b f\big)^2 \Big] + CL^{-d} \left\{ \sum_{x \in \Lambda_L} \left[E_\mu\left(\frac{\partial f}{\partial \eta_x} \right)^2 \right]^{1/2} \right\}^2.$$

We can divide $\mathbb{Z}^d$ into cubes of size L and index them by κ. For each cube κ, we have the previous estimate. Hence we can sum over κ to have

$$E_\mu\left[\sum_{x \in \mathbb{Z}^d} \left(\frac{\partial f}{\partial \eta_x} \right)^2 \right] \leq CL^2 \sum_{b \in \mathbb{Z}^d} E_\mu\Big[\big(T_b f\big)^2 \Big] + CL^{-d} \sum_{\kappa} \left\{ \sum_{x \in \Lambda_L(\kappa)} \left[E_\mu\left(\frac{\partial f}{\partial \eta_x} \right)^2 \right]^{1/2} \right\}^2.$$

By the inequality $\Sigma_o u_\alpha^2 \le (\Sigma_\alpha u_\alpha)^2$, we have

$$E_\mu\left[\sum_{x\in\mathbb{Z}^d}\left(\frac{\partial f}{\partial \eta_x}\right)^2\right] \le CL^2 \sum_{b\in\mathbb{Z}^d} E_\mu\left[(T_b f)^2\right] + CL^{-d}\left\{\sum_{x\in\mathbb{Z}^d}\left[E_\mu\left(\frac{\partial f}{\partial \eta_x}\right)^2\right]^{1/2}\right\}^2.$$

Optimizing L, we obtained (B.1). Hence a Nash inequality follows from combining with the spectral gap of the Glauber dynamics, as discussed previously in this section.

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