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A STRONG LAW FOR THE LONGEST EDGE OF THE MINIMAL SPANNING TREE

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Suppose $X_1, X_2, X_3, \ldots$ are independent random points in $\mathbf{R}^d$, $d \geq 2$, with common density f, having connected compact support Ω with smooth boundary $\partial\Omega$, with $f|_\Omega$ continuous. Let M_n denote the smallest r such that the union of balls of diameter r centered at the first n points is connected. Let θ denote the volume of the unit ball. Then as $n \to \infty$,

$$n\theta M_n^d/\log n \to \max\left(\left(\min_{\Omega} f\right)^{-1}, 2(1-1/d)\left(\min_{\partial\Omega} f\right)^{-1}\right), \quad \text{a.s.}$$

1. Introduction. Suppose S is a finite set in a metric space. For any $r > 0$, let $G(S; r)$ be the graph obtained by taking S as the vertex set and including edges between each pair x, y of distinct points in S with $\mathrm{dist}(x, y) \leq r$. We are interested here in the quantity $M(S)$, defined to be the minimum r such that $G(S; r)$ is connected. Following Appel and Russo [1], we call $M(S)$ the *connectivity distance* for S. In particular, we are interested in this quantity when S is a random set obtained by taking independent identically distributed random elements in $\mathbf{R}^d$.

A related concept is that of a *minimal spanning tree* on S, that is, a connected graph with vertex set S and minimal total edge length. If τ is a minimal spanning tree on S, then the connectivity distance equals the *maximum* edge length of τ. This is not hard to see; for any spanning tree on S with maximum edge length greater than the connectivity distance $M(S)$, a spanning tree with a smaller total length can be found by removing the longest edge and replacing it by an edge with length at most $M(S)$. In view of this relationship, the connectivity distance is the subject of papers by the author [7] and by Tabakis [10], as well as in [1]. In those papers and the present one, the connectivity distance for random subsets of $\mathbf{R}^d$ is studied.

Let $d \geq 2$ be an integer. Let the metric on $\mathbf{R}^d$ be given by an arbitrary norm $\|\cdot\|$, and let the volume of the unit ball in that norm be denoted θ. Let $X_1, X_2, X_3, \ldots$ be independent d-dimensional random variables with common density function f, and let $\mathscr{X}_n$ be the point process $\{X_1, \ldots, X_n\}$. In this paper we study the random variable $M(\mathscr{X}_n)$; general motivation for doing so is given in [7]. Appel and Russo [1] obtained a strong law in the special case of uniformly distributed points on the unit cube, with the L^∞ norm. Weak convergence results for $M(\mathscr{X}_n)$ were obtained for f normal or uniform on a

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cube in [6, 7, 8]. Here we obtain a strong law for $M(\mathscr{X}_n)$ for a general class of functions f.

Let Ω denote the (closed) support of f, with boundary denoted $\partial\Omega$. We assume throughout that Ω is bounded and connected in $\mathbf{R}^d$ and $\partial\Omega$ is a $(d-1)$-dimensional C^2 submanifold of $\mathbf{R}^d$. We assume also that the discontinuity set of the restriction of f to Ω is Lebesgue null and contains no element of $\partial\Omega$. Set

$$f_0 := \operatorname*{ess\,inf}_{\Omega} f \quad \text{and} \quad f_1 := \inf_{\partial\Omega} f$$

and assume that $f_0 > 0$. The main result of this paper goes as follows.

THEOREM 1.1. *Under the assumptions above on f, with probability* 1,

$$\lim_{n\to\infty} \left(n\theta(M(\mathscr{X}_n))^d / \log n \right) = \max(1/f_0, 2(d-1)/(df_1)). \tag{1.1}$$

The specific motivation is a conjecture in the statistical literature [10]. Suppose $Y_1, Y_2, Y_3, \dots$ are independent d-dimensional variables with common unknown density g, and for $\delta > 0$ set $A_\delta := g^{-1}((\delta,\infty])$ and $q_\delta := \int_{A_\delta} g(x)\,dx$. Tabakis proposed to look at $M(\mathscr{Y}_n \cap A_\delta^n)$, where $\mathscr{Y}_n = \{Y_1,\dots,Y_n\}$ and A_δ^n is some nonparametric estimate for A_δ. The purpose, among other things, was to test for unimodality. As an approximation to $M(\mathscr{Y}_n \cap A_\delta^n)$, consider $M(\mathscr{Y}_n \cap A_\delta)$. The point process $\mathscr{Y}_n \cap A_\delta$ is a sample of asymptotic size nq_δ from the density $f_\delta(\cdot) := g(\cdot) I_{A_\delta}(\cdot)/q_\delta$. Typically, the support of this density will be compact, and under a null hypothesis of unimodality, the minimum of f_δ over A_δ will be achieved on the boundary. Then, assuming sufficient smoothness of the boundary, Theorem 1.1 gives us

$$\lim_{n\to\infty} nq_\delta (M(\mathscr{Y}_n \cap A_\delta))^d / \log n = 2(d-1)/(d\theta\delta) \quad \text{a.s.}$$

Tabakis conjectured that $\delta(n/\log n)(M(\mathscr{Y}_n \cap A_\delta))^d$ would converge to a constant independent of f or δ; the above shows that in fact the limit depends on these via q_δ, which can be estimated by $n^{-1}\operatorname{card}(\mathscr{Y}_n \cap A_\delta^n)$.

A lower bound for $M(\mathscr{X}_n)$ is given by the *largest nearest neighbour link* for $\mathscr{X}_n$. This is a number $M_1(\mathscr{X}_n)$ defined by

$$M_1(\mathscr{X}_n) = \max_{x\in\mathscr{X}_n} \left(\min_{y\in\mathscr{X}_n\setminus\{x\}} \operatorname{dist}(x,y) \right).$$

In [9], the author proved that under the hypotheses of Theorem 1.1,

$$\lim_{n\to\infty} \left(n\theta(M_1(\mathscr{X}_n))^d / \log n \right) = \max(1/f_0, 2(d-1)/(df_1)) \quad \text{a.s.} \tag{1.2}$$

Since clearly $M(\mathscr{X}_n) \ge M_1(\mathscr{X}_n)$, it follows from (1.2) that

$$\liminf_{n\to\infty} \left(n\theta(M(\mathscr{X}_n))^d / \log n \right) \ge \max(1/f_0, 2(d-1)/(df_1)) \quad \text{a.s.} \tag{1.3}$$

So the proof of Theorem 1.1 amounts to proving an inequality the other way for the limit superior. This is carried out in Sections 2 and 3. Note that if Ω were *not* connected, then (1.2) would still hold but (1.1) would not.

In [9], certain extensions to (1.2) are described. For example, (1.2) also holds with $M_1(X_n)$ replaced by the smallest k-nearest neighbor distance. Also, (1.2) holds when $f_0 = 0$, and a formula is given in [9] for the case where Ω is a product of intervals. Analogous results should hold for the connectivity distance in each case, although we have not written out the details. For example, if $M^{(k)}(\mathscr{X}_n)$ denotes the smallest r for which $G(\mathscr{X}_n; r)$ is k-connected, it should be possible to combine the result on k-nearest neighbor distances in [9] with the methods of this paper and of [8] to show that the asymptotics for $M^{(k)}(\mathscr{X}_n)$ are the same as for $M(\mathscr{X}_n)$.

2. Notation and geometrical results. For $x = (x_1, \dots, x_d) \in \mathbf{R}^d$ and $a > 0$, let $B(x; a)$ denote the ball $\{y \in \mathbf{R}^d: \|y - x\| \le a\}$, and let $C(x; a)$ denote the hypercube of side a centered at x:

$$C(x; a) = \prod_{i=1}^{d} [x_i - a/2, x_i + a/2).$$

Let c_1 denote the diameter, in terms of the norm $\|\cdot\|$, of the unit cube,

$$c_1 = \sup\{\|x - y\|: x, y \in C(0; 1)\}. \tag{2.1}$$

For any $S \subset \mathbf{R}^d$ we define its diameter in L^∞ terms by

$$\operatorname{diam}(S) = \inf\{a > 0: \exists\, x \in \mathbf{R}^d \text{ with } S \subset C(x; a)\}.$$

Write xy for the usual L^2 dot product of vectors $x, y \in \mathbf{R}^d$, and write $|x|$ for $(xx)^{1/2}$, the Euclidean norm. Let $S^{d-1} = \{x \in \mathbf{R}^d: |x| = 1\}$. Given $x \in \mathbf{R}^d$, $r > 0$, $e \in S^{d-1}$ and $\eta > 0$, define $B^+(x; r, \eta, e)$ by top-slicing the ball $B(x; r)$ as follows:

$$B^+(x; r, \eta, e) := \{y \in B(x; r): (y - x)e > \eta r\}. \tag{2.2}$$

Also, set $B^-(x; r, \eta, e) = B^+(x; r, \eta, -e)$. See the more darkly shaded "caps" in Figure 1 for examples.

For $x \in \mathbf{R}^d$ and $e \in S^{d-1}$, let $D(x; r, e)$ denote the cylinder of L^2 height $2r$ and radius r, centered at x, pointing in the direction of e; that is, set

$$D(x; r, e) := \{x + y + \lambda e: ye = 0, -r < \lambda < r, |y| < r\}.$$

Also let $D_0(x; r, e)$ be the disk through the center of $D(x; r, e)$; that is,

$$D_0(x; r, e) := \{x + y: ye = 0, |y| < r\}.$$

For $\eta > 0$, define a cylinder $D^+(x; r, \eta, e)$ analogously to $B^+(x; r, \eta, e)$ by

$$D^+(x; r, \eta, e) := \{y \in D(x; r, e): (y - x)e > \eta r\}, \tag{2.3}$$

and set $D^-(x; r, \eta, e) := D^+(x; r, \eta, -e)$. Also, define the line $L(x; e)$ by $L(x; e) = \{x + \lambda e: \lambda \in \mathbf{R}\}$.

For $r > 0$ and $U \subset \mathbf{R}^d$, let the *r-covering number* of U, denoted $\kappa(U; r)$, be the minimum n such that there exists a collection of n balls of the form $B(x; r)$ with $x \in U$, whose union contains U.

The following geometrical results help us to deal with boundary effects. The first one also features in [9].

LEMMA 2.1. $\limsup_{r\downarrow 0} r^{d-1}\kappa(\partial\Omega; r) < \infty$.

PROOF. We give only a sketch. Clearly, if U is a bounded subset of a $(d-1)$-dimensional subspace of $\mathbf{R}^d$, then $\limsup_{r\downarrow 0} r^{d-1}\kappa(U; r) < \infty$. However, by assumption, $\partial\Omega$ is locally diffeomorphic to a $(d-1)$-dimensional subspace of $\mathbf{R}^d$, and by Taylor's theorem, the diffeomorphism maps small balls in $\mathbf{R}^d$ to sets of comparable diameter. The proof is achieved using these facts and a compactness argument. □

The next result is an interpretation of the "locally flat" nature of $\partial\Omega$.

LEMMA 2.2. *Suppose $x \in \partial\Omega$, and $\eta > 0$. Then there exist $e \in S^{d-1}$ and $\delta > 0$, such that*

$$B^{+}(y; r, \eta, e) \subset \Omega \qquad \forall\, y \in B(x;\delta) \cap \Omega\ \forall\, r \in (0,\delta) \tag{2.4}$$

and

$$B^{-}(y; r, \eta, e) \subset \Omega^{c} \qquad \forall\, y \in B(x;\delta) \cap \partial\Omega\ \forall\, r \in (0,\delta). \tag{2.5}$$

PROOF. We give a sketch; see also the proof of Lemma 2.3 of [9], which is similar but not the same. There exists a neighborhood U of x and a C^2 diffeomorphism ϕ: $U \to \phi(U) \subset \mathbf{R}^d$, such that if π_d denotes projection onto the dth coordinate,

$$U \cap \Omega = (\pi_d \circ \phi)^{-1}([0,\infty)) \quad \text{and} \quad U \cap \Omega^{c} = (\pi_d \circ \phi)^{-1}((-\infty, 0)). \tag{2.6}$$

The derivative $\phi'(x)$ is a linear isomorphism on $\mathbf{R}^d$, and the composition $\pi_d \circ \phi'(x)$ is the L^2 inner product with some vector, denoted be with $b > 0$ and e an L^2-unit vector. For y close to x, the linear map $\phi'(y)$ is close to $\phi'(x)$. By Taylor's theorem, there is a constant $M > 0$ such that for y close to x and z close to y,

$$|\phi(z) - \phi(y) - \phi'(x)(z-y)| \le M\big(\|z-y\|^2 + \|y-x\| \times \|z-y\|\big). \tag{2.7}$$

If $\max(\|y-x\|, \|z-y\|) \le b\eta/(2M)$, the right-hand side of (2.7) is at most $b\eta\|z-y\|$.

Suppose y is close to x and z is close to y as above and also suppose $y \in \Omega$ so that $\pi_d \circ \phi(y) \ge 0$ by (2.6). If also $\pi_d \circ \phi'(x)(z-y) \ge \eta b\|z-y\|$, then $\pi_d \circ \phi(z) \ge 0$ by (2.7), and hence $z \in \Omega$ by (2.6). Hence, if $(z-y)e \ge \eta\|z-y\|$ then $z \in \Omega$, and (2.4) follows. The proof of (2.5) is similar. □

It is straightforward to recast Lemma 2.2 in terms of cylinders as follows.

COROLLARY 2.1. *Suppose $x \in \partial\Omega$, and $\eta > 0$. Then there exists $e \in S^{d-1}$ and $\delta > 0$, such that*

$$D^{+}(y; r, \eta, e) \subset \Omega \qquad \forall\, y \in D(x;\delta) \cap \Omega\ \forall\, r \in (0,\delta) \tag{2.8}$$

and

$$D^{-}(y; r, \eta, e) \subset \Omega^{c} \qquad \forall\, y \in D(x;\delta) \cap \partial\Omega\ \forall\, r \in (0,\delta). \tag{2.9}$$

PROPOSITION 2.1. *There exists a constant $\delta_1 > 0$, and a finite collection of pairs $\{(\xi_i, e_i), i = 1, 2, \dots, \mu\}$ with $\xi_i \in \partial\Omega$ and $e_i \in S^{d-1}$, such that*

$$\partial\Omega \subset \bigcup_{i=1}^{\mu} D(\xi_i; \delta_1, e_i),$$

and for $1 \le i \le \mu$,

$$D^+(y; r, 0.1, e_i) \subset \Omega \qquad \forall\, y \in D(\xi_i; 10\delta_1, e_i) \cap \Omega\ \forall\, r \in (0, 10\delta_1) \tag{2.10}$$

and

$$D^-(y; r, 0.1, e_i) \subset \Omega^c \qquad \forall\, y \in D(\xi_i; 10\delta_1, e_i) \cap \partial\Omega\ \forall\, r \in (0, 10\delta_1). \tag{2.11}$$

Moreover, if $x \in \mathbf{R}^d$ with $L(x; e_i) \cap D(\xi_i; 10\delta_1, e_i) \neq \emptyset$, there is a unique point denoted $\psi_i(x)$ of the line $L(x; e_i)$ which is in $D(\xi_i; 10\delta_1, e_i) \cap \partial\Omega$.

Finally, there exists a constant $c_2 > 0$ such that for all $i \le \mu$, if $u, v \in D(\xi_i, 10\delta_1, e_i)$ with $|v - u| < 5\delta_1$, then $\|\psi_i(v) - \psi_i(u)\| \le c_2 \|v - u\|$.

PROOF. By Corollary 2.1, for all $x \in \partial\Omega$ there exists $e = e(x) \in S^{d-1}$ and $\delta = \delta(x) > 0$, such that (2.8) and (2.9) hold with $\eta = 0.1$. By compactness we can take a finite collection of points $x_j \in \partial\Omega$ such that the cylinders $D(x_j; \delta(x_j)/10, e(x_j))$ cover $\partial\Omega$. For each j write $\tilde{e}_j$ for the vector $e(x_j)$. Let δ_1 denote the minimum of the numbers $\delta(x_j)/12$.

Again by compactness, for each j there is a finite collection of points $\zeta_{jk} \in \partial\Omega \cap D(x_j; \delta(x_j)/10, \tilde{e}_j)$ such that the cylinders $D(\zeta_{jk}; \delta_1, \tilde{e}_j)$ cover $D(x_j; \delta(x_j)/10, \tilde{e}_j) \cap \partial\Omega$. The $(\zeta_{jk}, \tilde{e}_j)$, relabelled as (ξ_i, e_i), are the pairs required, since if $y \in D(\zeta_{jk}; 10\delta_1, \tilde{e}_j)$, then $y \in D(x_j; (\frac{10}{12} + \frac{1}{10})\delta(x_j), \tilde{e}_j)$.

Let $i \le \mu$. If $y \in D(\xi_i; 10\delta_1, e_i) \cap \Omega$, then it follows from (2.10) that $y + \lambda e_i \in \Omega$ for all $\lambda \in (0, 10\delta_1)$. Hence, for $x \in D(\xi_i; 10\delta_1, e_i)$ there cannot be more than one point of $L(x; e_i)$ in $D(\xi_i; 10\delta_1, e_i) \cap \partial\Omega$. The existence of such a point follows from the fact that $D^+(\xi_i; 10\delta_1, 0.1, e_i) \subset \Omega$, but $D^-(\xi_i; 10\delta_1, 0.1, e_i) \subset \Omega^c$.

Finally, suppose $u, v \in D(\xi_i; 10\delta_1, e_i)$ with $|v - u| < 5\delta_1$. Then $v \in D(u; 2|v - u|, e_i)$, and since $D(\psi_i(u); 2|v - u|, e_i)$ contains points of the line $L(v; e_i)$ both in Ω and in Ω^c, it must also contain the point $\psi_i(v)$. Hence $|\psi_i(v) - \psi_i(u)| \le 4|v - u|$, and by the comparability of norms there exists c_2 such that $\|\psi_i(v) - \psi_i(u)\| \le c_2 \|v - u\|$. □

With δ_1 and the pairs (ξ_i, e_i) given by Proposition 2.1, define the "interior" set Ω^I by

$$\Omega^I = \Omega \setminus \bigcup_{i=1}^{\mu} D(\xi_i; \delta_1, e_i). \tag{2.12}$$

Then Ω^I is nonempty; in fact, for each $i \le \mu$, we assert that

$$D_0(\xi_i + 4\delta_1 e_i; \delta_1, e_i) \subset \Omega^I. \tag{2.13}$$

This is because any point in the disk $D_0(\xi_i + 4\delta_1 e_i; \delta_1, e_i)$ lies at an L^2 distance at least $3\delta_1$ from Ω^c, since $D^+(\xi_i; 10\delta_1, 0.1, e_i) \subset \Omega$, whereas for all

$j \le \mu$, any point in $D(\xi_j; \delta_1, e_j)$ lies an L^2 distance at most $2\delta_1$ from Ω^c, since $D^-(\xi_j, \delta_1, 0.1, e_j) \subset \Omega^c$.

Let $\mathscr{A}_m$ denote the class of sets Γ of the form $\Gamma = \bigcup_{i=1}^{j} \{C(2^{-m} z_i; 2^{-m})\}$, with $\{z_1, \dots, z_m\} \subset \mathbf{Z}^d$ and such that Γ has connected interior; that is, $\{z_1, \dots, z_m\}$ is a *lattice animal* in $\mathbf{Z}^d$; see [4]. For $\Gamma \in \mathscr{A}_m$ let $\overline{\Gamma}$ denote the closure of Γ, and let Γ^o be the interior of Γ. Let Ω^o be the interior of Ω.

Pick $x_0 \in \Omega^I$. For integer m, let Γ_m be the maximal element Γ of $\mathscr{A}_m$ (possibly the empty set) such that $x_0 \in \Gamma$ and $\overline{\Gamma} \subset \Omega^o$. Then $\Gamma_1 \subset \Gamma_2 \subset \Gamma_3 \subset \cdots$ (the inclusions are not strict), and we assert that the union of the sets Γ_m^o is Ω^o. This can be proved along the following lines. Since Ω is connected, so is Ω^o. Any open connected set in $\mathbf{R}^d$ is path connected, hence so is Ω^o. But any path in Ω^o with one end at x_0 is bounded away from $\partial\Omega$, so lies in the union of the sets Γ_m^o, $m \ge 1$.

Since Ω^I is a compact set contained in Ω^o, there exists m_1 with $\Omega^I \subset \Gamma_{m_1}^o$. The set Γ_{m_1} is a connected finite union of hypercubes with $\overline{\Gamma}_{m_1} \subset \Omega^o$. Also, we can and do take $m_3 > m_2 > m_1$ such that $\overline{\Gamma}_{m_1} \subset \Gamma_{m_2}^o$ and $\overline{\Gamma}_{m_2} \subset \Gamma_{m_3}^o$. We relabel these sets as follows for later reference:

$$\Omega_1 := \Gamma_{m_1}, \qquad \Omega_2 := \Gamma_{m_2}, \qquad \Omega_3 := \Gamma_{m_3} \tag{2.14}$$

3. Proof of Theorem 1.1. For each $n > 0$, define

$$\rho_n := (\log n/n)^{1/d}. \tag{3.1}$$

Recall that θ denotes the volume of the unit ball. For the duration of this section, we fix t satisfying

$$\theta t^d > \max(1/f_0, 2(d-1)/(df_1)). \tag{3.2}$$

We shall prove that with probability 1, $M(\mathscr{X}_n) \le t\rho_n$ for large enough n. Since t satisfying (3.2) is arbitrary, this, along with (1.3), will suffice to prove Theorem 1.1.

In this paper, by a *separating set* for $\mathscr{X}_n$, we mean a nonempty proper subset U of $\mathscr{X}_n$ such that U is the vertex set of a connected component of $G(\mathscr{X}_n; t\rho_n)$. Suppose $M(\mathscr{X}_n) > t\rho_n$; then there exist at least two disjoint separating sets for $\mathscr{X}_n$. In that case, given $K > 0$, either there is at least one separating set of diameter at most $K\rho_n$, or there are at least two separating sets of diameter greater than $K\rho_n$. So Theorem 1.1 follows from Propositions 3.1 and 3.2 below.

PROPOSITION 3.1. *Let $K > 0$. Let $E_n(K)$ be the event that there exists a separating set U for $\mathscr{X}_n$ with* $\mathrm{diam}(U) \le K\rho_n$. *Then with probability* 1, *events $E_n(K)$ occur for only finitely many n.*

PROPOSITION 3.2. *For $K > 0$, let $H_n(K)$ be the event that there exist disjoint separating sets U and V for $\mathscr{X}_n$ with* $\min(\mathrm{diam}(U), \mathrm{diam}(V)) > K\rho_n$. *Then there exists $K > 0$ such that with probability* 1, *the events $H_n(K)$ occur for only finitely many n.*

Before proving these we describe some more notation and a further geometric lemma. For $S \subset \mathscr{X}_n$ and $r > 0$, set $S_r := \bigcup_{x \in S} B(x; r)$, the r-neighborhood of S. For $A \subset \mathbf{R}^d$, write $\mathscr{X}_n[A]$ for the number of points of $\mathscr{X}_n$ in A. Note that a nonempty proper subset U of $\mathscr{X}_n$ is separating, if and only if $\mathscr{X}_n[U_{t\rho_n} \setminus U] = 0$ and $U_{t\rho_n/2}$ is connected.

With t fixed and satisfying (3.2), pick numbers $f_2 \in (0, f_1)$ and $q < r < s < t$ such that

$$\theta q^d > \max(1/f_0, 2(d-1)/(d f_2)). \tag{3.3}$$

With δ_1 given by Proposition 2.1 and with c_1 defined by (2.1), fix $\varepsilon > 0$ satisfying

$$\varepsilon < \min((t-s)/c_1, s/(2c_1), \delta_1/d) \tag{3.4}$$

and

$$\frac{(t/2) - d\varepsilon}{(t/2) + d\varepsilon} \geq 0.1 \tag{3.5}$$

and also, for all $x \in \mathbf{R}^d$ and $e \in S^{d-1}$, satisfying

$$\mathrm{Leb}(B^+(x; s, d\varepsilon/s, e)) \geq \theta r^d/2. \tag{3.6}$$

For integer $n > 0$, let $\mathscr{L}_n$ denote the $\varepsilon\rho_n$-lattice, that is, the set of points of the form $\varepsilon\rho_n z$ with $z \in \mathbf{Z}^d$, regarded as a subset of $\mathbf{R}^d$. For $z \in \mathscr{L}_n$, define the hypercube

$$C_n(z) = C(z; \varepsilon\rho_n).$$

The proof is based on discretization; instead of the precise configuration $\mathscr{X}_n$, one considers the set of $z \in \mathscr{L}_n$ for which $\mathscr{X}_n(C_n(z)) > 0$ and applies counting arguments to those possibilities for this set which are compatible with the existence of separating sets.

By (3.3), we can (and do) choose a positive integer α such that $\alpha(f_0\theta q^d - 1) > 1$, and

$$\beta := \alpha\left(\frac{f_2\theta q^d}{2} - \frac{d-1}{d}\right) > 1. \tag{3.7}$$

For $m = 1, 2, 3, \ldots,$ let $\nu(m) = m^\alpha$. The purpose of working with this subsequence is to get almost sure convergence; we shall prove that the probability of existence of a "small" separating set for $\mathscr{X}_n$ tends to zero sufficiently fast to ensure that by Borel–Cantelli, there are "small" separating sets for $\mathscr{X}_{\nu(m)}$ only finitely often, almost surely. The gaps between the numbers $\nu(m)$ will be filled in using the fact that $\rho_{\nu(m+1)}/\rho_{\nu(m)} \to 1$. This subsequence trick is adapted from [2].

We work toward a proof of Proposition 3.1. For $K > 0$, let $\mathscr{T}_m(K)$ be the collection of all subsets τ of $\mathscr{L}_{\nu(m)}$ such that for all $z \in \tau$, the cube $C_{\nu(m)}(z)$ has nonempty intersection with Ω, and such that $\mathrm{diam}(\tau) \leq (K + \varepsilon)\rho_{\nu(m)}$.

Given $\tau \in \mathscr{T}_m(K)$, define the "annulus-like" set $A_m(\tau)$ by

$$A_m(\tau) := \Big(\bigcup_{z \in \tau} B(z; s\rho_{\nu(m+1)}) \Big) \setminus \Big(\bigcup_{z \in \tau} C_{\nu(m)}(z) \Big).$$

See Figure 1 for an example with card$(\tau) = 4$ and $A_m(\tau)$ shaded (both kinds of shading). Define the event

$$F_m(\tau) := \big\{ \mathscr{X}_{\nu(m)}[A_m(\tau)] = 0 \big\}. \tag{3.8}$$

The purpose of these definitions is demonstrated by the next result.

LEMMA 3.1. *There exists m_0 such that if $m \geq m_0$ and $\nu(m) \leq n < \nu(m+1)$, then the event that there is a separating set U for $\mathscr{X}_n$ with* diam$(U) \leq K\rho_n$ *is contained in the union of the events $F_m(\tau)$, $\tau \in \mathscr{T}_m(K)$.*

PROOF. By (3.4), we can choose m_0 so that if $m \geq m_0$, then

$$c_1 \varepsilon \rho_{\nu(m)} + s\rho_{\nu(m+1)} \leq t\rho_{\nu(m+1)}. \tag{3.9}$$

Suppose $m \geq m_0$ and $\nu(m) \leq n < \nu(m+1)$. Given $U \subset \mathscr{X}_n$, let $\tau(U)$ denote the discretization of U in $\mathscr{L}_{\nu(m)}$, that is, the set of $z \in \mathscr{L}_{\nu(m)}$ with $U \cap C_{\nu(m)}(z) \neq \emptyset$. If diam$(U) \leq K\rho_n$, then diam$(\tau(U)) \leq (K + \varepsilon)\rho_{\nu(m)}$, so that $\tau(U) \in \mathscr{T}_m(K)$. If also U is a separating set for $\mathscr{X}_n$, then there are no points of $\mathscr{X}_n$ in $U_{t\rho_n} \setminus U$ and hence none in $U_{t\rho_{\nu(m+1)}} \setminus U$. But by (3.9) and the triangle inequality, $A_m(\tau(U)) \subset U_{t\rho_{\nu(m+1)}} \setminus U$. So if U is a separating set for $\mathscr{X}_n$, then $\mathscr{X}_n[A_m(\tau(U))] = 0$. $\square$

Let $\mathscr{T}_m^I(K)$ be the collection of all $\tau \in \mathscr{T}_m(K)$ such that $A_m(\tau) \subset \Omega$ (the I stands for "interior"), and let $\mathscr{T}_m^B(K) = \mathscr{T}_m(K) \setminus \mathscr{T}_m^I(K)$ (the B stands for "boundary"). Let $|\mathscr{T}_m^I(K)|$ and $|\mathscr{T}_m^B(K)|$ be the cardinalities of $\mathscr{T}_m^I(K)$ and $\mathscr{T}_m^B(K)$, respectively.

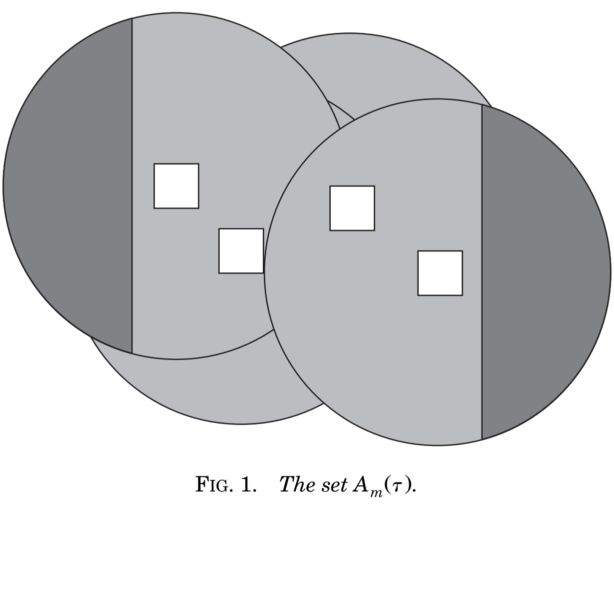

FIG. 1. *The set $A_m(\tau)$.*

LEMMA 3.2. *Let $K > 0$. There exists $c_3 > 0$ depending on K and ε, such that for all large enough m,*

$$|\mathscr{T}_m^I(K)| \le c_3\,\rho_{\nu(m)}^{-d} = c_3 m^\alpha/(\alpha \log m) \tag{3.10}$$

and

$$|\mathscr{T}_m^B(K)| \le c_3\,\rho_{\nu(m)}^{1-d} = c_3\big(m^\alpha/(\alpha \log m)\big)^{(d-1)/d}. \tag{3.11}$$

PROOF. Given $\tau \in \mathscr{T}_m(K)$, let $|\tau|$ denote its cardinality and list the elements of τ in order of increasing distance from $\partial\Omega$ as $z_1(\tau), z_2(\tau), \dots, z_{|\tau|}(\tau)$; in the case of ties, use the lexicographic ordering. If $\tau \in \mathscr{T}_m^I(K)$, then, since $\mathscr{L}_{\nu(m)}$ is a grid of side $\varepsilon\rho_{\nu(m)}$ and Ω is bounded, the number of possibilities for $z_1(\tau)$ is bounded by a constant times $\rho_{\nu(m)}^{-d}$. Also the set

$$\tau' := \Big\{\big(\varepsilon\rho_{\nu(m)}\big)^{-1}\big(z_i(\tau) - z_1(\tau)\big) : 1 \le i \le |\tau|\Big\}$$

is a subset of $\mathbf{Z}^d \cap C(0; K/\varepsilon + 1)$, and the number of such subsets is independent of m. Since $z_1(\tau)$ and τ' together determine τ, (3.10) follows.

If $\tau \in \mathscr{T}_m^B(K)$, then $\operatorname{dist}(z_1, \partial\Omega) \le s\rho_{\nu(m+1)}$. By Lemma 2.1, and the fact that $\rho_{\nu(m+1)}/\rho_{\nu(m)} \to 1$, the number of balls of radius $s\rho_{\nu(m+1)}$ centered at points of $\partial\Omega$, required to cover $\partial\Omega$, is $O(\rho_{\nu(m)}^{1-d})$. The number of points of $\mathscr{L}_{\nu(m)}$ lying in any ball of radius $2s\rho_{\nu(m+1)}$ is bounded by a constant independent of m, and it follows that for $\tau \in \mathscr{T}_m^B(K)$, the number of possibilities for $z_1(\tau)$ is $O(\rho_{\nu(m)}^{1-d})$. Using this fact, the proof of (3.11) is concluded in the same manner as that of (3.10). $\square$

LEMMA 3.3. *Define the event $G_m^I = \bigcup_{\tau \in \mathscr{T}_m^I(K)} F_m(\tau)$. With probability 1, G_m^I occurs for only finitely many m.*

PROOF. Let e_1 be the L^2-unit vector $(1, 0, \dots, 0)$. Let π_1 denote projection onto the first coordinate. Given $\tau \in \mathscr{T}_m^I(K)$, pick z^+ in τ with maximal first component, that is, with $\pi_1(z^+) \ge \pi_1(z)$ for all $z \in \tau$. Similarly, pick z^- in τ with minimal first component. Assume m is large enough so that $\rho_{\nu(m)} \le 2\rho_{\nu(m+1)}$. Then for $x \in C_{\nu(m)}(0)$, we have $|xe_1| \le \varepsilon\rho_{\nu(m+1)}$, so by (2.2), the sets $B^+(z^+; s\rho_{\nu(m+1)}, \varepsilon/s, e_1)$ and $B^-(z^-; s\rho_{\nu(m+1)}, \varepsilon/s, e_1)$, represented by the more darkly shaded "caps" in Figure 1, are both contained in $A_m(\tau)$. Hence,

$$\operatorname{Leb}(A_m(\tau)) \ge 2\rho_{\nu(m+1)}^d \operatorname{Leb}\big(B^+(0; s, \varepsilon/s, e_1)\big),$$

and by the condition (3.6) on ε and the definition (3.1) of ρ_n, for m large enough,

$$\operatorname{Leb}(A_m(\tau)) \ge \theta\big(r\rho_{\nu(m+1)}\big)^d \ge \theta q^d \alpha \log m/\nu(m).$$

If $\tau \in \mathscr{T}_m^I(K)$, then $A_m(\tau) \subset \Omega$. Hence, there exists m_4 such that if $m \ge m_4$ and $\tau \in \mathscr{T}_m^I(K)$, then by (3.8),

$$\begin{aligned} P[F_m(\tau)] &\le \big(1 - f_0\theta q^d \alpha \log m/\nu(m)\big)^{\nu(m)} \\ &\le \exp\big(-f_0\theta q^d\alpha \log m\big) = m^{-\alpha f_0 \theta q^d}. \end{aligned} \tag{3.12}$$

By subadditivity and (3.10), for large enough m,

$$P[G_m^I] \le |\mathscr{T}_m^I(K)| m^{-\alpha f_0 \theta q^d} \le c_3 m^{\alpha(1-f_0\theta q^d)}/(\alpha \log m),$$

which is summable in m by the choice of α [see just before (3.7)]. The result follows by the Borel–Cantelli lemma.

LEMMA 3.4. *Define the event* $G_m^B = \bigcup_{\tau \in \mathscr{T}_m^B(K)} F_m(\tau)$. *With probability* 1, G_m^B *occurs for only finitely many* m.

PROOF. Recall that $f_2 < f_1$. By a compactness argument, there exists $\gamma > 0$, such that for almost all $x \in \Omega$ with $B(x;\gamma) \cap \Omega^c \neq \emptyset$, we have $f(x) > f_2$. By Lemma 2.2 and another compactness argument, there exists a finite collection of triples (ζ_i, δ_i, e_i), $1 \le i \le \mu'$, with $\zeta_i \in \partial\Omega$, $\delta_i > 0$ and e_i a unit vector for each i, such that for all $x \in B(\zeta_i; \delta_i) \cap \Omega$ and $h < \delta_i$, we have $B^+(x; h, d\varepsilon/(8s), e_i) \subset \Omega$, and such that $\partial\Omega \subset \bigcup_{i=1}^{\mu'} B(\zeta_i; \delta_i/2)$.

Suppose $\tau \in \mathscr{T}_m^B(K)$. Then, provided m is large enough, $f(x) \ge f_2$ for $x \in A_m(\tau) \cap \Omega$, and also $A_m(\tau) \subset B(\zeta_i; \delta_i)$ for some $i = i(\tau) \le \mu'$, and also $\rho_{\nu(m)} < (3/2)\rho_{\nu(m+1)}$. For $z \in \tau$, pick $y \in C_{\nu(m)}(z) \cap \Omega$. Then by (2.1), $\|y - z\| \le c_1 \varepsilon \rho_{\nu(m)} \le (3/2) c_1 \varepsilon \rho_{\nu(m+1)}$ and since $2c_1\varepsilon < s$ by (3.4), $\|y - z\| \le s\rho_{\nu(m+1)}$, so that

$$B(z; s\rho_{\nu(m+1)}) \subset B(y; 2s\rho_{\nu(m+1)}). \tag{3.13}$$

Also $|y - z| \le (1/2) d\varepsilon\rho_{\nu(m)} \le (3/4) d\varepsilon \rho_{\nu(m+1)}$, so that if $(w - z)e_i \ge d\varepsilon\rho_{\nu(m+1)}$ then $(w - y)e_i \ge d\varepsilon\rho_{\nu(m+1)}/4$. Combining this observation with (3.13), we have

$$B^+(z; s\rho_{\nu(m+1)}, d\varepsilon/s, e_i) \subset B^+(y; 2s\rho_{\nu(m+1)}, d\varepsilon/(8s), e_i) \subset \Omega.$$

For $x \in C_{\nu(m)}(z)$ we have $|x - z| \le (d/2)\varepsilon\rho_{\nu(m)} \le d\varepsilon\rho_{\nu(m+1)}$. Hence, if $z \in \tau$ is chosen to maximize $e_i z$, then $B^+(z; s\rho_{\nu(m+1)}, d\varepsilon/s, e_i) \subset A_m(\tau)$; here one can envisage a diagram similar to Figure 1 with a single "cap" oriented in the direction of e_i. Hence for this choice of z,

$$B^+(z; s\rho_{\nu(m+1)}, d\varepsilon/s, e_i) \subset A_m(\tau) \cap \Omega.$$

By the condition (3.6) on ε and the definition (3.1) of ρ_n, it follows that

$$\mathrm{Leb}(A_m(\tau) \cap \Omega) \ge \theta(r\rho_{\nu(m+1)})^d/2 \ge \theta q^d \alpha \log m/(2\nu(m)).$$

Hence, by the inequality $1 - x \le e^{-x}$, $x > 0$,

$$P[F_m(\tau)] \le \left(1 - f_2\theta q^d \alpha \log m/(2\nu(m))\right)^{\nu(m)} \le m^{-\alpha f_2 \theta q^d/2}.$$

Thus, by (3.11) and (3.7),

$$P[G_m^B] \le |\mathscr{T}_m^B(K)| m^{-\alpha\theta f_2 q^d/2} \le c_3 m^{-\beta}/(\alpha \log m)^{(d-1)/d},$$

which is summable in m. The result follows by Borel–Cantelli. $\square$

The proof of Proposition 3.1 is immediate from Lemmas 3.1, 3.3 and 3.4.

It remains to prove Proposition 3.2, dealing with the possibility of "large" separating sets. The key observation is that if U is a separating set, then a region near the boundary of $U_{t\rho_n/2}$ is void of points of $\mathscr{X}_n$. We discretize this region into cubes of side $\varepsilon\rho_n$, and count the number of possibilities for the discretized region using a Peierls argument; see, for example, [4] for an introduction to Peierls arguments.

Matters are complicated by the fact that part of the discretized boundary region can lie outside Ω. However, we show that the proportion of the boundary region lying inside Ω is bounded away from zero, which is sufficient to get the Peierls argument to work.

The proof uses notation from Section 2. The constant δ_1 and the finite collection of pairs (ξ_i, e_i), $1 \le i \le \mu$ are given by Proposition 2.1. The sets Ω_1, Ω_2 and Ω_3 are as defined by (2.14). Recall that $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \Omega$, with the boundaries of these sets all being disjoint. Recall also that Ω_1 is nonempty and Ω_3 is a connected finite union of dyadic hypercubes with common side-length 2^{-m_3}; we write η_1 for 2^{-m_3}.

We say a set $\sigma \subset \mathscr{L}_n$ is $*$-connected if the union of the closures of the cubes $C_n(z)$, $z \in \sigma$, is connected. Given $\eta > 0$, let $\mathscr{C}_{n,i}(\eta)$ denote the collection of $*$-connected sets $\sigma \subset \mathscr{L}_n(\varepsilon)$ of cardinality i, such that at least ηi of the points z of σ satisfy $C_n(z) \subset \Omega$. The main step in proving Proposition 3.2 is the following topological lemma. In this, ε is the same constant chosen earlier to satisfy (3.4)–(3.6).

LEMMA 3.5. *There exists a constant $\eta_2 > 0$ and a positive integer n_1, such that for all $n \ge n_1$, if U and V are disjoint separating sets for $\mathscr{X}_n$, then there exists $\sigma \in \mathscr{C}_{n,i}(\eta_2)$ with $\mathscr{X}_n[\bigcup_{z\in\sigma} C_n(z)] = 0$, for some i with*

$$i\varepsilon\rho_n \ge \min\bigl(\mathrm{diam}(U), \mathrm{diam}(V), \eta_1/2\bigr).$$

Taking Lemma 3.5 as read for now, we complete the proof of Proposition 3.2. By a Peierls argument (see Klarner [5], Lemma 3), there are constants $\gamma = \gamma(d) > 0$ and $c > 0$, such that $\mathrm{card}(\mathscr{C}_{n,i}(\eta_2)) \le c(n/\log n)\exp(\gamma i)$ for all n. Choose K so that $K\eta_2 f_0 \varepsilon^{d-1} > 2$. If $H_n(K)$ occurs, there exist disjoint separating sets U, V for $\mathscr{X}_n$, with $\min(\mathrm{diam}(U), \mathrm{diam}(V)) \ge K\rho_n$. If also n is large enough so that $K\rho_n \le \eta_1/2$, and also $n \ge n_1$ with n_1 appearing in Lemma 3.5, then by that result there exists $\sigma \in \mathscr{C}_{n,i}(\eta_2)$ with $\mathscr{X}_n[\bigcup_{z\in\sigma} C_n(z)] = 0$, for some i with $i\varepsilon\rho_n \ge K\rho_n$. Hence for large n,

$$\begin{aligned} P[H_n(K)] &\le \sum_{i\ge K/\varepsilon} \sum_{\sigma\in\mathscr{C}_{n,i}(\eta_2)} P\left[\mathscr{X}_n\left[\bigcup_{z\in\sigma} C_n(z)\right] = 0\right] \\ &\le c\sum_{i\ge K/\varepsilon} (n/\log n)\exp(\gamma i)\bigl(1 - f_0\eta_2 i(\rho_n\varepsilon)^d\bigr)^n \\ &\le c(n/\log n)\sum_{i\ge K/\varepsilon} \exp\bigl(i(\gamma - \eta_2 f_0 \varepsilon^d \log n)\bigr) \\ &\le 2c(n/\log n)\bigl(n^{-\eta_2 f_0 \varepsilon^d} e^{\gamma}\bigr)^{K/\varepsilon}. \end{aligned} \tag{3.14}$$

By the choice of K above, this expression is summable in n, so the result follows by Borel–Cantelli. □

PROOF OF LEMMA 3.5. Suppose U and V are disjoint separating sets for $\mathscr{X}_n$. First consider the case with $U \cap \Omega_2 \neq \emptyset$ and $V \cap \Omega_2 \neq \emptyset$. The sets $U_{t\rho_n/2}$ and $V_{t\rho_n/2}$ are disjoint connected subsets of $\mathbf{R}^d$. So $\mathbf{R}^d \setminus U_{t\rho_n/2}$ has a connected component which contains $V_{t\rho_n/2}$; denote this component V', and let $U' := \mathbf{R}^d \setminus V'$. Then the closures of U' and V' are connected and their union is $\mathbf{R}^d$, so their intersection, a part of the boundary of $U_{t\rho_n/2}$ denoted ∂U, is connected by the unicoherence of $\mathbf{R}^d$; see [3]. Also, $U \subset U'$ and $V \subset V'$, so *any path from a point of U to a point of V must pass through ∂U*. We claim that

$$\operatorname{diam}(\partial U) \geq \min(\operatorname{diam}(U), \operatorname{diam}(V)). \tag{3.15}$$

To see this, assume the contrary. Then there would exist $x_0 \in \mathbf{R}^d$ and $b < \min(\operatorname{diam}(U), \operatorname{diam}(V))$, such that $\partial U \subset C(x_0; b)$. By the condition on b there would be points $X \in U$ and $Y \in V$, which were not in $C(x_0; b)$; it would then be possible to get from X to Y by a path avoiding the box $C(x_0; b)$, and hence avoiding ∂U, a contradiction.

Also, $\partial U \cap \Omega_2 \neq \emptyset$, since by assumption we can pick $\tilde{X} \in U \cap \Omega_2$ and $\tilde{Y} \in V \cap \Omega_2$ and take a path in Ω_2 from $\tilde{X}$ to $\tilde{Y}$. Pick $x_1 \in \partial U \cap \Omega_2$, and let $\partial_1 U$ denote the component including x_1 of $\partial U \cap C(x_1; \eta_1)$. Since ∂U is connected, if $\partial_1 U \subset C(x_1; \eta_3)$ for some $\eta_3 < \eta_1$, then $\partial_1 U = \partial U$ by an exercise in topology. Hence by (3.15),

$$\operatorname{diam}(\partial_1 U) \geq \min(\operatorname{diam}(U), \operatorname{diam}(V), \eta_1/2). \tag{3.16}$$

Let DU denote the set of $z \in \mathscr{L}_n$ such that $C_n(z)$ has nonempty intersection with $\partial_1 U$. Then DU is $*$-connected, and $\bigcup_{z \in DU} C_n(z) \subset \Omega_3 \subset \Omega$. Also, since $\operatorname{dist}(x, \mathscr{X}_n) = t\rho_n/2$ for each $x \in \partial U$, the condition $c_1 \varepsilon < t/2$ from (3.4) and the triangle inequality ensure that

$$\mathscr{X}_n[C_n(z)] = 0 \qquad \forall\, z \in DU. \tag{3.17}$$

Finally, $\varepsilon\rho_n \operatorname{card}(DU) \geq \operatorname{diam}(\partial_1 U)$, and by (3.16), the conclusion of the lemma follows for this case.

The other more complicated case to be considered is when $U \cap \Omega_2$ and $V \cap \Omega_2$ are not both nonempty. Assume without loss of generality that $U \cap \Omega_2 = \emptyset$. Let V' be the component of $\mathbf{R}^d \setminus U_{t\rho_n/2}$, which includes Ω_1. Let $U' := \mathbf{R}^d \setminus V'$. Then the closures of U' and V' are connected and their union is $\mathbf{R}^d$, so their intersection, a part of the boundary of $U_{t\rho_n/2}$ denoted ∂U, is connected by unicoherence. Let DU denote the set of $z \in \mathscr{L}_n$ such that $C_n(z)$ has nonempty intersection with ∂U. Then DU is $*$-connected, and since $\operatorname{diam}(\Omega_1) \geq \eta_1$, by a similar argument to that for (3.15),

$$\varepsilon\rho_n \operatorname{card}(DU) \geq \operatorname{diam}(\partial U) \geq \min(\operatorname{diam}(U), \eta_1).$$

Also, (3.17) holds for the same reasons as in the previous case. We shall show that the proportion of DU lying inside Ω is bounded away from zero.

For $z \in DU$ with $C_n(z) \cap \Omega^c \neq \emptyset$, we shall define $\phi(z) \in \mathscr{L}_n(\varepsilon)$ in such a way that $\phi(z) \in DU$ and $C_n(\phi(z)) \subset \Omega$. The general idea goes as follows: the reader should refer to Figure 2, in which the horizontal line represents part of the upper boundary of Ω and the shaded region represents $U_{t\rho_n/2}$. Given z

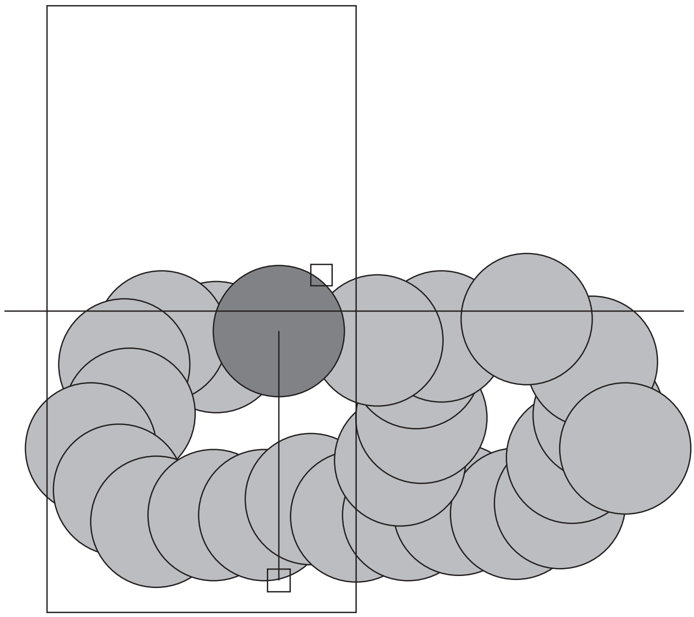

FIG. 2. *The set $U_{t\rho_n/2}$ and mapping ϕ.*

[the center of the higher small square, representing $C_n(z)$, in Figure 2], look for a nearby point X of U (the center of the more darkly shaded disk), which must be in Ω but near the boundary, and hence in one of the cylinders $D(\xi_i, \delta_1, e_i)$ defined in Proposition 2.1. This cylinder is represented by the large vertical rectangle in Figure 2. Move from X in the direction of e_i (that is, towards the interior of Ω), until the last exit within the cylinder from $U_{t\rho_n/2}$ (there is a last exit because U is assumed to lie entirely near the boundary of Ω). The nearest point of $\mathscr{L}_n$ to this exit point (the center of the lower small square in Figure 2) is $\phi(z)$.

Here is the formal definition of $\phi(z)$, given $z \in DU$ with $C_n(z) \cap \Omega^c \neq \emptyset$. Pick $y = y(z) \in C_n(z) \cap \partial U$. Then pick a point $X = X(z)$ of U with $\|X - y\| = t\rho_n/2$. If there are several possible choices for y or for X, make the choice according to some arbitrary prespecified deterministic ordering on $\mathbf{R}^d$, say the lexicographic ordering.

By the assumption on U, $X \notin \Omega_2$, so by the definition (2.12) of Ω^I and the fact that $\Omega^I b7 \subset \Omega_1 \subset \Omega_2$, X lies in a cylinder $D(\xi_i; \delta_1, e_i)$ for some $i \leq \mu$; let $i(z)$ be the smallest such i. Take $\lambda_1(z) \in (0, 5\delta_1]$ such that $X + \lambda_1(z)e_{i(z)}$ is in the disk $D_0(\xi_{i(z)} + 4\delta_1 e_{i(z)}; \delta_1, e_{i(z)})$. Then by (2.13), $X + \lambda_1(z)e_{i(z)} \in \Omega_I \subset U^c_{t\rho_n/2}$. Let

$$\lambda(z) = \max\left\{\lambda \in \left(0, \lambda_1(z)\right] : X(z) + \lambda e_{i(z)} \in U_{t\rho_n/2}\right\}$$

and let $w(z) = X(z) + \lambda(z)e_{i(z)}$. Let $\phi(z)$ be the point $u \in \mathscr{L}_n$ such that $w(z) \in C_n(u)$.

Clearly $w(z)$ lies on the boundary of $U_{t\rho_n/2}$, and we claim that additionally $w(z)$ is on the boundary of V'. Indeed, $w(z)$ is connected by a path in the complement of $U_{t\rho_n/2}$ to $X + \lambda_1(z)e_{i(z)}$, which lies in Ω^I and hence in Ω_1. Hence, $w(z) \in \partial U$ and $\phi(z) \in DU$.

We assert that $C_n(\phi(z)) \subset \Omega$. To prove this, let $x \in C_n(\phi(z))$ and set $i = i(z)$. Then $|x - w(z)| \le d\varepsilon\rho_n$, so

$$((t\rho_n/2)e_i + (x - w(z)))e_i \ge ((t/2) - d\varepsilon)\rho_n.$$

Setting $X' := X + (t\rho_n/2)e_i + (x - w(z))$, by the condition (3.5) on ε, we have $(X' - X)e_i \ge 0.1(t/2 + d\varepsilon)\rho_n$, so that by (2.3),

$$X' \in D^+(X; (d\varepsilon + t/2)\rho_n, 0.1, e_i),$$

and hence, provided n is big enough, $X' \in \Omega$ by (2.10). Also $\lambda(z) \ge t\rho_n/2$, so that $x = X' + (\lambda(z) - t\rho_n/2)e_i \in \Omega$ by the second part of Proposition 2.1, proving the assertion.

The mapping ϕ is many-to-one, but there is a uniform bound on the number of points z which ϕ can map to the same point u, as we now show. Fix $u \in \mathscr{L}_n$ and $i \le \mu$ and suppose $z \in DU$ satisfies $C_n(z) \cap \Omega^c \ne \varnothing$ and $\phi(z) = u$, and $i(z) = i$. Pick $v_1 \in \partial\Omega$ with $\|v_1 - X(z)\| \le (c_1\varepsilon + t/2)\rho_n$.

Let $v = w(z) + (v_1 - X(z))$. Then $\|v - u\| \le (2c_1\varepsilon + t/2)\rho_n$. Also $v_1 \in \partial\Omega$ and $v_1 - v$ is a multiple of e_i, so $v_1 = \psi_i(v)$ with ψ_i defined in Proposition 2.1. So by the last part of Proposition 2.1,

$$\|v_1 - \psi_i(u)\| \le c_2(2c_1\varepsilon + t/2)\rho_n,$$

and hence

$$\begin{aligned}\|z - \psi_i(u)\| &\le \|z - X(z)\| + \|X(z) - v_1\| + \|v_1 - \psi_i(u)\| \\ &\le 2(c_1\varepsilon + t/2)\rho_n + c_2(2c_1\varepsilon + t/2)\rho_n.\end{aligned}$$

The number of points $z \in \mathscr{L}_n$ satisfying this inequality is bounded by a constant, denoted c_4, independent of n or u; hence, the number of points z mapped by ϕ to u is bounded by $c_4\mu$, where μ is the number of cylinders in Proposition 2.1. Therefore, the proportion of points u of DU satisfying $C_n(u) \subset \Omega$ is at least η_2, where we set $\eta_2 = 1/(c_4\mu + 1)$. Thus DU is the required set σ. □

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