The Annals of Probability
2002, Vol. 30, No. 1, 270–292

GENERALIZED INTEGRATION AND STOCHASTIC ODEs[1]

BY FRANCO FLANDOLI AND FRANCESCO RUSSO

Università di Pisa and Université de Paris 13

Stochastic forward integrals for processes more general than semimartingales are shown to exist, generalized forms of Itô–Wentzell formula and covariation formula are proved, and one-dimensional stochastic equations driven by finite quadratic variation processes and semimartingales are solved. This generalized stochastic calculus is motivated by applications to uniqueness and dependence on parameters for stochastic equations with nonregular drift.

1. Introduction. The aim of this paper is to develop a generalized stochastic calculus for a class of finite quadratic variation processes. It is well known that the stochastic calculus for semimartingales cannot be extended to more general processes unless restrictions on the integrands are imposed; see [4]. Föllmer [7] defines integrals of the form $\int_0^t f(A_s)\,dA_s$ for finite quadratic variation processes (A_t) and C^1 functions $f\colon \mathbb{R} \to \mathbb{R}$. Other directions have been explored by Lyons and Zhang [12] (see also [13, 16]), Lyons [11], Bertoin [2], Zähle [27, 28], Russo and Vallois [20], Wolf [24, 25].

Here, given a finite quadratic variation processes (A_t), we introduce a class of processes $\mathcal{A}_2$ constructed from (A_t), such that the forward integral (see [20–22])

$$\int_0^t V_s\,d^-U_s \tag{1}$$

exists for all $U, V \in \mathcal{A}_2$, and belongs to $\mathcal{A}_2$. In fact we can take the integrand V in a larger class $\mathcal{A}_1 \supset \mathcal{A}_2$. The class $\mathcal{A}_2$ contains for instance the semimartingales and the C^2-functions of A (and $\mathcal{A}_1$ the C^1-functions of A). The set $\mathcal{A}_k$ will be defined as the set of all processes of the form

$$V_t = X_t(A_t)$$

where $(X_t(x),\ t \in [0,T],\ x \in \mathbb{R})$ is a C^k Itô field driven by some semimartingale $N = (N^1, \dots, N^m)$, such that (A, N) has all its mutual brackets (see more detailed definitions below). The procedure to define the forward integral (1) is a refinement of the method of Föllmer [7]. We make use of a Itô–Wentzell type formula (Section 3), and of various formulas for the brackets of certain processes

Received March 2000; revised March 2001.

[1]Supported in part by MURST.

AMS 2000 subject classifications. 60H05, 60H10

Key words and phrases. Forward stochastic integration, generalized Itô–Wentsell formula, finite quadratic variation process.

(Sections 2.3 and 4). These tools are extensions of previous results of Russo and Vallois [20–22].

When the basic stochastic calculus for processes of class $\mathcal{A}_2$ is developed, we solve non-linear SDEs of the form

$$X_t = x + \int_0^t \sigma(X_s)\, d^- A_s + \int_0^t b(s, X_s)\, dN_s. \tag{2}$$

We prove that, under proper conditions on the coefficients, this equation has a unique solution in $\mathcal{A}_2$.

Apart from the intrinsic interest in a generalization of the classical calculus for semimartingales, one of our main motivations was the development of a new approach, based on these tools to the analysis of the SDE

$$\begin{cases} dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t & \text{for } t \in [0, T], \\ X_0 = x, \end{cases} \tag{3}$$

with $b, \sigma\colon [0, T] \times \mathbb{R} \to \mathbb{R}$, and (W_t) a one-dimensional Brownian motion. We are interested in the case when b is not (locally) Lipschitz continuous in x. Of course on this subject there is a large literature, we recall in particular [1, 3, 5, 10, 15, 17, 19, 23, 26] or [29]. In the forthcoming paper [6], by means of the generalized stochastic calculus developed here, we show that one can perform computations on equation (3) and prove results of uniqueness and regular dependence on parameters, along lines close to the case of Lipschitz continuous drift. Roughly speaking, these results can be obtained by certain linearizations of equation (3), yielding linear equations like

$$V_t = 1 + \int_0^t V_s\, d^- A_s + \int_0^t \frac{\partial \sigma}{\partial x}(s, X_s)\, dW_s \tag{4}$$

where

$$A_t = \int_0^t \frac{\partial b}{\partial x}(s, X_s)\, ds.$$

In spite of the lack of regularity of b, it is possible to define properly this process A, which has finite quadratic variation but is not a semimartingale. Then one can define the generalized integral $\int_0^t V_s\, d^- A_s$ and study equation (4) in the class $\mathcal{A}_2$.

2. Forward, backward integrals and covariation.

2.1. *Preliminaries.* We first recall some basic concepts. For simplicity all the processes will be supposed to be continuous. Let $(X_t)_{t \geq 0}$, $(Y_t)_{t \geq 0}$, be real stochastic processes. We recall the definition of, respectively, the forward, backward integrals and brackets:

$$\int_0^\cdot Y\, d^- X = \lim_{\varepsilon \to 0^+} \int_0^\cdot Y_s \frac{X_{s+\varepsilon} - X_s}{\varepsilon}\, ds,$$

$$\int_0^{\cdot} Y d^+ X = \lim_{\varepsilon \to 0^+} \int_0^{\cdot} Y_s \frac{X_s - X_{(s-\varepsilon)\vee 0}}{\varepsilon} \, ds,$$

$$[X, Y] = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_0^{\cdot} (X_{s+\varepsilon} - X_s)(Y_{s+\varepsilon} - Y_s) \, ds,$$

provided the limits, taken in the uniform convergence in probability on each compact interval (ucp) sense, exist. We recall that a sequence X^n of real processes converge ucp to some process X if $\sup_{t \le T} |X_t^n - X_t|$ converges to zero in probability for all $T > 0$. We denote $[X, X]$ also by $[X]$. We recall some basic concepts from [20, 21].

If X is such that $[X, X]$ exists, it is called a finite quadratic variation process. We denote $[X, X]$ also by $[X]$. A vector process $(X^1, \dots, X^m)$ is said to have all its mutual brackets if $[X^i, X^j]$ exists for every $1 \le i \le j \le m$. In this case $[X^i, X^j]$ has bounded variation. We recall that, without the previous condition, a bracket $[X^1, X^2]$ may exist and not have bounded variation: if $f \in C^0$, B is a classical Brownian motion, $[f(B), B]$ is well defined but it does not have bounded variation. For this sake, we can consult for instance [8, 21] and [14] which develop Itô formula for functions $f \in C^1$, or less regular, of the Brownian motion or more general semimartingale.

REMARK 1.

$$[X, Y] = \int_0^{\cdot} Y \, d^+ X - \int_0^{\cdot} Y \, d^- X$$

provided that two of the three previous terms exist.

REMARK 2. If X is a finite quadratic variation process and A is such that $[A, A] = 0$, then the bracket $[X, A]$ exists and $[X, A] = 0$.

REMARK 3. Let $X = (X^1, \dots, X^d)$ be a vector of real processes having all its mutual brackets, $F, G \in C^1(\mathbb{R}^d)$. Then the bracket $[F(X), G(X)]$ exists and is given by

$$[F(X), G(X)] = \sum_{i,j=1}^{d} \int_0^{\cdot} \partial_i F(X) \partial_j G(X) \, d[X^i, X^j].$$

This includes the case $d = 2$, $F(x_1, x_2) = f(x_1)$, $G(x_1, x_2) = g(x_2)$, $f, g \in C^1(\mathbb{R})$.

REMARK 4 (Classical Itô formula). Let $X = (X^1, \dots, X^d)$ be a vector of real processes having all its mutual brackets such that X^i, $2 \le i \le d$, are either semimartingales, or finite quadratic variation processes such that the forward integrals $\int_0^t \partial_i F(X) \, d^- X^i$ exist (resp. bounded variation processes), X^1 is a finite quadratic variation process, $F \in C^2(\mathbb{R}^d)$ (resp. $F \in C^{2,1}(\mathbb{R} \times \mathbb{R}^{d-1})$). Then the forward integral $\int_0^t \partial_1 F(X)\, d^- X^1$ exists and we have

$$F(X_t) = F(X_0) + \sum_{i=2}^{d} \int_0^t \partial_i F(X)\, d^- X^i + \int_0^t \partial_1 F(X)\, d^- X^1 + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \partial_{i,j}^2 F(X)\, d[X^i, X^j] \tag{5}$$

(resp.

$$F(X_t) = F(X_0) + \sum_{i=2}^{d} \int_0^t \partial_i F(X)\, dX^i + \int_0^t \partial_1 F(X)\, d^- X^1 + \frac{1}{2} \int_0^t \partial_{1,1}^2 F(X)\, d[X^1, X^1] \Big). \tag{6}$$

A similar formula with the additional term $\int_0^t \frac{\partial F}{\partial t}(s, X_s)\, ds$ is true if F depends also on t, of class C^1.

2.2. *Itô–Dirichlet fields.* Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ be a filtration. Let $N = (N^1, \ldots, N^m)$ be a continuous $\mathcal{F}$-semimartingale (resp. $\mathcal{F}$-local martingale). We shall say that a random field $(X_t(x), t \in [0,T], x \in \mathbb{R}^d)$ is a C^0 *Itô field* (resp. a C^0 *Itô martingale field*) *driven by* N if it is a.s. continuous, there are $f\colon \mathbb{R}^d \times \Omega \to \mathbb{R}$, $a^i\colon [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ a.s. continuous, f being $\mathcal{F}_0$-measurable and a^i being $\mathcal{F}$-adapted for every x, such that

$$X_t(x) = f(x) + \sum_{i=1}^{m} \int_0^t a^i(s,x)\, dN_s^i. \tag{7}$$

Given a continuous $\mathcal{F}$-local martingale $N = (N^1, \ldots, N^m)$, a random field $(X_t(x), t \in [0,T], x \in \mathbb{R}^d)$ is said to be a C^0 *Itô–Dirichlet field driven by* N if it can be written as

$$X_t(x) = M_t(x) + Z_t(x)$$

where $(M_t(x))$ is a C^0 Itô martingale field driven by N and $(Z_t(x))$ is a *strict zero quadratic variation process* in the following sense:

$$\sup_{|x| \leq R} \int_0^T \big(Z_{t+\varepsilon}(x) - Z_t(x)\big)^2 \frac{dt}{\varepsilon} \to 0 \quad \text{ucp} \tag{8}$$

for all $R > 0$. We also assume that $(Z_t(x))$ is a.s. continuous in (t, x), and it is $\mathcal{F}$-adapted for every x.

EXAMPLE 5. If Z is a Hölder continuous process in $t \in [0,T]$ of parameter $\alpha > \frac{1}{2}$, uniformly with respect to x on compact sets, then (8) is verified.

EXAMPLE 6. If

$$Z_t(x) = \sum_{i=1}^{l} \int_0^t b^i(s,x)\, dV_s^i$$

where b^i are continuous fields and $(V_t^i)_{t\in[0,T]}$, $i = 1, \dots, l$, are bounded variation processes, then (8) is verified. In particular, a C^0 Itô field is also a C^0 Itô–Dirichlet field.

Finally, we shall be interested in more regular Itô fields. Given a positive integer k, a C^0 Itô field $(X_t(x),\ t \in [0,T],\ x \in \mathbb{R}^d)$ driven by N of the form (7) will be called a C^k *Itô field driven by* N if:

(i) $(X_t(x))$ is a.s. of class C^k in x, with derivatives in x up to order k a.s. continuous in (t,x);
(ii) a^i is a.s. of class C^k in x, with derivatives in x up to order k a.s. continuous in (t,x);
(iii) f is a.s. of class C^k;
(iv) for every multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$ with $|\alpha| \le k$ we have

$$\frac{\partial^\alpha X_t(x)}{\partial x^\alpha} = \frac{\partial^\alpha f(x)}{\partial x^\alpha} + \sum_{i=1}^{m} \int_0^t \frac{\partial^\alpha a^i(s,x)}{\partial x^\alpha}\, dN_s^i.$$

In fact, in the definition of C^k Itô field driven by N we could assume only that a^i and f are C^{k-1} in x with Hölder continuous $(k-1)$-derivatives: the k-derivatives of the coefficients do not appear explicitly in the statements and proofs, and the Hölder continuity of the $(k-1)$-derivatives is used to apply substitution arguments in some proofs.

The definition of C^k Itô martingale field is similar, with the only difference that N is a local martingale.

2.3. *Brackets of Itô–Dirichlet fields.* The following proposition extends the result of Remark 3. This is the only result of this paper involving the concept of Itô–Dirichlet fields. In most of the work we only need Itô fields. However, on one side the next result on brackets is more properly posed in the most general framework of Itô–Dirichlet fields. On the other side, in other applications, see [6], it is necessary to compute the bracket of an Itô–Dirichlet field which is not an Itô field.

PROPOSITION 7. *Let* $X = (X_t(x),\ t \in [0,T],\ x \in \mathbb{R}^d)$, $Y = (Y_t(x),\ t \in [0,T], x \in \mathbb{R}^d)$ *be two* C^0 *Itô–Dirichlet fields driven by a local martingale* $N = (N^1, \dots, N^m)$, *with decompositions* $M^1 + Z^1$, $M^2 + Z^2$. *We denote by* a_j^i, *the coefficients of* M^j, $j = 1, 2$, $1 \le i \le m$. *We suppose moreover that* M^1, *and* M^2 *are* C^1 *Itô martingale fields. Let* $A = (A^1, \dots, A^d)$ *be a process such that* (A, N)

has all its mutual brackets. Then the bracket $[X_t(A_t^1), Y_t(A_t^2)]$ *exists and it is given by*

$$
\begin{aligned}
[X(A), Y(A)]_t = {} & \sum_{i,j=1}^{m} \int_0^t a_1^i(s, A_s) \cdot a_2^j(s, A_s)\, d[N^i, N^j]_s \\
& + \sum_{i,j=1}^{d} \int_0^t \frac{\partial X_s}{\partial x_i}(A_s) \cdot \frac{\partial Y_s}{\partial x_j}(A_s)\, d[A^i, A^j]_s \\
& + \sum_{i=1}^{d} \sum_{j=1}^{m} \int_0^t \frac{\partial X_s}{\partial x_i}(A_s) \cdot a_2^j(s, A_s)\, d[A^i, N^j]_s \\
& + \sum_{i=1}^{m} \sum_{j=1}^{d} \int_0^t a_1^i(s, A_s) \cdot \frac{\partial Y_s}{\partial x_j}(A_s)\, d[N^i, A^j]_s.
\end{aligned} \tag{9}
$$

EXAMPLE 8. If $(N^1, \dots, N^m)$ is a standard Brownian motion $(W^1, \dots, W^m)$ then the integral sum in the first line can be replaced by

$$\sum_{i=1}^{m} \int_0^t a_1^i(s, A_s) \cdot a_2^i(s, A_s)\, ds.$$

PROOF OF PROPOSITION 7. For simplicity of notation we give the proof in the case $d = m = 1$, $X = Y$. The proof in the general case does not contain essential differences. We have to prove

$$
\begin{aligned}
[X(A), X(A)]_t = {} & \int_0^t (a(s, A_s))^2\, d[N, N]_s \\
& + \int_0^t \left(\frac{\partial X_s}{\partial x}(A_s) \right)^2 d[A, A]_s \\
& + 2 \int_0^t \frac{\partial X_s}{\partial x}(A_s) \cdot a(s, A_s)\, d[N, A]_s.
\end{aligned} \tag{10}
$$

We can write

$$
\begin{aligned}
X_{s+\varepsilon}(A_{s+\varepsilon}) - X_s(A_s) = {} & X_{s+\varepsilon}(A_{s+\varepsilon}) - X_{s+\varepsilon}(A_s) \\
& + X_{s+\varepsilon}(A_s) - X_s(A_s)
\end{aligned}
$$

so that for $t \in [0, T]$,

$$\frac{1}{\varepsilon} \int_0^t \big(X_{s+\varepsilon}(A_{s+\varepsilon}) - X_s(A_s) \big)^2 ds = I_1 + I_2 + I_3$$

where

$$I_1 = \frac{1}{\varepsilon} \int_0^t \big(X_{s+\varepsilon}(A_{s+\varepsilon}) - X_{s+\varepsilon}(A_s) \big)^2 ds,$$

$$I_2 = \frac{2}{\varepsilon}\int_0^t \big(X_{s+\varepsilon}(A_{s+\varepsilon}) - X_{s+\varepsilon}(A_s)\big)\cdot\big(X_{s+\varepsilon}(A_s) - X_s(A_s)\big)\,ds,$$
$$I_3 = \frac{1}{\varepsilon}\int_0^t \big(X_{s+\varepsilon}(A_s) - X_s(A_s)\big)^2\,ds.$$

Now I_1 does not create any problem, since it equals

$$\frac{1}{\varepsilon}\int_0^t \left(\int_0^1 \frac{\partial X_{s+\varepsilon}}{\partial x}\big(A_s + \alpha(A_{s+\varepsilon} - A_s)\big)\,d\alpha\right)^2 (A_{s+\varepsilon} - A_s)^2\,ds$$

and by usual uniform continuity arguments (see, for instance, the proof of Proposition 2.1 in [20]), it converges to

$$\int_0^t \left(\frac{\partial X_s}{\partial x}(A_s)\right)^2 d[A, A]_s.$$

Concerning I_3, assumption (8) implies that it converges if and only if it converges to

$$\frac{1}{\varepsilon}\int_0^t \big(M_{s+\varepsilon}(A_s) - M_s(A_s)\big)^2\,ds.$$

This equals

$$\int_0^t \left(\frac{1}{\sqrt{\varepsilon}}\int_s^{s+\varepsilon} a(u,x)\,dN_u\right)^2\bigg|_{x=A_s} ds.$$

Since $a(u,x)$ is Hölder continuous in u, we can apply a substitution argument (see for instance [21], Lemma 3.3) and have

$$\int_0^t \left(\frac{1}{\sqrt{\varepsilon}}\int_s^{s+\varepsilon} a(u,A_s)\,dN_u\right)^2 ds.$$

It remains to show that

$$\int_0^t \left\{\left(\frac{1}{\sqrt{\varepsilon}}\int_s^{s+\varepsilon} a(u,A_s)\,dN_u\right)^2 - \frac{1}{\varepsilon}(N_{s+\varepsilon} - N_s)^2\big(a(s,A_s)\big)^2\right\} ds \tag{11}$$

converges ucp to zero.

Using localization techniques, we can suppose that

$$\sup_{s\in[0,T]} |a(s,A_s)|, \qquad \sup_{s\in[0,T]} |[N,N]_s|, \tag{12}$$

are bounded random variables. For classical typical localization arguments the reader can consult [20–22]. Now (11) is equal to

$$
\begin{aligned}
&\int_0^t \left\{ \frac{1}{\sqrt{\varepsilon}} \int_s^{s+\varepsilon} a(u, A_s)\, dN_u - \frac{1}{\sqrt{\varepsilon}} (N_{s+\varepsilon} - N_s) a(s, A_s) \right\} \\
&\qquad \times \left\{ \left(\frac{1}{\sqrt{\varepsilon}} \int_s^{s+\varepsilon} a(u, A_s)\, dN_u - \frac{1}{\sqrt{\varepsilon}} (N_{s+\varepsilon} - N_s) a(s, A_s) \right) \right. \\
&\qquad\qquad\qquad\qquad \left. + \frac{2}{\sqrt{\varepsilon}} (N_{s+\varepsilon} - N_s) a(s, A_s) \right\} ds \\
&\quad \le \left(\int_0^T \left| \frac{1}{\sqrt{\varepsilon}} \int_s^{s+\varepsilon} \big(a(u, A_s) - a(s, A_s) \big)\, dN_u \right|^2 ds \right)^{1/2} \\
&\qquad \times \left(2 \int_0^T \left| \frac{1}{\sqrt{\varepsilon}} \int_s^{s+\varepsilon} \big(a(u, A_s) - a(s, A_s) \big)\, dN_u \right|^2 ds \right. \\
&\qquad\qquad\qquad \left. + \frac{8}{\varepsilon} \int_0^T (N_{s+\varepsilon} - N_s)^2 a^2(s, A_s)\, ds \right)^{1/2}.
\end{aligned}
$$

Now

$$
\int_0^T \frac{(N_{s+\varepsilon} - N_s)^2}{\varepsilon} a^2(s, A_s)\, ds \le C \int_0^T \frac{(N_{s+\varepsilon} - N_s)^2}{\varepsilon}\, ds \to [N, N]_T.
$$

In order to conclude the proof that (11) converges to zero, it remains to prove the following result:

$$
E \int_0^T \left| \frac{1}{\sqrt{\varepsilon}} \int_s^{s+\varepsilon} \big(a(u, A_s) - a(s, A_s) \big)\, dN_u \right|^2 ds \to 0. \tag{13}
$$

To do this, recall (12). We have

$$
\begin{aligned}
&E \int_0^T \left| \frac{1}{\sqrt{\varepsilon}} \int_s^{s+\varepsilon} \big(a(u, A_s) - a(s, A_s) \big)\, dN_u \right|^2 ds \\
&\qquad = \frac{1}{\varepsilon} \int_0^T E\left[\int_s^{s+\varepsilon} \big(a(u, A_s) - a(s, A_s) \big)^2\, d[N, N]_u \right] ds \\
&\qquad = E \int_0^T \int_{u-\varepsilon}^{u} \big(a(u, A_s) - a(s, A_s) \big) 2\, \frac{ds}{\varepsilon}\, d[N, N]_u
\end{aligned}
$$

which clearly converges to zero because of Lebesgue dominated convergence theorem.

At this level we have proved the convergence of I_3 to the first term of (10).

We still have to prove the convergence of I_2. It can be written as

$$\frac{2}{\varepsilon}\int_0^t \left(\int_0^1 \frac{\partial X_{s+\varepsilon}}{\partial x}\big(A_s+\alpha(A_{s+\varepsilon}-A_s)\big)\,d\alpha\right) \times (A_{s+\varepsilon}-A_s)\cdot\big(X_{s+\varepsilon}(A_s)-X_s(A_s)\big)\,ds.$$

By usual uniform continuity arguments, it is sufficient to prove that

$$\frac{1}{\varepsilon}\int_0^\cdot (A_{s+\varepsilon}-A_s)\cdot\big(X_{s+\varepsilon}(A_s)-X_s(A_s)\big)\,ds \to \int_0^\cdot a(s,A_s)d[A,N]_s \quad \text{ucp}. \tag{14}$$

Moreover, it is easy to see that

$$\frac{1}{\varepsilon}\int_0^\cdot (A_{s+\varepsilon}-A_s)\cdot\big(Z_{s+\varepsilon}(A_s)-Z_s(A_s)\big)\,ds \to 0$$

so in (14) we can replace X by M.

Similarly as for the proof of the convergence of I_3, we have

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_0^t (A_{s+\varepsilon}-A_s)\big(M_{s+\varepsilon}(A_s)-M_s(A_s)\big)\,ds\\
&\quad=\frac{1}{\varepsilon}\int_0^t (A_{s+\varepsilon}-A_s)\left(\int_s^{s+\varepsilon} a(u,x)\,dN_u\right)\bigg|_{x=A_s} ds\\
&\quad=\frac{1}{\varepsilon}\int_0^t (A_{s+\varepsilon}-A_s)a(s,A_s)(N_{s+\varepsilon}-N_s)\,ds\\
&\qquad+\frac{1}{\varepsilon}\int_0^t (A_{s+\varepsilon}-A_s)\left(\int_s^{s+\varepsilon} a(u,x)\,dN_u - a(s,x)(N_{s+\varepsilon}-N_s)\right)\bigg|_{x=A_s} ds\\
&\quad=J_1+J_2
\end{aligned}$$

where

$$J_1 \to \int_0^\cdot a(s,A_s)\,d[A,N]_s \quad \text{ucp}$$

and by substitution as above,

$$\begin{aligned}
J_2 &= \int_0^t \left(\frac{1}{\sqrt{\varepsilon}}\int_s^{s+\varepsilon}\big(a(u,A_s)-a(s,A_s)\big)\,dN_u\right)\frac{1}{\sqrt{\varepsilon}}(A_{s+\varepsilon}-A_s)\,ds\\
&\le \left(\int_0^T \left(\frac{1}{\sqrt{\varepsilon}}\int_s^{s+\varepsilon}\big(a(u,A_s)-a(s,A_s)\big)\,dN_u\right)^2 ds\right)^{1/2}\\
&\quad\times\left(\int_0^T \frac{1}{\varepsilon}(A_{s+\varepsilon}-A_s)^2\,ds\right)^{1/2}.
\end{aligned}$$

This converges to zero in accordance with (13). The proof is complete. $\square$

3. Itô–Wentzell formula. In our applications we need in fact a generalization of the classical Itô formula stated in Remark 4. It will be of Itô–Wentzell type; see, for example, [9].

PROPOSITION 9. *Let $F = (F(t,x),\ t \in [0,T],\ x \in \mathbb{R}^d)$ be a C^2 Itô field driven by a semimartingale N, of the form* (7). *Let $A = (A^1, \dots, A^d)$ be a $\mathcal{F}$-adapted process such that (A, N) has all its mutual brackets. Assume that the forward integrals $\int_0^t \frac{\partial F}{\partial x_i}(s, A_s)\, d^- A_s^i$ exist for $i = 2, \dots, d$. Then we have*

$$\begin{aligned} F(t, A_t) = F(0, A_0) &+ \sum_{i=1}^m \int_0^t a^i(s, A_s)\, dN_s^i + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(s, A_s)\, d^- A_s^i \\ &+ \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(s, A_s)\, d[A^i, A^j]_s \\ &+ \sum_{i=1}^m \sum_{j=1}^d \int_0^t \frac{\partial a^i}{\partial x_j}(s, A_s)\, d[N^i, A^j]_s. \end{aligned} \tag{15}$$

REMARK 10. In particular, $\int_0^\cdot \frac{\partial F}{\partial x_1}(s, A_s)\, d^- A_s^1$ exists. A similar formula would hold for $\int_0^\cdot \frac{\partial F}{\partial x_1}(s, A_s)\, d^+ A_s^1$.

PROOF. We give the proof for $m = d = 1$ for simplicity of notation. Let $s \in]0, T[$; we have

$$F(s+\varepsilon, A_{s+\varepsilon}) - F(s, A_s) = I_1(s,\varepsilon) + I_2(s,\varepsilon)$$

where

$$\begin{aligned} I_1(s,\varepsilon) &= F(s+\varepsilon, A_{s+\varepsilon}) - F(s+\varepsilon, A_s), \\ I_2(s,\varepsilon) &= F(s+\varepsilon, A_s) - F(s, A_s). \end{aligned}$$

On one side,

$$\int_0^t \frac{F(s+\varepsilon, A_{s+\varepsilon}) - F(s, A_s)}{\varepsilon}\, ds$$

converges ucp to $F(t, A_t) - F(0, A_0)$. On the other side, since

$$I_2(s,\varepsilon) = \int_s^{s+\varepsilon} a(u, A_s)\, dN_u$$

(because A is adapted and because a is Hölder continuous in x to apply a substitution argument; see [21], Lemma 3.3), it is not difficult to show that

$$\int_0^t \frac{I_2(s,\varepsilon)}{\varepsilon}\, ds$$

converges ucp to

$$\int_0^t a(u, A_u)\, dN_u.$$

It remains to discuss $I_1(s, \varepsilon)$. This gives

$$I_1(s, \varepsilon) = J_1(s, \varepsilon) + J_2(s, \varepsilon) + J_3(s, \varepsilon) + J_4(s, \varepsilon)$$

where

$$\begin{aligned}
J_1(s, \varepsilon) &= \frac{\partial F}{\partial x}(s, A_s)(A_{s+\varepsilon} - A_s),\\
J_2(s, \varepsilon) &= \left(\frac{\partial F}{\partial x}(s+\varepsilon, A_s) - \frac{\partial F}{\partial x}(s, A_s)\right)(A_{s+\varepsilon} - A_s)\\
J_3(s, \varepsilon) &= \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(s, A_s)(A_{s+\varepsilon} - A_s)^2,\\
J_4(s, \varepsilon) &= \int_0^1 (1-\alpha)\left(\frac{\partial^2 F}{\partial x^2}\big(s+\varepsilon, A_s + \alpha(A_{s+\varepsilon} - A_s)\big) - \frac{\partial^2 F}{\partial x^2}(s, A_s)\right) d\alpha\\
&\quad \times (A_{s+\varepsilon} - A_s)^2.
\end{aligned}$$

Integrating from 0 to t the previous expressions and using similar arguments to those of the proof of Proposition 7, we get

$$\lim_{\varepsilon\to 0}\int_0^t J_2(s, \varepsilon)\,\frac{ds}{\varepsilon} = \int_0^t \frac{\partial a}{\partial x}(s, A_s)\, d[N, A]_s.$$

Using classical uniform continuity arguments for $\frac{\partial^2 F}{\partial x^2}$ and pathwise weak convergence of $\int_0^t (A_{s+\varepsilon} - A_s)^2 \frac{ds}{\varepsilon}$ on $[0, T]$, we also obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^t J_3(s, \varepsilon)\,\frac{ds}{\varepsilon} = \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(s, A_s)\, d[A, A]_s,\\
&\lim_{\varepsilon\to 0}\int_0^t J_4(s, \varepsilon)\,\frac{ds}{\varepsilon} = 0.
\end{aligned}$$

This forces in particular the convergence of

$$\int_0^t J_1(s, \varepsilon)\,\frac{ds}{\varepsilon}$$

whose limit is by definition $\int_0^t \frac{\partial F}{\partial x}(s, A_s)\, d^- A_s$. The proof is complete. $\square$

4. Existence of forward integrals. From this section we restrict ourselves to $d = 1$. Thus A will be a scalar process with finite quadratic variation, and the Itô fields $X_t(x)$ will have $x \in \mathbb{R}$.

With the help of the Itô–Wentzell formula, in this section we first show the existence of the forward integral

$$\int_0^t X_s(A_s)\,d^- A_s$$

for suitable random fields $(X_t(x),\ t \in [0,T],\ x \in \mathbb{R})$; then with the aid of this forward integral we show the existence of a more general forward integral

$$\int_0^t X_s(A_s)\,d^- Y_s(A_s) \tag{16}$$

for suitable random fields $(X_s(x))$ and $(Y_s(x))$. The only assumption on A imposed below is that the process (A, N) has all its mutual bracket, N being the semimartingale driving X and Y. Since A in general will not be a semimartingale, the class of integrand $X_s(A_s)$ is a restricted one, but still including semimartingales and functions of A itself. Apparently, the integration defined in this section is not a particular case of the known stochastic integration theories.

4.1. *Forward integral with integrator A.* In this section it is always understood that (A_t) is a continuous $\mathcal{F}$-adapted process.

DEFINITION 11. Given (A_t), we denote by $\mathcal{A}_k$ the set of processes V_t of the form

$$V_t = X_t(A_t)$$

where $(X_t(x),\ t \in [0,T],\ x \in \mathbb{R})$ is a C^k Itô field driven by some continuous $\mathcal{F}$-semimartingale $N = (N^1, \dots, N^m)$ such that (A, N) has all its mutual brackets.

REMARK 12. The set $\mathcal{A}_k$ is an algebra, since the space of all C^k Itô fields $(X_t(x))$ is an algebra (by classical stochastic calculus for semimartingales).

REMARK 13. If $(X_t(x))$ is a C^2 Itô field of the form (7), so $V_t = X_t(A_t)$ belongs to $\mathcal{A}_2$, we can use Itô–Wentzell formula to express V_t as

$$\begin{aligned} V_t = V_0 &+ \sum_{i=1}^m \int_0^t a^i(s, A_s)\,dN_s^i + \int_0^t \frac{\partial X_s}{\partial x}(A_s)\,d^- A_s \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 X_s}{\partial x^2}(A_s)\,d[A,A]_s + \sum_{i=1}^m \int_0^t \frac{\partial a^i}{\partial x}(s, A_s)\,d[N^i, A]_s. \end{aligned}$$

In particular, the forward integral $\int_0^t \frac{\partial X_s}{\partial x}(A_s)\,d^- A_s$ exists.

DEFINITION 14. Given a C^k Itô field $X_t(x)$ of the form (7), we denote by $X_t^*(x)$ the C^{k+1} Itô field defined as

$$X_t^*(x) = \int_0^x X_t(x')\,dx'$$

having the representation

$$X_t^*(x) = f^*(x) + \sum_{i=1}^m \int_0^t a^{i*}(s,x)\,dN_s^i$$

with

$$f^*(x) = \int_0^x f(x')\,dx', \qquad a^{i*}(t,x) = \int_0^x a^i(t,x')\,dx'.$$

PROPOSITION 15. *The forward integral*

$$\int_0^t V_s\,d^- A_s$$

exists for all $V \in \mathcal{A}_1$. *If* $V_t = X_t(A_t)$, $(X_t(x))$ *of the form* (7), *then we have the formula*

$$\begin{aligned}\int_0^t V_s\,d^-A_s &= X_t^*(A_t) - f^*(A_0) - \sum_{i=1}^m \int_0^t a^{i*}(s,A_s)\,dN_s^i \\ &\quad - \frac{1}{2}\int_0^t \frac{\partial X_s}{\partial x}(A_s)\,d[A,A]_s - \sum_{i=1}^m \int_0^t a^i(s,A_s)\,d[N^i,A]_s.\end{aligned} \tag{17}$$

In particular, *we have*

$$\int_0^\cdot V_s\,d^-A_s \in \mathcal{A}_2.$$

PROOF. It is a direct consequence of the Itô–Wentzell formula and the previous definitions and remarks. $\square$

REMARK 16. As announced, the previous proposition proves the existence of a forward integral for processes more general than semimartingales, which does not seem a particular case of other theories of generalized integration. Let us notice that by similar reasonings we can define the backward integral

$$\int_0^t V_s\,d^+A_s. \tag{18}$$

REMARK 17. Let $\mathcal{I}$ be the linear operator

$$\mathcal{I}\colon \mathcal{A}_1 \to L^0\big(\Omega; C([0,T];\mathbb{R})\big)$$

defined as

$$(\mathcal{I}V)_t = \int_0^t V_s\, d^- A_s.$$

Assume that A has zero quadratic variation. Then

$$(\mathcal{I}V)_t = X_t^*(A_t) - f^*(A_0) - \sum_{i=1}^m \int_0^t a^{i*}(s, A_s)\, dN_s^i.$$

Therefore, the mapping $\mathcal{I}$ can be extended to $\mathcal{A}_0$, and even to processes $V_t = X_t(A_t)$ where $X_t(x)$ is the derivative in x of a C^0 Itô field (thus $X_t(.)$ is a distribution). However in this case we do not know whether $(\mathcal{I}V)_t$ is a forward integral.

We complete this subsection with the computation of the bracket of the previous integrals.

LEMMA 18. *Let $U, V \in \mathcal{A}_1$. Then*

$$\left[\int_0^\cdot U_s\, d^- A_s, \int_0^\cdot V_s\, d^- A_s\right]_t = \int_0^t U_s V_s\, d[A, A]_s. \tag{19}$$

PROOF. We can write $V_t = X_t(A_t)$, $U_t = Y_t(A_t)$, where $X_t(x)$ and $Y_t(x)$ are C^1-Itô fields given by

$$X_t(x) = f(x) + \sum_{i=1}^m \int_0^t a^i(s, x)\, dN_s^i, \tag{20}$$

$$Y_t(x) = g(x) + \sum_{i=1}^m \int_0^t \alpha^i(s, x)\, dN_s^i \tag{21}$$

(it is not restrictive to take the same driving semimartingales). First we recall that bounded variation processes give zero contributions in brackets calculations. Formula (17) implies that the left-hand side of (19) equals

$$\left[X^*(A) - \sum_{i=1}^m \int_0^\cdot a^{i*}(s, A_s)\, dN_s^i,\ Y^*(A) - \sum_{i=1}^m \int_0^\cdot \alpha^{i*}(s, A_s)\, dN_s^i\right]_t.$$

Proposition 7 and the bilinearity of covariation gives the right member of (19), after a few easy calculations (see next lemma for similar ones). $\square$

LEMMA 19. *Let $V \in \mathcal{A}_1$, M a $\mathcal{F}$-local martingale, R a continuous $\mathcal{F}$-adapted process. Assume that $[A, M]$ exists. Then*

$$\left[\int_0^\cdot V_s\, d^- A_s, \int_0^\cdot R_s\, dM_s\right]_t = \int_0^t V_s R_s\, d[A, M]_s. \tag{22}$$

PROOF. We assume the representation (20), with $V_t = X_t(A_t)$. From (17) we have that the left-hand side of (22) equals

$$\left[X^*(A) - \sum_{i=1}^m \int_0^\cdot a^{i*}(s, A_s) dN_s^i, \int_0^\cdot R_s \, dM_s\right]_t.$$

To $[X^*(A), \int_0^\cdot R_s \, dM_s]_t$ we apply Proposition 7 and have

$$\sum_{i=1}^m \int_0^t a^{i*}(s, A_s) R_s \, d[N^i, M]_s + \int_0^t X_s(A_s) R_s \, d[A, M]_s.$$

The sum is

$$\int_0^t X_s(A_s) R_s \, d[A, M]_s$$

completing the proof [since $V = X(A)$]. □

From the previous facts we can deduce an interesting result.

COROLLARY 20. *If A is a Dirichlet process, that is, $A = M + \tilde{A}$, with M a local martingale and $\tilde{A}$ a process with zero quadratic variation, then for all $V \in \mathcal{A}_1$ the forward integral $\int_0^\cdot V_s \, d^- A_s$ is a Dirichlet process.*

PROOF. From the definition of forward integral, the forward integral $\int_0^\cdot V_s \, d^- \tilde{A}_s$ exists (by linearity), equal to

$$\int_0^\cdot V_s \, d^- A_s - \int_0^\cdot V_s \, dM_s.$$

Then it is sufficient to apply the bilinearity of the covariation and the previous two lemmata, to show that $\int_0^\cdot V_s \, d^- \tilde{A}_s$ has zero quadratic variation. Since $\int_0^\cdot V_s \, dM_s$ is a local martingale, the corollary is proved. □

4.2. *Forward integral with more general integrator.* The following proposition states simultaneously the existence of the announced forward integral $\int_0^t X_s(A_s) \, d^- Y_s(A_s)$ and a generalization of Itô–Wentzell formula to a chain rule. Setting $V = 1$, in the next formula, we obtain the previous Itô–Wentzell formula.

PROPOSITION 21 (Itô–Wentzell chain rule). *For every $V \in \mathcal{A}_1$ and $U \in \mathcal{A}_2$, the forward integral*

$$\int_0^t V_s \, d^- U_s \tag{23}$$

exists and belongs to $\mathcal{A}_2$. *If* $V_t = X_t(A_t)$, $U_t = Y_t(A_t)$, *where* $(X_t(x))$ *and* $(Y_t(x))$ *have the representations* (20), (21), *we have the formula*

$$\begin{aligned}\int_0^t V_s\,d^-U_s &= \sum_{i=1}^m \int_0^t V_s\alpha^i(s,A_s)\,dN_s^i + \int_0^t V_s\,\frac{\partial Y_s}{\partial x}(A_s)\,d^-A_s\\ &\quad + \frac{1}{2}\int_0^t V_s\frac{\partial^2 Y_s}{\partial x^2}(A_s)\,d[A,A]_s\\ &\quad + \sum_{i=1}^m\int_0^t V_s\frac{\partial\alpha^i}{\partial x}(s,A_s)\,d[N^i,A]_s.\end{aligned} \tag{24}$$

REMARK 22. We already know from the previous subsection that the forward integral

$$\int_0^t R_s\,d^-A_s$$

exists if $R \in \mathcal{A}_1$. In particular here

$$\int_0^t V_s\,\frac{\partial Y_s}{\partial x}(A_s)\,d^-A_s$$

exists because $V \cdot \frac{\partial Y}{\partial x}(A) \in \mathcal{A}_1$.

PROOF. The proof is very similar to the case $V = 1$. One has to prove the ucp convergence of

$$\int_0^t V_s\,\frac{U_{s+\varepsilon}(A_{s+\varepsilon}) - U_s(A_s)}{\varepsilon}\,ds.$$

The existence of $\int_0^t V_s\,\frac{\partial Y_s}{\partial x}(A_s)\,d^-A_s$, known a priori from the previous section, is needed in the proof. We omit the details. $\square$

The next corollary expresses a nontrivial substitution property.

COROLLARY 23. *Let* $V, Z \in \mathcal{A}_1$. *We set* $U_t = \int_0^t Z_s\,d^-A_s$ (*which belongs to* $\mathcal{A}_2$, *so* $\int_0^t V_s\,d^-U_s$ *is well defined*). *Then*

$$\int_0^t V_s\,d^-U_s = \int_0^t V_s Z_s\,d^-A_s. \tag{25}$$

PROOF. We have to use (24), so we need a representation of U as $U_t = Y_t(A_t)$, Y having a representation of the form (21). However, we have the additional information $U_t = \int_0^t Z_s d^-A_s$, which provides formulas for Y and α^i in terms of the representation of $Z \in \mathcal{A}_1$. Substituting these formulas into (24) we shall find the result.

Assume $Z_t = X_t(A_t)$, X with representation

$$X_t(x) = f(x) + \sum_{i=1}^{m_0} \int_0^t a^i(s,x)\, dN_s^{0,i}.$$

Then we have (17), with $m = m_0$ and $N^i = N^{0,i}$, so $U_t = Y_t(A_t)$ with

$$\begin{aligned} Y_t(x) &= X_t^*(x) - f^*(A_0) - \sum_{i=1}^{m_0} \int_0^t a^{i*}(s, A_s)\, dN_s^{0,i} \\ &\quad - \frac{1}{2}\int_0^t \frac{\partial X_s}{\partial x}(A_s)\, d[A,A]_s - \sum_{i=1}^{m_0} \int_0^t a^i(s, A_s)\, d[N^{0,i}, A]_s. \end{aligned}$$

To compare this representation of $Y_t(x)$ with (21) we have to take $m = 2m_0 + 1$,

$$N^i = N^{0,i} \quad \text{and} \quad \alpha^i(s,x) = a^{i*}(s,x) - a^{i*}(s, A_s) \quad \text{for } i = 1, \dots, m_0,$$

$$N^{m_0+1} = [A, A], \qquad \alpha^{m_0+1}(s,x) = -\frac{1}{2}\frac{\partial X_s}{\partial x}(A_s),$$

$$N^i = [N^{0,i}, A] \quad \text{and} \quad \alpha^i(s,x) = -a^i(s, A_s) \quad \text{for } i = m_0+2, \dots, 2m_0+1,$$

$$g(x) = f^*(x) - f^*(A_0).$$

Therefore, in (25) we have to substitute

$$\alpha^i(s,x) = 0 \quad \text{for } i = 1, \dots, m_0,$$
$$\frac{\partial Y_s}{\partial x}(A_s) = X_s(A_s) = Z_s;$$

and notice that in (25)

$$\int_0^t V_s \alpha^{m_0+1}(s, A_s)\, dN_s^{m_0+1} + \frac{1}{2}\int_0^t V_s \frac{\partial^2 Y_s}{\partial x^2}(A_s)\, d[A,A]_s = 0,$$
$$\sum_{i=m_0+2}^{2m_0+1} \int_0^t V_s \alpha^i(s, A_s)\, dN_s^i + \sum_{i=1}^{m_0} \int_0^t \frac{\partial \alpha^i}{\partial x}(s, A_s)\, d[N^i, A]_s = 0.$$

Therefore (24) reduces to (25). The proof is complete. $\square$

Finally, let us compute the covariation of these new integrals.

PROPOSITION 24. *Let* $U, V \in \mathcal{A}_1$, $\Phi, \Psi \in \mathcal{A}_2$. *Then*

$$\left[\int_0^\cdot U_s\, d^-\Phi_s, \int_0^\cdot V_s\, d^-\Psi_s\right]_t = \int_0^t U_s V_s\, d[\Phi, \Psi]_s.$$

The proof is a not difficult exercise based on the bilinearity of the covariation, the representation (24), Lemma 18, Lemma 19 and Proposition 7.

4.3. *Processes of class $\mathcal{A}_2$.* We have introduced processes of the form $V_t = X_t(A_t)$ and we have developed some rules of calculus for these processes. In comparison with the classical Itô calculus, it would be interesting to have a representation of these new processes as sum of stochastic integrals and to have rules of calculus based on this representation.

LEMMA 25. *A process V is of class $\mathcal{A}_2$ if and only if it has a representation of the form*

$$V_t = V_0 + \sum_{i=1}^{m} \int_0^t \alpha^i(s)\, dN_s^i + \int_0^t \gamma(s)\, d^- A_s \tag{26}$$

where α^i and γ are continuous $\mathcal{F}$-adapted processes, with $\gamma \in \mathcal{A}_1$, V_0 is $\mathcal{F}_0$-measurable and $N = (N^1, \dots, N^m)$ is a continuous $\mathcal{F}$-semimartingale, (A, N) with all its mutual brackets. Similarly, we can say that a process V is of class $\mathcal{A}_2$ if and only if it has a representation of the form

$$V_t = V_0 + Y_t + \int_0^t \gamma(s)\, d^- A_s$$

where Y is a continuous $\mathcal{F}$-semimartingale, (A, Y) with all its mutual brackets.

This result is a rewriting of formula (17) in one direction, and of Proposition 15 in the opposite direction (the semimartingales may change from one representation to the other).

REMARK 26. The decomposition of a process V of class $\mathcal{A}_2$ in general is not unique.

REMARK 27. If A is a Dirichlet process, then every process of class $\mathcal{A}_2$ is a Dirichlet process. This follows from Corollary 20 and the representation (26).

Some basic rules of calculus based on this representation are expressed by the following proposition. The assumption $F \in C^{1,4}$ is stronger than the classical one. It is needed, in view of the following remark, to have that $\frac{\partial F}{\partial x_i}(X_t^1(A_t), \dots, X_t^n(A_t))$, where $X_t^i(A_t) = V_t^i$, is of the form $Y_t(A_t)$ with $(Y_t(x))$ an Itô field [we need to apply Itô formula to the composition $\frac{\partial F}{\partial x_i}(X_t^1(x), \dots, X_t^n(x))$].

REMARK 28. If $X^1(A), \dots, X^n(A) \in \mathcal{A}_2$, and $\varphi \in C^3(\mathbb{R}^n)$, then

$$\varphi\bigl(X^1(A), \dots, X^n(A)\bigr) \in \mathcal{A}_1.$$

Indeed, by Itô formula, $(t, x) \mapsto \varphi(X_t^1(x), \dots, X_t^n(x))$ is a C^1 Itô field [we omit the full computation; for instance, the term $\frac{\partial^2 \varphi}{\partial y^k \partial y^h}(X_s^1(x), \dots, X_s^n(x))$ appears in the Itô representation of $\varphi(X_t^1(x),\dots,X_t^n(x))$, and this term is of class C^1 in x since φ is of class C^3 and $X_t^1(x),\dots,X_t^n(x)$ are of class C^2 in x]. In the next proposition we apply this remark to $\varphi = \frac{\partial F}{\partial x_i}$.

PROPOSITION 29. *Let $V^1,\dots,V^n \in \mathcal{A}_2$ and $F(t,x_1,\dots,x_n)$ of class $C^{1,4}$. Then*

$$\begin{aligned} F(t,V_t^1,\dots,V_t^n) &= F(0,V_0^1,\dots,V_0^n) \\ &\quad + \sum_{i=1}^n \int_0^t \frac{\partial F}{\partial x_i}(s,V_s^1,\dots,V_s^n)\, d^- V_s^i \\ &\quad + \frac{1}{2}\sum_{i,j=1}^n \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(s,V_s^1,\dots,V_s^n)\, d[V^i,V^j]_s \end{aligned} \tag{27}$$

where the brackets exist by Proposition 7, *and the forward integrals exist by Proposition* 21 (*the integrators belong to $\mathcal{A}_2$, the integrands to $\mathcal{A}_1$ by the previous remark*). *Moreover*, *as in the case of semimartingales*, *we can make the substitution in the forward integrals*:

$$\begin{aligned} &\int_0^t \frac{\partial F}{\partial x_i}(s,V_s^1,\dots,V_s^n)\, d^- V_s^k \\ &\quad = \sum_{i=1}^m \int_0^t \frac{\partial F}{\partial x_i}(s,V_s^1,\dots,V_s^n)\alpha^{i,k}(s)\, dN_s^i \\ &\qquad + \int_0^t \frac{\partial F}{\partial x_i}(s,V_s^1,\dots,V_s^n)\gamma^k(s)\, d^- A_s, \end{aligned} \tag{28}$$

where $\alpha^{i,k}$ and γ^k are the coefficients of V^k in a representation of the form (26).

The Itô formula (27) is an application of the classical Itô formula of Remark 4: the process $(V^1,\dots,V^n)$ has all its mutual brackets and the forward integrals exist. Equation (28) is a consequence of (26) and Corollary 23.

5. On a SDE driven by a finite quadratic variation process and a semimartingale. We are interested in the well posedness of the following SDE:

$$X_t = x + \int_0^t \sigma(X_s) d^- A_s + \int_0^t b(s,X_s)\, dN_s \tag{29}$$

where N is a $\mathcal{F}$-semimartingale, A is a continuous $\mathcal{F}$-adapted process such that (A,N) has all its mutual brackets. We make also the following assumptios on σ and b:

(i) $\sigma \in C^4(\mathbb{R})$, σ', σ'' bounded;

(ii) b: $\mathbb{R}_+ \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a.s. continuous and Lipschitz continuous in the second argument, and $\mathcal{F}$-adapted.

The main result of this section is the following theorem.

THEOREM 30. *There is a unique $\mathcal{F}$-adapted solution in $\mathcal{A}_2$ to equation* (29).

We only give an outline of the proof, since it is rather classical as soon as the previous stochastic calculus is established. The main tool in the proof is the so called Doss–Sussman transform as in [22]. Let F: $\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be defined as the unique solution to

$$\begin{cases} \dfrac{\partial F}{\partial r}(r, x) = \sigma\big(F(r, x)\big), \\ F(0, x) = x. \end{cases}$$

We have that $F(r, .)$ is a C^4-diffeomorphism on $\mathbb{R}$. We set

$$H(r, x) = F^{-1}(r, x),$$

the inverse being taken with respect to the second variable x. H is again of class C^4.

The basic idea of the proof of uniqueness in Theorem 30 is the application of Itô formula to $H(A_t, X_t)$. This will involve mutual brackets between A and X where X is a solution to (29).

LEMMA 31. *If $X \in \mathcal{A}_2$ satisfies* (29), *then*

$$\begin{aligned} [X, X]_t = {} & \int_0^t \sigma^2(X_s)\, d[A, A]_s \\ & + 2\int_0^t \sigma(X_s) b(s, X_s)\, d[A, N]_s + \int_0^t b^2(s, X_s)\, d[N, N]_s, \\ [X, A]_t = {} & \int_0^t \sigma(X_s)\, d[A, A]_s + \int_0^t b(s, X_s)\, d[A, N]_s. \end{aligned}$$

PROOF. In Section 4.3 we showed that X admits a suitable decomposition (26). In particular, since X solves equation (29), we use the decomposition given by the right-hand side of (29). Then the claims follow easily from Lemma 18 and Lemma 19. $\square$

Let us outline the proof of uniqueness. Let X be a solution to (29) and let $Y_t = H(A_t, X_t)$. By the classical Itô formula (remark 4), lemma 31, and direct computations on the partial derivatives of H, it is not hard to prove that

$$
\begin{aligned}
Y_t = Y_0 &+ \int_0^t \frac{b(s, F(A_s, Y_s))}{\frac{\partial F}{\partial x}(A_s, Y_s)}\, dN_s - \frac{1}{2}\int_0^t \frac{\sigma\sigma'(F(A, Y))}{\frac{\partial F}{\partial x}(A, Y)}\, d[A, A] \\
&- \int_0^t b\big(s, F(A_s, Y_s)\big)\frac{\sigma'(F(A_s, Y_s))}{\frac{\partial F}{\partial x}(A_s, Y_s)}\, d[A, N]_s \\
&- \frac{1}{2}\int_0^t \left(\int_0^{A_s} \sigma''\big(F(u, Y_s)\big)\, du\right)\frac{b^2(s, F(A_s, Y_s))}{(\frac{\partial F}{\partial x}(A_s, Y_s))^2}\, d[N, N]_s.
\end{aligned}
\tag{30}
$$

Therefore Y solves a classical semimartingale driven SDE, treated for instance in [18]. Equation (30) admits a unique adapted solution and therefore uniqueness for (29) is established.

Existence follows through similar arguments applying Itô formula to $X_t = F(A_t, Y_t)$, Y being the unique solution to equation (30). This completes the proof of Theorem 30.

REMARK 32. The result of Theorem 30 can be extended to equation (29) generalized to the case when there is a finite number of semimartingales, say

$$X_t = x + \int_0^t \sigma(X_s)\, d^- A_s + \sum_{i=1}^n \int_0^t b^i(s, X_s)\, dN_s^i.$$

An important particular case is given in the affine case: $\sigma(x) = x$, $N^1 = N$, $N_t^2 = t$, $b^1(s, x) = x$, $b^2(s, x) = \gamma_s$, where γ is a given continuous $\mathcal{F}$-adapted process.

Next result is directly used in [6], and it can be derived from Theorem 30, according to the previous remark. However, with the rules of stochastic calculus proved above, a proof of this theorem can be given along the lines of the proof for the semimartingale case.

THEOREM 33. *Consider the linear* (*or more precisely affine*) *SDE*

$$X_t = x + \int_0^t X_s\, d^- Y_s + \int_0^t g_s\, ds \tag{31}$$

where $Y = A + N$, with N a continuous $\mathcal{F}$-semimartingale and A a continuous $\mathcal{F}$-adapted process such that (N, A) has all its mutual brackets, and g is an adapted process with integrable paths. Then there is a unique solution in $\mathcal{A}_2$, given by

$$X_t = \mathcal{U}_t\left(x + \int_0^t \mathcal{U}_s^{-1} g_s\, ds\right) \tag{32}$$

where

$$\mathcal{U}_t = \exp\big(Y_t - Y_0 - \tfrac{1}{2}[Y]_t\big). \tag{33}$$

Acknowledgments. The authors are grateful to an anonymous referee who suggested some relevant improvements. The second author is grateful to Professor Giuseppe Da Prato for the kind invitation to the Scuola Normale Superiore of Pisa and for the stimulating atmosphere created. The first author is grateful to Université de Paris 13 for the kind hospitality.

REFERENCES

[1] BARLOW, M. T. (1988). Skew Brownian motion and a one dimensional stochastic differential equation. *Stochastics* **25** 1–2.

[2] BERTOIN, J. (1989). Sur une integrale pour les processus a α-variation bornée. *Ann. Probab.* **17** 1521–1535.

[3] CHITASHVILI, R. J. and TORONJADZE, T. A. (1981). On one-dimensional stochastic differential equations with unit diffusion coefficient. Structure of solutions. *Stochastics* **4** 281–315.

[4] DELLACHERIE, C. and MEYER, P.-A. (1980.) *Probabilités et Potentiels. Théorie des Martingales*, Chapter 8. Hermann, Paris.

[5] ENGELBERT, H. J. (1991). On the theorem of T. Yamada and S. Watanabe. *Stochastics Stochastic Rep.* **36** 205–216.

[6] FLANDOLI, F. and RUSSO, F. (2001). Generalized calculus and SDEs with non regular drift. *Stochastics Stochastic Rep.* To appear.

[7] FÖLLMER, H. (1981). Calcul d'Itô sans probabilités. *Séminaire de Probabilités XV. Lecture Notes in Math.* **850** 143–150. Springer, Berlin.

[8] FÖLLMER, H., PROTTER, P. and SHIRYAYEV, A. N. (1995). Quadratic covariation and an extension of Itô's formula. *Bernoulli* **1** 149–169.

[9] KUNITA, H. (1984). *Stochastic Differential Equations and Stochastic Flows of Diffeomorphisms. Ecole d'été de St. Flour XII. Lecture Notes in Math.* **1097**. Springer, Berlin.

[10] LE GALL, J. F. (1983). Applications du temps local aux equations differentielles stochastiques unidimensionelles. *Séminaire de Probabilités XVII. Lecture Notes in Math.* **986** 15–31. Springer, Berlin.

[11] LYONS, T. J. (1998). Differential equations driven by rough signals. *Revista Mat. Iberoamericana* **14** 215–310.

[12] LYONS, T. J. and ZHANG, T. S. (1994). Decomposition of Dirichlet processes and its applications. *Ann. Probab.* **22** 494–524.

[13] LYONS, T. J. and ZHENG, W. (1988). A crossing estimate for the canonical process on a Dirichlet space and tightness result. *Astérisque* **157–158** 249–271.

[14] MORET, S. and NUALART, D. (2000). Quadratic covariation and Itô formula. *J. Theoret. Probab.* **3** 193–224.

[15] NAKAO, S. (1972). On the pathwise uniqueness of solutions of one-dimensional stochastic differential equations. *Osaka J. Math.* **9** 513–518.

[16] NAKAO, S. (1985). Stochastic calculus for continuous additive functionals of zero energy. *Wahrsch. Verw. Gebiete* **68** 557–578.

[17] PERKINS, E. (1982). Local times and pathwise uniqueness for stochastic differential equations. *Séminaire de Probabilités XVI. Lecture Notes in Math.* **920** 201–208. Springer, Berlin.

[18] PROTTER, P. (1992). *Stochastic Integration and Differential Equations*. Springer, New York.

[19] REVUZ, D. and YOR, M. (1994). *Continuous Martingales and Brownian Motion*, 2nd ed. Springer, Berlin.

[20] RUSSO, F. and VALLOIS, P. (1995). The generalized covariation process and Itô formula. *Stochastic Process. Appl.* **59** 81–104.

[21] RUSSO, F. and VALLOIS, P. (1996). Itô formula for C^1-functions of semimartingales. *Probab. Theory Related Fields* **104** 27–41.
[22] RUSSO, F. and VALLOIS, P. (2000). Stochastic calculus with respect to a finite quadratic variation process. *Stochastics Stochastic Rep.* **70** 1–40.
[23] VERETENNIKOV, A. YU. (1981). On strong solutions and explicit formulas for solutions of stochastic integral equations. *Math. USSR Sbornik* **39** 387–403.
[24] WOLF, J. (1997). Transformations of semimartingales and local Dirichlet processes. *Stochastics Stochastic Rep*. **62** 95–101.
[25] WOLF, J. (1997). An Itô formula for continuous local Dirichlet processes. *Stochastics Stochastic Rep*. **62** 103–115.
[26] YAMADA, T. and WATANABE, S. (1971). On the uniqueness of solutions of stochastic differential equations. *J. Math. Kyoto Univ.* **11** 155–167.
[27] ZÄHLE, M. (1998). Integration with respect to fractal functions and stochastic calculus. I. *Probab. Theory Related Fields* **111** 333–374.
[28] ZÄHLE, M. (2001). On the link between fractional and stochastic calculus. Part II. *Math. Nachr.* **225** 145–183.
[29] ZVONKIN, A. K. (1975). A transformation of the phase space of a diffusion process which annihilates the drift. *Math. USSR Sbornik* **22** 129–149.

DIPARTIMENTO DI MATEMATICA APPLICATA
UNIVERSITÀ DI PISA
VIA BONANNO 25B
56126 PISA
ITALY

INSTITUT GALILÉE, MATHÉMATIQUES
UNIVERSITÉ DE PARIS 13
99, AV. J.-B. CLÉMENT
93430 VILLETANEUSE
FRANCE