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STABILITY OF THE OVERSHOOT FOR LÉVY PROCESSES

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We give equivalences for conditions like $X(T(r))/r \to 1$ and $X(T^*(r))/r \to 1$, where the convergence is in probability or almost sure, both as $r \to 0$ and $r \to \infty$, where X is a Lévy process and $T(r)$ and $T^*(r)$ are the first exit times of X out of the strip $\{(t, y) : t > 0, |y| \le r\}$ and half-plane $\{(t, y) : t > 0,\ y \le r\}$, respectively. We also show, using a result of Kesten, that $X(T^*(r))/r \to 1$ a.s. as $r \to 0$ is equivalent to X "creeping" across a level.

1. Introduction. Our Lévy process has exponent $\Psi(\theta)$, so that $Ee^{i\theta X(t)} = e^{-t\Psi(\theta)}$ where

$$\Psi(\theta) = \tfrac{1}{2}\sigma^2\theta^2 - i\gamma\theta + \int_{(-\infty,\infty)} \big(1 - e^{i\theta x} + i\theta x 1_{\{|x|\le 1\}}\big)\Pi(dx), \qquad \theta \in \mathbb{R}, \tag{1.1}$$

and Π is a measure satisfying

$$\int_{(-\infty,\infty)} \{x^2 \wedge 1\}\Pi(dx) < \infty. \tag{1.2}$$

We use the functions (all on $x > 0$)

$$N(x) = \Pi\big((x,\infty)\big), \qquad M(x) = \Pi\big((-\infty,-x)\big),$$
$$L(x) = N(x) + M(x), \qquad D(x) = N(x) - M(x),$$
$$A(x) = \gamma + D(1) - \int_x^1 D(y)\,dy = \gamma + \int_{(x,1]} y\,dD(y) + xD(x)$$

and

$$U(x) = \sigma^2 + 2\int_0^x yL(y)\,dy.$$

As is explained in Doney and Maller (2002), A and U play the rôles that the truncated mean and truncated variance do in the random walk situation.

We assume throughout that $L(0+) > 0$; in the contrary case X is either a pure drift, or a Brownian motion, and our results are essentially trivial.

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2. Stability of X at 0. We begin with weak stability. Define the "two-sided" exit time

$$T(r) = \inf\{t > 0 : |X(t)| > r\}, \qquad r > 0. \tag{2.1}$$

THEOREM 1. *We have*

$$\frac{|X(T(r))|}{r} \xrightarrow{P} 1 \qquad \text{as } r \downarrow 0, \tag{2.2}$$

if and only if

$$X \in D_0(N) \cup RS \quad (\text{at } 0). \tag{2.3}$$

To explain (2.3): $D_0(N)$ (at 0) is the *domain of attraction of uncentered X to normality*, as $t \downarrow 0$; precisely, $X \in D_0(N)$ (at 0) if there is a non-stochastic (measurable) function $b(t) > 0$ such that $X(t)/b(t) \xrightarrow{D} N(0, 1)$ as $t \downarrow 0$.

RS (at 0) is the class of processes *relatively stable* at 0; $X \in RS$ (at 0) if there is a non-stochastic $b(t) > 0$ such that $X(t)/b(t) \xrightarrow{P} \pm 1$ as $t \downarrow 0$. Equivalent analytic conditions for $D_0(N)$ and RS (both at 0 and at ∞) along with further discussion are in Doney and Maller (2002). In particular under our assumption that $L(0+) > 0$, we have that $X \in D_0(N)$ (at 0) if and only if

$$\frac{U(x)}{x^2 L(x) + x|A(x)|} \to \infty \qquad \text{as } x \downarrow 0, \tag{2.4}$$

and when this happens $U(x)$ is slowly varying as $x \downarrow 0$. Also $X \in RS$ (at 0) if and only if $\sigma^2 = 0$ and

$$\frac{|A(x)|}{xL(x)} \to \infty \text{ or, equivalently, } \frac{x|A(x)|}{U(x)} \to \infty \qquad \text{as } x \downarrow 0, \tag{2.5}$$

and in this case $A(x)$ is slowly varying as $x \downarrow 0$.

$D_0(N)$ (at ∞) and RS (at ∞) are defined in an exactly analogous way, and we have $X \in D_0(N)$ (at ∞) if and only if (2.4) holds (with $x \to \infty$ rather than $x \downarrow 0$), and $X \in RS$ (at ∞) if and only if (2.5) holds (with $x \to \infty$ rather than $x \downarrow 0$).

We mention a result related to Theorem 1 which crops up during the proof of the theorem:

THEOREM 2.

$$\frac{X(T(r))}{r} \quad \text{is tight as } r \downarrow 0, \tag{2.6}$$

if and only if

$$\lim_{\lambda \to \infty} \limsup_{x \to 0+} \frac{x^2 L(x\lambda)}{x|A(x)| + U(x)} = 0. \tag{2.7}$$

The next theorem characterizes a.s. stability.

THEOREM 3. *We have*

$$\frac{|X(T(r))|}{r} \overset{a.s.}{\to} 1 \qquad as\ r \downarrow 0, \tag{2.8}$$

if and only if

$$\sigma^2 > 0 \quad or \quad \sigma^2 = 0, \quad \int_0^1 L(x)\,dx < \infty \quad and \quad \delta = \lim_{x \to 0+} A(x) \neq 0. \tag{2.9}$$

REMARK 1. It should be noted that the two situations in which (2.9) hold are completely different. In the first case the probability that X exits the interval at the top tends to $1/2$, whereas in the second case X has bounded variation, δ is its drift coefficient, and $X(T(r))/r \overset{\text{a.s.}}{\to} 1$ if $\delta > 0$ and $\overset{\text{a.s.}}{\to} -1$ if $\delta < 0$.

Now define the "one-sided" exit time

$$T^*(r) = \inf\{t > 0 : X(t) > r\}, \qquad r > 0. \tag{2.10}$$

Let Z_+ be the upward ladder height subordinator associated with X, and let δ_+ be its drift. (Similarly Z_- will be the ladder height process for $-X$, and δ_- its drift.) Define

$$T_+^*(r) = \inf\{t > 0 : Z_+(t) > r\}, \qquad r > 0. \tag{2.11}$$

Then clearly

$$X(T^*(r)) = Z_+(T_+^*(r)). \tag{2.12}$$

THEOREM 4. *We have*

$$\frac{X(T^*(r))}{r} \overset{a.s.}{\to} 1 \qquad as\ r \downarrow 0, \tag{2.13}$$

if and only if $\delta_+ > 0$.

REMARK 2. The process X is said to creep across the height $r_0 > 0$ if its overshoot $X(T^*(r_0)) - r_0$ at r_0 is zero with positive probability, that is,

$$P\{X(T^*(r_0)) = r_0\} > 0. \tag{2.14}$$

Millar [(1973), Corollary 3.3] shows (under some basic assumptions) that X creeps across (some) $r_0 > 0$ if and only if

$$P\{X(T^*(r)) = r\} \to 1 \qquad \text{as } r \downarrow 0 \tag{2.15}$$

[and then (2.14) holds for all $r_0 > 0$]. This suggests a connection of creeping with the (weak) stability of $X(T^*(r))$. In fact, the connection is closer with strong stability, as Kesten (1969) showed that X creeps (at some, hence all, $r > 0$) if and only if $\delta_+ > 0$. Thus we have:

COROLLARY 1. *X creeps if and only if* (2.13) *holds*.

So far we do not have an analytic equivalence for (2.13), but one is conjectured in the next section.

REMARK 3. The above discussion suggests saying that *the modulus creeps* across the level $r_0 > 0$ if

$$P\{|X(T(r_0))| = r_0\} > 0. \tag{2.16}$$

An interesting problem is that of finding a necessary and sufficient condition, in terms of the characteristics of X, for (2.16) to hold. In view of the one-sided results, it is tempting to think that (2.16) is equivalent to (2.8). However there are examples of Lévy processes which have infinite variation and no Brownian component [so that (2.9), and hence (2.8) fail] which have $\delta_+ > 0$, and then (2.16) holds.

3. Stability of X at ∞. The next three results parallel Theorems 1–3 of Section 2; their proofs are merely sketched in Section 6.

THEOREM 5. *We have*

$$\frac{|X(T(r))|}{r} \xrightarrow{P} 1 \qquad \textit{as } r \to \infty, \tag{3.1}$$

if and only if

$$X \in D_0(N) \cup RS \quad (\textit{at } \infty). \tag{3.2}$$

THEOREM 6.

$$\frac{X(T(r))}{r} \quad \textit{is tight as } r \to \infty, \tag{3.3}$$

if and only if

$$\lim_{\lambda\to\infty} \limsup_{x\to\infty} \frac{x^2 L(x\lambda)}{x|A(x)| + U(x)} = 0. \tag{3.4}$$

THEOREM 7. *We have*

$$\frac{|X(T(r))|}{r} \xrightarrow{a.s.} 1 \qquad \textit{as } r \to \infty, \tag{3.5}$$

if and only if

$$EX^2 < \infty \quad \textit{and} \quad EX = 0 \quad \textit{or} \quad 0 < |EX| \le E|X| < \infty. \tag{3.6}$$

REMARK 4. It should again be noted that the two situations in which (3.6) hold are completely different. In the first case the probability that X exits the interval at the top tends to 1/2, whereas in the second case $X(T(r))/r \stackrel{\text{a.s.}}{\to} 1$ if $EX > 0$ and $\stackrel{\text{a.s.}}{\to} -1$ if $EX < 0$.

Finally we consider one-sided stability at ∞.

THEOREM 8. *Assume X oscillates as $t \to \infty$. Then the following are equivalent*:

$$\frac{X(T^*(r))}{r} \stackrel{a.s.}{\to} 1 \qquad \textit{as } r \to \infty; \tag{3.7}$$

$$X\big(T^*(r)\big) - r \quad \textit{is tight as } r \to \infty; \tag{3.8}$$

$$X\big(T^*(r)\big) - r \quad \textit{has a (proper) limiting distribution as } r \to \infty; \tag{3.9}$$

$$E\big(Z_+(1)\big) < \infty; \tag{3.10}$$

$$E\big(X\big(T^*(r)\big)\big) < \infty \qquad \textit{for all } r > 0; \tag{3.11}$$

$$E\left(\frac{X(T^*(r))}{r}\right) \to 1 \qquad \textit{as } r \to \infty; \tag{3.12}$$

$$\begin{aligned} &E|X(1)| < \infty, \quad E(X(1)) = 0 \quad \textit{and} \\ &J := \int_{[1,\infty)} \frac{xN(x)\,dx}{\int_0^x dy \int_y^\infty M(z)\,dz} < \infty. \end{aligned} \tag{3.13}$$

Alternatively, if $X(t) \to \infty$ a.s. as $t \to \infty$, then (3.7)–(3.12) hold if and only if $0 < EX(1) \le E|X(1)| < \infty$.

REMARK 5. Note that when X oscillates $M(0+) > 0$, so the denominator in (3.13) is positive. The oscillatory behavior of X as $t \to \infty$ is well understood. Rogozin (1966) shows that $\lim_{t\to\infty} X_t = \infty$ a.s., $\lim_{t\to\infty} X_t = -\infty$ a.s., or

$$-\infty = \liminf_{t\to\infty} X_t < \limsup_{t\to\infty} X_t = +\infty \quad \text{a.s.}$$

Integral tests for these [based, essentially, on results of Bertoin (1997)], are in Doney and Maller (2002).

REMARK 6. We conjecture that a variant of the integral in (3.13) is the right one to test for "creeping," that is, a.s. stability of $X(T^*(r))/r$, as $r \downarrow 0$; namely

$$\int_0^1 \frac{N(x)\,dx}{\int_x^1 M(y)\,dy} < \infty. \tag{3.14}$$

4. Preliminary results for the proofs. Our first result is similar, in content and proof, to Proposition 2 of Bertoin [(1996), page 76]. Let

$$U_r(dy) = \int_0^\infty P\left\{\sup_{0\le u<t} |X(u)| \le r, X(t-) \in dy\right\} dt.$$

LEMMA 1. *For* $0 \le |y| \le r < |z|$ *we have*

$$P\{X(T(r)-) \in dy, X(T(r)) \in dz\} = U_r(dy)\Pi(dz-y). \tag{4.1}$$

PROOF. Write $\Delta_t = X(t) - X(t-)$ for the jump in X at time t [we will sometimes write $\Delta(t)$]. We proceed as in Bertoin, using the "compensation formula" [Bertoin (1996), page 7] to get, for Borel functions $f(\cdot)$ and $g(\cdot)$, and $r > 0$,

$$\begin{aligned}
&E\big(f(X(T(r)-))g(X(T(r)))\big)\\
&\quad = E\left(\sum_{t\ge 0} f(X(t-))g(X(t))1\{T(r)=t\}\right)\\
&\quad = E\left(\sum_{t\ge 0} f(X(t-))g(X(t-)+\Delta_t)\right.\\
&\qquad\qquad \left.\times 1\left\{\sup_{0\le u<t}|X(u)|\le r, |X(t-)+\Delta_t|>r\right\}\right)\\
&\quad = \int_0^\infty dt\, E\left(f(X(t-))1\left\{\sup_{0\le u<t}|X(u)|\le r\right\}\right.\\
&\qquad\qquad \left.\times \int_{-\infty}^\infty g(X(t-)+s)1\{|X(t-)+s|>r\}\Pi(ds)\right)\\
&\quad = \int_0^\infty dt \int_{|y|\le r} f(y)\int_{s:|y+s|>r} g(y+s)\Pi(ds)\\
&\qquad\qquad \times P\left\{\sup_{0\le u<t}|X(u)|\le r,\ X(t-)\in dy\right\}\\
&\quad = \int_{|y|\le r}\int_{|z|>r} f(y)g(z)\Pi(dz-y)\int_0^\infty P\left\{\sup_{0\le u<t}|X(u)|\le r, X(t-)\in dy\right\}dt\\
&\quad = \int_{|y|\le r}\int_{|z|>r} f(y)g(z)\Pi(dz-y)\, U_r(dy),
\end{aligned}$$

and the result follows. □

COROLLARY 2. *For* $|z| > r > 0$,

$$P\{X(T(r)) \in dz\} = \int_{|y|\le r} \Pi(dz-y)U_r(dy). \tag{4.2}$$

COROLLARY 3. *For all* $\lambda > 0,\ r > 0,$

$$P\{|X(T(r))| - r > \lambda\}$$
$$= \int_{|y|\le r} \{\Pi((\lambda + r - y, \infty)) + \Pi((-\infty, -(\lambda + r + y)))\} U_r(dy) \tag{4.3}$$
$$= \int_{|y|\le r} \{N(\lambda + r - y) + M(\lambda + r + y)\} U_r(dy).$$

Corollary 3 generalizes Proposition 3.1 of Griffin and McConnell (1992).

COROLLARY 4. *For all* $r > 0,\ \lambda > 1,$

$$L((\lambda + 1)r)E(T(r)) \le P\left\{\frac{|X(T(r))|}{r} > \lambda\right\} \le L((\lambda - 1)r)E(T(r)). \tag{4.4}$$

PROOF. Replace λ by $(\lambda - 1)r$, where $\lambda > 1$, in Corollary 3 to get

$$\begin{aligned} P\{|X(T(r))| > \lambda r\} &= \int_{|y|\le r} (N(\lambda r - y) + M(\lambda r + y)) U_r(dy) \\ &\le (N((\lambda - 1)r) + M((\lambda - 1)r)) U_r([-r, r]) \\ &= L((\lambda - 1)r)E(T(r)), \end{aligned}$$

because

$$\begin{aligned} U_r([-r, r]) &= \int_0^\infty P\left\{\sup_{0\le u<t} |X(u)| \le r, |X(t)| \le r\right\} dt \\ &= \int_0^\infty P\left\{\sup_{0\le u\le t} |X(u)| \le r\right\} dt \\ &= \int_0^\infty P\{T(r) > t\}\, dt = E(T(r)). \end{aligned} \tag{4.5}$$

Similarly,

$$P\{|X(T(r))| > \lambda\} \ge L((\lambda + 1)r)E(T(r)),$$

and the result follows. □

The next lemma is based on inequalities of Pruitt (1981). Let

$$\overline{X}(t) = \sup_{0\le s\le t} |X(s)|$$

and write

$$k(x) = x^{-1}|A(x)| + x^{-2}U(x), \qquad x > 0. \tag{4.6}$$

LEMMA 2. *There are positive constants c_1, c_2, c_3, c_4 such that, for all $r > 0$, $t > 0$,*

$$P\{\overline{X}(t) \geq r\} \leq c_1 t k(r), \qquad P\{\overline{X}(t) \leq r\} \leq \frac{c_2}{t k(r)} \tag{4.7}$$

and

$$\frac{c_3}{k(r)} \leq E\big(T(r)\big) \leq \frac{c_4}{k(r)}. \tag{4.8}$$

Moreover,

$$\frac{1}{\lambda^3} \leq \frac{k(\lambda x)}{k(x)} \leq 3 \qquad \textit{for all } x > 0 \textit{ and } \lambda > 1. \tag{4.9}$$

PROOF. Pruitt gives (4.7), but using a certain function $h(x)$ in place of our $k(x)$, and he also takes $\sigma^2 = 0$; assume this for the moment. A straightforward calculation shows that Pruitt's $h(x)$ is, in our notation,

$$h(x) = \frac{|A(x) - xD(x)|}{x} + \frac{U(x)}{x^2} = \frac{|\gamma + \int_{(x,1]} y\, dD(y)|}{x} + \frac{U(x)}{x^2}.$$

Since $|D(x)| \leq N(x) + M(x) = L(x) \leq U(x)/x^2$, we have immediately that

$$h(x) \leq 2\left(\frac{|A(x)|}{x} + \frac{U(x)}{x^2}\right) = 2k(x).$$

Also

$$\begin{aligned} k(x) &= \frac{|A(x)|}{x} + \frac{U(x)}{x^2} \\ &\leq \frac{|\gamma + \int_{(x,1]} y\, dD(y)|}{x} + xL(x) + \frac{U(x)}{x^2} \\ &\leq \frac{|\gamma + \int_{(x,1]} y\, dD(y)|}{x} + \frac{2U(x)}{x^2} \leq 2h(x). \end{aligned}$$

In other words, for all $x > 0$,

$$\tfrac{1}{2}k(x) \leq h(x) \leq 2k(x).$$

Thus indeed (4.7) holds, provided we can insert σ^2 in the definition of $U(x)$, as we have it. A perusal of Pruitt's proof shows that we can do this (just replace his process X^1 by $X^1 + \sigma^2 B$, where B is an independent BM). Hence (4.7) holds.

Now for $\lambda > 1$,

$$
\begin{aligned}
k(\lambda x) &= \frac{|A(x\lambda)|}{x\lambda} + \frac{U(x\lambda)}{(x\lambda)^2} \\
&\le \frac{|A(x)|}{x\lambda} + \frac{(\lambda-1)}{\lambda}L(x) + \frac{U(x)}{(x\lambda)^2} + \frac{(\lambda^2-1)L(x)}{\lambda^2} \qquad (4.10)\\
&\le \frac{|A(x)|}{x} + \frac{3U(x)}{x^2} \le 3k(x).
\end{aligned}
$$

Also

$$
\begin{aligned}
k(x) &\le \frac{|A(\lambda x)|}{x} + (\lambda-1)L(x) + \frac{U(x)}{x^2} \\
&\le \lambda\frac{|A(\lambda x)|}{\lambda x} + \frac{(\lambda-1)U(x)}{x^2} + \frac{U(x)}{x^2} \\
&\le \lambda\frac{|A(\lambda x)|}{\lambda x} + \lambda^3\frac{U(\lambda x)}{\lambda^2 x^2} \le \lambda^3 k(\lambda x).
\end{aligned}
$$

This gives (4.9).

The proof of (4.8) is essentially the same as that of Pruitt's Theorem 1. First note that for $y > t/2$, $\widetilde{X}(y) = X(y) - X(t/2)$ is independent of $X(t/2)$, so for $r > 0$,

$$
\begin{aligned}
P\{\overline{X}(t) \le r\} &= P\left\{\overline{X}(t/2) \le r, \sup_{t/2<y\le t}\left|X(t/2) + \widetilde{X}(y)\right| \le r\right\} \\
&\le P\{\overline{X}(t/2) \le r\}P\left\{\sup_{t/2<y\le t}\left|X(y) - X(t/2)\right| \le 2r\right\} \\
&\le P\{\overline{X}(t/2) \le r\}P\{\overline{X}(t/2) \le 2r\} \le \left(P\{\overline{X}(t/2) \le 2r\}\right)^2.
\end{aligned}
$$

So, for any $N > 0$,

$$
\begin{aligned}
E(T(r)) &= \int_0^\infty P\{T(r) > t\}\,dt = \int_0^\infty P\{\overline{X}(t) \le r\}\,dt \\
&\le N + \int_N^\infty P\{\overline{X}(t) \le r\}\,dt \\
&\le N + \int_N^\infty \left(P\{\overline{X}(t/2) \le 2r\}\right)^2 dt \\
&= N + 2\int_{N/2}^\infty \left(P\{\overline{X}(t) \le 2r\}\right)^2 dt \le N + 2\int_{N/2}^\infty \frac{c_2^2}{t^2k^2(2r)}\,dt \\
&= N + \frac{4c_2^2}{Nk^2(2r)}.
\end{aligned}
$$

Now choose $N = 1/k(2r)$ and use (4.9) to get

$$E\big(T(r)\big) \le \frac{1+4c_2^2}{k(2r)} \le \frac{c_4}{k(r)}. \tag{4.11}$$

For a lower bound, note that $t \le \{2c_1 k(r)\}^{-1}$ implies $P\{\overline{X}(t) > r\} \le 1/2$ by (4.7), so

$$E\big(T(r)\big) \ge \int_0^{\{2c_1k(r)\}^{-1}} P\big\{\overline{X}(t) \le r\big\}\,dt \ge \int_0^{\{2c_1k(r)\}^{-1}} \frac{1}{2}\,dt = \frac{1}{4c_1k(r)},$$

which together with (4.11) gives (4.8). □

For the one-sided case, let

$$U_r^*(dy) = \int_{t=0}^{\infty} P\big\{X(t-) \in dy, X^*(t-) \le r\big\}\,dt,$$

where

$$X^*(t) = \sup_{0\le u\le t} X(u).$$

Similar working to Lemma 1 gives:

LEMMA 3. *Suppose $U_r^*((-\infty, r]) < \infty$ for all $r \ge 0$. Then for $-\infty < y \le r < z$,*

$$P\big\{X\big(T^*(r)-\big) \in dy, X\big(T^*(r)\big) \in dz\big\} = \Pi(dz-y)U_r^*(dy). \tag{4.12}$$

COROLLARY 5. *Suppose $U_r^*((-\infty, r]) < \infty$ for all $r \ge 0$. Then for $z > r > 0$,*

$$P\big\{X\big(T^*(r)\big) \in dz\big\} = \int_{y\le r} \Pi(dz-y)U_r^*(dy) \tag{4.13}$$

and

$$P\big\{X\big(T^*(r)\big) > z\big\} = \int_{y\le r} N(z-y)U_r^*(dy). \tag{4.14}$$

5. Proofs for Section 2.

PROOF OF THEOREM 1. From (4.4) and Lemma 2 we see that there are constants $c_3 > 0, c_4 > 0$ such that for all $\lambda > 1, r > 0$,

$$\frac{c_3r^2L((\lambda+1)r)}{r|A(r)|+U(r)} \le P\left\{\frac{|X(T(r))|}{r} > \lambda\right\} \le \frac{c_4r^2L((\lambda-1)r)}{r|A(r)|+U(r)}. \tag{5.1}$$

Hence we obtain using (4.9): $|X(T(r))|/r \xrightarrow{P} 1$ as $r \downarrow 0$ if and only if

$$\frac{r|A(r)| + U(r)}{r^2 L(r)} \to \infty \qquad \text{as } r \downarrow 0. \tag{5.2}$$

The next lemma, based on Proposition 3.1 of Griffin and McConnell (1995) [see also Lemma 2.1 of Kesten and Maller (1998)] allows us to break (5.2) up into component parts [cf. Griffin and Maller (1998)].

LEMMA 4. *In the following*, (5.3) *implies* (5.4) *and* (5.5), *and* (5.5) *implies* (5.3):

$$\frac{x|A(x)| + U(x)}{x^2 L(x)} \to \infty \qquad \textit{as } x \downarrow 0; \tag{5.3}$$

$$\frac{x|A(x)|}{U(x)} \to 0 \quad \textit{as } x \downarrow 0 \quad \textit{or} \quad \liminf_{x\downarrow 0} \frac{x|A(x)|}{U(x)} > 0; \tag{5.4}$$

$$\frac{|A(x)|}{xL(x)} \to \infty \quad \textit{as } x \downarrow 0 \quad \textit{or} \quad \frac{U(x)}{x|A(x)| + x^2 L(x)} \to \infty \quad \textit{as } x \downarrow 0. \tag{5.5}$$

PROOF. Let (5.3) hold. Suppose (5.4) fails, so

$$\limsup_{x\downarrow 0} \frac{x|A(x)|}{U(x)} > 0 \quad \text{and} \quad \liminf_{x\downarrow 0} \frac{x|A(x)|}{U(x)} = 0. \tag{5.6}$$

Then, by the continuity of A and U, we can find $\varepsilon > 0$ and $r_k \downarrow 0$, $s_k \downarrow 0$ with $r_k < s_k$ such that

$$\frac{r_k|A(r_k)|}{U(r_k)} = \varepsilon > \frac{s_k|A(s_k)|}{U(s_k)} = \frac{\varepsilon}{2} \tag{5.7}$$

and

$$\frac{y|A(y)|}{U(y)} \le \varepsilon \qquad \text{for all } y \in [r_k, s_k]. \tag{5.8}$$

Take a further subsequence if necessary so that

$$\frac{s_k}{r_k} \to \lambda \in [1, \infty]. \tag{5.9}$$

Choose a constant $D > 1$ as follows: if $\lambda < \infty$ take $D = \lambda + 1$, otherwise $D > 1$ is arbitrary. Then choose $\eta = \eta(\varepsilon, D) > 0$ so that

$$\frac{1 + \eta\varepsilon + \eta\varepsilon^2}{1 - \eta - \eta\varepsilon} < D \wedge 3\lambda/2. \tag{5.10}$$

By (5.3) we can assume r_k so small that

$$r_k L(r_k) \le \frac{\eta\varepsilon}{D^2}\left(\frac{U(r_k)}{r_k} + |A(r_k)|\right). \tag{5.11}$$

Take any $y \in [r_k, Dr_k]$. Then

$$\begin{aligned}
|A(y)| &= \left|\gamma + D(1) - \int_y^1 D(z)\,dz\right| \ge |A(r_k)| - \left|\int_{r_k}^y D(z)\,dz\right| \\
&\ge |A(r_k)| - yL(r_k) \ge |A(r_k)| - Dr_k L(r_k) \\
&\ge |A(r_k)| - \frac{\eta\varepsilon}{D}\left(\frac{U(r_k)}{r_k} + |A(r_k)|\right) \quad \text{[by (5.11)]} \\
&\ge |A(r_k)| - \eta\varepsilon|A(r_k)|(1/\varepsilon + 1) \quad \text{[by(5.7) and } D > 1] \\
&= (1 - \eta - \eta\varepsilon)|A(r_k)|.
\end{aligned}$$

Next,

$$\begin{aligned}
U(y) &= \sigma^2 + 2\int_0^y zL(z)\,dz = U(r_k) + 2\int_{r_k}^y zL(z)\,dz \\
&\le U(r_k) + y^2 L(r_k) \le U(r_k) + D^2 r_k^2 L(r_k) \\
&\le U(r_k) + \eta\varepsilon\big(U(r_k) + r_k|A(r_k)|\big) \quad \text{[by(5.11)]} \\
&= (1 + \eta\varepsilon + \eta\varepsilon^2)U(r_k) \quad \text{[by (5.7)]}.
\end{aligned}$$

Consequently, for $y \in [r_k, Dr_k]$,

$$\frac{y|A(y)|}{U(y)} \ge \left(\frac{y}{r_k}\right)\frac{(1-\eta-\eta\varepsilon)r_k|A(r_k)|}{(1+\eta\varepsilon+\eta\varepsilon^2)U(r_k)} = \left(\frac{y}{r_k}\right)\frac{(1-\eta-\eta\varepsilon)}{(1+\eta+\eta\varepsilon^2)}\varepsilon. \tag{5.12}$$

If $\lambda < \infty$ take $y = s_k$ in (5.12) to get [by (5.7)]

$$\frac{\varepsilon}{2} = \frac{s_k|A(s_k)|}{U(s_k)} \ge \left(\frac{s_k}{r_k}\right)\left(\frac{1-\eta-\eta\varepsilon}{1+\eta+\eta\varepsilon^2}\right)\varepsilon \to \lambda\frac{(1-\eta-\eta\varepsilon)}{(1+\eta+\eta\varepsilon^2)}\varepsilon. \tag{5.13}$$

But by (5.10) the last quantity exceeds $2\varepsilon/3$, giving a contradiction. If $\lambda = \infty$ take $y = Dr_k$ in (5.12) to get

$$\frac{Dr_k|A(Dr_k)|}{U(Dr_k)} \ge D\frac{(1-\eta-\eta\varepsilon)}{(1+\eta+\eta\varepsilon^2)}\varepsilon > \varepsilon \qquad \text{[by (5.10)]}. \tag{5.14}$$

But since $Dr_k \le s_k$ for large k (since $s_k/r_k \to \infty$), $Dr_k \in [r_k, s_k]$, so (5.8) implies that the left-hand side of (5.14) is no greater than ε. Again this is a contradiction. Hence (5.3) implies (5.4).

If (5.3) holds then $x|A(x)| = o(U(x))$ from the left-hand side of (5.4), which together with (5.3) gives the right-hand side of (5.5), or else $U(x) = O(x|A(x)|)$

from the right-hand side of (5.4) which together with (5.3) gives the left-hand side of (5.5). Finally, (5.5) obviously implies (5.3).

Theorem 1 is now immediate from (5.2), Lemma 4 and (2.4)–(2.5). □

PROOF OF THEOREM 2. This is immediate from (4.4) and (4.8). □

Next we aim to prove Theorem 3. The following results are needed.

LEMMA 5. *We have*

$$\int_0^1 \frac{xL(x)\,dx}{x|A(x)|+U(x)} < \infty \tag{5.15}$$

if and only if

$$\sigma^2 > 0 \quad or \quad \sigma^2 = 0, \quad \int_0^1 L(x)\,dx < \infty \quad and \quad \delta = \lim_{x\downarrow 0} A(x) \neq 0. \tag{5.16}$$

PROOF. Clearly (5.16) implies (5.15). Let (5.15) hold. If $\sigma^2 > 0$ then (5.16) holds, so suppose $\sigma^2 = 0$. Our first step is to show that (5.3) holds. To this end note that, by (4.9), $y \le x$ implies $k(x) \le 3k(y)$, thus, given $\varepsilon > 0$, we can choose y so small that

$$\begin{aligned}
\varepsilon &\geq \int_y^{2y} \frac{L(x)\,dx/x}{|A(x)|/x + U(x)/x^2} \\
&\geq \frac{1}{3k(y)} \int_y^{2y} \frac{L(x)\,dx}{x} \geq \frac{L(2y)\log 2}{3k(y)} \asymp \frac{L(2y)}{k(2y)}.
\end{aligned}$$

[We use $f(y) \asymp g(y)$ to mean that $f(y)/g(y)$ is bounded away from 0 and ∞ for all sufficiently small (or large) y.] Hence (5.3) holds.

Now (5.3) implies, by Lemma 4, that (5.5) holds. If the right-hand side of (5.5) holds, so that $x|A(x)| = o(U(x))$, and since $\sigma^2 = 0$, (5.15) gives

$$\infty > \int_0^1 \frac{xL(x)\,dx}{U(x)} = \int_0^1 \frac{xL(x)\,dx}{2\int_0^x yL(y)\,dy},$$

which is impossible. Thus the left-hand side of (5.5) holds, and $\sigma^2 = 0$, so $U(x) = o(xA(x))$ by (2.5). Then (5.15) gives

$$\int_0^1 \frac{L(x)}{|A(x)|} < \infty. \tag{5.17}$$

Suppose $\int_0^1 L(x)\,dx = \infty$. Then

$$|A(x)| \leq |\gamma + D(1)| + \int_x^1 L(y)\,dy \sim \int_x^1 L(y)\,dy \qquad \text{as } x \downarrow 0.$$

But this is impossible, according to (5.17). Hence $\int_0^1 L(x)\,dx < \infty$ and so $\delta = \lim_{x\downarrow 0} A(x)$ exists. If $\delta = 0$ then

$$|A(x)| = \left|\delta + \int_0^x D(y)\,dy\right| = \left|\int_0^x D(y)\,dy\right| \le \int_0^x L(y)\,dy,$$

which again is impossible by (5.17). Thus $\delta \ne 0$ and again (5.16) holds. □

LEMMA 6. *For* $r > 0$ *and* $0 < \varepsilon < 2$,

$$P\{|\Delta(T(r))| > \varepsilon r\} \le L(\varepsilon r) U_r([-r, r]). \tag{5.18}$$

For $r > 0$ *and* $\eta > 2$,

$$P\{|\Delta(T(r))| > \eta r\} = L(\eta r) U_r([-r, r]). \tag{5.19}$$

PROOF. By Lemma 1,

$$\begin{aligned} P\{|\Delta(T(r))| > x\} &\\ = P\{X(T(r)) > X(T(r)-) + x\}& \\ + P\{X(T(r)) < X(T(r)-) - x\}& \\ = \int_{|y|\le r} \left(\int_{z>(y+x)\vee r} + \int_{z<(y-x)\wedge(-r)}\right)& \Pi(dz - y) U_r(dy). \end{aligned} \tag{5.20}$$

By evaluating the inner integral we check the obvious fact that if $x \ge 2r$,

$$\begin{aligned} P\{|\Delta(T(r))| > x\} &= M(x) U_r([-r, r]) + N(x) U_r([-r, r]) \\ &= L(x) U_r([-r, r]), \end{aligned}$$

which establishes (5.19). When $0 < x < 2r$, the same calculation gives

$$\begin{aligned} P\{|\Delta(T(r))| > x\} = \int_{-r\le y\le r-x} & N(r-y) U_r(dy) + N(x) U_r((r-x, r]) \\ &+ M(x) U_r([-r, -r+x]) + \int_{-r+x<y\le r} M(r+y) U_r(dy). \end{aligned}$$

Now put $x = \varepsilon r$, $0 < \varepsilon < 2$. Then the right-hand side of this does not exceed

$$\begin{aligned} &N(\varepsilon r) U_r([-r, (1-\varepsilon)r]) + M(\varepsilon r) U_r((-(1-\varepsilon)r, r]) \\ &\qquad + N(\varepsilon r) U_r((1-\varepsilon)r, r]) + M(\varepsilon r) U_r([-r, -(1-\varepsilon)r]) \\ &= L(\varepsilon r) U_r([-r, r]), \end{aligned}$$

which establishes (5.18). □

PROOF OF THEOREM 3. Let (2.9) hold, so by Lemma 5,

$$\int_0^1 \frac{L(x)\,dx}{xk(x)} = \int_0^1 \frac{xL(x)\,dx}{x|A(x)| + U(x)} < \infty. \tag{5.21}$$

Choose $0 < \lambda < 1$ and $0 < \varepsilon < 1$. By Lemma 2, $U_r([-r, r]) = E(T(r)) \asymp 1/k(r)$, and by (4.9), $k(\varepsilon r) \asymp k(r)$. So by (5.18), for some $c > 0$,

$$\begin{aligned}\sum_{n\ge 0} P\{|\Delta(T(\lambda^n))| > \varepsilon\lambda^n\} &\le c \sum_{n\ge 0} \frac{L(\varepsilon\lambda^n)}{k(\varepsilon\lambda^n)} \le c \sum_{n\ge 0} \frac{\lambda^{-n}}{1-\lambda} \int_{\lambda^{n+1}}^{\lambda^n} \frac{L(\varepsilon y)\,dy}{k(\varepsilon\lambda^n)} \\ &\le \frac{3c\lambda^{-3}}{1-\lambda} \sum_{n\ge 0} \int_{\lambda^{n+1}}^{\lambda^n} \frac{L(\varepsilon y)}{yk(\varepsilon y)}\,dy = \frac{3c\lambda^{-3}}{1-\lambda} \int_0^1 \frac{L(\varepsilon y)\,dy}{yk(\varepsilon y)} \\ &= \frac{3c\lambda^{-3}}{1-\lambda} \int_0^\varepsilon \frac{L(y)\,dy}{yk(y)} < \infty.\end{aligned}$$

Thus

$$\frac{\Delta(T(\lambda^n))}{\lambda^n} \to 0 \quad \text{a.s., as } n \to \infty. \tag{5.22}$$

It follows that, for all large n,

$$\lambda^n < |X(T(\lambda^n))| = |X(T(\lambda^n)-) + \Delta(T(\lambda^n))| \le \lambda^n + o(\lambda^n) \quad \text{a.s.,}$$

so

$$|X(T(\lambda^n))|/\lambda^n \to 1 \quad \text{a.s., as } n \to \infty.$$

Then, given $r > 0$, choose $n = n(\lambda, r)$ such that $\lambda^{n+1} \le r < \lambda^n$. This gives

$$1 < \frac{|X(T(r))|}{r} \le \frac{|X(T(\lambda^n))|}{\lambda^{n+1}} \to \frac{1}{\lambda} \quad \text{a.s., as } r \to 0+.$$

Then let $\lambda \uparrow 1$ to get (2.8).

Conversely, let (2.8) hold. Then, as $n \to \infty$,

$$\frac{|\Delta(T(2^{-n}))|}{2^{-n}} = \frac{|X(T(2^{-n})) - X(T(2^{-n})-)|}{2^{-n}} \overset{\text{a.s.}}{\le} (1+\varepsilon) + 1 = 2 + \varepsilon.$$

Thus

$$P\{|\Delta(T(2^{-n}))| > (2+\varepsilon)2^{-n} \text{ i.o.}\} = 0. \tag{5.23}$$

Suppose we have

$$\sum P\{|\Delta(T(2^{-n}))| > (2+\varepsilon)2^{-n}\} < \infty. \tag{5.24}$$

Then, using (5.19) and the same sort of manipulations as in the first part of the proof, we get

$$\int_0^1 \frac{xL(x)\,dx}{x|A(x)| + U(x)} < \infty; \tag{5.25}$$

hence, via Lemma 5, we have (2.9).

So we need to deduce (5.30) from (5.23). We do this using a version of the Borel–Cantelli lemma, modifying the working of Griffin and Maller (1998). With $\varepsilon > 0$ fixed put $r_n = 2^{-n}$, $x_n = (2+\varepsilon)r_n$ and $B_n = \{|\Delta(T(r_n))| > x_n\}$, so that (5.30) is $\sum P(B_n) < \infty$. Then for $1 \le m < l < \infty$ we have $P(B_m \cap B_l) = \Theta_1 + \Theta_2$, where

$$\Theta_1 = E\left(\sum_{0\le t<u<\infty} 1\{T(r_l) = t, T(r_m) = u, |\Delta_t| > x_l, |\Delta_u| > x_m\}\right) \tag{5.26}$$

and

$$\Theta_2 = E\left(\sum_{0\le t<\infty} 1\{T(r_l) = t, T(r_m) = t, |\Delta_t| > x_m\}\right). \tag{5.27}$$

To estimate Θ_1, note that on the event $\{T(r_l) = t, T(r_m) = u\}$ we have $\overline{X}(t-) \le r_l$, and $\sup_{t\le z<u} |X(z) - X(t)| \le 2r_m$. Thus Θ_1 is not bigger than

$$E\left(\sum_{0\le t<u<\infty} 1\left\{\overline{X}(t-) \le r_l, \sup_{t\le z<u} |X(z) - X(t)| \le 2r_m, |\Delta_t| > x_l, |\Delta_u| > x_m\right\}\right),$$

and this equals

$$\begin{aligned}
&\sum_{0\le t<u<\infty} P\{\overline{X}(t-) \le r_l, |\Delta_t| > x_l\} P\left\{\sup_{0\le z<u-t} |X(z)| \le 2r_m, |\Delta_{u-t}| > x_m\right\} \\
&\quad = \left(\int_0^\infty P\{\overline{X}(t-) \le r_l\}\, dt\, L(x_l)\right)\left(\int_0^\infty P\{\overline{X}(v-) \le 2r_m\}\, dv\, L(x_m)\right) \\
&\quad = E(T(r_l))L(x_l)E(T(2r_m))L(x_m).
\end{aligned} \tag{5.28}$$

Similarly, on the event $\{T(r_l) = T(r_m) = t\}$ we have $\overline{X}(t-) \le r_l$, and $|\Delta_t| > r_m - r_l$, so

$$\begin{aligned}
\Theta_2 &\le E\left(\sum_{t\ge 0} 1\{\overline{X}(t-) \le r_l, |\Delta_t| > x_m \wedge (r_m - r_l)\}\right) \\
&\le E(T(r_l))L(r_m/2).
\end{aligned} \tag{5.29}$$

Since $E(T(r)) \asymp 1/k(r)$ as $r \to 0$ by (4.8), we can find bounds for (5.29). In fact we have, for some $c > 0$,

$$\Theta_1 \le \frac{cL(x_l)L(x_m)}{k(r_m)k(r_l)} \asymp P(B_m)P(B_l) \qquad \text{[by (5.19)]}. \tag{5.30}$$

At this point we note that (2.8) implies the weak stability of $|X(T(r))|$ at 0, so by Theorem 1, X is in RS (at 0) or in $D_0(N)$ (at 0). In the first case $A(x)$ is positive (say) and $A(x)/x$ dominates $U(x)/x^2$ (by (2.5)), while in the second case $U(x)/x^2$ dominates $A(x)/x$ at 0 (by (2.4)). In other words $k(x)$ is asymptotically equivalent to $A(x)/x$ or to $U(x)/x^2$ as $x \downarrow 0$. Now, as mentioned in Section 2, in the first case $A(x)$ is slowly varying and in the second case $U(x)$ is slowly varying, so $k(x)$ is regularly varying with index -1, or regularly varying with index -2, as $x \downarrow 0$. So, for $\lambda > 0$,

$$\frac{k(\lambda x)}{k(x)} \to \lambda^{-1} \text{ or } \lambda^{-2} \qquad \text{as } x \downarrow 0.$$

Thus by well known uniform bounds for regularly varying functions, certainly there is a $c > 0$ such that for all $\lambda > 1$, $x > 0$,

$$\frac{k(\lambda x)}{k(x)} \le \frac{c}{\sqrt{\lambda}}. \tag{5.31}$$

From this it follows that

$$\frac{L(r_m/2)}{k(r_l)} \le \frac{cL(r_m/2)}{k(r_m/2)\sqrt{2^{l-m}}} \le \frac{c'}{\sqrt{2^{l-m}}} P(B_m) \qquad \text{[by (5.19)]}. \tag{5.32}$$

Combining (5.29), (5.30) and (5.32) gives

$$P(B_m \cap B_l) = \Theta_1 + \Theta_2 \le c_1 P(B_m)P(B_l) + \frac{c_2}{\sqrt{2^{l-m}}} P(B_m).$$

Thus, for $N > 1$,

$$\begin{aligned} &\sum_{m=1}^{N-1} \sum_{l=m+1}^{N} P(B_m \cap B_l) \\ &\quad \le c_1 \left(\sum_{m=1}^{N} P(B_m) \right)^2 + c_2 \sum_{m=1}^{N} P(B_m) \sum_{l=m+1}^{N} \frac{1}{\sqrt{2^{l-m}}}. \end{aligned} \tag{5.33}$$

The second term on the right-hand side of (5.33) is $O(\sum_{m=1}^{N} P(B_m))$. Thus if we assume $\sum_{n\ge 1} P(B_m) = \infty$ we have

$$\sum_{m=1}^{N-1} \sum_{l=m+1}^{N} P(B_m \cap B_l) \le (c_1 + o(1)) \left(\sum_{m=1}^{N} P(B_m) \right)^2. \tag{5.34}$$

By Spitzer [(1976), page 317] this implies $P(B_n \text{ i.o.}) > 0$, which is impossible if (5.23) holds. Hence $\sum_{n\ge 1} P(B_n) < \infty$ and we have (5.30). $\square$

PROOF OF THEOREM 4. Simply use (2.12) to see that (2.13) is equivalent to $Z_+(T_+^*(r))/r \to 1$ a.s. Since $Z_+(\cdot) \ge 0$ a.s., $T_+^*(r)$ is also the two-sided exit time for $Z_+(\cdot)$, so (2.9) holds for $Z_+(\cdot)$, and conversely. This is only possible if $\delta_+ > 0$.

6. Proofs for Section 3.

PROOF OF THEOREMS 5–7. These proofs are very similar to corresponding results for random walks given in Griffin and Maller (1998), and can be obtained by modifying the results in Section 2 of the present paper. In fact, (5.1) shows that (3.1) holds if and only if (5.2) holds with $r \to \infty$ rather than $r \downarrow 0$; then similar working as in Lemma 4 gives the equivalence of (3.1) and (3.2). Theorem 6 is immediate from (5.1). For Theorem 7, we need a variant of Lemma 5 which shows that

$$\int_1^\infty \frac{xL(x)\,dx}{x|A(x)| + U(x)} < \infty \tag{6.1}$$

if and only if

$$\begin{aligned} &U(\infty) < \infty \quad \text{and} \quad A(\infty) = 0, \\ \text{or} \quad &\int_1^\infty L(x)\,dx < \infty \quad \text{and} \quad A(\infty) \neq 0. \end{aligned} \tag{6.2}$$

This shows that (6.1) is equivalent to (3.6). Its proof proceeds using (3.2) and similar working as in Lemma 5. The equivalence of (3.5) and (6.1) is proved as in Theorem 3 and Griffin and Maller (1998). □

Next we turn to the proof of Theorem 8. Using (2.12) we can reduce most of the proof to a result concerning subordinators. Thus, let S be a *subordinator* with *infinite lifetime*. Let $\Pi_s(\cdot)$ be the Lévy measure and δ_s the drift of S and put

$$T_s^*(r) = \inf\{t > 0 : S(t) > r\}, \qquad r \geq 0.$$

LEMMA 7. *The following are equivalent*:

$$\frac{S(T_s^*(r))}{r} \to 1 \quad \textit{a.s. as } r \to \infty; \tag{6.3}$$

$$S(T_s^*(r)) - r \quad \textit{is tight as } r \to \infty; \tag{6.4}$$

$$S(T_s^*(r)) - r \textit{ has a (proper) limiting distribution as } r \to \infty; \tag{6.5}$$

$$E(S(1)) < \infty; \tag{6.6}$$

$$E(S(T_s^*(r))) < \infty \qquad \textit{for all } r > 0, \tag{6.7}$$

$$\frac{E(S(T_s^*(r)))}{r} \to 1 \quad \textit{as } r \to \infty. \tag{6.8}$$

PROOF. We first prove that (6.3) holds if and only if (6.6) holds. Let $\mu_s = E(S(1)) \in (0, \infty]$. Let $\tau_0 = 0$ and $\tau_n, n \geq 1$ be the times at which S jumps by at least 1 [i.e., $S(\tau_n) - S(\tau_n-) \geq 1$] [assume $L(1) > 0$, where $L(x) = \Pi_s((x, \infty))$] and let $J_n = S(\tau_n) - S(\tau_n-) \geq 1$ be the corresponding jumps. Let $\Delta_n = S(\tau_n-) - S(\tau_{n-1}), n = 1, 2, \ldots$. Then:

(i) $J_1, J_2, \ldots$ are i.i.d. with $P\{J_1 > x\} = L(x)/L(1),\ x \geq 1$;
(ii) $(\tau_n)_{n\geq 1}$ are the points of a Poisson $(L(1))$ process, independent of the J's;
(iii) $\Delta_1, \Delta_2, \ldots$ are i.i.d. with the distribution of $S^{(1)}(\tau_1)$, where $S^{(1)}$ is a subordinator which results from eliminating the jumps bigger than 1 in S, and closing up the gaps. $S^{(1)}$ is independent of τ_1.

Thus the Laplace exponent of $S^{(1)}$ is

$$\delta_s \lambda + \int_{(0,1]} (1 - e^{-\lambda x}) \Pi_s(dx) \quad (\lambda > 0).$$

Then

$$E\big(S^{(1)}(t)\big) = t\bigg(\delta_s + \int_{(0,1]} x\, \Pi_s(dx)\bigg) = t\mu^{(1)},$$

say, and hence

$$E(\Delta_1) = \mu^{(1)}/L(1) < \infty.$$

Now

$$\sup_{S(\tau_n)\leq x < S(\tau_{n+1}-)} \left(\frac{S(T_s^*(x)) - x}{x}\right) \leq \frac{1}{S(\tau_n)} \to 0 \quad \text{a.s.,} \tag{6.9}$$

because no overshoot in $[S(\tau_n),\ S(\tau_{n+1}-))$ can exceed 1. Also

$$\sup_{S(\tau_{n+1}-)\leq x < S(\tau_{n+1})} \left(\frac{S(T_s^*(x)) - x}{x}\right) = \frac{J_{n+1}}{\sum_{i=1}^{n+1} \Delta_i + \sum_{i=1}^{n} J_i}. \tag{6.10}$$

If $\mu_s < \infty$ then $E(J_1) < \infty$ so $J_{n+1}/\sum_{i=1}^n J_i \to 0$ a.s. by the strong law. Then (6.9) and (6.10) show that (6.6) implies (6.3).

Conversely, if $\mu_s = \infty$ then $(\sum_{i=1}^n \Delta_i + \sum_{i=1}^n J_i)/\sum_{i=1}^n J_i \to 1$ a.s. by the strong law, since $\sum_{i=1}^n \Delta_i/n \to E\Delta_1 < \infty$ a.s. Thus the right-hand side of (6.10) has limsup equal to ∞ a.s. by Kesten (1970). Thus (6.3) cannot hold. So we see that (6.3) and (6.6) are equivalent.

Again let (6.6) hold. Write the Laplace exponent of the subordinator S as

$$\Psi(\lambda) = \delta_s \lambda + \int_{(0,\infty)} (1 - e^{-\lambda x}) \Pi_s(dx).$$

Let

$$U_s(dx) = \int_0^\infty P\{S(t) \in dx\}\, dt$$

be the potential measure of S. This is in fact a renewal measure, see Bertoin and Doney (1994). Now by (6.6) and Wald's lemma,

$$E\big(S(\tau_1)\big) = E\big(S(1)\big)E(\tau_1) = \mu_s/L(1) < \infty,$$

so we have from Bertoin [(1996), Proposition 2, page 76] and the renewal theorem that

$$\begin{aligned} P\big\{S\big(T_s^*(x)\big) - x > \lambda\big\} &= \int_{[0,x)} \Pi_s(\lambda + x - y)U_s(dy) \\ &\to \frac{1}{\mu_s}\int_0^\infty L_s(\lambda + y)\,dy = \frac{1}{\mu_s}\int_\lambda^\infty L_s(y)\,dy. \end{aligned}$$

Thus $S(T^*(x)) - x$ converges in this case to the distribution with density

$$\frac{L_s(y)}{\delta_s + \int_0^\infty L_s(y)\,dy} \qquad \text{on } (0, \infty)$$

and with mass

$$\frac{\delta_s}{\delta_s + \int_0^\infty L_s(y)\,dy} \qquad \text{at } 0.$$

Consequently (6.5) holds.

Clearly (6.5) implies (6.4). Next we show that if (6.6) fails, i.e. $\mu_s = \infty$, then (6.4) also fails. First note that the process $\{S(\tau_n), n \geq 0\}$ is a renewal process with $E\{S(\tau_1)\} = \mu_s/L(1) = \infty$, by assumption. Thus if $\{O_x, x \geq 0\}$ is the corresponding overshoot (or unexpired lifetime) process, we know that $O_x \xrightarrow{P} \infty$ as $x \to \infty$. [See, e.g., Feller (1971), Section XI.4.] Now $S(T_s^*(x)) - x = O_x$ whenever

$$\sum_1^{n-1} J_i + \sum_1^n \Delta_i \leq x < \sum_1^n J_i + \sum_1^n \Delta_i \qquad \text{for some } n \geq 1,$$

and since $E(\Delta_1) < \infty$ it is easily seen that the probability of this event converges to 1 as $x \to \infty$. In other words $S(T_s^*(x)) - x \xrightarrow{P} \infty$ as $x \to \infty$, contradicting (6.4). Thus (6.4) implies (6.6).

Finally (6.6) and (6.7) are equivalent by Wald's lemma, and (6.8) is equivalent to (6.6) by the renewal theorem. □

PROOF OF THEOREM 8. When X either oscillates or $\to +\infty$ a.s. as $t \to \infty$ the ladder height process Z_+ is a subordinator with infinite lifetime, and so the equivalence of (3.7)–(3.12) follows directly from Lemma 7, using (2.12). In the case of drift to $+\infty$ the ladder height process Z_- has finite lifetime, and it is an easy deduction from the Wiener–Hopf factorization of the exponent Ψ [see Bertoin (1996), page 166] that (3.10) holds if and only if $0 < EX(1) \leq E|X(1)| < \infty$. In the oscillatory case Z_- has infinite lifetime, and now the factorization shows that

(3.10) implies $E|X(1)| < \infty$, and hence that $E(X(1)) = 0$, since otherwise X drifts to $\pm\infty$ as $t \to \infty$.

It remains only to show that, when $E|X(1)| < \infty$ and $E(X(1)) = 0$, then (3.7)–(3.12) hold if and only if the integral J in (3.13) is finite. Let σ_n be the nth time at which X has a jump whose absolute value exceeds 1, and put $\sigma_0 = 0$. Let the random walk $Z = (Z_n, n \geq 0)$ be defined by

$$Z_n = X(\sigma_n), \qquad n \geq 0,$$

and let H be the first (strict, increasing) ladder height in this walk. We start by showing that $J < \infty \Leftrightarrow E(H) < \infty$. To see this note that we can write

$$Z_n = \sum_1^n Y_j = \sum_1^n \{W_j + V_j\},$$

where the W_j and the V_j are each i.i.d. sequences, independent of each other and with $W_1 = X(\sigma_1-)$ and $V_1 = X(\sigma_1) - X(\sigma_1-)$. Note that the σ_j form a Poisson process (independent of the V_j) with parameter $L(1)$, and the distribution of the V_j is given by

$$F_V(dx) = P\{V_j \in dx\} = \frac{\Pi(dx)}{L(1)}, \qquad |x| > 1.$$

With this notation, $J < \infty$ is equivalent to $I < \infty$, where

$$I = \int_0^\infty \frac{(1 - F_V(z))z\,dz}{\int_0^z \int_y^\infty F_V(-x)\,dx\,dy}. \tag{6.11}$$

The denominator in (6.11) can be written as

$$\int_0^z yF_V(-y)\,dy + z\int_z^\infty F_V(-y)\,dy = \int_{[0,\infty)} y(y \wedge z)|dF_V(-y)|, \tag{6.12}$$

and hence is positive for $z > 0$, since $M(0+) > 0$ when $EX(1) = 0$.

Also, $E|X(1)| < \infty$ implies $EV_1 < \infty$, and, since W_1 is the value at the exponential time σ_1 of a Lévy process with no jumps exceeding 1 in absolute value, it is easily seen that $E(W_1^2) < \infty$. It follows that $0 < E|Z_1| < \infty$, and, since σ_1 is a stopping time, $E(Z_1) = E(X(\sigma_1)) = 0$.

Next note that if we fix z_0 with $\omega = P\{W_1 \in (0, z_0)\} > 0$, and write F_Y for the distribution function of $Y_1 = V_1 + W_1$, for $z \geq z_0$ we have

$$\begin{aligned} 1 - F_Y(z) = P\{V_1 + W_1 > z\} &\geq \int_0^{z_0} P\{W_1 \in dy\}P\{V_1 > z - y\} \\ &\geq \omega P\{V_1 > z\} = \omega\big(1 - F_V(z)\big). \end{aligned}$$

Also

$$1 - F_Y(z) = P\{V_1 + W_1 > z\} \leq P\{V_1 > z/2\} + P\{W_1 > z/2\},$$

and $\int_0^\infty z P\{W_1 > z/2\}\, dz < \infty$. It follows that we can replace $1 - F_V(z)$ by $1 - F_Y(z)$ in the numerator of (6.11). Similarly, using the fact that

$$\int_0^\infty \int_y^\infty P\{W_1 < x\}\, dx\, dy < \infty,$$

we see that we can replace $F_V(-x)$ by $F_Y(-x)$. Thus we conclude that $I < \infty$ if and only if $\widetilde{I} < \infty$, where

$$\widetilde{I} = \int_0^\infty \frac{z(1 - F_Y(z))\, dz}{\int_0^z \int_y^\infty F_Y(-x)\, dx\, dy}. \tag{6.13}$$

Using the same kind of representation as in (6.12), it follows from Chow (1986) that $E(H) < \infty$ if and only if $\widetilde{I} < \infty$. Thus $J < \infty$ if and only if $E(H) < \infty$.

Introduce the overshoots in X and Z by putting $O_r = X(T^*(r)) - r$, and $\hat{O}_r = Z(\hat{T}_r) - r$, where $\hat{T}_r = \min\{n : Z_n > r\}$. Now assume $E(H) < \infty$. Since any overshoot in X which exceeds 1 is automatically an overshoot in Z, then for any $\lambda > 1$,

$$P\{O_r > \lambda\} = P\{O_r = \hat{O}_r, \hat{O}_r > \lambda\} \le P\{\hat{O}_r > \lambda\}.$$

But a standard renewal theoretic result is that, when $E(H) < \infty$, $\hat{O}_r$ has a nondegenerate limiting distribution, and hence is tight as $r \to \infty$. Thus (3.8) holds, and hence (3.7)–(3.12) hold.

Suppose now that $E(H) = \infty$, and write $H_1 = H$ and $H_1 + H_2 + \cdots + H_n$ for the nth ladder height in Z. Then the $H's$ are i.i.d., so by the strong law and Kesten (1970) we have

$$\lim_{n\to\infty} \frac{H_1 + H_2 + \cdots + H_{n-1}}{n} \overset{\text{a.s.}}{=} \infty$$

and

$$\limsup_{n\to\infty} \frac{H_n}{H_1 + H_2 + \cdots + H_{n-1}} \overset{\text{a.s.}}{=} \infty.$$

So given any $\Delta \in (0, \infty)$ with probability one there exist random integers $n_i \uparrow \infty$ with

$$H_{n_i} > (4\Delta)(H_1 + H_2 + \cdots + H_{n_i-1})$$

and

$$H_1 + H_2 + \cdots + H_{n_i-1} > (1 + \Delta)n_i, \qquad i \ge 1.$$

Consequently if $r_i = H_1 + H_2 + \cdots + H_{n_i-1}$ we have $r_i > \Delta n_i$ and $r_{i+1}/r_i \overset{\text{a.s.}}{\to} \infty$. Putting $\widetilde{r}_i = r_i(1 + \frac{3}{2}\Delta)$ we will show that this implies that a.s.

$$O_{\widetilde{r}_i} > \widetilde{r}_i \tag{6.14}$$

for all sufficiently large i, and hence that (3.7) [and hence also (3.8)–(3.12)] fails. To see this put $N_0 = 0$ and for $i \ge 1$ let N_i denote the ith ladder index in Z, so that

$$H_1 + H_2 + \cdots + H_i = Z_{N_i} = X(\sigma_{N_i}), \qquad i \ge 1.$$

Also introduce the i.i.d. variables

$$M_i = \sup_{\sigma_{N_{i-1}} \le t < \sigma_{N_i}} \{X(t) - X(\sigma_{N_{i-1}})\},$$

and assume for the moment that $E(M_1) < \infty$. Then $\sum_1^\infty P\{M_i > i\Delta/2\} < \infty$, and so a.s. there exists $i^* < \infty$ such that $M_i \le i\Delta/2$ for all $i \ge i^*$. Put $M^* = \sup_{n < i^*} M_i$ and note that for $n \ge i^*$

$$\begin{aligned}\sup_{0 \le t < \sigma_{N_n}} X(t) &\le \max_{1 \le j \le n} X(\sigma_{N_{j-1}}) + M^* + \max_{i^* \le j \le n} M_j \\ &\le H_1 + H_2 + \cdots + H_{n-1} + M^* + \frac{n\Delta}{2}.\end{aligned}$$

Applying this with $n = n_i$ and taking $\Delta > 1$ shows that for all large enough i

$$\begin{aligned}\sup_{0 \le t < \sigma_{N_{n_i}}} X(t) &\le r_i + M^* + \frac{n_i \Delta}{2} \le \frac{3r_i}{2} + M^* \\ &\le \left(1 + \frac{3\Delta}{2}\right) r_i = \widetilde{r}_i.\end{aligned}$$

Since

$$X(\sigma_{N_{n_i}}) = Z(N_{n_i}) = r_i + H_{n_i} > (1 + 4\Delta) r_i > 2\widetilde{r}_i,$$

we have established (6.14), and, noting that $\beta := M(1) + N(1) = 0$ is incompatible with $E(H) = \infty$, the proof is concluded by the following.

LEMMA 8. *Whenever* $E|X(1)| < \infty, E(X) = 0$ *and* $\beta > 0$ *it holds that* $E(M_1) < \infty$.

PROOF. We prove a stronger result, namely that $P\{M_1 > x\} \le c_0 e^{-\lambda_0 x}$ for some $c_0, \lambda_0 > 0$ and all $x \ge 0$. To see this we write

$$X_t = X_t^{(1)} + X_t^{(2)}, \qquad t \ge 0,$$

where $X^{(2)}$ is the compound Poisson process defined by

$$X_t^{(2)} = Z_n = X(\sigma_n) \quad \text{for } t \in [\sigma_n, \sigma_{n+1}), \qquad n = 0, 1, \ldots.$$

Of course $X^{(1)}$ is a Lévy process, independent of $X^{(2)}$, whose characteristics are the same as those of X, except that its Lévy measure is the restriction of Π to $[-1, 1]$. Introduce

$$m_j = \sup_{\sigma_{j-1} \le t < \sigma_j} X(t), \qquad j \ge 1,$$

and note that since $X(\sigma_{j-1}) \le 0$ for $j \le N_1$ we have

$$M_1 \le \max_{1\le j\le N_1} m_j.$$

Here $m_1, m_2, \dots$ are i.i.d., and we can write

$$m_1 = \sup_{0\le t<\sigma_1} X^{(1)}(t),$$

where σ_1, being the time of the first jump in $X^{(2)}$, is independent of $X^{(1)}$, and has the Exp(β) distribution. Also $X_t^{(1)} = X_t^{(0)} + \gamma t$, where $X^{(0)}$ has Lévy exponent

$$\Psi^{(0)}(\theta) = \frac{\sigma^2\theta^2}{2} + \int_{-1}^{1}(1 - e^{i\theta x} + i\theta x)\Pi(dx).$$

Thus $X^{(0)}$ is a martingale whose moment generating function exists, and is given for all real λ by $E(e^{\lambda X_t^{(0)}}) = e^{t\phi^{(0)}(\lambda)}$, where

$$\phi^{(0)}(\lambda) = -\Psi^{(0)}(-i\lambda) = \frac{\sigma^2\lambda^2}{2} + \int_{-1}^{1}(e^{\lambda x} - 1 - \lambda x)\Pi(dx).$$

By a standard martingale inequality

$$P\Big\{\sup_{0\le s<t} X^{(0)}(s) > x\Big\} \le e^{-\lambda x}E\big(e^{\lambda X_t^{(0)}}\big), \qquad \lambda > 0, x \ge 0,$$

and hence

$$P\Big\{\sup_{0\le s<t} X^{(1)}(s) > x\Big\} \le P\Big\{\sup_{0\le s<t} X^{(0)}(s) > x - t|\gamma|\Big\}$$
$$\le e^{-\lambda x}e^{t\{\phi^{(0)}(\lambda)+|\gamma|\lambda\}}.$$

Since $\phi^{(0)}(\lambda) \to 0$ as $\lambda \downarrow 0$, we can choose λ_0 such that $\phi^{(0)}(\lambda_0) + |\gamma|\lambda_0 \le \beta/2$ to conclude, by conditioning on σ_1, that

$$P\{m_1 > x\} \le \int_0^\infty \beta e^{-\beta t}e^{-\lambda_0 x}e^{t\beta/2}\,dt = 2e^{-\lambda_0 x}.$$

Finally, it is clear that

$$P\{M_1 > x|N_1, Z_i, i < N_1\} \le \sum_{j=i}^{N_1} P\{m_j > x - Z_{j-1}\}.$$

Using the duality lemma [see Feller (1971), Section XV.3], we conclude that

$$P\{M_1 > x\} \le \int_0^\infty P\{m_1 > x + z\}G^-(dz),$$

where G^- is the renewal function in the weak increasing ladder process of $-Z$. It follows that

$$P\{M_1 > x\} \le \int_0^\infty 2e^{-\lambda_0(z+x)}G^-(dz) := c_0 e^{-\lambda_0 x},$$

and we are finished. $\square$

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