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# STRICT POSITIVITY OF THE DENSITY FOR SIMPLE JUMP PROCESSES USING THE TOOLS OF SUPPORT THEOREMS. APPLICATION TO THE KAC EQUATION WITHOUT CUTOFF

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Consider the one-dimensional solution $X = \{X_t\}_{t\in[0,T]}$ of a possibly degenerate stochastic differential equation driven by a (non compensated) Poisson measure. We denote by $\mathcal{M}$ a set of deterministic integer-valued measures associated with the considered Poisson measure. For $m \in \mathcal{M}$, we denote by $S(m) = \{S_t(m)\}_{t\in[0,T]}$ the skeleton associated with $X$. We assume some regularity conditions, which allow to define a sort of "derivative" $DS_t(m)$ of $S_t(m)$ with respect to $m$. Then we fix $t \in ]0, T]$, $y \in \mathbb{R}$, and we prove that as soon there exists $m \in \mathcal{M}$ such that $S_t(m) = y$, $DS_t(m) \neq 0$ and $\Delta S_t(m) = 0$, the law of $X_t$ is bounded below by a nonnegative measure admitting a continuous density not vanishing at $y$. In the case where the law of $X_t$ admits a continuous density $p_t$, this means that $p_t(y) > 0$. We finally apply the described method in order to prove that the solution to a Kac equation without cutoff does never vanish.

**1. Introduction.** Consider the following one-dimensional stochastic differential equation on $[0, T]$:

$$X_t = x_0 + \int_0^t \int_O h(X_{s-}, z)N(ds, dz) + \int_0^t g(X_s)\,ds \tag{1.1}$$

where $O$ is an open subset of $\mathbb{R}$, $N$ is a Poisson measure on $\mathbb{R}^+ \times O$ with intensity measure $\nu(ds, dz) = \varphi(z)\,ds\,dz$. The $C^1$ function $\varphi$: $O \mapsto \mathbb{R}^+$ is supposed to be strictly positive.

The problem we study in the present paper is the following: at which points $y \in \mathbb{R}$ is the law of $X_t$ (for some fixed $t > 0$) bounded below by a measure admitting a continuous density $\theta$ satisfying $\theta(y) > 0$? In other words, if $\mathcal{L}(X_t)$ admits a continuous density $p_t$, we would like to characterize the set $\{p_t > 0\}$.

In [8], a partial answer is given in the more general case where the Poisson measure is compensated: under a strong non-degeneracy assumption, the law of $X_t$ is bounded below by a measure admitting a continuous strictly positive density on $\mathbb{R}$. This result is not optimal. First, it allows to consider almost only the case where $X$ has infinite variations: the non-degeneracy assumption is very strong. Furthermore, we obviously cannot, with such a method, study the case where $X_t$

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is increasing, or a.s. nonnegative: either the density is positive everywhere, or the method used in [8] fails.

This method was adapted from a work of Bally and Pardoux [2], who were dealing with the strict positivity of the density of Wiener functionnals, and from the work of Bichteler, Gravereaux and Jacod [4], who were interested in the stochastic calculus of variations for Poisson functionals.

We now would like to transpose to the Poisson context the ideas of Ben Arous and Léandre [3]; see also Aida, Kusuoka and Stroock [1] and Millet and Sanz [19]. Considering the solution $Y_t$ of a Gaussian stochastic differential equation, they characterize the set of the points of strictly positive density of $Y_t$ by using the usual tools of support theorems. Indeed, they consider the associated "skeleton" $S_t(h)$, for $h$ in an appropriate Cameron–Martin space. Then, instead of "differentiating" $Y_t(\omega)$ with respect to $\omega$, they "differentiate" $S_t(h)$ with respect to $h$. Then they just have to deal with deterministic objects: they prove that the density $p_t$ of $Y_t$ does not vanish at $y \in \mathbb{R}$ if and only if there exists $h$ such that $y = S_t(h)$ and $\frac{\partial}{\partial h} S_t(h) \neq 0$.

We will see that in the Poisson context, the transposed method is quite convincing, since it drives to natural assumptions, and no non-degeneracy condition is needed. We will use the Malliavin calculus for jump processes developped by Bichteler, Gravereaux and Jacod [5] and [4], and the main ideas of Simon [22], who deals with support theorems for jump processes (see also [12]).

We consider here processes with finite variations for two reasons: first, it drives to easier computations, and second, the case with infinite variations is often contained in [8].

Let us mention that to our knowledge almost all the works about lowerbounds of the density for Poisson functionals concern asymptotically small time: see Léandre [17], Ishikawa [14] and Picard [20].

The only known result is that of Léandre [18], who deals with the simpler case where the process $X$ can be written as the sum of its jumps. He also assumes a non-degeneracy condition, which implies that the law of $X_t$ admits a smooth density. However, our method follows the same scheme.

The main motivation of this work is the study of spatially homogeneous Boltzmann equations. Tanaka [23], showed an ingenious way to relate the solution $f(t, v)$ of a Boltzmann equation to the solution $V_t$ of a Poisson driven (nonclassical) S.D.E.: the law of $V_t$ is given by $f(t, v)\, dv$. Using this approach and the Malliavin calculus for jump processes, Graham and Méléard [13], have recently proved some existence and regularity results for the solution of a Kac equation, which is a one-dimensional "caricature" of the Boltzmann equation. These results have been extended to the 2-dimensional case in [9].

Analysts and theoritical physicists are interested in the strict positivity of $f$. In particular, it allows them to deal "rigorously" with the entropy of $f$, and it seems to be usefull for proving the convergence to equilibrium. Pulvirenti and Wennberg have proved in [21] a Maxwellian lowerbound for $f$, by using analytic methods, under a cutoff assumption corresponding to the case where the process $V_t$ has a finite number of jumps a.s. But this assumption is not physically reasonnable, and the method used in [21] breaks down in the non-cutoff case. We have applied, in [10] and [11], the method of [8] in order to prove that when $V_t$ has infinite variations, $f$ does never vanish. Thus a case is still open: what does happen when $V_t$ has finite variations, but an infinite number of jumps? The present method will apply.

This paper is organized as follows. In Section 2, we state our assumptions and main result, and we deal with remarks and examples of applications. In Section 3, we introduce some notations and definitions. Then we state a "support type" proposition, and we prove our main result. The "support type" proposition is proved in Section 4. In Section 5, we use the described method, in order to prove the strict positivity of the solution to a Kac equation without cutoff. Finally, a "jump" version of Gronwall's Lemma is stated and proved in the Appendix.

**2. Statement of the main result.** First of all, let us state our hypothesis.

ASSUMPTION (H). The function $g$ is $C^3$ on $\mathbb{R}$, and its derivatives of order 1 to 3 are bounded. The function $h(x,z)$ is of class $C^3$ on $\mathbb{R}\times O$. The partial derivatives $h^{(n+q)}_{x^n z^q}$ (with $n+q\le 3$) are bounded as soon as $q\ge 1$, and there exists a function $\eta\in L^1(O,\varphi(z)\,dz)$ such that

$$|h(0,z)|+|h'_x(x,z)|+|h''_{xx}(x,z)|+|h'''_{xxx}(x,z)|\le\eta(z). \tag{2.1}$$

Under Assumption (H), equation (1.1) clearly admits a unique solution $X=\{X_t\}_{t\in[0,T]}$, adapted, belonging a.s. to the set of càdlàg functions $\mathbb{D}_T=\mathbb{D}([0,T],\mathbb{R})$, and satisfying

$$E\Big(\sup_{[0,T]}|X_t|\Big)<\infty. \tag{2.2}$$

We now would like to build a skeleton associated with equation (1.1), by following the ideas of Simon [22]. By "skeleton," we mean a family $\{S_.(m)\}_{m\in\mathcal{M}}$ of solutions to ordinary differential equations with jumps, obtained by replacing the Poisson random measure $N$ by deterministic integer-valued measures $m\in\mathcal{M}$ in equation (1.1). This way, we will obtain a rigorous version of the following assertion: let $t\ge 0$ and $y\in\mathbb{R}$ be fixed:

$$\begin{aligned}&\text{there exists } \omega\in\Omega \text{ such that } X_t(\omega)=y\\ &\text{if and only if there exists } m\in\mathcal{M} \text{ such that } S_t(m)=y.\end{aligned} \tag{2.3}$$

This will allow us to know where the law of $X_t$ (for $t$ fixed) may be bounded below.

We first consider an increasing sequence of open subsets $O_p\subset O$, such that $\bigcup_{p\ge 1}O_p=O$ and such that for each $p$, $\int_{O_p}\varphi(z)\,dz<\infty$. (If $\int_O\varphi(z)\,dz<\infty$, then we simply set $O_p = O$.) For each $p$, we consider the set of deterministic integer-valued measures

$$\mathcal{M}_p = \left\{ \sum_{i=1}^{n} \delta_{(t_i, z_i)} \;\middle|\; n \in \mathbb{N},\ 0 < t_1 < \cdots < t_n < T,\ z_i \in O_p \right\} \tag{2.4}$$

with the convention $\sum_1^0 = 0$, and we set

$$\mathcal{M} = \bigcup_p \mathcal{M}_p. \tag{2.5}$$

For each $m = \sum_{i=1}^n \delta_{(t_i, z_i)} \in \mathcal{M}$, we denote by $S_t(m)$ the unique solution of the following deterministic differential equation on $[0, T]$:

$$\begin{aligned} S_t(m) &= x_0 + \int_0^t \int_O h\big(S_{s-}(m), z\big) m(ds, dz) + \int_0^t g\big(S_s(m)\big)\, ds \\ &= x_0 + \sum_{i=1}^n h\big(S_{t_i -}(m), z_i\big) 1_{\{t \ge t_i\}} + \int_0^t g\big(S_s(m)\big)\, ds. \end{aligned} \tag{2.6}$$

Under Assumption (H), one can prove that this equation admits a unique solution belonging to $\mathbb{D}_T$, by applying standard arguments on each time interval $[0, t_1[, [t_1, t_2[, \ldots, [t_n, T]$.

In order to deal with the density of $X_t$, we have to introduce a sort of derivative of $S_t(m)$ with respect to $m$. This will replace the usual "derivative" of $X_t(\omega)$ with respect to $\omega$ (see [5], [4], [8], etc.). To this aim, we introduce some "directions" in which we will be able to "perturb" $S_t(m)$, and then to differentiate the obtained expression.

NOTATION 2.1. $\partial O$ denotes the boundary of $O$ in $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.

DEFINITION 2.2.

1. Let $\alpha(z)$ be a $C^1$ positive function on $O$, going to 0 as $z$ tends to $\partial O$, and such that $|\alpha'| < 1$. Then the following functions are well-defined on $O$:

$$\begin{aligned} \phi_\alpha(z) &= \frac{1}{\varphi(z)} \times \sup\big\{|\varphi'(w)|;\ |w - z| \le |\alpha(z)|\big\}, \\ \xi_\alpha(z) &= |\alpha'(z)| + 3|\alpha(z)|\phi_\alpha(z). \end{aligned} \tag{2.7}$$

We say that $\alpha$ belongs to the class $\mathcal{D}$ if for some contant $c < 1$,

$$|\alpha| + \xi_\alpha \in L^1\big(O, \varphi(z)\, dz\big) \cap L^\infty\big(O, \varphi(z)\, dz\big); \qquad \xi_\alpha(z) \le c. \tag{2.8}$$

2. If $\alpha \in \mathcal{D}$, we set, for each $\lambda \in [-1, 1]$,

$$\gamma_\alpha^\lambda(z) = z + \lambda\alpha(z). \tag{2.9}$$

One easily deduces from the supposed properties that for all $\lambda \in [-1, 1]$, the map $z \mapsto \gamma_\alpha^\lambda(z)$ is an increasing bijection from $O$ into itself. This allows us to define, for each $m \in \mathcal{M}$, the new integer-valued measure $\gamma_\alpha^\lambda(m) \in \mathcal{M}$ by

$$\gamma_\alpha^\lambda(m)(A) = \int_0^T \int_O 1_A\big(s, \gamma_\alpha^\lambda(z)\big) m(ds, dz). \tag{2.10}$$

In other words, if $m = \sum_{i=1}^n \delta_{(t_i, z_i)}$, then $\gamma_\alpha^\lambda(m) = \sum_{i=1}^n \delta_{(t_i, \gamma_\alpha^\lambda(z_i))}$.

We will see in the next section (see Proposition 3.4) that under Assumption (H), for all $m \in \mathcal{M}$, all $\alpha \in \mathcal{D}$ and all $t > 0$, the map $\lambda \mapsto S_t(\gamma_\alpha^\lambda(m))$ is twice differentiable on $[-1, 1]$, thanks to Assumption (H). In particular, $D_\alpha S_t(m) = \frac{\partial}{\partial\lambda} S_t(\gamma_\alpha^\lambda(m))|_{\lambda=0}$ satisfies the linear deterministic equation

$$\begin{aligned} D_\alpha S_t(m) = &\int_0^t \int_O h'_x\big(S_{s-}(m), z\big) D_\alpha S_{s-}(m) m(ds, dz) \\ &+ \int_0^t g'\big(S_s(m)\big) D_\alpha S_s(m)\, ds \\ &+ \int_0^t \int_O h'_z\big(S_{s-}(m), z\big)\alpha(z) m(ds, dz). \end{aligned} \tag{2.11}$$

We now can state our main result.

THEOREM 2.3. *Assume* (H), *and let* $t_0 \in ]0, T]$, $y_0 \in \mathbb{R}$ *be fixed. Assume that there exists* $m_0 \in \mathcal{M}$ *such that, for some* $\alpha \in \mathcal{D}$,

$$y_0 = S_{t_0}(m_0); \qquad m_0\big(\{t_0\} \times O\big) = 0; \qquad D_\alpha S_{t_0}(m_0) \neq 0. \tag{2.12}$$

*Then the law of* $X_{t_0}$ *is bounded below by a nonnegative measure admitting a continuous density* $\theta_{y_0}(y)$ *on* $\mathbb{R}$, *satisfying* $\theta_{y_0}(y_0) > 0$.

*In the case where* $\mathcal{L}(X_{t_0})$ *admits a continuous density* $p_{t_0}$, *this means that* $p_{t_0}(y_0) > 0$.

Let us comment on this result. First notice that, for $t_0 > 0$ fixed, the only points $y$ which may be some points of positive density for $X_{t_0}$ are those $y$ which belong to the interior of the support of the law of $X_{t_0}$. We will prove [see Proposition (3.5)] that the support of the law of $X_.$ is the closure, in $\mathbb{D}_T$ endowed with the Skorokhod topology, of the set $\{S_.(m),\ m \in \mathcal{M}\}$. But we will only deduce that the support of the law of $X_{t_0}$ contains $\{S_{t_0}(m),\ m \in \mathcal{M}, m(\{t_0\} \times O) = 0\}$. This comes from the fact that the application $t_0 \mapsto x(t_0)$, from $\mathbb{D}_T$ into $\mathbb{R}$, is not continuous on $\mathbb{D}_T$, except at the points $x \in \mathbb{D}_T$ not jumping at $t_0$. The condition $m(\{t_0\} \times O) = 0$ implies that $\Delta S_{t_0}(m) = 0$. This explains the two first conditions in (2.12).

Roughly speaking, the last condition in (2.12) implies the existence of some $\varepsilon > 0$ and of a neighborhood $\mathcal{V}$ of $m_0$ in $\mathcal{M}$ such that the map $m \mapsto S_{t_0}(m)$ is a submersion from $\mathcal{V}$ into $[y_0 - \varepsilon, y_0 + \varepsilon]$. More and more heuristically, in view of (2.3) this implies that $\omega \mapsto X_{t_0}(\omega)$ is a local submersion into $[y_0 - \varepsilon, y_0 + \varepsilon]$. Hence, for all $\eta < \varepsilon$, the quantity $P(|X_{t_0} - y_0| < \eta)$ will be (at least) of order $\eta$, which implies that the density of $X_{t_0}$ at $y_0$, obtained as the limit of $\frac{1}{\eta} P(|X_{t_0} - y_0| < \eta)$, is strictly positive.

Let us now deal with remarks which might allow to apply easily Theorem 2.3.

REMARK 2.4. Let $t_0 > 0$ be fixed, and let $]a, b[\subset \mathbb{R}$ ($a$ and $b$ may be infinite). Assume that for each $y_0 \in ]a, b[$, the assumptions of Theorem 2.3 are satisfied. Then the law of $X_{t_0}$ is bounded below by nonnegative measure admitting a continuous density $\theta_{t_0}(y)$ on $\mathbb{R}$, never vanishing on $]a, b[$.

PROOF. Let us write $]a, b[= \bigcup_n K_n$, where $K_n$ is an increasing sequence of compact subsets of $]a, b[$. Then it is not hard to deduce from Theorem 2.3 that for each $n$, there exists a constant $c_n > 0$ such that $\mathcal{L}(X_{t_0})(dy) \geq c_n 1_{K_n}(y)\, dy$. The sequence $c_n$ may be chosen decreasing to 0. Then one can build a continuous function $\theta_{t_0}$ on $\mathbb{R}$, such that for $y \in K_n / K_{n-1}$, $\theta_{t_0}(y) \in [c_{n+1}, c_n]$ and $\theta_{t_0}(y) = 0$ for $y$ outside of $]a, b[$. Then $\mathcal{L}(X_{t_0})(dy) \geq \theta_{t_0}(y)\, dy$, and the remark is proved. $\square$

The second remark shows a simple way to choose the "directions" $\alpha \in \mathcal{D}$.

REMARK 2.5. Let $\tilde{\alpha}$ be a $C^1$ function on $O$, such that $\operatorname{supp} \tilde{\alpha} \subset \bigcup_{i=1}^n [a_i, b_i]$, where $[a_i, b_i]$ are disjoint compact subsets of $O$. Then there exists a constant $\varepsilon > 0$ such that $\varepsilon\tilde{\alpha}$ belongs to $\mathcal{D}$.

The last remark deals with an explicit computation of $D_\alpha S_t(m)$, and the proof is contained in Jacod [15], Jacod and Shiryaev [16] (who consider much more complicated equations).

REMARK 2.6. Let $m = \sum_{i=1}^n \delta_{(t_i, z_i)} \in \mathcal{M}$. Consider the following linear (deterministic) equation:

$$A_t(m) = 1 + \int_0^t \int_O h'_x(S_{s-}(m), z) A_{s-}(m) m(ds, dz) + \int_0^t g'(S_s(m)) A_s(m)\, ds.$$

Then

$$A_t(m) = \exp\biggl(\int_0^t g'(S_s(m))\, ds\biggr) \times \prod_{i=1}^n \bigl(1 + h'_x(S_{t_i -}(m), z_i) 1_{\{t \geq t_i\}}\bigr). \tag{2.13}$$

Assume now that for all $i \in \{1, \dots, n\}$, $1 + h'_x(S_{t_i-}(m), z_i) \neq 0$. Then $A(m)$ does never vanish, and the solution of (2.11) can be written as

$$D_\alpha S_t(m) = A_t(m) \int_0^t \int_O \frac{h'_z(S_{s-}(m), z)}{A_{s-}(m)(1 + h'_x(S_{s-}(m), z))} \alpha(z) m(ds, dz). \tag{2.14}$$

In particular, if for some $i \in \{1, \dots, n\}$,

$$h'_z(S_{t_i-}(m), z_i) \neq 0 \quad \text{ and } \quad \forall\, j \neq i,\ z_j \neq z_i$$

then there exists $\alpha \in \mathcal{D}$ such that $D_\alpha S_t(m) \neq 0$ for all $t > t_i$. [It suffices to choose any $\alpha \in \mathcal{D}$ such that $\alpha(z_i) \neq 0$, but $\alpha(z_j) = 0$ for all $j \neq i$.]

We now give some examples of applications.

EXAMPLE 1. We consider the following SDE:

$$X_t = x_0 + \int_0^t \int_0^1 a(X_{s-}) z N(ds, dz) \tag{2.15}$$

with $\varphi(z) = z^\beta$, for some $\beta > -2$, on $O = ]0, 1[$. If $a$ is $C_b^3$ on $\mathbb{R}$, Assumption (H) is clearly met. Assume now that for some $a_0 > 0$, $a(x) \geq a_0$ for all $x$. Then for all $t > 0$, the law of $X_t$ is bounded below by a positive measure admitting a continuous density $\theta_t$ on $\mathbb{R}$, such that $\theta_t$ does never vanish on $]x_0, +\infty[$. This result is optimal, since for all $t > 0$, $X_t \geq x_0$ a.s.

Indeed, let $t_0 > 0$ and $y_0 > x_0$. Then it is clear, since $a(x) \geq a_0 > 0$ and since $O = ]0, 1[$, that there exists $m_0 = \sum_{i=1}^n \delta_{(t_i, z_i)}$, such that $0 < t_1 < \dots < t_n < t_0$, such that the $z_i$ are distincts, and such that $y_0 = S_{t_0}(m_0)$. Of course, $m(\{t_0\} \times O) = 0$. We thus just have to check that there exists $\alpha \in \mathcal{D}$ such that $D_\alpha S_{t_0}(m_0) \neq 0$. Since the $z_i$ are distincts, there exists $\varepsilon > 0$ such that for all $i \in \{1, \dots, n-1\}$, $z_i \notin ]z_n - \varepsilon, z_n + \varepsilon[$ and such that $]z_n - \varepsilon, z_n + \varepsilon[ \subset ]0, 1[$. We choose $\alpha \in \mathcal{D}$ in such a way that $\alpha(z_n) \neq 0$, and $\operatorname{supp} \alpha \subset [z_n - \varepsilon/2, z_n + \varepsilon/2]$. This way,

$$S_{t_0}(\gamma_\alpha^\lambda(m_0)) = x_0 + \sum_{i=1}^{n-1} a(S_{t_i-}(m_0)) \eta(z_i) + a(S_{t_n-}(m_0)) \times (z_n + \lambda \alpha(z_n))$$

which implies that

$$D_\alpha S_{t_0}(m_0) = a(S_{t_n-}(m_0)) \alpha(z_n) \neq 0. \tag{2.16}$$

Remark 2.4 allows us to conclude.

EXAMPLE 2. We consider the case of the following SDE:

$$X_t = x_0 + \int_0^t X_s\, ds + \int_0^t \int_O a(X_{s-}) \eta(z) N(ds, dz); \tag{2.17}$$

$h(x, z) = a(x)\eta(z)$ is supposed to be nonnegative and to satisfy Assumption (H). We assume that $a(x_0) > 0$, that $\eta'$ does never vanish, and that $\{\eta(z);\ z \in O\} = ]0, +\infty[$. Then for each $t > 0$, the law of $X_t$ is bounded below by a nonnegative measure admitting a continuous density $\theta_t$ on $\mathbb{R}$, never vanishing on $]x_0 e^{t_0}, +\infty[$. This result is optimal, since for all $t > 0$, $X_t \geq x_0 e^t$ a.s.

Let $t_0 > 0$ and $y_0 > x_0 e^{t_0}$ be fixed. One can easily check that if $m = \delta_{(t_1, z_1)} \in \mathcal{M}$, with $t_1 < t_0$, then

$$S_{t_0}(m) = x_0 e^{t_0} + a(x_0 e^{t_1})\eta(z_1) e^{t_0 - t_1}. \tag{2.18}$$

Since $a(x_0) > 0$, since $a$ is continuous, we can choose $t_1 \in ]0, t_0[$ small enough, in order to obtain that $a(x_0 e^{t_1}) > 0$. We thus can choose $z_1 \in O$ such that $\eta(z_1) = (y_0 - x_0 e^{t_0})/(a(x_0 e^{t_1}) e^{t_0 - t_1})$. Then, if $m_0 = \delta_{(t_1, z_1)}$, $S_{t_0}(m_0) = y_0$ and $m_0(\{t_0\} \times O) = 0$. Furthermore, one can easily check that if $\alpha \in \mathcal{D}$, with $\alpha(z_1) \neq 0$,

$$S_{t_0}\big(\gamma_\alpha^\lambda(m_0)\big) = x_0 e^{t_0} + a(x_0 e^{t_1})\eta\big(z_1 + \lambda\alpha(z_1)\big) e^{t_0 - t_1} \tag{2.19}$$

and thus

$$D_\alpha S_{t_0}(m_0) = a(x_0 e^{t_1}) e^{t_0 - t_1} \eta'(z_1)\alpha(z_1) \neq 0. \tag{2.20}$$

Remark 2.4 allows to conclude.

Of course, in every of these particular cases, there may exist simpler arguments, but Theorem 2.3 unifies the proofs.

## 3. Framework.

First we introduce some notation.

NOTATION 3.1. Let $\alpha$ belong to $\mathcal{D}$, and $\lambda \in [-1, 1]$. Recall that the map $\gamma_\alpha^\lambda$ was defined by (2.10). For each $\omega \in \Omega$, we define the new integer-valued random measure $\gamma_\alpha^\lambda(N(\omega))$ on $[0, T] \times O$ by

$$\gamma_\alpha^\lambda\big(N(\omega)\big)(A) = \int_0^T \int_O 1_A\big(s, \gamma_\alpha^\lambda(z)\big) N(\omega, ds, dz). \tag{3.1}$$

We denote by $\mathcal{T}_\alpha^\lambda\colon\ \Omega \mapsto \Omega$ the shift defined (and entirely defined) by $N \circ \mathcal{T}_\alpha^\lambda = \gamma_\alpha^\lambda(N)$.

We will use the following criterion of positivity.

THEOREM 3.2. *Let $X$ be a real-valued random variable on $\Omega$ and let $y_0 \in \mathbb{R}$. Assume that for some $\alpha$ of class $\mathcal{D}$, the map $\lambda \mapsto X \circ \mathcal{T}_\alpha^\lambda$ is a.s. twice differentiable on $[-1, 1]$. Assume that there exists $c > 0$, $\delta > 0$, and $k < \infty$, such that for all $r \in ]0, 1]$,*

$$P\big(\Lambda(r)\big) > 0 \tag{3.2}$$

*where*

$$\Lambda(r)=\left\{|X-y_0|<r,\ \left|\frac{\partial}{\partial\lambda}X\circ\mathcal{T}_\alpha^\lambda\Big|_{\lambda=0}\right|\geq c,\right.$$

(3.3)

$$\left.\sup_{|\lambda|\leq\delta}\left[\left|\frac{\partial}{\partial\lambda}X\circ\mathcal{T}_\alpha^\lambda\right|+\left|\frac{\partial^2}{\partial\lambda^2}X\circ\mathcal{T}_\alpha^\lambda\right|\right]\leq k\right\}.$$

*Then there exists a continuous function* $\theta_{y_0}(.)\colon \mathbb{R}\mapsto\mathbb{R}^+$ *such that* $\theta_{y_0}(y_0)>0$ *and such that for all* $f\in C_b^+(\mathbb{R})$,

$$E\big(f(X)\big)\geq\int_{\mathbb{R}}f(y)\theta_{y_0}(y)\,dy. \tag{3.4}$$

This result is a particular case of Theorem 3.3 in [8]. Let us however give an idea of the proof.

PROOF. Thanks to the definition of the class $\mathcal{D}$, one can check, using the Girsanov Theorem for random measures (see Jacod and Shiryaev [16]), the existence, for each $\lambda$, each $\alpha\in\mathcal{D}$, of a Doléans–Dade martingale $G_t^\lambda>0$ such that $(G_T^\lambda.P)\circ(\mathcal{T}_\alpha^\lambda)^{-1}=P$. Furthermore, $G_T^\lambda$ is a.s. continuous in $\lambda$. Let $f\geq 0$ be a continuous function on $\mathbb{R}$. Then

$$E\big(f(X)\big)=E\big(f(X\circ\mathcal{T}_\alpha^\lambda)G_T^\lambda\big)\geq\frac{1}{2}E\bigg(\int_{-1}^{1}f(X\circ\mathcal{T}_\alpha^\lambda)G_T^\lambda\,d\lambda\,1_{\Lambda(r)}\bigg). \tag{3.5}$$

Using a "uniform version" of the local inverse theorem, one can check the existence of $\beta>0$, $R>0$ (as small as we want) such that for each $\omega\in\Lambda(r)$, the map $\lambda\mapsto\mathcal{T}_\alpha^\lambda(\omega)$ is a diffeomorphism from $V(\omega)\subset\,]{-R},R[$ into $]X\circ\mathcal{T}_\alpha^0(\omega)-\beta, X\circ\mathcal{T}_\alpha^0(\omega)+\beta[\,=\,]X(\omega)-\beta,X(\omega)-\beta[$. We choose $r>0$ in such a way that $r<\beta$. This way, using the substitution $y=X\circ\mathcal{T}_\alpha^\lambda(\omega)$ for each $\omega\in\Lambda(r)$, we obtain

$$\begin{aligned}E\big(f(X)\big)&\geq\frac{1}{2}E\bigg(\int_V f(X\circ\mathcal{T}_\alpha^\lambda)G_T^\lambda\,d\lambda\,1_{\Lambda(r)}\bigg)\\&\geq\frac{1}{2}E\bigg(\int_{X-\beta}^{X+\beta}f(y)\times\frac{G_T^{(X\circ\mathcal{T}_\alpha^\cdot)^{-1}(y)}}{\frac{\partial}{\partial\lambda}[X\circ\mathcal{T}_\alpha^\cdot]((X\circ\mathcal{T}_\alpha^\cdot)^{-1}(y))}\,dy\,1_{\Lambda(r)}\bigg)\\&\geq\int_{\mathbb{R}}f(y)\theta(y)\,dy\end{aligned} \tag{3.6}$$

where, if $\psi$ is a continuous function on $\mathbb{R}$ such that $1_{[0,r]}\leq\psi\leq 1_{[0,\beta]}$,

$$\theta(y)=\frac{1}{2}E\bigg[\psi(|X-y|)\bigg\{1\wedge\frac{G_T^{(X\circ\mathcal{T}_\alpha^\cdot)^{-1}(y)}}{\frac{\partial}{\partial\lambda}[X\circ\mathcal{T}_\alpha^\cdot]((X\circ\mathcal{T}_\alpha^\cdot)^{-1}(y))}\bigg\}1_{\Lambda(r)}\bigg]. \tag{3.7}$$

It is clear that $\theta(y_0) > 0$, and one can prove that $\theta$ is continuous by using the Lebesgue theorem. $\square$

Our aim is of course to apply this result to the solution $X_t$ of equation (1.1). We thus have to check that for all $\alpha \in \mathcal{D}$, all $t \in [0, T]$, the map $\lambda \mapsto X_t \circ \mathcal{T}_\alpha^\lambda$ is sufficiently regular.

PROPOSITION 3.3. *Assume* (H). *Let $X$ be the solution of equation* (1.1), *and let $\alpha \in \mathcal{D}$. Then for all $t \in [0, T]$, the map $\lambda \mapsto X_t^{\lambda,\alpha} = X_t \circ \mathcal{T}_\alpha^\lambda$ is a.s. twice differentiable on* $[-1, 1]$. *For each $\lambda$ fixed, the processes $X_t^{\lambda,\alpha}$, $\frac{\partial}{\partial\lambda} X_t^{\lambda,\alpha}$ and $\frac{\partial^2}{\partial\lambda^2} X_t^{\lambda,\alpha}$ belong a.s. to $\mathbb{D}_T$, and satisfy the following S.D.E.s*:

$$X_t^{\lambda,\alpha} = x_0 + \int_0^t \int_O h\big(X_{s-}^{\lambda,\alpha}, \gamma_\alpha^\lambda(z)\big) N(ds, dz) + \int_0^t g\big(X_s^{\lambda,\alpha}\big)\, ds, \tag{3.8}$$

$$\begin{aligned}\frac{\partial}{\partial\lambda} X_t^{\lambda,\alpha} = &\int_0^t \int_O h'_x\big(X_{s-}^{\lambda,\alpha}, \gamma_\alpha^\lambda(z)\big) \frac{\partial}{\partial\lambda} X_{s-}^{\lambda,\alpha} N(ds, dz) \\ &+ \int_0^t g'\big(X_s^{\lambda,\alpha}\big) \frac{\partial}{\partial\lambda} X_s^{\lambda,\alpha}\, ds \\ &+ \int_0^t \int_O h'_z\big(X_{s-}^{\lambda,\alpha}, \gamma_\alpha^\lambda(z)\big) \alpha(z) N(ds, dz),\end{aligned} \tag{3.9}$$

$$\begin{aligned}\frac{\partial^2}{\partial\lambda^2} X_t^{\lambda,\alpha} = &\int_0^t \int_O h'_x\big(X_{s-}^{\lambda,\alpha}, \gamma_\alpha^\lambda(z)\big) \frac{\partial^2}{\partial\lambda^2} X_{s-}^{\lambda,\alpha} N(ds, dz) \\ &+ \int_0^t g'\big(X_s^{\lambda,\alpha}\big) \frac{\partial^2}{\partial\lambda^2} X_s^{\lambda,\alpha}\, ds \\ &+ \int_0^t \int_O h''_{xx}\big(X_{s-}^{\lambda,\alpha}, \gamma_\alpha^\lambda(z)\big) \bigg(\frac{\partial}{\partial\lambda} X_{s-}^{\lambda,\alpha}\bigg)^2 N(ds, dz) \\ &+ \int_0^t g''\big(X_s^{\lambda,\alpha}\big) \bigg(\frac{\partial}{\partial\lambda} X_s^{\lambda,\alpha}\bigg)^2 ds \\ &+ 2\int_0^t \int_O h''_{zx}\big(X_{s-}^{\lambda,\alpha}, \gamma_\alpha^\lambda(z)\big) \frac{\partial}{\partial\lambda} X_{s-}^{\lambda,\alpha} \alpha(z) N(ds, dz) \\ &+ \int_0^t \int_O h''_{zz}\big(X_{s-}^{\lambda,\alpha}, \gamma_\alpha^\lambda(z)\big) \alpha^2(z) N(ds, dz).\end{aligned} \tag{3.10}$$

This proposition is quite easy to check, using the positivity of the measure $N$. If $N$ is a finite Poisson measure, that is, if $\int_O \varphi(z)\, dz < \infty$, then one can prove, using Assumption (H), Lemma A.1 and equations (3.8), (3.9), (3.10), the existence of a.s. finite random variables $A(\omega)$ and $B(\omega)$ such that for all $t \in [0, T]$, all $\lambda, \lambda + \mu \in [-1, 1]$:

$$\left| X_t^{\lambda+\mu,\alpha} - X_t^{\lambda,\alpha} - \mu \frac{\partial}{\partial \lambda} X_t^{\lambda,\alpha} \right| \le A \times \mu^2, \tag{3.11}$$

$$\left| \frac{\partial}{\partial \lambda} X_t^{\lambda+\mu,\alpha} - \frac{\partial}{\partial \lambda} X_t^{\lambda,\alpha} - \mu \frac{\partial^2}{\partial \lambda^2} X_t^{\lambda,\alpha} \right| \le B \times \mu^2, \tag{3.12}$$

which allows to conclude. If $N$ is infinite, one has to approximate $N$ with a sequence of finite Poisson measures, and to prove the convergences. See [8] for a similar (but more difficult) problem.

We also have to differentiate the skeleton.

PROPOSITION 3.4. *Assume* (H). *Let $m \in \mathcal{M}$ and $\alpha \in \mathcal{D}$ be fixed. Then for all $t \in [0, T]$, the map $\lambda \mapsto S_t(\gamma_\alpha^\lambda(m))$ is twice differentiable on $[-1, 1]$. For each $\lambda$ fixed, the functions $S_t(\gamma_\alpha^\lambda(m))$, $\frac{\partial}{\partial \lambda} S_t(\gamma_\alpha^\lambda(m))$ and $\frac{\partial^2}{\partial \lambda^2} S_t(\gamma_\alpha^\lambda(m))$ belong to $\mathbb{D}_T$ and satisfy the following equations*:

$$\begin{aligned} S_t(\gamma_\alpha^\lambda(m)) = x_0 &+ \int_0^t \int_O h(S_{s-}(\gamma_\alpha^\lambda(m)), \gamma_\alpha^\lambda(z)) m(ds, dz) \\ &+ \int_0^t g(S_{s-}(\gamma_\alpha^\lambda(m)))\, ds, \end{aligned} \tag{3.13}$$

$$\begin{aligned} \frac{\partial}{\partial \lambda} S_t(\gamma_\alpha^\lambda(m)) = &\int_0^t \int_O h'_x(S_{s-}(\gamma_\alpha^\lambda(m)), \gamma_\alpha^\lambda(z)) \frac{\partial}{\partial \lambda} S_{s-}(\gamma_\alpha^\lambda(m)) m(ds, dz) \\ &+ \int_0^t g'(S_s(\gamma_\alpha^\lambda(m))) \frac{\partial}{\partial \lambda} S_{s-}(\gamma_\alpha^\lambda(m))\, ds \\ &+ \int_0^t \int_O h'_z(S_{s-}(\gamma_\alpha^\lambda(m)), \gamma_\alpha^\lambda(z)) \alpha(z) m(ds, dz), \end{aligned} \tag{3.14}$$

$$\begin{aligned} \frac{\partial^2}{\partial \lambda^2} S_t(\gamma_\alpha^\lambda(m)) = &\int_0^t \int_O h'_x(S_{s-}(\gamma_\alpha^\lambda(m)), \gamma_\alpha^\lambda(z)) \frac{\partial^2}{\partial \lambda^2} S_{s-}(\gamma_\alpha^\lambda(m)) m(ds, dz) \\ &+ \int_0^t g'(S_s(\gamma_\alpha^\lambda(m))) \frac{\partial^2}{\partial \lambda^2} S_s(\gamma_\alpha^\lambda(m))\, ds \\ &+ \int_0^t \int_O h''_{xx}(S_{s-}(\gamma_\alpha^\lambda(m)), \gamma_\alpha^\lambda(z)) \left( \frac{\partial}{\partial \lambda} S_{s-}(\gamma_\alpha^\lambda(m)) \right)^2 \\ &\qquad \times m(ds, dz) \\ &+ \int_0^t g''(S_s(\gamma_\alpha^\lambda(m))) \left( \frac{\partial}{\partial \lambda} S_s(\gamma_\alpha^\lambda(m)) \right)^2 ds \end{aligned} \tag{3.15}$$

$$+2\int_0^t \int_O h''_{zx}\big(S_{s-}(\gamma_\alpha^\lambda(m)), \gamma_\alpha^\lambda(z)\big)\frac{\partial}{\partial\lambda}S_{s-}\big(\gamma_\alpha^\lambda(m)\big)\alpha(z)$$
$$\times m(ds, dz)$$
$$+\int_0^t \int_O h''_{zz}\big(S_{s-}(\gamma_\alpha^\lambda(m)), \gamma_\alpha^\lambda(z)\big)\alpha^2(z) m(ds, dz).$$

The proof of this proposition is quite easy: it suffices to use the definition of the differentiability, and to show inequalities as (3.11) and (3.12), by using Assumption (H) and Lemma A.1.

As a final tool, we recall the definition of the Skorokhod distance on $\mathbb{D}_T$. First, the set of the changes of times is defined by

$$\Lambda = \{\psi(t) \in C([0, T], [0, T]) \mid \psi(0) = 0,\ \psi(T) = T,\ \psi \text{ is strictly increasing}\}.$$

The norm on $\Lambda$ is defined by

$$|||\psi||| = \sup_{0\le s<t\le T}\left|\ln\left(\frac{\psi(t)-\psi(s)}{t-s}\right)\right|. \tag{3.16}$$

Finally, if $x$ and $y$ belong to $\mathbb{D}_T$, the distance between $x$ and $y$ is given by

$$\delta(x, y) = \inf_{\psi\in\Lambda}\left\{\sup_{[0,T]}|x(t) - y\circ\psi(t)| + |||\psi|||\right\}. \tag{3.17}$$

Our main result will be proved as a consequence of Theorem 3.2 and of the "support type" proposition below, that will be checked in the next section.

PROPOSITION 3.5. *Let $m \in \mathcal{M}$ and $\alpha \in \mathcal{D}$ be fixed. For all $\varepsilon > 0$, the set*

$$\begin{aligned}\Omega_m^\alpha(\varepsilon) = \Bigg\{ &\sup_{|\lambda|\le 1}\delta\big(X^{\lambda,\alpha}, S(\gamma_\alpha^\lambda(m))\big) \le \varepsilon;\\ &\sup_{|\lambda|\le 1}\delta\left(\frac{\partial}{\partial\lambda}X^{\lambda,\alpha}, \frac{\partial}{\partial\lambda}S(\gamma_\alpha^\lambda(m))\right) \le \varepsilon;\\ &\sup_{|\lambda|\le 1}\delta\left(\frac{\partial^2}{\partial\lambda^2}X^{\lambda,\alpha}, \frac{\partial^2}{\partial\lambda^2}S(\gamma_\alpha^\lambda(m))\right) \le \varepsilon\Bigg\}\end{aligned} \tag{3.18}$$

*has a strictly positive probability.*

Assuming for a moment that this proposition holds, we prove our main result. In order to apply Theorem 3.2, we need two lemmas. The first one is probably a well-known fact about the Skorokhod distance, and can be easily proved.

LEMMA 3.6.

(1) *For all $x$, $y$ in $\mathbb{D}_T$, $\|x\|_\infty \le \|y\|_\infty + \delta(x, y)$.*

(2) *Let $y \in \mathbb{D}_T$ be fixed. Assume that for some $t_0 \in [0, T]$, $\Delta y(t_0) = 0$. Then for all $\varepsilon > 0$, there exists $r(\varepsilon) > 0$ such that for all $x \in \mathbb{D}_T$ satisfying $\delta(x, y) \le r(\varepsilon)$, the following inequality holds*:

$$|x(t_0) - y(t_0)| \le \varepsilon. \tag{3.19}$$

The second one deals with a technical property of the skeleton.

LEMMA 3.7. *Assume* (H). *For all $m \in \mathcal{M}$, $\alpha \in \mathcal{D}$,*

$$\sup_{|\lambda|\le 1,\ 0\le t\le T} \left\{ \left|S_t\big(\gamma_\alpha^\lambda(m)\big)\right| + \left|\frac{\partial}{\partial\lambda} S_t\big(\gamma_\alpha^\lambda(m)\big)\right| + \left|\frac{\partial^2}{\partial\lambda^2} S_t\big(\gamma_\alpha^\lambda(m)\big)\right| \right\} < \infty. \tag{3.20}$$

PROOF. We will only prove that $\sup_{\lambda,t} |\frac{\partial}{\partial\lambda} S_t(\gamma_\alpha^\lambda(m))| < \infty$, because the other cases can be checked similarly. We thus use equation (3.14), an we write $m$ as $\sum_{i=1}^n \delta_{(t_i,z_i)}$. Our aim is to apply Lemma A.1 for each $\lambda$.

First of all, notice that thanks to Assumption (H) and (2.9), for all $x \in \mathbb{R}$, $z \in O$, all $|\lambda| \le 1$,

$$\begin{aligned} \left|h'_x\big(x, \gamma_\alpha^\lambda(z)\big)\right| &\le \left|h'_x(x, z)\right| + \left|\gamma_\alpha^\lambda(z) - z\right| \times \|h''_{xz}\|_\infty \\ &\le K\big\{\eta(z) + |\alpha(z)|\big\}. \end{aligned} \tag{3.21}$$

Hence, for all $i \in \{1, \dots, n\}$,

$$\left|h'_x\big(x, \gamma_\alpha^\lambda(z_i)\big)\right| \le K \sup_{k\in\{1,\dots,n\}} \big\{\eta(z_k) + |\alpha(z_k)|\big\} \le C. \tag{3.22}$$

Thus

$$\begin{aligned} \left|\frac{\partial}{\partial\lambda} S_t\big(\gamma_\alpha^\lambda(m)\big)\right| &\le C \sum_{i=1}^n \left|\frac{\partial}{\partial\lambda} S_{t_i-}\big(\gamma_\alpha^\lambda(m)\big)\right| 1_{\{t\ge t_i\}} \\ &\qquad + \|g'\|_\infty \int_0^t \left|\frac{\partial}{\partial\lambda} S_s\big(\gamma_\alpha^\lambda(m)\big)\right| ds + n\|h'_z\|_\infty \|\alpha\|_\infty \\ &\le K_1 + K_2 \int_0^t \left|\frac{\partial}{\partial\lambda} S_s\big(\gamma_\alpha^\lambda(m)\big)\right| ds \\ &\qquad + K_3 \sum_{i=1}^n \left|\frac{\partial}{\partial\lambda} S_{t_i-}\big(\gamma_\alpha^\lambda(m)\big)\right| 1_{\{t\ge t_i\}} \end{aligned} \tag{3.23}$$

where the constants $K_i$ do not depend on $\lambda$. Applying Lemma A.1, we deduce the existence of a constant $C$, not depending on $\lambda$, such that

$$\sup_{[0,T]} \left|\frac{\partial}{\partial\lambda} S_t\big(\gamma_\alpha^\lambda(m)\big)\right| \le C. \tag{3.24}$$

This concludes the proof. $\square$

We finally prove our main result.

PROOF OF THEOREM 2.3. We consider $t_0 \in ]0,T]$, $m_0 \in \mathcal{M}$, and $y_0 = S_{t_0}(m_0)$. We assume that $m_0(\{t_0\} \times O) = 0$. We know, by assumption, that

$$c_0 = \left|\left\{\frac{\partial}{\partial\lambda} S_{t_0}\big(\gamma_\alpha^\lambda(m)\big)\Big|_{\lambda=0}\right\}\right| > 0 \tag{3.25}$$

for some $\alpha \in \mathcal{D}$, which we now consider. Thanks to Lemma 3.7,

$$\begin{aligned} k_0 &= \sup_{|\lambda|\le 1, 0\le t\le T} \left\{\big|S_t\big(\gamma_\alpha^\lambda(m_0)\big)\big| + \left|\frac{\partial}{\partial\lambda} S_t\big(\gamma_\alpha^\lambda(m_0)\big)\right| + \left|\frac{\partial^2}{\partial\lambda^2} S_t\big(\gamma_\alpha^\lambda(m_0)\big)\right|\right\} \\ &< \infty. \end{aligned} \tag{3.26}$$

Our aim is to prove that for all $r > 0$, there exists $\varepsilon > 0$ such that

$$\Omega_{m_0}^\alpha(\varepsilon) \subset \left\{ \begin{array}{c} |X_{t_0} - y_0| < r; \left|\left\{\dfrac{\partial}{\partial\lambda} X_{t_0}^{\lambda,\alpha}\Big|_{\lambda=0}\right\}\right| \ge c_0/2; \\ \sup\limits_{|\lambda|\le 1}\left[\left|\dfrac{\partial}{\partial\lambda} X_{t_0}^{\lambda,\alpha}\right| + \left|\dfrac{\partial^2}{\partial\lambda^2} X_{t_0}^{\lambda,\alpha}\right|\right] \le k_0 + 1 \end{array}\right\} \tag{3.27}$$

where $\Omega_{m_0}^\alpha(\varepsilon)$ is defined in Proposition 3.5. This will suffice, thanks to Theorem 3.2 and Proposition 3.5.

Let us now check (3.27). Let $\omega \in \Omega_{m_0}^\alpha(\varepsilon)$, for some $\varepsilon > 0$. Since $m_0(\{t_0\} \times O) = 0$, it is clear from equations (3.13) and (3.14) that the càdlàg functions $t \mapsto S_t(m_0)$ and $t \mapsto \frac{\partial}{\partial\lambda} S_t(\gamma_\alpha^\lambda(m_0))|_{\lambda=0}$ are continuous at $t_0$. We thus deduce from Lemma 3.6(2) and the fact that $\omega \in \Omega_{m_0}^\alpha(\varepsilon)$ the existence of a decreasing to 0 function $\zeta(\varepsilon)$, such that

$$|X_{t_0} - y_0| = |X_{t_0} - S_{t_0}(m_0)| = \big|X_{t_0}^{0,\alpha} - S_{t_0}\big(\gamma_\alpha^0(m_0)\big)\big| \le \zeta(\varepsilon) \tag{3.28}$$

and

$$\left|\left\{\frac{\partial}{\partial\lambda} X_{t_0}^{\lambda,\alpha}\Big|_{\lambda=0}\right\}\right| \ge \left|\left\{\frac{\partial}{\partial\lambda} S_{t_0}\big(\gamma_\alpha^\lambda(m_0)\big)\Big|_{\lambda=0}\right\}\right| - \zeta(\varepsilon) \ge c_0 - \zeta(\varepsilon). \tag{3.29}$$

On the other hand, thanks to Lemma 3.6(1), since $\omega \in \Omega_{m_0}^\alpha(\varepsilon)$, it is clear that for all $|\lambda| \le 1$,

$$\begin{aligned} &\sup_{t\in[0,T]}\left[\left|\frac{\partial}{\partial\lambda} X_t^{\lambda,\alpha}\right| + \left|\frac{\partial^2}{\partial\lambda^2} X_t^{\lambda,\alpha}\right|\right] \\ &\quad\le \sup_{t\in[0,T]}\left[\left|\frac{\partial}{\partial\lambda} S_t\big(\gamma_\alpha^\lambda(m)\big)\right| + \left|\frac{\partial^2}{\partial\lambda^2} S_t\big(\gamma_\alpha^\lambda(m)\big)\right|\right] + 2\varepsilon \\ &\quad\le k_0 + 2\varepsilon. \end{aligned} \tag{3.30}$$

We now choose $\varepsilon \in ]0, 1/2]$ small enough, in order that $\zeta(\varepsilon) \le r \wedge (c_0/2)$. This way, (3.27) is clearly satisfied, and this concludes the proof. $\square$

**4. Proof of the "support type" proposition.** Our aim in this section is to prove Proposition 3.5. Thus, in the whole sequel, $p$, $m=\sum_{i=1}^n \delta_{(t_i,z_i)} \in \mathcal{M}_p$ and $\alpha \in \mathcal{D}$ are fixed, and Assumption (H) is assumed. For simplicity, we denote $\gamma^\lambda = \gamma_\alpha^\lambda$ (see (2.9)), $X_t^\lambda = X_t^{\lambda,\alpha}$, and $S_t^\lambda = S_t(\gamma^\lambda(m))$. All the constants $C$ and $K$ below will depend only on the functions $g$ and $h$, on $m$, $\alpha$ and $T$.

We set $t_0=0$, $t_{n+1}=T$ and

$$\zeta_0 = \inf_{i\in\{0,\dots,n\}} |t_{i+1}-t_i|; \qquad d_0 = \inf_{i\in\{1,\dots,n\}} d(z_i, \partial O_p). \tag{4.1}$$

We also set $N_p = N|_{[0,T]\times O_p}$, which is a finite Poisson measure, by $0<T_1<T_2<\cdots<T_\mu<T$ its successive times of jump, and by $Z_1, Z_2, \dots, Z_\mu \in O_p$ the size of its jumps. In other words,

$$N_p(\omega) = \sum_{i=1}^{\mu(\omega)} \delta_{(T_i(\omega), Z_i(\omega))}. \tag{4.2}$$

Finally, we denote by $X^{p,\lambda}$, $\frac{\partial}{\partial\lambda} X^{p,\lambda}$ and $\frac{\partial^2}{\partial\lambda^2} X^{p,\lambda}$ the solutions of equations (3.8), (3.9) and (3.10), where $N$ has been replaced by $N_p$, $X^\lambda$ by $X^{\lambda,p}$, $\frac{\partial}{\partial\lambda}X^\lambda$ by $\frac{\partial}{\partial\lambda}X^{\lambda,p}$, and $\frac{\partial^2}{\partial\lambda^2}X^\lambda$ by $\frac{\partial^2}{\partial\lambda^2}X^{\lambda,p}$.

We begin with a lemma.

LEMMA 4.1. *Let $a \in ]0,\zeta_0/10[$, $b\in ]0, d_0/2[$ and $c>0$ be fixed. Consider the sets*

$$\Gamma_1(a,b) = \{\omega\in\Omega \mid \mu = n, \text{ and } \forall i, t_i - a \le T_i \le t_i, |z_i - Z_i| \le b\}, \tag{4.3}$$

$$\Gamma_2(c) = \left\{\omega\in\Omega \;\Big|\; \int_0^T \int_{O/O_p} (\eta(z)+|\alpha(z)|) N(ds,dz) \le c\right\}. \tag{4.4}$$

*Then the set $\Gamma_1(a,b)\cap\Gamma_2(c)$ has a strictly positive probability.*

PROOF. First of all, notice that $\Gamma_1(a,b)\in\sigma(N_p)$ is clearly independent of $\Gamma_2(c)\in\sigma(N|_{[0,T]\times(O/O_p)})$. On the other hand, it is well known that $\Gamma_1(a,b)$ has a strictly positive probability. We thus just have to check that $P(\Gamma_2(c))>0$.

Consider, for $q\ge p$, the following random variables:

$$\begin{aligned} Z_p^q &= \int_0^T \int_{O_q/O_p} (\eta(z)+|\alpha(z)|) N(ds,dz),\\ Z_q &= \int_0^T \int_{O/O_q} (\eta(z)+|\alpha(z)|) N(ds,dz). \end{aligned} \tag{4.5}$$

We see that $Z_p = Z_p^q + Z_q$ for any $q$. For all $q$, $P(Z_p^q = 0)>0$, because $N|_{[0,T]\times O_q/O_p}$ is a finite Poisson measure. When $q$ goes to infinity, $Z_q$ goes to

0 in $L^1$ (and thus in probability) because $\eta + |\alpha| \in L^1(O, \varphi(z)\, dz)$. But clearly, $Z_p^q$ is independent of $Z_q$ for all $q > p$. Hence for all $q$,

$$P\big(\Gamma_1(c)\big) = P(Z_p \le c) \ge P(Z_p^q = 0) P(Z_q \le c). \tag{4.6}$$

Choosing $q$ large enough, we obtain $P(Z_q \le c) > 0$, and the lemma follows. $\square$

The following lemma proves Proposition 3.5 in the case where $N$ is a finite Poisson measure.

LEMMA 4.2. *For all $\varepsilon > 0$, there exists $a_\varepsilon \in ]0, \zeta_0/10[$ and $b_\varepsilon \in ]0, d_0/2[$ such that*

$$\Gamma_1(a_\varepsilon, b_\varepsilon) \subset \Lambda_1(\varepsilon) \tag{4.7}$$

*where*

$$\begin{aligned} \Lambda_1(\varepsilon) = \bigg\{ &\sup_{|\lambda|\le 1} \delta\big(X^{p,\lambda}, S^\lambda\big) \le \varepsilon;\ \sup_{|\lambda|\le 1} \delta\bigg(\frac{\partial}{\partial\lambda} X^{p,\lambda}, \frac{\partial}{\partial\lambda} S^\lambda\bigg) \le \varepsilon; \\ &\qquad\qquad \sup_{|\lambda|\le 1} \delta\bigg(\frac{\partial^2}{\partial\lambda^2} X^{p,\lambda}, \frac{\partial^2}{\partial\lambda^2} S^\lambda\bigg) \le \varepsilon \bigg\}. \end{aligned} \tag{4.8}$$

PROOF. Let $a \in ]0, \zeta_0/10[$ and $b \in ]0, d_0/2[$ be fixed. We consider $\gamma \in ]2a, \zeta_0/5[$, to be chosen later. The element $\omega \in \Gamma_1(a, b)$ is now fixed.

First of all, we consider the polygonal change of time $\psi \in \Lambda$ defined by

$$\begin{aligned} &\psi(0) = 0, \\ &\forall i \in \{1, \dots, n\}, \quad \psi(T_i - \gamma) = T_i - \gamma,\ \psi(T_i) = t_i,\ \psi(T_i + \gamma) = T_i + \gamma, \\ &\psi(T) = T. \end{aligned} \tag{4.9}$$

Simple computations show that

$$\sup_{[0,T]} |\psi(t) - t| \le a, \qquad |||\psi||| \le 2a/\gamma \tag{4.10}$$

and

$$\int_0^T 1_{\{\psi(s) \ne s\}}\, ds \le 2n\gamma = C\gamma. \tag{4.11}$$

This change of time will allow us to prove the lemma. Indeed, we will check the existence of a constant $K < \infty$, not depending on $a$, $b$, on $\omega \in \Gamma_1(a, b)$ nor on $\lambda \in [-1, 1]$, such that

$$\sup_{t \in [0,T]} \big|X_t^{p,\lambda} - S_{\psi(t)}^\lambda\big| \le K(b + \gamma), \tag{4.12}$$

$$\sup_{t\in[0,T]}\left|\frac{\partial}{\partial\lambda}X_t^{p,\lambda}-\frac{\partial}{\partial\lambda}S_{\psi(t)}^{\lambda}\right|\le K(b+\gamma), \tag{4.13}$$

$$\sup_{t\in[0,T]}\left|\frac{\partial^2}{\partial\lambda^2}X_t^{p,\lambda}-\frac{\partial^2}{\partial\lambda^2}S_{\psi(t)}^{\lambda}\right|\le K(b+\gamma). \tag{4.14}$$

This way, we will obtain, for all $\omega\in\Gamma_1(a,b)$,

$$\begin{aligned}\sup_{|\lambda|\le 1}&\left[\delta\big(X^{p,\lambda},S^{\lambda}\big)+\delta\left(\frac{\partial}{\partial\lambda}X^{p,\lambda},\frac{\partial}{\partial\lambda}S^{\lambda}\right)+\delta\left(\frac{\partial^2}{\partial\lambda^2}X^{p,\lambda},\frac{\partial}{\partial\lambda}S^{\lambda}\right)\right]\\&\le 3K(b+\gamma)+6a/\gamma.\end{aligned}$$

Choosing $b_\varepsilon<(\varepsilon/3K)\wedge(d_0/2)$, $\gamma<(\varepsilon/3K)\wedge(\zeta_0/5)$ and $a_\varepsilon<(\varepsilon\gamma/18)\wedge(\zeta_0/10)\wedge(\gamma/2)$, we will obtain (4.7). We thus just have to prove (4.12), (4.13), and (4.14). Since the three proofs are similar, we will only check (4.13). We thus assume that (4.12) is proved. Then we set $\Delta_t^{\lambda}=\frac{\partial}{\partial\lambda}X_t^{p,\lambda}-\frac{\partial}{\partial\lambda}S_t^{\lambda}$. A direct computation, using equations (3.9), (3.14) and the fact that $1_{\{\psi(t)\ge t_i\}}=1_{\{t\ge T_i\}}$, shows that for all $\omega\in\Gamma_1(a,b)$,

$$\begin{aligned}|\Delta_t^{\lambda}|\le&\sum_{i=1}^{n}\big|h'_x\big(X_{T_i-}^{p,\lambda},\gamma^{\lambda}(Z_i)\big)\big|\times\left|\frac{\partial}{\partial\lambda}X_{T_i-}^{p,\lambda}-\frac{\partial}{\partial\lambda}S_{t_i-}^{\lambda}\right|\times 1_{\{t\ge T_i\}}\\&+\sum_{i=1}^{n}\left|\frac{\partial}{\partial\lambda}S_{t_i-}^{\lambda}\right|\times\left|h'_x\big(X_{T_i-}^{p,\lambda},\gamma^{\lambda}(Z_i)\big)-h'_x\big(S_{t_i-}^{\lambda},\gamma^{\lambda}(Z_i)\big)\right|\times 1_{\{t\ge T_i\}}\\&+\sum_{i=1}^{n}\left|\frac{\partial}{\partial\lambda}S_{t_i-}^{\lambda}\right|\times\left|h'_x\big(S_{t_i-}^{\lambda},\gamma^{\lambda}(Z_i)\big)-h'_x\big(S_{t_i-}^{\lambda},\gamma^{\lambda}(z_i)\big)\right|\times 1_{\{t\ge T_i\}}\\&+\sum_{i=1}^{n}\left|h'_z\big(X_{T_i-}^{p,\lambda},\gamma^{\lambda}(Z_i)\big)\right|\times|\alpha(Z_i)-\alpha(z_i)|\times 1_{\{t\ge T_i\}}\\&+\sum_{i=1}^{n}|\alpha(z_i)|\times\left|h'_z\big(X_{T_i-}^{p,\lambda},\gamma^{\lambda}(Z_i)\big)-h'_z\big(S_{t_i-}^{\lambda},\gamma^{\lambda}(Z_i)\big)\right|\times 1_{\{t\ge T_i\}}\\&+\sum_{i=1}^{n}|\alpha(z_i)|\times\left|h'_z\big(S_{t_i-}^{\lambda},\gamma^{\lambda}(Z_i)\big)-h'_z\big(S_{t_i-}^{\lambda},\gamma^{\lambda}(z_i)\big)\right|\times 1_{\{t\ge T_i\}}\\&+\int_t^{\psi(t)}\big|g'(S_s^{\lambda})\big|\times\left|\frac{\partial}{\partial\lambda}S_s^{\lambda}\right|ds\\&+\int_0^t\left|g'(S_{\psi(s)}^{\lambda})\frac{\partial}{\partial\lambda}S_{\psi(s)}^{\lambda}-g'(S_s^{\lambda})\frac{\partial}{\partial\lambda}S_s^{\lambda}\right|ds\end{aligned} \tag{4.15}$$

$$
\begin{aligned}
&+\int_0^t |g'(X_s^{p,\lambda})| \times \left|\frac{\partial}{\partial\lambda} S^\lambda_{\psi(s)} - \frac{\partial}{\partial\lambda} X_s^{p,\lambda}\right| ds \\
&+\int_0^t \left|\frac{\partial}{\partial\lambda} S^\lambda_{\psi(s)}\right| \times |g'(S^\lambda_{\psi(s)}) - g'(X_s^{p,\lambda})| \, ds \\
\le{}& A_t^\lambda + B_t^\lambda + \cdots + J_t^\lambda.
\end{aligned}
$$

We study these terms one by one. First notice that thanks to Assumption (H), since $\alpha$ belongs to $\mathcal{D}$ and $\omega \in \Gamma_1(a,b)$, for all $x$ in $\mathbb{R}$, all $i$ in $\{1,\dots,n\}$,

$$
\begin{aligned}
\bigl|h'_x\bigl(x,\gamma^\lambda(Z_i)\bigr)\bigr| &\le |h'_x(x,z_i)| + \|h''_{zx}\|_\infty |\gamma^\lambda(Z_i) - z_i| \\
&\le \sup_k \eta(z_k) + K\bigl(|Z_i - z_i| + \|\alpha\|_\infty\bigr) \le K + K(b+K) \le K.
\end{aligned} \tag{4.16}
$$

This way, we obtain, since $t_i = \psi(T_i)$,

$$A_t^\lambda \le K \sum_{i=1}^n \bigl|\Delta^\lambda_{T_i-}\bigr| \times 1_{\{t \ge T_i\}}. \tag{4.17}$$

Using Lemma 3.7, we know that for some $k_0 < \infty$,

$$\sup_{t,\lambda}\left[|S_t^\lambda| + \left|\frac{\partial}{\partial\lambda} S_t^\lambda\right| + \left|\frac{\partial^2}{\partial\lambda^2} S_t^\lambda\right|\right] \le k_0. \tag{4.18}$$

Furthermore, one can check as previously [see (4.16)] that for all $x$ in $\mathbb{R}$, all $i$ in $\{1,\dots,n\}$,

$$\bigl|h''_{xx}\bigl(x,\gamma^\lambda(Z_i)\bigr)\bigr| \le K. \tag{4.19}$$

Since $t_i = \psi(T_i)$, we deduce that

$$B_t^\lambda \le k_0 K \sum_{i=1}^n |X^{p,\lambda}_{T_i-} - S^\lambda_{\psi(T_i)-}| \times 1_{\{t \ge T_i\}}. \tag{4.20}$$

Finally, using (4.12), we obtain

$$B_t^\lambda \le K(b+\gamma). \tag{4.21}$$

It is clear that

$$
\begin{aligned}
C_t^\lambda &\le k_0 \|h''_{zx}\|_\infty \sum_{i=1}^n \bigl|\gamma^\lambda(Z_i) - \gamma^\lambda(z_i)\bigr| \\
&\le K \sum_{i=1}^n \bigl\{|Z_i - z_i| + \|\alpha'\|_\infty |Z_i - z_i|\bigr\} \le Kb
\end{aligned}
$$

and that

$$D_t^\lambda \le \|h_z'\|_\infty \sum_{i=1}^n \|\alpha'\|_\infty |Z_i - z_i| \le Kb. \tag{4.22}$$

Using (4.12) again, we see that

$$E_t^\lambda \le \|\alpha\|_\infty \|h_{zx}''\|_\infty \sum_{i=1}^n |X_{T_i-}^{p,\lambda} - S_{\psi(T_i)-}^\lambda| \le K(b+\gamma). \tag{4.23}$$

One can also check that

$$F_t^\lambda \le \|\alpha\|_\infty \|h_{zz}''\|_\infty \sum_{i=1}^n \big|\gamma^\lambda(Z_i) - \gamma^\lambda(z_i)\big| \le Kb. \tag{4.24}$$

Using (4.18) and (4.10), we obtain

$$G_t^\lambda \le \|g'\|_\infty k_0 |\psi(t) - t| \le Ka. \tag{4.25}$$

Due to (4.18) and (4.11), we see that

$$H_t^\lambda \le 2\|g'\|_\infty k_0 \int_0^T 1_{\{\psi(s)\neq s\}}\, ds \le K\gamma. \tag{4.26}$$

It is immediate that

$$I_t^\lambda \le \|g'\|_\infty \int_0^t |\Delta_s^\lambda|\, ds. \tag{4.27}$$

We finally obtain, thanks to (4.12) and (4.18),

$$J_t^\lambda \le k_0 T \|g''\|_\infty K(b+\gamma) \le K(b+\gamma). \tag{4.28}$$

We thus have proved, since $2a \le \gamma$, that

$$|\Delta_t^\lambda| \le K_1(b+\gamma) + K_2 \int_0^t |\Delta_s^\lambda|\, ds + K_3 \sum_{i=1}^n |\Delta_{T_i-}^\lambda| 1_{\{t \ge T_i\}} \tag{4.29}$$

where the constants $K_i$ do not depend on $\lambda \in [-1, 1]$ nor on $\omega \in \Gamma_1(a, b)$. We now apply Lemma A.1,which yields the existence of a constant $K_4$, such that

$$\sup_{[0,T]} |\Delta_s^\lambda| \le K_4(b+\gamma). \tag{4.30}$$

Hence, for all $\omega \in \Gamma_1(a, b)$, all $\lambda \in [-1, 1]$, (4.13) holds, and the lemma is proved. □

Our aim is now to establish the following result, which will allow to conclude the proof of Proposition 3.5.

LEMMA 4.3. *For all $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that for all $a \in ]0, \zeta_0/10[$, all $b \in ]0, d_0/2[$,*

$$\Gamma_1(a,b) \cap \Gamma_2(c_\varepsilon) \subset \Lambda_2(\varepsilon) \tag{4.31}$$

*where*

$$\begin{aligned}\Lambda_2(\varepsilon) = \Bigg\{ &\sup_{|\lambda|\le 1} \|X^\lambda - X^{p,\lambda}\|_\infty \le \varepsilon;\ \sup_{|\lambda|\le 1} \left\| \frac{\partial}{\partial\lambda} X^\lambda - \frac{\partial}{\partial\lambda} X^{p,\lambda} \right\|_\infty \le \varepsilon; \\ &\sup_{|\lambda|\le 1} \left\| \frac{\partial^2}{\partial\lambda^2} X^\lambda - \frac{\partial^2}{\partial\lambda^2} X^{p,\lambda} \right\|_\infty \le \varepsilon \Bigg\}.\end{aligned} \tag{4.32}$$

In order to prove this result, we have to begin with a technical lemma.

LEMMA 4.4. *There exists $K_0 < \infty$ and $c_0 > 0$ such that for all $a \in ]0, \zeta_0/10[$, all $b \in ]0, d_0/2[$, and all $c < c_0$,*

$$\Gamma_1(a,b) \cap \Gamma_2(c) \subset \left\{ \sup_{|\lambda|\le 1, 0\le t\le T} \left[ |X_t^\lambda| + \left| \frac{\partial}{\partial\lambda} X_t^\lambda \right| + \left| \frac{\partial^2}{\partial\lambda^2} X_t^\lambda \right| \right] \le K_0 \right\}. \tag{4.33}$$

PROOF. A direct computation, using equation (3.8), shows that for all $\lambda \in [-1,1]$, all $\omega \in \Gamma_1(a,b) \cap \Gamma_2(c)$,

$$\begin{aligned}|X_t^\lambda| \le |x_0| &+ \sum_{i=1}^n \left| h\left(X_{T_i-}^\lambda, \gamma^\lambda(Z_i)\right) \right| 1_{\{t \ge T_i\}} \\ &+ \int_0^t \int_{O/O_p} \left| h\left(X_{s-}^\lambda, \gamma^\lambda(z)\right) \right| N(ds,dz) + \int_0^t \left| g(X_s^\lambda) \right| ds.\end{aligned} \tag{4.34}$$

Using Assumption (H), and the fact that $\omega \in \Gamma_1(a,b)$, one easily checks that for all $i \in \{1, \dots, n\}$,

$$\left| h\left(X_{T_i-}^\lambda, \gamma^\lambda(Z_i)\right) \right| \le K\left(1 + |X_{T_i-}^\lambda|\right). \tag{4.35}$$

It is also clear, thanks to Assumption (H) and (2.9), that

$$\left| h\left(X_{s-}^\lambda, \gamma^\lambda(z)\right) \right| \le \left| h(X_{s-}^\lambda, z) \right| + \left| \gamma^\lambda(z) - z \right| \sup_u \left| h'_z(X_{s-}^\lambda, u) \right|, \tag{4.36}$$

$$\le K\left(1 + |X_{s-}^\lambda|\right)\left(\eta(z) + |\alpha(z)|\right) \tag{4.37}$$

and

$$\left| g(X_{s-}^\lambda) \right| \le K\left(1 + |X_{s-}^\lambda|\right). \tag{4.38}$$

Hence

$$|X_t^\lambda| \leq K + K \sum_{i=1}^{n} |X_{T_i-}^\lambda| 1_{\{t \geq T_i\}} + K \int_0^t |X_s^\lambda|\, ds$$
$$+ K \int_0^t \int_{O/O_p} \big(\eta(z) + |\alpha(z)|\big) N(ds, dz)$$
$$+ K \sup_{[0,t]} |X_s^\lambda| \times \int_0^t \int_{O/O_p} \big(\eta(z) + |\alpha(z)|\big) N(ds, dz).$$

But, since the left hand side member is increasing, and since $\omega \in \Gamma_2(c)$, we deduce that if $c \leq 1$,

$$\sup_{[0,t]} |X_s^\lambda| \leq K + K \sum_{i=1}^{n} |X_{T_i-}^\lambda| 1_{\{t \geq T_i\}} + K \int_0^t |X_s^\lambda|\, ds + Kc \times \sup_{[0,t]} |X_s^\lambda|. \tag{4.39}$$

Thus, if $c_0^1 = (1/2K) \wedge 1$, we deduce that as soon as $c \leq c_0^1$,

$$2 \sup_{[0,t]} |X_s^\lambda| \leq K + K \sum_{i=1}^{n} |X_{T_i-}^\lambda| 1_{\{t \geq T_i\}} + K \int_0^t |X_s^\lambda|\, ds. \tag{4.40}$$

Lemma A.1 allows to conclude the existence of a constant $K_0^1$, not depending on $a \in ]0, \zeta_0/10[$, $b \in ]0, d_0/2[$, $c \leq c_0^1$, $\lambda \in [-1, 1]$ nor on $\omega \in \Gamma_1(a, b) \cap \Gamma_2(c)$ such that

$$\sup_{[0,T]} |X_s^\lambda| \leq K_0^1. \tag{4.41}$$

One can check in the same way the existence of $c_0^2 > 0$ and $K_0^2 < \infty$ such that if $c \leq c_0^2$, for all $\lambda \in [-1, 1]$ and all $\omega \in \Gamma_1(a, b) \cap \Gamma_2(c)$,

$$\sup_{[0,T]} \left[ \left| \frac{\partial}{\partial \lambda} X_s^\lambda \right| + \left| \frac{\partial^2}{\partial \lambda^2} X_s^\lambda \right| \right] \leq K_0^2. \tag{4.42}$$

Choosing $c_0 = c_0^1 \wedge c_0^2$ and $K_0 = K_0^1 + K_0^2$ concludes the proof of the lemma. □

We are now able to prove Lemma 4.3.

PROOF OF LEMMA 4.3. First, we consider $a \in ]0, \zeta_0/10[$, $b \in ]0, d_0/2[$, and $c \in ]0, c_0[$. We work with an element $\omega$ of $\Gamma_1(a, b) \cap \Gamma_2(c)$. We have to check that

$$\sup_{\lambda \in [-1,1]} \sup_{t \in [0,T]} |X_t^\lambda - X_t^{p,\lambda}| \leq Kc, \tag{4.43}$$

$$\sup_{\lambda \in [-1,1]} \sup_{t \in [0,T]} \left| \frac{\partial}{\partial \lambda} X_t^\lambda - \frac{\partial}{\partial \lambda} X_t^{p,\lambda} \right| \leq Kc, \tag{4.44}$$

$$\sup_{\lambda\in[-1,1]}\sup_{t\in[0,T]}\left|\frac{\partial^2}{\partial\lambda^2}X_t^\lambda-\frac{\partial^2}{\partial\lambda^2}X_t^{p,\lambda}\right|\le Kc. \tag{4.45}$$

As usual, the proofs of the three inequalities are similar, and we will only check (4.44). We thus assume that (4.43) holds. From now on, $\lambda\in[-1,1]$ is fixed, and we set $V_t^\lambda=\frac{\partial}{\partial\lambda}X_t^\lambda-\frac{\partial}{\partial\lambda}X_t^{p,\lambda}$. One obtains, since $\omega\in\Gamma_1(a,b)\cap\Gamma_2(c)$,

$$\begin{aligned}
|V_t^\lambda| &\le \sum_{i=1}^n\left|\frac{\partial}{\partial\lambda}X_{T_i-}^\lambda\right|\times\left|h_x'\big(X_{T_i-}^\lambda,\gamma^\lambda(Z_i)\big)-h_x'\big(X_{T_i-}^{p,\lambda},\gamma^\lambda(Z_i)\big)\right|\times 1_{\{t\ge T_i\}}\\
&\quad+\sum_{i=1}^n\left|h_x'\big(X_{T_i-}^{p,\lambda},\gamma^\lambda(Z_i)\big)\right|\times|V_{T_i-}^\lambda|\times 1_{\{t\ge T_i\}}\\
&\quad+\int_0^t\int_{O/O_p}\left|h_x'\big(X_{s-}^\lambda,\gamma^\lambda(z)\big)\right|\times\left|\frac{\partial}{\partial\lambda}X_{s-}^\lambda\right|N(ds,dz)\\
&\quad+\int_0^t\left|\frac{\partial}{\partial\lambda}X_s^\lambda\right|\times\left|g'(X_s^\lambda)-g'(X_s^{p,\lambda})\right|ds\\
&\quad+\int_0^t\left|g'(X_s^{p,\lambda})\right|\times|V_s^\lambda|\,ds\\
&\quad+\sum_{i=1}^n|\alpha(Z_i)|\times\left|h_z'\big(X_{T_i-}^\lambda,\gamma^\lambda(Z_i)\big)-h_z'\big(X_{T_i-}^{p,\lambda},\gamma^\lambda(Z_i)\big)\right|\times 1_{\{t\ge T_i\}}\\
&\quad+\int_0^t\int_{O/O_p}\left|h_z'\big(X_{s-}^\lambda,\gamma^\lambda(z)\big)\right|\times|\alpha(z)|N(ds,dz)\\
&\le A_t^\lambda+B_t^\lambda+\cdots+G_t^\lambda.
\end{aligned}\tag{4.46}$$

Let's compute. Thanks to Lemma 4.4, using Assumption (H), the fact that $\omega\in\Gamma_1(a,b)\cap\Gamma_2(c)$, and (4.43), one easily checks that $A_t^\lambda\le Kc$, and that

$$B_t^\lambda\le K\sum_{i=1}^n|V_{T_i-}^\lambda|\times 1_{\{t\ge T_i\}}. \tag{4.47}$$

For the same reasons, we obtain

$$C_t^\lambda+D_t^\lambda+F_t^\lambda+G_t^\lambda\le Kc \tag{4.48}$$

and

$$E_t^\lambda \le K \int_0^t |V_s^\lambda|\, ds. \tag{4.49}$$

We finally can write, for all $\omega \in \Gamma_1(a,b) \cap \Gamma_2(c)$, with $a < \zeta_0/10$, $b < d_0/2$, $c < c_0$,

$$|V_t^\lambda| \le Kc + K\int_0^t |V_s^\lambda|\, ds + K\sum_{i=1}^n |V_{T_i-}^\lambda| \times 1_{\{t \ge T_i\}} \tag{4.50}$$

where $K$ does not depend on $\omega, \lambda, a, b$ nor $c$. Using Lemma A.1 allows to conclude that (4.44) holds, and the lemma is proved. □

We finally are able to conclude.

PROOF OF PROPOSITION 3.5. It is a simple association of the previous lemmas. Let $\varepsilon > 0$ be fixed. Then, thanks to lemmas 4.2 and 4.3,

$$\Gamma_1(a_{\varepsilon/2}, b_{\varepsilon/2}) \cap \Gamma_2(c_{\varepsilon/2}) \subset \Lambda_1(\varepsilon/2) \cap \Lambda_2(\varepsilon/2) \subset \Omega_m^\alpha(\varepsilon). \tag{4.51}$$

Thanks to Lemma 4.1, we deduce that $P(\Omega_m^\alpha(\varepsilon)) > 0$, and the proposition is proved. □

**5. Strict positivity of a solution to a Kac equation.** The Kac equation deals with the density of particles in a gaz, and is a one-dimensional "caricature" of the famous spatially homogeneous Boltzmann equation. We denote by $f(t,v)$ the density of particles which have the velocity $v \in \mathbb{R}$ at the instant $t > 0$. Then

$$\frac{\partial f}{\partial t}(t,v) = \int_{v_* \in \mathbb{R}} \int_{\theta=-\pi}^{\pi} \bigl[f(t,v')f(t,v'_*) - f(t,v)f(t,v_*)\bigr]\beta(\theta)\, d\theta\, dv_* \tag{5.1}$$

where

$$v' = v\cos\theta - v_*\sin\theta, \qquad v'_* = v\sin\theta + v_*\cos\theta \tag{5.2}$$

are the post-collisional velocities. The "cross section" $\beta$ is an even and positive function on $[-\pi,\pi]\backslash\{0\}$ exploding near 0 because of an accumulation of "grazing collisions," but satisfying the physically reasonable assumption

$$\int_0^\pi \theta^2 \beta(\theta)\, d\theta < \infty. \tag{5.3}$$

We are interested in the strict positivity of the solution to (5.1). In the case with cutoff, namely when $\int_0^\pi \beta(\theta)\, d\theta < \infty$, the analysts Pulvirenti and Wennberg have proved in [21] a Maxwellian lowerbound for $f$. It is also proved in [10] that $f$ does never vanish if $\int_0^\pi \theta\beta(\theta)\, d\theta = \infty$. We now would like to study the case where $\int_0^\pi \beta(\theta)\, d\theta = \infty$, but $\int_0^\pi \theta\beta(\theta)\, d\theta < \infty$.

First, we will consider solutions in the following (weak) sense, which is obtained by using a standard integration by parts.

DEFINITION 5.1. Let $P_0$ be a probability measure on $\mathbb{R}$ that admits a moment of order 2. A positive function $f$ on $]0,+\infty[\times\mathbb{R}$ is a weak solution of Eq. (5.1) with initial distribution $P_0$ if for every test function $\phi \in C_b^2(\mathbb{R})$,

$$\begin{aligned}\int_{v\in\mathbb{R}} \phi(v) f(t,v)\,dv = \int_{v\in\mathbb{R}} \phi(v) P_0(dv) & \\ + \int_0^t \int_{v\in\mathbb{R}} \int_{v_*\in\mathbb{R}} \int_{-\pi}^{\pi} \{\phi(v\cos\theta - v_*\sin\theta) - \phi(v)\} & \\ \times f(s,v) f(s,v_*)\beta(\theta)\,d\theta\,dv_*\,dv\,ds. &\end{aligned} \tag{5.4}$$

We now state our assumption.

ASSUMPTION (K). (1) The initial distribution $P_0$ admits a moment of order 2, and $\int_0^\pi \theta\beta(\theta)\,d\theta < \infty$.

(2) $P_0$ is not a Dirac mass at 0. The cross section splits into $\beta = \beta_0 + \beta_1$, where $\beta_1$ is even and positive on $[-\pi,\pi]\backslash\{0\}$, and there exists $k_0 > 0$, $\theta_0 \in ]0,\pi[$ and $r \in ]1,2[$ such that $\beta_0(\theta) = (k_0/|\theta|^r) 1_{[-\theta_0,\theta_0]}(\theta)$.

Following Graham and Méléard [13], we build the following random elements.

NOTATION 5.2. We denote by $N_0$ and $N_1$ two independant Poisson measures on $\mathbb{R}_+ \times [0,1] \times [-\pi,\pi]$, with intensity measures

$$\nu_0(ds,d\alpha,d\theta) = \beta_0(\theta)\,ds\,d\alpha\,d\theta, \qquad \nu_1(ds,d\alpha,d\theta) = \beta_1(\theta)\,ds\,d\alpha\,d\theta. \tag{5.5}$$

We will write $N = N_0 + N_1$. We consider a real-valued random variable $V_0$ independant of $N_0$ and $N_1$, of which the law is $P_0$. We also assume that our probability space is the canonical one associated with the independent random elements $V_0$, $N_0$ and $N_1$:

$$\begin{aligned}&\big(\Omega,\mathcal{F},\{\mathcal{F}_t\},P\big) \\ &\quad = \big(\Omega',\mathcal{F}',\{\mathcal{F}'\},P'\big) \otimes \big(\Omega^0,\mathcal{F}^0,\{\mathcal{F}_t^0\},P^0\big) \otimes \big(\Omega^1,\mathcal{F}^1,\{\mathcal{F}_t^1\},P^1\big).\end{aligned}$$

We will consider $[0,1]$ as a probability space, denote by $d\alpha$ the Lebesgue measure on $[0,1]$, by $E_\alpha$ and $\mathcal{L}_\alpha$ the expectation and law on $([0,1],\mathcal{B}([0,1]),d\alpha)$.

The following results are proved by Desvillettes, Graham and Méléard [6], Theorem 3.6 and Graham and Méléard [13], Theorem 1.6 and Corollary 1.8.

THEOREM 5.3. (i) *Assume* (K)(1). *There exists a càdlàg adapted process* $\{V_t(\omega)\}$ *on* $\Omega$ *and a process* $\{W_t(\alpha)\}$ *on* $[0,1]$ *such that*

$$V_t(\omega) = V_0(\omega) + \int_0^t \int_0^1 \int_{-\pi}^{\pi} \big[(\cos\theta - 1)V_{s-}(\omega) - (\sin\theta)W_{s-}(\alpha)\big] \times N(\omega, ds, d\alpha, d\theta), \tag{5.6}$$

$$\mathcal{L}_\alpha(W) = \mathcal{L}(V) \qquad \text{and} \qquad E\Big(\sup_{[0,T]} V_t^2\Big) < \infty.$$

*The uniqueness in law holds, in the sense that* $\mathcal{L}(V) = \mathcal{L}_\alpha(W)$ *is unique.*

(ii) *Assume* (K). *Then for each* $t > 0$, *the law of* $V_t$ *admits a density* $f(t,.)$ *with respect to the Lebesgue measure on* $\mathbb{R}$. *The obtained function* $f$ *is a solution to the Kac equation* (5.1) *in the sense of Definition* 5.1.

(iii) *Assume, furthermore, that* $P_0$ *admits some moments of all orders. Then for each* $t > 0$, *the function* $f(t,v)$ *is of class* $C^\infty$ *in* $v \in \mathbb{R}$.

The result we will prove in this section is the following.

THEOREM 5.4. *Assume* (K), *and consider the solution* $f$ *in the sense of Definition* 5.1 *of equation* (5.1) *built in Theorem* 5.3. *Then there exists a strictly positive function* $g(t,v)$ *on* $]0,+\infty[\times\mathbb{R}$, *continuous in* $v$, *such that for all* $t > 0$, $f(t,v)\,dv \ge g(t,v)\,dv$.

*If* $f(t,v)$ *is continuous in* $v$ (*e.g., if* $P_0$ *admits some moments of all orders*), *this means that* $f(t,v)$ *does never vanish.*

In the sequel, we will sketch the proof of this result by applying the method described in the previous sections to the solution process $\{V_t\}$ as constructed of equation (5.6) in Theorem 5.3(i). We will always work on the finite time interval $[0,T]$, for some $T > 0$ fixed, which of course suffices. In a first subsection, we will introduce the skeleton associated with $\{V_t\}$, we will define the "directions" associated with $N$, and state an intermediate result, looking like Theorem 2.3. We will sketch the proof of this result in a second subsection. Finally, we will conclude in the last subsection, by studying the skeleton.

We give the following lemma that will be frequently used.

LEMMA 5.5. *Assume* (K)(1). *For all* $t \ge 0$, $\operatorname{supp} P_0 \subset \operatorname{supp} \mathcal{L}(V_t) = \operatorname{supp} \mathcal{L}_\alpha(W_t)$.

The main idea of the proof is very simple. If $N$ were a finite Poisson measure, it would be immediate. One thus has to approximate $N$ with a sequence of finite Poisson measures $N^p$. Then, $V_t$ will be close to $V_0$ on the set where $N^p = 0$ (of which the probability is strictly positive), and $N - N^p$ will go to 0 in a certain sense. One concludes by using the independence, for each $p$, of $V_0$, $N^p$ and $N - N^p$. See [11], Lemma 1.6 for the rigorous proof of a very similar lemma.

5.1. *An intermediate result.* First of all, we introduce the skeleton associated with $\{V_t\}$. Notice that instead of one random element [in the case of equation (1.1)], we have to deal with three: $V_0$, $N_0$, $N_1$. Inspired by Lemma 5.5 and the form of equation (5.6), we consider

$$\mathcal{V}_0 = \operatorname{supp} P_0; \qquad \mathcal{M}_0 = \bigcup_p \mathcal{M}_0^p; \qquad \mathcal{M}_1 = \bigcup_p \mathcal{M}_1^p \tag{5.7}$$

where

$$\mathcal{M}_0^p = \left\{ m = \sum_{i=1}^n \delta_{(t_i,w_i,\theta_i)} \;\middle|\; \begin{array}{l} n \in \mathbb{N},\ 0 < t_1 < \cdots < t_n < T, \\ w_i \in \mathcal{V}_0,\ |\theta_i| \in ]1/p, \theta_0[ \end{array} \right\},$$

$$\mathcal{M}_1^p = \left\{ q = \sum_{i=1}^n \delta_{(t_i,w_i,\theta_i)} \;\middle|\; \begin{array}{l} n \in \mathbb{N},\ 0 < t_1 < \cdots < t_n < T, \\ w_i \in \mathcal{V}_0,\ |\theta_i| \in \operatorname{supp} \beta_1 \cap ]1/p, \pi[ \end{array} \right\}.$$

Then, for $v_0 \in \mathcal{V}_0$, $m \in \mathcal{M}_0$ and $q \in \mathcal{M}_1$, we denote by $S(v_0, m, q)$ the unique solution of the deterministic equation

$$\begin{aligned} S_t(v_0, m, q) = v_0 + \int_0^t \int_{\mathbb{R}} \int_{-\pi}^{\pi} &\{S_{s-}(v_0, m, q)(\cos\theta - 1) - w \sin\theta\} \\ &\times (m + q)(ds, dw, d\theta). \end{aligned} \tag{5.8}$$

We also introduce the following directions in which we will "differentiate" $S(v_0, m, q)$ with respect to $m$.

DEFINITION 5.6.

1. Let $\alpha$ be a $C^1$ function on $[-\theta_0, \theta_0]$. We say that $\alpha$ belongs to $\mathcal{D}$ if $|\alpha(\theta)| \le |\theta|/2$, if $\alpha(-\theta_0) = \alpha(\theta_0) = 0$, if $\xi(\theta) \le 1/2$, and if $\xi \in L^1(\beta_0(\theta)d\theta)$, where

$$\xi(\theta) = |\alpha'(\theta)| + 3r \times 2^{r+1} \frac{|\alpha(\theta)|}{|\theta|}. \tag{5.9}$$

2. If $\alpha \in \mathcal{D}$, we set, for each $\lambda \in [-1, 1]$, $\gamma_\alpha^\lambda(\theta) = \theta + \lambda\alpha(\theta)$, which is an increasing bijection from $]$–$\theta_0, \theta_0[\backslash\{0\}$ into itself. For any $m \in \mathcal{M}_0$, the new integer-valued measure $\gamma_\alpha^\lambda(m)$ still belongs to $\mathcal{M}_0$.

REMARK 5.7. If $\alpha \in \mathcal{D}$, then the assumptions of Definition 2.2 are satisfied in the particular case where $O = ]$–$\theta_0, \theta_0[/\{0\}$, and $\phi(\theta) = \beta_0(\theta)$.

PROOF. First, it is clear that $\alpha$ goes to 0 when $\theta$ goes to $\partial O = \{-\theta_0, 0, \theta_0\}$. Thanks to (5.9), one can check that $|\alpha'| \le 1/2$. Then, for example, for $\theta \in ]0, \theta_0[$,

$$\begin{aligned}\phi_\alpha(\theta) &= \frac{1}{\beta_0(\theta)} \sup_{\bar\theta \in [\theta - |\alpha(\theta)|, \theta + |\alpha(\theta)|]} |\beta_0'(\bar\theta)| \\ &\le \frac{\theta^r}{k_0} \sup_{\bar\theta \in [\theta - |\alpha(\theta)|, \theta + |\alpha(\theta)|]} \frac{rk_0}{\bar\theta^{r+1}} \\ &\le \frac{r\theta^r}{(\theta - |\alpha(\theta)|)^{r+1}} \le \frac{r2^{r+1}}{\theta}\end{aligned} \tag{5.10}$$

where the last inequality comes from the fact that $|\alpha(\theta)| \le |\theta|/2$. Hence,

$$\xi_\alpha(\theta) \le |\alpha'(\theta)| + 3|\alpha(\theta)|r2^{r+1}/\theta \le \xi(\theta) \tag{5.11}$$

where $\xi$ is defined by (5.9). $\square$

One easily checks that for all $v_0 \in \mathcal{V}_0$, all $m \in \mathcal{M}_0$, all $q \in \mathcal{M}_1$, and each $t \ge 0$, the map $\lambda \mapsto S_t(v_0, \gamma_\alpha^\lambda(m), q)$ is twice differentiable on $[-1, 1]$, and that $D_\alpha S_t(v_0, m, q) = \frac{\partial}{\partial\lambda} S_t(v_0, \gamma_\alpha^\lambda(m), q)|_{\lambda=0}$, satisfies the following linear equation:

$$\begin{aligned}D_\alpha S_t(v_0, m, q) = \int_0^t \int_{\mathbb{R}} \int_{-\pi}^{\pi} & D_\alpha S_{s-}(v_0, m, q)(\cos\theta - 1) \\ & \times (m+q)(ds, dw, d\theta) \\ - \int_0^t \int_{\mathbb{R}} \int_{-\pi}^{\pi} & \{S_{s-}(v_0, m, q)\sin\theta + w\cos\theta\} \\ & \times \alpha(z) m(ds, dw, d\theta).\end{aligned} \tag{5.12}$$

The following result will be proved by following the method described in the previous sections.

THEOREM 5.8. *Let $t > 0$, and $y \in \mathbb{R}$ be fixed. Assume that there exists $v_0 \in \mathcal{V}_0$, $m \in \mathcal{M}_0$, $q \in \mathcal{M}_1$ and $\alpha \in \mathcal{D}$, such that*

$$\begin{aligned}& y = S_t(v_0, m, q); \qquad (m+q)(\{t\} \times \mathbb{R} \times [-\pi, \pi]) = 0; \\ & D_\alpha S_t(v_0, m, q) \ne 0.\end{aligned} \tag{5.13}$$

*Then the law of $V_t$ is bounded below by a nonnegative measure admitting a continuous density not vanishing at $y$.*

5.2. *Sketch of the proof of Theorem* 5.8. We first give a criterion of strict positivity. As usual, we define for all $\lambda \in [-1,1]$, $\alpha \in \mathcal{D}$, $\omega \in \Omega$, the perturbed Poisson measure $\gamma_\alpha^\lambda(N_0)$. Then we consider the shift $\mathcal{T}_\alpha^\lambda$ on $\Omega$ defined by

$$V_0 \circ \mathcal{T}_\alpha^\lambda = V_0; \qquad N_0 \circ \mathcal{T}_\alpha^\lambda = \gamma_\alpha^\lambda(N_0); \qquad N_1 \circ \mathcal{T}_\alpha^\lambda = N_1. \tag{5.14}$$

In this situation, Theorem 3.2 still holds (this is a particular case of Theorem 2.3 in [10]). Furthermore, one can check (see [10]) that for all $t > 0$, the map $t \mapsto V_t^{\lambda,\alpha} = V_t \circ \mathcal{T}_\alpha^\lambda$ is a.s. twice differentiable on $[-1,1]$. The following equations are satisfied:

$$\begin{aligned} V_t^{\lambda,\alpha} = V_0 &+ \int_0^t \int_0^1 \int_{-\pi}^{\pi} \Big[ \big(\cos\gamma_\alpha^\lambda(\theta) - 1\big) V_{s-}^{\lambda,\alpha} - \sin\gamma_\alpha^\lambda(\theta) W_{s-}(\alpha) \Big] \\ &\qquad \times N_0(ds, d\alpha, d\theta) \\ &+ \int_0^t \int_0^1 \int_{-\pi}^{\pi} \Big[ (\cos\theta - 1) V_{s-}^{\lambda,\alpha} - \sin\theta W_{s-}(\alpha) \Big] N_1(ds, d\alpha, d\theta), \end{aligned} \tag{5.15}$$

$$\begin{aligned} \frac{\partial}{\partial\lambda} V_t^{\lambda,\alpha} &= \int_0^t \int_0^1 \int_{-\pi}^{\pi} \big(\cos\gamma_\alpha^\lambda(\theta) - 1\big) \frac{\partial}{\partial\lambda} V_{s-}^{\lambda,\alpha} N_0(ds, d\alpha, d\theta) \\ &\quad + \int_0^t \int_0^1 \int_{-\pi}^{\pi} (\cos\theta - 1) \frac{\partial}{\partial\lambda} V_{s-}^{\lambda,\alpha} N_1(ds, d\alpha, d\theta) \\ &\quad - \int_0^t \int_0^1 \int_{-\pi}^{\pi} \Big[ \sin\gamma_\alpha^\lambda(\theta) V_{s-}^{\lambda,\alpha} + \cos\gamma_\alpha^\lambda(\theta) W_{s-}(\alpha) \Big] \\ &\qquad\qquad \times \alpha(\theta) N_0(ds, d\alpha, d\theta), \end{aligned} \tag{5.16}$$

$$\begin{aligned} \frac{\partial^2}{\partial\lambda^2} V_t^{\lambda,\alpha} &= \int_0^t \int_0^1 \int_{-\pi}^{\pi} \big(\cos\gamma_\alpha^\lambda(\theta) - 1\big) \frac{\partial^2}{\partial\lambda^2} V_{s-}^{\lambda,\alpha} N_0(ds, d\alpha, d\theta) \\ &\quad + \int_0^t \int_0^1 \int_{-\pi}^{\pi} (\cos\theta - 1) \frac{\partial^2}{\partial\lambda^2} V_{s-}^{\lambda,\alpha} N_1(ds, d\alpha, d\theta) \\ &\quad - 2 \int_0^t \int_0^1 \int_{-\pi}^{\pi} \sin\gamma_\alpha^\lambda(\theta) \frac{\partial}{\partial\lambda} V_{s-}^{\lambda,\alpha} \alpha(\theta) N_0(ds, d\alpha, d\theta) \\ &\quad - \int_0^t \int_0^1 \int_{-\pi}^{\pi} \Big[ \cos\gamma_\alpha^\lambda(\theta) V_{s-}^{\lambda,\alpha} - \sin\gamma_\alpha^\lambda(\theta) W_{s-}(\alpha) \Big] \\ &\qquad\qquad \times \alpha^2(\theta) N_0(ds, d\alpha, d\theta). \end{aligned} \tag{5.17}$$

The skeleton is also regular enough. For each $\lambda \in [-1,1]$. the càdlàg functions $S_t(v_0, \gamma_\alpha^\lambda(m), q)$, $\frac{\partial}{\partial\lambda} S_t(v_0, \gamma_\alpha^\lambda(m), q)$ and $\frac{\partial^2}{\partial\lambda^2} S_t(v_0, \gamma_\alpha^\lambda(m), q)$ satisfy equations as

(5.15), (5.16) and (5.17), where $V_0$, $N_0$ and $N_1$ have been replaced by $v_0$, $m$ and $q$. It thus suffices, as in Section 3, to prove the following proposition.

PROPOSITION 5.9. *Let $v_0 \in \mathcal{V}_0$, $m \in \mathcal{M}_0$, $q \in \mathcal{M}_1$ and $\alpha \in \mathcal{D}$ be fixed. Then for all $\varepsilon > 0$, the set*

$$\begin{aligned}\Omega^{\alpha}_{v_0,m,q}(\varepsilon) = \Big\{ &\sup_{|\lambda|\le 1} \delta\big(V^{\lambda,\alpha}, S\big(v_0, \gamma^{\lambda}_{\alpha}(m), q\big)\big) \le \varepsilon; \\ &\sup_{|\lambda|\le 1} \delta\Big(\frac{\partial}{\partial\lambda} V^{\lambda,\alpha}, \frac{\partial}{\partial\lambda} S\big(v_0, \gamma^{\lambda}_{\alpha}(m), q\big)\Big) \le \varepsilon; \\ &\sup_{|\lambda|\le 1} \delta\Big(\frac{\partial^2}{\partial\lambda^2} V^{\lambda,\alpha}, \frac{\partial^2}{\partial\lambda^2} S\big(v_0, \gamma^{\lambda}_{\alpha}(m), q\big)\Big) \le \varepsilon \Big\}\end{aligned} \tag{5.18}$$

*has a strictly positive probability.*

We now would like to give an idea of the proof of this proposition. We thus fix

$$v_0 \in \mathcal{V}_0, \qquad m = \sum_{i=1}^{n_0} \delta_{(t_i^0, w_i^0, \theta_i^0)} \in \mathcal{M}_0^p, \qquad q = \sum_{i=1}^{n_1} \delta_{(t_i^1, w_i^1, \theta_i^1)} \in \mathcal{M}_1^p \tag{5.19}$$

and $\alpha \in \mathcal{D}$. For simplicity, we denote $V^{\lambda} = V^{\lambda,\alpha}$ and $S_t^{\lambda} = S_t(v_0, \gamma^{\lambda}_{\alpha}(m), q)$. We also consider the finite Poisson measures $N^p = N_0^p + N_1^p$, where

$$N_0^p = N_0|_{[0,T]\times[0,1]\times\{[-\theta_0,\theta_0]\setminus[-1/p,1/p]\}} = \sum_{i=1}^{\mu_0} \delta_{(T_i^0, \alpha_i^0, \phi_i^0)}, \tag{5.20}$$

$$N_1^p = N_1|_{[0,T]\times[0,1]\times\{\operatorname{supp}\beta_1\setminus[-1/p,1/p]\}} = \sum_{i=1}^{\mu_1} \delta_{(T_i^1, \alpha_i^1, \phi_i^1)}. \tag{5.21}$$

We denote by $V^{\lambda,p}$ the solution of equation (5.15) where $N_0$ and $N_1$ have been replaced by $N_0^p$ and $N_1^p$. Then we consider the following sets:

$$\Gamma_0(\varepsilon) = \{\omega \in \Omega \mid |V_0 - v_0| \le \varepsilon\},$$

$$\begin{aligned}\Gamma_1^0(a,b,c) = \Big\{&\omega \in \Omega \mid \mu_0 = n_0, \\ &\forall i,\ t_i^0 - a \le T_i^0 \le t_i^0, \big|W_{T_i^0}(\alpha_i^0) - w_i^0\big| \le b;\ \big|\phi_i^0 - \theta_i^0\big| \le c\Big\},\end{aligned}$$

$$\begin{aligned}\Gamma_1^1(a,b,c) = \Big\{&\omega \in \Omega \mid \mu_1 = n_1, \\ &\forall i,\ t_i^1 - a \le T_i^1 \le t_i^1, \big|W_{T_i^1}(\alpha_i^1) - w_i^1\big| \le b;\ \big|\phi_i^1 - \theta_i^1\big| \le c\Big\},\end{aligned}$$

$$\Gamma_2^0(d) = \biggl\{\omega \in \Omega \Bigm| \int_0^T \int_0^1 \int_{-1/p}^{1/p} \bigl(|\theta| + |\alpha(\theta)|\bigr)\bigl(1 + |W_s(\alpha)|\bigr) \times N_0(ds, d\alpha, d\theta) \le d\biggr\},$$

$$\Gamma_2^1(d) = \biggl\{\omega \in \Omega \Bigm| \int_0^T \int_0^1 \int_{-1/p}^{1/p} |\theta|\bigl(1 + |W_s(\alpha)|\bigr) \times N_1(ds, d\alpha, d\theta) \le d\biggr\}.$$

Then one can check that for all $\varepsilon > 0$, $a > 0$, $b > 0$, $c > 0$, $d > 0$, $a' > 0$, $b' > 0$, $c' > 0$, $d' > 0$, small enough,

$$P\bigl\{\Gamma_0(\varepsilon) \cap \Gamma_1^0(a, b, c) \cap \Gamma_1^1(a', b', c') \cap \Gamma_2^0(d) \cap \Gamma_2^1(d')\bigr\} > 0. \tag{5.22}$$

It suffices to use the some independence arguments, Lemma 5.5, the same arguments as in the proof of Lemma 4.1, and the facts that $(|\theta| + |\alpha(\theta)|)(1 + |W_s(\alpha)|) \in L^1(\beta_0(\theta)\, d\theta\, d\alpha\, ds)$ and $|\theta|(1 + |W_s(\alpha)|) \in L^1(\beta_1(\theta)\, d\theta\, d\alpha\, ds)$. Indeed, for example,

$$\begin{aligned} &\int_0^T \int_0^1 \int_{-\pi}^{\pi} \bigl(|\theta| + |\alpha(\theta)|\bigr) \times \bigl(1 + |W_s(\alpha)|\bigr)\beta_0(\theta)\, d\theta\, d\alpha\, ds \\ &\qquad \le T\biggl(1 + E_\alpha\Bigl(\sup_{[0,T]} |W_t|\Bigr)\biggr) \int_{-\pi}^{\pi} \bigl(|\theta| + |\alpha(\theta)|\bigr)\beta_0(\theta)\, d\theta < \infty. \end{aligned} \tag{5.23}$$

This shows the way to prove Proposition 5.9: following the ideas of Lemma 4.2, one can check that for $\beta > 0$ fixed, then for $\varepsilon > 0$, $a > 0$, $b > 0$, $c > 0$ small enough,

$$\begin{aligned} &\Gamma_0(\varepsilon) \cap \Gamma_1^0(a, b, c) \cap \Gamma_1^1(a, b, c) \\ &\quad \subset \biggl\{ \sup_{|\lambda| \le 1} \delta\bigl(V^{\lambda, p}, S\bigl(v_0, \gamma_\alpha^\lambda(m), q\bigr)\bigr) \le \beta; \\ &\qquad \sup_{|\lambda| \le 1} \delta\biggl(\frac{\partial}{\partial\lambda} V^{\lambda, p}, \frac{\partial}{\partial\lambda} S\bigl(v_0, \gamma_\alpha^\lambda(m), q\bigr)\biggr) \le \beta; \\ &\qquad \sup_{|\lambda| \le 1} \delta\biggl(\frac{\partial^2}{\partial\lambda^2} V^{\lambda, p}, \frac{\partial^2}{\partial\lambda^2} S\bigl(v_0, \gamma_\alpha^\lambda(m), q\bigr)\biggr) \le \beta \biggr\}. \end{aligned} \tag{5.24}$$

Then, following the ideas of Lemma 4.3, we see that for $\zeta > 0$ fixed, we obtain, if $\varepsilon > 0$, $a > 0$, $b > 0$, $c > 0$ and $d > 0$ are small enough,

$$
\begin{aligned}
&\Gamma_0(\varepsilon) \cap \Gamma_1^0(a,b,c) \cap \Gamma_1^1(a,b,c) \cap \Gamma_2^0(d) \cap \Gamma_2^1(d) \\
&\quad \subset \Biggl\{ \sup_{|\lambda|\le 1} \|V^\lambda - V^{\lambda,p}\|_\infty \le \zeta; \\
&\qquad \sup_{|\lambda|\le 1} \left\| \frac{\partial}{\partial\lambda} V^\lambda - \frac{\partial}{\partial\lambda} V^{\lambda,p} \right\|_\infty \le \zeta; \\
&\qquad \sup_{|\lambda|\le 1} \left\| \frac{\partial^2}{\partial\lambda^2} V^\lambda - \frac{\partial^2}{\partial\lambda^2} V^{\lambda,p} \right\|_\infty \le \zeta \Biggr\}.
\end{aligned}
\tag{5.25}
$$

This concludes the proof of Proposition 5.9.

The sketch of proof of Theorem 5.8 is complete. □

5.3. *Proof of Theorem* 5.4. We now have to study the skeleton, in order to check that under Assumption (K) every $y$ in $\mathbb{R}$ satisfies the assumptions of Theorem 5.8. Thus (K) is assumed. Hence $\mathcal{V}_0$ contains (at least) one point $v_0 \neq 0$. Since the support of $\beta_1$ might be the empty set, and since the support of $P_0$ might contain only $v_0$, we will only study the skeletons of the form $S(m) = S(v_0, m, 0)$, for $m \in \mathcal{N}_0$, where $\mathcal{N}_0$ is the following subset of $\mathcal{M}$:

$$\mathcal{N}_0 = \left\{ m = \sum_{i=1}^{n} \delta_{(t_i, v_0, \theta_i)} \in \mathcal{M}_0 \right\}. \tag{5.26}$$

We will prove the following proposition:

PROPOSITION 5.10. *Let $y \in \mathbb{R}$, and let $t > 0$ be fixed. There exists $m \in \mathcal{N}_0$, and $\alpha \in \mathcal{D}$, such that*

$$S_t(m) = y; \qquad m(\{t\} \times \mathbb{R} \times [-\theta_0, \theta_0]) = 0; \qquad D_\alpha S_t(m) \neq 0. \tag{5.27}$$

This proposition, composed with Theorem 5.8, drives immediately to Theorem 5.4. Thus, the whole sequel is devoted to the proof of this proposition.

*We will assume that* $v_0 = 1$. We may do so. Indeed, assume that $v_0 \neq 1$. Then we notice that $(V/v_0, W/v_0)$ satisfy (5.6) with initial condition $V_0/v_0$. The support of the law of $V_0/v_0$ contains 1. One concludes easily by using the uniqueness in law for (5.6).

We now consider the set

$$\mathcal{E} = \{(n, \theta_1, \dots, \theta_n) \mid n \in \mathbb{N},\ \theta_i \in ]{-\theta_0}, \theta_0[\backslash\{0\}\} \tag{5.28}$$

and the function $F$ from $\mathcal{E}$ into $\mathbb{R}$, defined recursively by

$$F(0)=1; \qquad F(n+1,\theta_1,\dots,\theta_n,\theta_{n+1}) = F(n,\theta_1,\dots,\theta_n)\cos\theta_{n+1} - \sin\theta_{n+1}.$$

The main idea is that we have to prove that $F$ is surjective. Indeed, for any $t>0$ fixed, choosing $m=\sum_{i=1}^n \delta_{(t_i,1,\theta_i)} \in \mathcal{N}_0$, with $0<t_1<\cdots<t_n<t$, we see that $S_t(m)=F(n,\theta_1,\dots,\theta_n)$. We first prove a lemma showing that $F$ can go to infinity.

LEMMA 5.11. *There exists a sequence $\varphi_n^0$ in $]{-\theta_0},0[\,\cup\,]0,\theta_0[$ such that the sequence $F(n,\varphi_1^0,\dots,\varphi_n^0)$ increases to infinity as $n$ tends to infinity.*

PROOF. First notice that for any $u>0$, the function $g_u(\theta)=u\cos\theta-\sin\theta$ on $[-\pi,\pi]$ reaches its maximum at $\theta^u=-\arctan 1/u$, and that $g_u(\theta^u)=\sqrt{1+u^2}$.

Assume first that $\theta_0>\pi/4$. We define recursively, for $n\geq 0$,

$$\varphi_{n+1}^0 = -\arctan\frac{1}{F(n,\varphi_1^0,\dots,\varphi_n^0)}. \tag{5.29}$$

We also set $u_n=F(n,\varphi_1^0,\dots,\varphi_n^0)$. Then $u_n$ grows to infinity, because $u_0=1$ and $u_{n+1}=\sqrt{1+u_n^2}$. We thus just have to prove that for all $i\geq 1$, $\varphi_i^0\in\,]{-\theta_0},\theta_0[\setminus\{0\}$. But $\varphi_1^0=-\arctan 1=-\pi/4\in\,]{-\theta_0},0[$, and, since $u_n$ increases to infinity, we deduce from (5.29) that $\varphi_n^0$ increases to 0, which allows to conclude that for all $i$, $\varphi_i^0\in\,]{-\theta_0},0[$.

Assume now that $\theta_0\leq\pi/4$, and consider the sequence $u'_n=F(n,-\theta_0/2,\dots,-\theta_0/2)$. Then $u'_0=1$, and $u'_{n+1}=(\cos\theta_0/2)u'_n+\sin\theta_0/2$, from which we deduce that for all $n\geq 0$,

$$u'_n=(\cos\theta_0/2)^n+\frac{\sin\theta_0/2}{1-\cos\theta_0/2}\bigl(1-(\cos\theta_0/2)^n\bigr). \tag{5.30}$$

Hence $u'_n$ increases to $(\sin\theta_0/2)/(1-\cos\theta_0/2)>1/\tan\theta_0$, and there exists $n_0\geq 0$ such that $\arctan 1/u'_{n_0}<\theta_0$. We thus set $\varphi_1^0=\cdots=\varphi_{n_0}^0=-\theta_0/2$, and recursively, for $n\geq n_0$,

$$\varphi_{n+1}^0 = -\arctan\frac{1}{F(n,\varphi_1^0,\dots,\varphi_n^0)}. \tag{5.31}$$

One concludes, as in the case where $\theta_0>\pi/4$, that $F(n,\varphi_1^0,\dots,\varphi_n^0)$ goes to infinity, and that for all $i$, $\varphi_i^0\in\,]{-\theta_0},0[$. $\square$

A second lemma, shows that $F$ can reach $-1$.

LEMMA 5.12. *There exists $m_0\in\mathbb{N}$, $\psi_1^0,\dots,\psi_{m_0}^0$ in $]0,\theta_0[$, such that for all $n\in\{0,\dots,m_0-1\}$, $F(n,\psi_1^0,\dots,\psi_n^0)\geq F(n+1,\psi_1^0,\dots,\psi_{n+1}^0)$ and*

$$F(m_0,\psi_1^0,\dots,\psi_{m_0}^0)=-1. \tag{5.32}$$

PROOF. Notice that the sequence $F(n, \theta_0/2, \dots, \theta_0/2)$ goes to $-\sin(\theta_0/2)/(1-\cos(\theta_0/2)) < -1$ (because $\theta_0 < \pi$). We denote by $m_0 \in \mathbb{N}$ the first $n \in \mathbb{N}$ such that $F(m_0, \theta_0/2, \dots, \theta_0/2) \le -1$. Then

$$\begin{aligned} &F(m_0 - 1, \theta_0/2, \dots, \theta_0/2)\cos 0 - \sin 0 \\ &\qquad > -1 \ge F(m_0 - 1, \theta_0/2, \dots, \theta_0/2)\cos\theta_0/2 - \sin\theta_0/2. \end{aligned} \tag{5.33}$$

Thus there exists $\psi^0_{m_0} \in ]0, \theta_0/2]$ such that $-1 = F(m_0 - 1, \theta_0/2, \dots, \theta_0/2)\cos\psi^0_{m_0} - \sin\psi^0_{m_0} = F(m_0, \theta_0/2, \dots, \theta_0/2, \psi^0_{m_0})$. We conclude by setting $\psi^0_1 = \cdots = \psi^0_{m_0-1} = \theta_0/2$. $\square$

PROOF OF PROPOSITION 5.10. We break the proof into several steps.

*Step* 1. We first prove that $F$ is surjective. Let $y > 1$. Thanks to Lemma 5.11, there exists $n \in \mathbb{N}$ such that $F(n, \varphi^0_1, \dots, \varphi^0_n) < y \le F(n+1, \varphi^0_1, \dots, \varphi^0_{n+1})$. This can also be written

$$\begin{aligned} F(n, \varphi^0_1, \dots, \varphi^0_n)\cos 0 - \sin 0 < y \le F(n, \varphi^0_1, \dots, \varphi^0_n)&\cos\varphi^0_{n+1} \\ &- \sin\varphi^0_{n+1}. \end{aligned} \tag{5.34}$$

Thus there exists $\theta \in [\varphi^0_{n+1}, 0[$ such that $y = F(n, \varphi^0_1, \dots, \varphi^0_n)\cos\theta - \sin\theta$. In other words, $y = F(n+1, \varphi^0_1, \dots, \varphi^0_n, \theta)$ and $F$ reaches $y$.

If $y \in [0, 1]$, one can use the same argument, using Lemma 5.12 instead of Lemma 5.11.

Assume now that $y \le 0$, and consider $n \in \mathbb{N}$, $\theta_1, \dots, \theta_n$ in $]-\theta_0, \theta_0[\setminus\{0\}$, such that $-y = F(n, \theta_1, \dots, \theta_n)$. One can check, using Lemma 5.12, that

$$y = F(m_0 + n, \psi^0_1, \dots, \psi^0_{m_0}, -\theta_1, \dots, -\theta_n) \tag{5.35}$$

and $F$ reaches $y$.

*Step* 2. Let now $y \in \mathbb{R}$ be fixed, and let $n \in \mathbb{N}$, $\theta_1, \dots, \theta_n$ in $]-\theta_0, \theta_0[\setminus\{0\}$ such that $y = F(n, \theta_1, \dots, \theta_n)$. One can easily check the existence of $\phi \in ]0, \theta_0[$, $\phi' \in ]0, \theta_0[$, as small as we want, such that

$$y = (y\cos\phi + \sin\phi)\cos\phi' - \sin\phi'. \tag{5.36}$$

We choose $\phi$ and $\phi'$ small enough, in order to obtain $\phi' < \inf\{|\theta_1|, \dots, |\theta_n|\}$, and such that $y \ne -1/\sin\phi'$.

Then it is clear that

$$y = F(n+2, \theta_1, \dots, \theta_n, -\phi, \phi'). \tag{5.37}$$

We consider any $0 < t_1 < \cdots < t_{n+2} < t$, and we set

$$m = \sum_{i=1}^n \delta_{(t_i, 1, \theta_i)} + \delta_{(t_{n+1}, 1, -\phi)} + \delta_{(t_{n+2}, 1, \phi')} \tag{5.38}$$

which belongs to $\mathcal{N}_0$. Then, $y = S_t(m)$, and $m(\{t\} \times \mathbb{R} \times ]-\theta_0, \theta_0[) = 0$. Furthermore, choosing any $\alpha \in \mathcal{D}$ in such a way that $\alpha(\phi') \neq 0$, but $\alpha(-\phi) = \alpha(\theta_1) = \cdots = \alpha(\theta_n) = 0$, we see that

$$
\begin{aligned}
D_\alpha S_t(m) &= \frac{\partial}{\partial \lambda}\big[(y\cos\phi + \sin\phi)\cos\big(\phi' + \lambda\alpha(\phi')\big) - \sin\big(\phi' + \lambda\alpha(\phi')\big)\big]\big|_{\lambda=0} \\
&= -\alpha(\phi')\big[(y\cos\phi + \sin\phi)\sin\phi' + \cos\phi'\big].
\end{aligned}
\tag{5.39}
$$

Thus $D_\alpha S_t(m) \neq 0$, except if $\phi' = -\arctan 1/(y\cos\phi + \sin\phi)$. But if so, $y\cos\phi + \sin\phi = -1/\tan\phi'$, and we deduce from (5.36) that

$$
y = -\cos\phi'/\tan\phi' - \sin\phi' = -1/\sin\phi' \tag{5.40}
$$

which was supposed to fail. Hence $D_\alpha S_t(m) \neq 0$, and this concludes the proof. □

The proof of Theorem 5.4 is complete.

## APPENDIX

We give in this Appendix an extended version of Gronwall's lemma.

LEMMA A.1. *Let $f$ be a positive càdlàg function on $[0, T]$. Assume that for some $a \geq 0$, $b \geq 0$, $c \geq 0$, and some $0 \leq t_1 < t_2 < \cdots < t_n \leq T$,*

$$
f(t) \leq a + b\int_0^t f(s)\,ds + c\sum_{i=1}^n f(t_i-)1_{\{t \geq t_i\}}. \tag{A.1}
$$

*Then there exists a constant $K$, depending only $b$, $c$, $n$, $T$, such that*

$$
\sup_{[0,T]} f(t) \leq K \times a. \tag{A.2}
$$

A somewhat more general version of this lemma can be found in the Appendix of Ethier and Kurtz [7]. We, however, give an idea of the proof.

PROOF OF LEMMA A.1. Thanks to Gronwall's Lemma, it is obvious that for all $t \in [0, t_1[$,

$$
f(t) \leq a \times e^{bt_1} \leq a \times e^{bT}. \tag{A.3}
$$

Hence $f(t_1-) \leq ae^{bT}$, and thus, for all $t \in [0, t_2[$,

$$
f(t) \leq (a + cae^{bT}) + b\int_0^t f(s)\,ds \tag{A.4}
$$

which implies, thanks to Gronwall's Lemma again, that for all $t \in [0, t_2[$,

$$f(t) \le (a + cae^{bT})e^{bt_2} \le a \times (1 + ce^{bT})e^{bT}. \tag{A.5}$$

Iterating the method, we obtain the result. □

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