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PRECISE ASYMPTOTICS IN THE LAW OF THE ITERATED LOGARITHM

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Let $X, X_1, X_2, \ldots$ be i.i.d. random variables with mean 0 and positive, finite variance σ^2, and set $S_n = X_1 + \cdots + X_n$, $n \geq 1$. Continuing earlier work related to strong laws, we prove the following analogs for the law of the iterated logarithm:

$$\lim_{\varepsilon \searrow \sigma\sqrt{2}} \sqrt{\varepsilon^2 - 2\sigma^2} \sum_{n \geq 3} \frac{1}{n} P(|S_n| \geq \varepsilon\sqrt{n \log\log n} + a_n) = \sigma\sqrt{2}$$

whenever $a_n = O(\sqrt{n}(\log\log n)^{-\gamma})$ for some $\gamma \geq 1/2$ (assuming slightly more than finite variance), and

$$\lim_{\varepsilon \searrow 0} \varepsilon^2 \sum_{n \geq 3} \frac{1}{n \log n} P(|S_n| \geq \varepsilon\sqrt{n \log\log n}) = \sigma^2.$$

1. Introduction and results. The aim of this paper is to continue the investigations begun in Spătaru (1999), and in Gut and Spătaru (2000). Throughout, let $X, X_1, X_2, \ldots$ be i.i.d. random variables with common distribution function F, mean 0 and positive, finite variance σ^2, and set $S_n = X_1 + \cdots + X_n$, $n \geq 1$.

The following result was proved in Baum and Katz (1965).

Theorem A. *Let $p < 2$ and $r \geq p$. Then*

$$\sum_{n \geq 1} n^{r/p-2} P(|S_n| \geq \varepsilon n^{1/p}) < \infty, \qquad \varepsilon > 0, \tag{1.1}$$

if and only if $E|X|^r < \infty$ and, when $r \geq 1$, $EX = 0$.

For $r = 2$ and $p = 1$, the result reduces to the theorem of Hsu and Robbins (1947) (sufficiency) and Erdős (1949, 1950) (necessity). For $r = p = 1$, we rediscover the famous theorem of Spitzer (1956).

In view of the fact that the sums tend to infinity as $\varepsilon \searrow 0$, it is of interest to find the rate, that is, one would be interested in finding appropriate normalizations in terms of functions of ε that yield nontrivial limits. Toward this end, Heyde (1975) proved that

$$\lim_{\varepsilon \searrow 0} \varepsilon^2 \sum_{n \geq 1} P(|S_n| \geq \varepsilon n) = EX^2,$$

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whenever $EX = 0$ and $EX^2 < \infty$ (thus corresponding to the result of Hsu, Robbins and Erdős mentioned above). Remaining values of r and p were later taken care of by Chen (1978), Spătaru (1999) and Gut and Spătaru (2000).

In view of the central limit theorem, there cannot be any analog for $p = 2$. However, by replacing $n^{1/p}$ in (1.1) by $\sqrt{n \log n}$, corresponding results have been given in Gut and Spătaru (2000, Theorems 3 and 4).

A natural question at this point is to try to establish similar results related to the law of the iterated logarithm. Sums analogous to those of (1.1) have been considered by Davis (1968) and Gut (1980). The following result holds; for the sufficiency, see Davis (1968, Theorem 4); for the necessity, see Gut (1980, Theorem 6.2).

THEOREM B. *Suppose that* $EX = 0$ *and that* $E\,X^2 = \sigma^2 < \infty$. *Then*

$$\sum_{n \geq 3} \frac{1}{n} P\left(|S_n| \geq \varepsilon\sqrt{n \log \log n}\right) < \infty, \qquad \varepsilon > \sigma\sqrt{2}. \tag{1.2}$$

Conversely, if the sum is finite for some ε, *then* $EX = 0$ *and* $E\,X^2 < \infty$.

Let Φ and φ denote the distribution function and the density function, respectively, of the standard normal distribution. Since

$$\left(\frac{1}{x} - \frac{1}{x^3}\right)\varphi(x) < 1 - \Phi(x) < \frac{1}{x}\varphi(x), \quad x > 0,$$

[see, e.g., Feller (1968), page 175], it is easily seen that the sum diverges for all $\varepsilon < \sigma\sqrt{2}$. Therefore, the asymptotics in terms of ε must involve limits as $\varepsilon \searrow \sigma\sqrt{2}$; a natural guess is that a normalizing function should be $(\varepsilon - \sigma\sqrt{2})$ raised to a suitable power or, equivalently, $(\varepsilon^2 - 2\sigma^2)$ raised to a suitable power.

Noting that the convergence of the series in (1.2) implies that

$$\sum_{n \geq 3} \frac{1}{n} P\left(|S_n| \geq \varepsilon\sqrt{n \log \log n} + a_n\right) < \infty$$

whenever $a_n = o(\sqrt{n \log \log n})$ and $\varepsilon > \sigma\sqrt{2}$, we are now ready to state our first result.

THEOREM 1. *Suppose that* $EX = 0$, *that* $EX^2 = \sigma^2$, *that* $E[X^2 \times (\log^+ \log^+ |X|)^{1+\delta}] < \infty$ *for some* $\delta > 0$, *and that* $a_n = O(\sqrt{n}(\log \log n)^{-\gamma})$ *for some* $\gamma > 1/2$. *Then*

$$\lim_{\varepsilon \searrow \sigma\sqrt{2}} \sqrt{\varepsilon^2 - 2\sigma^2} \sum_{n \geq 3} \frac{1}{n} P\left(|S_n| \geq \varepsilon\sqrt{n \log \log n} + a_n\right) = \sigma\sqrt{2}. \tag{1.3}$$

REMARK 1.1. We do not know if the assumption that $E[X^2 \times (\log^+ \log^+ |X|)^{1+\delta}] < \infty$ for some $\delta > 0$ is necessary for the conclusion to hold; cf. also Remark 2.1.

Following is a result of the above kind assuming no more than finite variance.

THEOREM 2. *Suppose that* $EX = 0$ *and that* $EX^2 = \sigma^2 < \infty$. *Then*

$$\lim_{\varepsilon \searrow 0} \varepsilon^2 \sum_{n\geq 3} \frac{1}{n \log n} P(|S_n| \geq \varepsilon\sqrt{n \log \log n}) = \sigma^2. \tag{1.4}$$

REMARK 1.2. It follows from Proposition 4.4 below that the sum in (1.4) converges for all $\varepsilon > 0$ (for $\varepsilon > \sigma\sqrt{2}$, it follows immediately from Theorem B above). It is, however, possible to show that the sum, in fact, converges under the weaker assumption that $E[X^2(\log^+ \log^+ |X|)^{-\eta}] < \infty$ for some $0 < \eta < 1$ and $EX = 0$. Note, in particular, that finite variance is not necessary. We do not know what the best necessary and sufficient condition for the sum to converge might be.

REMARK 1.3. By modifying the proofs in Section 2.1 below in the obvious manner, one finds that if the summands are normal with mean 0 and variance σ^2, then, for $\alpha > -1$, we have

$$\lim_{\varepsilon \searrow \sigma\sqrt{2(\alpha+1)}} \sqrt{\varepsilon^2 - 2(\alpha+1)\sigma^2} \sum_{n\geq 3} \frac{(\log n)^\alpha}{n} P(|S_n| \geq \varepsilon\sqrt{n \log\log n}) = \sigma\sqrt{\frac{2}{\alpha+1}}. \tag{1.5}$$

We leave it to the interested reader to find the appropriate conditions for (1.5) to hold for general distributions (for $\alpha = 0$, see Theorem 1). For $\alpha = 1$, it was shown in Gut [(1980), Theorem 6.1] that the sum in (1.5) converges for $\varepsilon > 2\sigma$, provided $E[X^2(\log^+|X|)(\log^+\log^+|X|)^{-1}] < \infty$, and, conversely, that if the sum is finite for some ε, then this moment condition must be satisfied (and the mean equal 0).

It would also be of interest to find conditions for one-sided versions (in terms of moments of $\max\{X, 0\}$) of the above results, that is, for limits of sums like $\sum_{n\geq 3} \frac{1}{n} P(S_n \geq \varepsilon\sqrt{n \log\log n})$ and $\sum_{n\geq 3}(1/n \log n) P(S_n \geq \varepsilon\sqrt{n \log\log n})$, properly normalized, as ε tends to the corresponding critical value.

The proofs consist essentially of two stages. The validity of (1.3) and (1.4) is verified first under the assumption that F is the normal distribution, after which the general case is treated via the central limit theorem. Theorem 1 is proved in Section 2 and Theorem 2 in Section 3.

Throughout the rest of the paper we assume, without restriction, that $\sigma^2 = 1$. Since $\varepsilon \searrow \sqrt{2}$ in Theorem 1, we suppose throughout the proof that $2 < \varepsilon^2 < 3$ (say). Similarly, since $\varepsilon \searrow 0$ in Theorem 2, we suppose that $0 < \varepsilon < 1/2$ (say) throughout that proof. Finally, C and K will denote absolute positive constants and absolute real constants, respectively, possibly varying from place to place, $[x]$ shall denote the largest integer $\leq x$, $\log^+ x = \max\{\log x, 1\}$ (this has already been used) and $\gamma > 1/2$.

2. Proof of Theorem 1.

2.1. *F is normal.* We thus assume, throughout this subsection, that F is the standard normal distribution function Φ, and set $\Psi(x) = 1 - \Phi(x) + \Phi(-x)$, $x \geq 0$. For $x \geq 3$, put $K(x) = K(\log\log x)^{-\gamma}$. (To simplify the exposition, we assume throughout this section that $\varepsilon\sqrt{\log\log x} + K(x) \geq 0$, $x \geq 3$.)

PROPOSITION 2.1. *We have,*

$$\lim_{\varepsilon \searrow \sqrt{2}} \sqrt{\varepsilon^2 - 2} \sum_{n \geq 3} \frac{1}{n} P(|S_n| \geq \varepsilon\sqrt{n \log\log n} + \sqrt{n} K(n))$$

$$= \lim_{\varepsilon \searrow \sqrt{2}} \sqrt{\varepsilon^2 - 2} \sum_{n \geq 3} \frac{1}{n} \Psi(\varepsilon\sqrt{\log\log n} + K(n)) = \sqrt{2}.$$

PROOF. We obtain, via the Euler-MacLaurin sum formula [see Cramér (1946), page 124],

$$\sum_{n \geq 3} \frac{1}{n} \Psi(\varepsilon\sqrt{\log\log n} + K(n))$$

$$= \int_3^\infty \frac{1}{x} \Psi(\varepsilon\sqrt{\log\log x} + K(x))\,dx + \frac{1}{6}\Psi(\varepsilon\sqrt{\log\log 3} + K(3))$$

$$- \int_3^\infty P_1(x)\,d\left[\frac{1}{x}\Psi(\varepsilon\sqrt{\log\log x} + K(x))\right],$$

where $P_1(x) = [x] - x + 1/2$. Continuing as in Spătaru (1997) and Gut and Spătaru (2000) yields

$$\left|\int_3^\infty P_1(x)\,d\left[\frac{1}{x}\Psi(\varepsilon\sqrt{\log\log x} + K(x))\right]\right|$$

$$\leq \int_3^\infty |P_1(x)| \frac{1}{x^2} \Psi(\varepsilon\sqrt{\log\log x} + K(x))\,dx$$

$$+ \frac{\varepsilon}{2} \int_3^\infty |P_1(x)| \frac{1}{x^2 \log x \sqrt{\log\log x}} |\Psi'(\varepsilon\sqrt{\log\log x} + K(x))|\,dx$$

$$+ |K|\gamma \int_3^\infty |P_1(x)| \frac{1}{x^2 \log x (\log\log x)^{\gamma+1}} |\Psi'(\varepsilon\sqrt{\log\log x} + K(x))|\,dx$$

$$\leq \frac{1}{2}\int_3^\infty \frac{dx}{x^2} + C\int_3^\infty \frac{dx}{x^2 \log x \sqrt{\log\log x}} + C \leq C,$$

from which it follows that

$$\lim_{\varepsilon \searrow \sqrt{2}} \sqrt{\varepsilon^2 - 2} \int_3^\infty P_1(x) d\left[\frac{1}{x}\Psi(\varepsilon\sqrt{\log\log x} + K(x))\right] = 0.$$

For the remaining part of the proof we first consider the case $K = 0$. By putting $y = \varepsilon\sqrt{\log\log x}$, and then by partial integration, we obtain

$$\begin{aligned}
\lim_{\varepsilon \searrow \sqrt{2}} \sqrt{\varepsilon^2 - 2} \sum_{n\geq 3} \frac{1}{n}\Psi(\varepsilon\sqrt{\log\log n}) &= \lim_{\varepsilon \searrow \sqrt{2}} \sqrt{\varepsilon^2 - 2} \int_3^\infty \frac{1}{x}\Psi(\varepsilon\sqrt{\log\log x}) dx \\
&= \lim_{\varepsilon \searrow \sqrt{2}} \sqrt{\varepsilon^2 - 2} \cdot \frac{2}{\varepsilon^2} \int_{\varepsilon\sqrt{\log\log 3}}^\infty y\Psi(y) e^{y^2/\varepsilon^2} dy \\
&= \lim_{\varepsilon \searrow \sqrt{2}} \sqrt{\varepsilon^2 - 2} \left[-\Psi(\varepsilon\sqrt{\log\log 3}) \log 3 - \int_{\varepsilon\sqrt{\log\log 3}}^\infty \Psi'(y) e^{y^2/\varepsilon^2} dy\right] \\
&= \lim_{\varepsilon \searrow \sqrt{2}} \sqrt{\varepsilon^2 - 2} \cdot \frac{2}{\sqrt{2\pi}} \int_{\varepsilon\sqrt{\log\log 3}}^\infty e^{-\frac{y^2(\varepsilon^2-2)}{2\varepsilon^2}} dy \\
&= \lim_{\varepsilon \searrow \sqrt{2}} 2\varepsilon(1 - \Phi(\sqrt{(\varepsilon^2 - 2)\log\log 3})) = \sqrt{2}.
\end{aligned}$$

In the general case we similarly obtain

$$\begin{aligned}
&\lim_{\varepsilon \searrow \sqrt{2}} \sqrt{\varepsilon^2 - 2} \sum_{n\geq 3} \frac{1}{n}\Psi(\varepsilon\sqrt{\log\log n} + K(n)) \\
&\quad = \lim_{\varepsilon \searrow \sqrt{2}} \sqrt{\varepsilon^2 - 2} \int_3^\infty \frac{1}{x}\Psi(\varepsilon\sqrt{\log\log x} + K(x)) dx \\
&\quad = \lim_{\varepsilon \searrow \sqrt{2}} \sqrt{\varepsilon^2 - 2} \cdot \frac{2}{\varepsilon^2} \int_{\varepsilon\sqrt{\log\log 3}}^\infty y\Psi(y + K(y^2/\varepsilon^2)^{-\gamma}) e^{y^2/\varepsilon^2} dy.
\end{aligned}$$

The conclusion now follows by Lagrange's theorem, since

$$\lim_{\varepsilon \searrow \sqrt{2}} \sqrt{\varepsilon^2 - 2} \cdot \frac{2}{\varepsilon^2} \int_{\varepsilon\sqrt{\log\log 3}}^\infty y\Big[\Psi(y + K(y^2/\varepsilon^2)^{-\gamma}) - \Psi(y)\Big] e^{y^2/\varepsilon^2} dy = 0. \qquad \square$$

2.2. *The general case.* We thus assume that $X, X_1, X_2, \ldots$ are i.i.d. random variables with mean 0 and variance 1. For $n \geq 1$, put $Z_{n,k} = X_k I\{|X_k| < \varepsilon\sqrt{n}\}$, $1 \leq k \leq n$, $U_n = Z_{n,1} + \cdots + Z_{n,n}$, $\sigma_n^2 = EZ_{n,1}^2 - (EZ_{n,1})^2 = E[X^2 I\{|X| < \varepsilon\sqrt{n}\}] - (E[XI\{|X| < \varepsilon\sqrt{n}\}])^2$ and $\rho_n = \sigma_n^{-3} E|Z_{n,1} - EZ_{n,1}|^3$. The following two propositions are technical steps.

PROPOSITION 2.2. *Assume that* $E[X^2(\log^+\log^+|X|)^{1+\delta}] < \infty$ *for some* $\delta > 0$. *Then*

$$\lim_{\varepsilon\searrow\sqrt{2}} \sqrt{\varepsilon^2-2} \sum_{n\geq 3} \frac{1}{n}\biggl(\Psi\biggl(\varepsilon\sqrt{\log\log n}+\frac{K}{(\log\log n)^\gamma}\biggr) - \Psi\biggl(\frac{\varepsilon\sqrt{\log\log n}}{\sigma_n}+\frac{K}{(\log\log n)^\gamma}\biggr)\biggr) = 0.$$

PROOF. We begin by noticing that, since $EX = 0$,

$$\begin{aligned}|EZ_{n,1}| \leq E[|X|I\{|X|\geq\sqrt{2n}\}] &\leq \frac{E[X^2(\log^+\log^+|X|)^{1+\delta}]}{\sqrt{2n}(\log^+\log^+\sqrt{2n})^{1+\delta}}\\ &\leq \frac{C}{\sqrt{n}(\log\log n)^{1+\delta}}, \quad n\geq 3.\end{aligned} \tag{2.1}$$

Also we see that $\sigma_n^2 \to 1$ uniformly with respect to ε. Indeed, we have

$$E[X^2 I\{|X|<\sqrt{2n}\}] - (E[|X|I\{|X|\geq\sqrt{2n}\}])^2 \leq \sigma_n^2 \leq 1.$$

Next, choose $n_0 > 3$ (independent of ε) such that $\sigma_n(1+\sigma_n) \geq 1$ for $n \geq n_0$. By Lagrange's theorem, for $n \geq n_0$, we then have

$$\begin{aligned}&\Psi\Bigl(\varepsilon\sqrt{\log\log n}+K(\log\log n)^{-\gamma}\Bigr)\\ &\quad -\Psi\Bigl(\varepsilon\sigma_n^{-1}\sqrt{\log\log n}+K(\log\log n)^{-\gamma}\Bigr)\\ &\quad = \varepsilon(1-\sigma_n^{-1})\sqrt{\log\log n}\,\Psi'(\theta_n)\\ &\quad = C\varepsilon\frac{1-\sigma_n^2}{\sigma_n(1+\sigma_n)}\sqrt{\log\log n}\,\exp(-\theta_n^2/2),\end{aligned} \tag{2.2}$$

where $\varepsilon\sqrt{\log\log n}+K(\log\log n)^{-\gamma} \leq \theta_n \leq \varepsilon\sigma_n^{-1}\sqrt{\log\log n}+K(\log\log n)^{-\gamma}$. Since

$$\frac{\theta_n^2}{2} \geq \frac{\varepsilon^2\log\log n}{2} + K\varepsilon(\log\log n)^{1/2-\gamma} \geq \frac{\varepsilon^2\log\log n}{2} - |K|\varepsilon,$$

(2.2) shows that

$$\begin{aligned}&\Psi\bigl(\varepsilon\sqrt{\log\log n}+K(\log\log n)^{-\gamma}\bigr) - \Psi\bigl(\varepsilon\sigma_n^{-1}\sqrt{\log\log n}+K(\log\log n)^{-\gamma}\bigr)\\ &\quad \leq C(1-\sigma_n^2)\sqrt{\log\log n}(\log n)^{-\varepsilon^2/2}, \quad n\geq n_0.\end{aligned}$$

Recalling that $EX^2 = 1$, and noticing that $|EZ_{n,1}| \leq 1/\sqrt{2n}$, an application of Fubini's theorem yields

$$\sum_{n\geq 3}\frac{1}{n}\Bigl(\Psi\bigl(\varepsilon\sqrt{\log\log n}+K(\log\log n)^{-\gamma}\bigr) - \Psi\bigl(\varepsilon\sigma_n^{-1}\sqrt{\log\log n}+K(\log\log n)^{-\gamma}\bigr)\Bigr)$$

$$\le n_0 + C\sum_{n\ge n_0}\frac{\sqrt{\log\log n}}{n(\log n)^{\varepsilon^2/2}}E[X^2I\{|X|\ge\varepsilon\sqrt{n}\}] + C\sum_{n\ge n_0}\frac{\sqrt{\log\log n}}{n^2(\log n)^{\varepsilon^2/2}}$$

$$\le n_0 + CE\bigg[X^2I\{X^2\ge\varepsilon^2 n_0\}\sum_{n=n_0}^{[X^2/\varepsilon^2]}\frac{\sqrt{\log\log n}}{n(\log n)^{\varepsilon^2/2}}\bigg] + C\sum_{n\ge n_0}\frac{1}{n^2}$$

$$\le C + CE\bigg[X^2I\{X^2\ge 8\}\sum_{n=4}^{[X^2/2]}\frac{\sqrt{\log\log n}}{n(\log n)^{\varepsilon^2/2}}\bigg]$$

$$\le C + CE\bigg[X^2I\{X^2\ge 8\}\int_3^{[X^2/2]}\frac{\sqrt{\log\log x}}{x(\log x)^{\varepsilon^2/2}}\,dx\bigg].$$

Now, by putting $\sqrt{\log\log x} = y$, we get

$$\sum_{n\ge 3}\frac{1}{n}\Big(\Psi(\varepsilon\sqrt{\log\log n} + K(\log\log n)^{-\gamma})$$
$$-\Psi\big(\varepsilon\sigma_n^{-1}\sqrt{\log\log n} + K(\log\log n)^{-\gamma}\big)\Big)$$
$$\le C + CE\bigg[X^2I\{X^2\ge 8\}\int_{\sqrt{\log\log 3}}^{\sqrt{\log\log[X^2/2]}} y^2\exp\big(-y^2(\varepsilon^2-2)/2\big)\,dy\bigg].$$

Further, assuming without any loss of generality that $\delta < 1/2$, the substitution $y\sqrt{\varepsilon^2-2} = z$ yields

$$\sum_{n\ge 3}\frac{1}{n}\Big(\Psi(\varepsilon\sqrt{\log\log n} + K(\log\log n)^{-\gamma})$$
$$-\Psi(\varepsilon\sigma_n^{-1}\sqrt{\log\log n} + K(\log\log n)^{-\gamma})\Big)$$
$$\le C + \frac{C}{(\varepsilon^2-2)^{3/2}}$$
$$\times E\bigg[X^2I\{X^2\ge 8\}\int_{\sqrt{(\varepsilon^2-2)\log\log 3}}^{\sqrt{(\varepsilon^2-2)\log\log[X^2/2]}} z^2\exp(-z^2/2)\,dz\bigg] \tag{2.3}$$
$$\le C + \frac{C}{(\varepsilon^2-2)^{3/2}}E\bigg[X^2I\{X^2\ge 8\}\int_0^{\sqrt{(\varepsilon^2-2)\log\log[X^2/2]}} z^{1+2\delta}\,dz\bigg]$$
$$= C + \frac{C}{(\varepsilon^2-2)^{1/2-\delta}}E\Big[X^2I\{X^2\ge 8\}\big(\log\log[X^2/2]\big)^{1+\delta}\Big]$$
$$\le C + \frac{C}{(\varepsilon^2-2)^{1/2-\delta}}E\Big[X^2\big(\log^+\log^+|X|\big)^{1+\delta}\Big].$$

Hence we have

$$\sqrt{\varepsilon^2-2}\sum_{n\ge 3}\frac{1}{n}\bigg(\Psi\bigg(\varepsilon\sqrt{\log\log n} + \frac{K}{(\log\log n)^{\gamma}}\bigg) - \Psi\bigg(\frac{\varepsilon\sqrt{\log\log n}}{\sigma_n} + \frac{K}{(\log\log n)^{\gamma}}\bigg)\bigg)$$
$$\le C\sqrt{\varepsilon^2-2} + C(\varepsilon^2-2)^{\delta}E\big[X^2(\log^+\log^+|X|)^{1+\delta}\big]\to 0 \quad\text{as}\quad \varepsilon\searrow\sqrt{2}. \qquad\square$$

Since $\sigma_n^2 \to 1$ uniformly with respect to ε, the next corollary is immediate.

COROLLARY 2.1. *We have*

$$\lim_{\varepsilon\searrow\sqrt{2}} \sqrt{\varepsilon^2-2}\sum_{n\geq 3}\frac{1}{n}\left(\Psi\left(\varepsilon\sqrt{\log\log n}+\frac{K}{(\log\log n)^\gamma}\right)\right.$$
$$\left.-\Psi\left(\frac{\varepsilon\sqrt{\log\log n}}{\sigma_n}+\frac{K}{\sigma_n(\log\log n)^\gamma}\right)\right)=0.$$

REMARK 2.1. In case the additional assumption "$E[X^2(\log^+\log^+|X|)^{1+\delta}]<\infty$ for some $\delta>0$" in Proposition 2.2 does not hold, we still have (see (2.3))

$$\sqrt{\varepsilon^2-2}\sum_{n\geq 3}\frac{1}{n}\left(\Psi\left(\varepsilon\sqrt{\log\log n}+\frac{K}{(\log\log n)^\gamma}\right)-\Psi\left(\frac{\varepsilon\sqrt{\log\log n}}{\sigma_n}+\frac{K}{(\log\log n)^\gamma}\right)\right)$$
$$\leq C\sqrt{\varepsilon^2-2}+\frac{C}{\varepsilon^2-2}E\left[X^2 I\{X^2\geq 8\}\int_0^{\sqrt{(\varepsilon^2-2)\log\log[X^2/2]}} z^2\exp(-z^2/2)\,dz\right].$$

Therefore,

$$\begin{aligned}&\lim_{\varepsilon\searrow\sqrt{2}} \sqrt{\varepsilon^2-2}\sum_{n\geq 3}\frac{1}{n}\left(\Psi\left(\varepsilon\sqrt{\log\log n}+\frac{K}{(\log\log n)^\gamma}\right)\right.\\ &\qquad\left.-\Psi\left(\frac{\varepsilon\sqrt{\log\log n}}{\sigma_n}+\frac{K}{(\log\log n)^\gamma}\right)\right)\\ &\quad\leq C\lim_{\varepsilon\searrow\sqrt{2}} E\left[X^2 I\{X^2\geq 8\}\frac{1}{\varepsilon^2-2}\int_0^{\sqrt{(\varepsilon^2-2)\log\log[X^2/2]}} z^2\exp(-z^2/2)\,dz\right].\end{aligned}\tag{2.4}$$

Now, by l'Hôpital's rule, for $X^2\geq 8$,

$$\lim_{\varepsilon\searrow\sqrt{2}}\frac{1}{\varepsilon^2-2}\int_0^{\sqrt{(\varepsilon^2-2)\log\log[X^2/2]}} z^2\exp(-z^2/2)\,dz$$
$$=\lim_{\varepsilon\searrow\sqrt{2}}\frac{(\log\log[X^2/2])^{3/2}}{2}\sqrt{\varepsilon^2-2}\exp\left(-\frac{(\varepsilon^2-2)\log\log[X^2/2]}{2}\right)=0.$$

This means that if "lim" and "E" could be interchanged in (2.4), then Proposition 2.2 would hold under the weaker assumption that $EX^2<\infty$. We have, however, not been able to justify the interchange.

PROPOSITION 2.3. *Set* $U_n^* = U_n - EU_n$. *We have*

$$\lim_{\varepsilon\searrow\sqrt{2}} \sqrt{\varepsilon^2-2}\sum_{n\geq 3}\frac{1}{n}\left|P\left(|U_n^*|\geq\varepsilon\sqrt{n\log\log n}+\frac{K\sqrt{n}}{(\log\log n)^\gamma}\right)\right.$$
$$\left.-\Psi\left(\frac{\varepsilon\sqrt{\log\log n}}{\sigma_n}+\frac{K}{\sigma_n(\log\log n)^\gamma}\right)\right|=0.$$

PROOF. First, choose $n_0 > 2$ (independent of ε) such that $\sigma_n^2 \geq 1/2$ for $n \geq n_0$. By the Berry–Esseen inequality [see, e.g., Petrov (1995), page 149], for $n \geq n_0$, we have

$$\begin{aligned}
&\left| P\left(|U_n^*| \geq \varepsilon\sqrt{n \log\log n} + \frac{K\sqrt{n}}{(\log\log n)^{\gamma}} \right) - \Psi\left(\frac{\varepsilon\sqrt{\log\log n}}{\sigma_n} + \frac{K}{\sigma_n (\log\log n)^{\gamma}} \right) \right| \\
&\quad \leq C\frac{\rho_n}{\sqrt{n}} \leq \frac{C}{\sqrt{n}} E|Z_{n,1} - EZ_{n,1}|^3 \\
&\quad \leq \frac{C}{\sqrt{n}} (E|Z_{n,1}|^3 + |EZ_{n,1}|^3) \\
&\quad \leq \frac{C}{\sqrt{n}} E|Z_{n,1}|^3 + \frac{C}{n^2},
\end{aligned}$$

where the last inequality comes from (2.1). Hence, by making use of Fubini's theorem, we may write

$$\begin{aligned}
&\sum_{n \geq 3} \frac{1}{n} \left| P\left(|U_n^*| \geq \varepsilon\sqrt{n \log\log n} + \frac{K\sqrt{n}}{(\log\log n)^{\gamma}} \right) - \Psi\left(\frac{\varepsilon\sqrt{\log\log n}}{\sigma_n} + \frac{K}{\sigma_n (\log\log n)^{\gamma}} \right) \right| \\
&\quad \leq n_0 + C \sum_{n \geq n_0} \frac{1}{n^{3/2}} E[|X|^3 I\{|X| < \varepsilon\sqrt{n}\}] + C \sum_{n \geq n_0} \frac{1}{n^3} \\
&\quad \leq C + C \sum_{n \geq n_0} \frac{1}{n^{3/2}} E[|X|^3 I\{|X| < \varepsilon\sqrt{2}\}] \\
&\qquad + C \sum_{n \geq n_0} \frac{1}{n^{3/2}} E[|X|^3 I\{\varepsilon\sqrt{2} \leq |X| < \varepsilon\sqrt{n}\}] \\
&\quad \leq C + C\varepsilon^3 + CE\left[|X|^3 I\{X^2 \geq 2\varepsilon^2\} \sum_{n \geq n_0, n > X^2/\varepsilon^2} \frac{1}{n^{3/2}} \right] \\
&\quad \leq C + CE\left[|X|^3 I\{X^2 \geq 2\varepsilon^2\} \sum_{n > X^2/\varepsilon^2} \frac{1}{n^{3/2}} \right] \\
&\quad \leq C + CE\left[|X|^3 I\{X^2 \geq 2\varepsilon^2\} / ([X^2/\varepsilon^2])^{1/2} \right] \\
&\quad \leq C + C\varepsilon EX^2 = C(1+\varepsilon),
\end{aligned}$$

which proves the proposition. □

We are now prepared to complete the proof of Theorem 1.

PROOF OF THEOREM 1. From Propositions 2.1 and 2.3 and Corollary 2.1, it follows that

$$\lim_{\varepsilon \searrow \sqrt{2}} \sqrt{\varepsilon^2 - 2} \sum_{n \geq 3} \frac{1}{n} P(|U_n - EU_n| \geq \varepsilon\sqrt{n \log\log n} + K\sqrt{n}(\log\log n)^{-\gamma}) = \sqrt{2}. \tag{2.5}$$

Put $\gamma' = (1+\delta) \wedge \gamma > 1/2$. On account of (2.1), we have

$$|EU_n| + |a_n| \leq C\sqrt{n}(\log\log n)^{-1-\delta} + C\sqrt{n}(\log\log n)^{-\gamma} \leq C\sqrt{n}(\log\log n)^{-\gamma'}.$$

Therefore, we get

$$\begin{aligned} &P(|U_n - EU_n| \geq \varepsilon\sqrt{n\log\log n} + C\sqrt{n}(\log\log n)^{-\gamma'}) \\ &\quad \leq P(|U_n - EU_n| \geq \varepsilon\sqrt{n\log\log n} + |EU_n| + |a_n|) \\ &\quad \leq P(|U_n| \geq \varepsilon\sqrt{n\log\log n} + a_n) \\ &\quad \leq P(|U_n - EU_n| \geq \varepsilon\sqrt{n\log\log n} - |EU_n| - |a_n|) \\ &\quad \leq P(|U_n - EU_n| \geq \varepsilon\sqrt{n\log\log n} - C\sqrt{n}(\log\log n)^{-\gamma'}). \end{aligned} \tag{2.6}$$

Now, from (2.5) and (2.6), we see that

$$\lim_{\varepsilon \searrow \sigma\sqrt{2}} \sqrt{\varepsilon^2 - 2} \sum_{n\geq 3} \frac{1}{n} P(|U_n| \geq \varepsilon\sqrt{n\log\log n} + a_n) = \sqrt{2},$$

which concludes the proof of the theorem, since

$$\sum_{n\geq 3} \frac{1}{n} P(S_n \neq U_n) \leq \sum_{n\geq 3} P(|X| \geq \varepsilon\sqrt{n}) \leq \frac{EX^2}{\varepsilon^2} = \frac{1}{\varepsilon^2}. \qquad \square$$

3. Proof of Theorem 2.

3.1. *F is normal.* We thus assume, once again, that F is the standard normal distribution function Φ, and set $\Psi(x) = 1 - \Phi(x) + \Phi(-x)$, $x \geq 0$.

PROPOSITION 3.1. *Equation* (1.4) *holds*.

PROOF. We have

$$\begin{aligned} &\sum_{n\geq 3} \frac{1}{n\log n} P\Big(|S_n| \geq \varepsilon\sqrt{n\log\log n}\Big) \\ &\quad = \int_3^\infty \frac{1}{x\log x} \Psi\Big(\varepsilon\sqrt{\log\log x}\Big)\,dx + \frac{1}{6\log 3}\Psi\Big(\varepsilon\sqrt{\log\log 3}\Big) \\ &\qquad - \int_3^\infty P_1(x)\,d\bigg[\frac{1}{x\log x}\Psi\Big(\varepsilon\sqrt{\log\log x}\Big)\bigg], \end{aligned}$$

where $P_1(x) = [x] - x + 1/2$. Continuing as above yields

$$\begin{aligned} &\bigg|\int_3^\infty P_1(x)\,d\bigg[\frac{1}{x\log x}\Psi\Big(\varepsilon\sqrt{\log\log x}\Big)\bigg]\bigg| \\ &\quad \leq \int_3^\infty |P_1(x)|\bigg(\frac{1}{x^2\log x} + \frac{1}{x^2(\log x)^2}\bigg)\Psi\Big(\varepsilon\sqrt{\log\log x}\Big)\,dx \\ &\qquad + \frac{\varepsilon}{2}\int_3^\infty |P_1(x)| \frac{1}{x^2(\log x)^2\sqrt{\log\log x}} \Big|\Psi'\Big(\varepsilon\sqrt{\log\log x}\Big)\Big|\,dx \\ &\quad \leq \int_3^\infty \frac{dx}{x^2} + \varepsilon C\int_3^\infty \frac{dx}{x^2(\log x)^2\sqrt{\log\log x}} \leq C(1+\varepsilon), \end{aligned}$$

from which it follows that

$$\lim_{\varepsilon\searrow 0}\varepsilon^2\int_3^\infty P_1(x)d\left[\frac{1}{x\log x}\Psi\left(\varepsilon\sqrt{\log\log x}\right)\right]=0.$$

By putting $y=\varepsilon\sqrt{\log\log x}$, we finally get

$$\begin{aligned}&\lim_{\varepsilon\searrow 0}\varepsilon^2\sum_{n\geq 3}\frac{1}{n\log n}P(|S_n|\geq\varepsilon\sqrt{n\log\log n})\\&=\lim_{\varepsilon\searrow 0}\varepsilon^2\int_3^\infty\frac{1}{x\log x}\Psi(\varepsilon\sqrt{\log\log x})dx\\&=\lim_{\varepsilon\searrow 0}\varepsilon^2\int_{\varepsilon\sqrt{\log\log 3}}^\infty\frac{2}{\varepsilon^2}y\Psi(y)dy=1.\end{aligned}$$

□

3.2. *The general case.* Now we assume that $X, X_1, X_2, \ldots$ are i.i.d. random variables with mean 0 and variance 1. Put $b(\varepsilon)=\exp\{\exp(M/\varepsilon^2)\}$, where $M>1$. We first establish the following fact.

PROPOSITION 3.2. *We have*

$$\lim_{\varepsilon\searrow 0}\varepsilon^2\sum_{n\leq b(\varepsilon)}\frac{1}{n\log n}\left|P(|S_n|\geq\varepsilon\sqrt{n\log\log n})-\Psi(\varepsilon\sqrt{\log\log n})\right|=0.$$

PROOF. Notice first that

$$\frac{1}{\log\log m}\sum_{n=1}^m\frac{\Delta_n}{n\log n}\to 0\qquad\text{as}\quad m\to\infty,\tag{3.1}$$

where $\Delta_n=\sup_x|P(|S_n|\geq\sqrt{n}x)-\Psi(x)|\to 0$ as $n\to\infty$. Hence, using (3.1), we obtain

$$\begin{aligned}&\varepsilon^2\sum_{n\leq b(\varepsilon)}\frac{1}{n\log n}\left|P(|S_n|\geq\varepsilon\sqrt{n\log\log n})-\Psi(\varepsilon\sqrt{\log\log n})\right|\\&\quad\leq\varepsilon^2\sum_{n\leq[b(\varepsilon)]}\frac{\Delta_n}{n\log n}=\varepsilon^2\log\log[b(\varepsilon)]\cdot\frac{1}{\log\log[b(\varepsilon)]}\sum_{n\leq[b(\varepsilon)]}\frac{\Delta_n}{n\log n}\\&\quad\leq M\cdot\frac{1}{\log\log[b(\varepsilon)]}\sum_{n\leq[b(\varepsilon)]}\frac{\Delta_n}{n\log n}\to 0\qquad\text{as}\quad\varepsilon\searrow 0.\end{aligned}$$

□

PROPOSITION 3.3. *We have, uniformly with respect to all sufficiently small* $\varepsilon>0$,

$$\lim_{M\to\infty}\varepsilon^2\sum_{n>b(\varepsilon)}\frac{1}{n\log n}\Psi(\varepsilon\sqrt{\log\log n})=0.$$

PROOF. There is an absolute constant $c>0$ such that $b(\varepsilon)-1\geq\sqrt{b(\varepsilon)}$ for $\varepsilon\leq c$. Thus, for $\varepsilon\leq c$, we have

$$\sum_{n>b(\varepsilon)}\frac{1}{n\log n}\Psi(\varepsilon\sqrt{\log\log n})\leq\int_{b(\varepsilon)-1}^{\infty}\frac{1}{x\log x}\Psi(\varepsilon\sqrt{\log\log x})\,dx$$

$$\leq\int_{\sqrt{b(\varepsilon)}}^{\infty}\frac{1}{x\log x}\Psi(\varepsilon\sqrt{\log\log x})\,dx.$$

Hence, by putting $y=\varepsilon\sqrt{\log\log x}$, for sufficiently small ε, we get

$$\varepsilon^2\sum_{n>b(\varepsilon)}\frac{1}{n\log n}\Psi(\varepsilon\sqrt{\log\log n})\leq 2\int_{\varepsilon\sqrt{\log(1/2)+M/\varepsilon^2}}^{\infty}y\Psi(y)\,dy$$

$$\leq 2\int_{\sqrt{M-1}}^{\infty}y\Psi(y)\,dy,$$

and the conclusion follows. □

PROPOSITION 3.4. *We have, uniformly with respect to all sufficiently small* $\varepsilon>0$,

$$\lim_{M\to\infty}\varepsilon^2\sum_{n>b(\varepsilon)}\frac{1}{n\log n}P(|S_n|\geq\varepsilon\sqrt{n\log\log n})=0.$$

LEMMA 3.1. *For any constant* $c>0$, *we have*

$$\sum_{n>b(\varepsilon)}\frac{1}{\log n}P(|X|\geq c\varepsilon\sqrt{n\log\log n})\leq M^{-1}c^{-2}<\infty.$$

PROOF. Since $k>b(\varepsilon)$ if and only if $k<M^{-1}\varepsilon^2 k\log\log k$, it follows that

$$\begin{aligned}
&\sum_{n>b(\varepsilon)}\frac{1}{\log n}P(|X|\geq c\varepsilon\sqrt{n\log\log n})\\
&\quad=\sum_{n>b(\varepsilon)}\frac{1}{\log n}\sum_{k\geq n}P\Big(c\varepsilon\sqrt{k\log\log k}\leq|X|<c\varepsilon\sqrt{(k+1)\log\log(k+1)}\Big)\\
&\quad=\sum_{k>b(\varepsilon)}\Bigg[\sum_{b(\varepsilon)<n\leq k}\frac{1}{\log n}\Bigg]P\Big(c\varepsilon\sqrt{k\log\log k}\leq|X|<c\varepsilon\sqrt{(k+1)\log\log(k+1)}\Big)\\
&\quad\leq\sum_{k>b(\varepsilon)}\exp\{-M/\varepsilon^2\}(k-b(\varepsilon))\\
&\qquad\qquad\times P\big(\varepsilon^2 k\log\log k\leq c^{-2}X^2<\varepsilon^2(k+1)\log\log(k+1)\big)\\
&\quad\leq\sum_{k>b(\varepsilon)}M^{-1}\varepsilon^2 k\log\log k\,P\big(\varepsilon^2 k\log\log k\leq c^{-2}X^2<\varepsilon^2(k+1)\log\log(k+1)\big)\\
&\quad\leq M^{-1}c^{-2}EX^2=M^{-1}c^{-2}.
\end{aligned}$$

□

Another important tool is the following lemma due to Spătaru (1999); see Lemma 2 there. It is, in turn, based on an inequality by Fuk and Nagaev (1971).

LEMMA 3.2. *For* $1<\beta\le 2$ *and* $x,y>0$, *we have*

$$P(|S_n|\ge x)\le nP(|X|\ge y)+2e^{x/y}\left(\frac{nE|X|^\beta}{nE|X|^\beta+xy^{\beta-1}}\right)^{x/y}$$
$$\le nP(|X|\ge y)+2n^{x/y}\left(\frac{eE|X|^\beta}{xy^{\beta-1}}\right)^{x/y}.$$

PROOF OF PROPOSITION 3.4. Lemma 3.2 with $x=\varepsilon\sqrt{n\log\log n}$, $y=\varepsilon\sqrt{n\log\log n}/2$ and $\beta=2$ yields

$$\sum_{n>b(\varepsilon)}\frac{1}{n\log n}P(|S_n|\ge\varepsilon\sqrt{n\log\log n})$$
$$\le\sum_{n>b(\varepsilon)}\frac{1}{\log n}P(|X|\ge\varepsilon\sqrt{n\log\log n}/2)+\frac{8e^2}{\varepsilon^4}\sum_{n>b(\varepsilon)}\frac{1}{n\log n(\log\log n)^2},$$

which, in view of Lemma 3.1, shows that, for small $\varepsilon>0$ (recall, in particular, that $\varepsilon<1/2$),

$$\varepsilon^2\sum_{n>b(\varepsilon)}\frac{1}{n\log n}P(|S_n|\ge\varepsilon\sqrt{n\log\log n})$$
$$\le\frac{4\varepsilon^2}{M}+\frac{C}{\varepsilon^2\log\log[b(\varepsilon)]}\le\frac{1}{M}+\frac{C}{M}=\frac{C}{M},$$

and we are done. □

Theorem 2 now follows from the propositions and the triangle inequality.

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