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RUIN PROBABILITY WITH CLAIMS MODELED BY A STATIONARY ERGODIC STABLE PROCESS[1]

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For a random walk with negative drift we study the exceedance probability (ruin probability) of a high threshold. The steps of this walk (claim sizes) constitute a stationary ergodic stable process. We study how ruin occurs in this situation and evaluate the asymptotic behavior of the ruin probability for a large variety of stationary ergodic stable processes. Our findings show that the order of magnitude of the ruin probability varies significantly from one model to another. In particular, ruin becomes much more likely when the claim sizes exhibit long-range dependence. The proofs exploit large deviation techniques for sums of dependent stable random variables and the series representation of a stable process as a function of a Poisson process.

1. Introduction. Let $X_1, X_2, \ldots$ be a stationary ergodic sequence of random variables with finite mean, and let $\mu > EX_1$ be a real number. Consider the random walk with negative drift

$$S_0 = 0\,, \quad S_n = X_1 + \cdots + X_n - n\mu\,, \quad n \geq 1\,,$$

generated from (X_n). The random quantity

$$\sup_{n\geq 0}\, S_n = \sup_{n\geq 0}\,(X_1 + \cdots + X_n - n\mu) \tag{1.1}$$

is then well defined. In various fields of applied probability theory it has different important interpretations. Traditionally, (1.1) has been considered in an insurance context as the largest ever excess of the total claim amount in an insurance portfolio when exceeding the loaded total premium; see, for example, [7], Chapter 1. In a queuing context, the quantity (1.1) represents the stationary workload in a stable queue; see, for example, [2].

Correspondingly, the exceedance probability

$$\psi(u) := P\left(\sup_{n\geq 0} S_n > u\right), \qquad u > 0,$$

can, at least in the insurance context, be thought of as *ruin probability with initial capital u*, or for short, as *ruin probability*. Moreover, (X_n) can be considered as the sequence of claim sizes in the portfolio. Obviously, we adopt

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here the language of insurance and keep using this language, however casually, throughout the paper. In the queuing context, however, the tail probability $\psi(u)$ of the stationary solution is often viewed as an overflow probability. We also mention that the tail probability of solutions to stochastic recurrence equations, including the tails of ARCH and GARCH processes, is closely related to the quantity $\psi(u)$; see [7], Section 8.4 and [9].

Initially, the research on ruin probabilities concentrated on the case of iid claim sizes. However, over the last few years the attention has turned to dependent claims, the main reason being the fact that in most applications the independence assumption is, clearly, unrealistic. For example, in queuing theory the difference between service times and inter-arrival times of successive customers is universally believed to be dependent. In addition, the case of dependent claim sizes leads to interesting theoretical questions, and it often gives new insight into the structure of the stationary processes underlying the claims. The present paper is an example of such a "reverse" effect.

A lot of interest and effort went into studying the case of "heavy-tailed" claim sizes. Even though different authors use a variety of definitions for "heavy tails," the general idea is that "very large" claims occur relatively often. It is precisely the extreme risk that banks, insurance companies, governmental institutions and others are trying to control, hence the theoretical interest in modeling heavy-tailed phenomena. Empirical evidence seems to indicate that their presence is almost universal. See, for example, [31] and [6] for the evidence of heavy tails in communication networks (file sizes, on-off times), [18] for a discussion and measurement of heavy tails in an insurance context and [16] for a description of heavy tails in financial markets.

The iid heavy-tailed ruin problem was finally solved by Embrechts and Veraverbeke (1982) in the greatest possible generality of subexponential claim sizes, following a series of less general results. It has been shown subsequently (e.g., [1]) that this result remains valid under certain departures from independence. Recently, [15], using a heavy-tailed linear process model for the claim sizes, have shown that the [8] result may fail if the claim sizes exhibit a certain dependence in the right tails.

There are two problems of obvious theoretical and practical importance. On the one hand, one has to understand what the connections between the dependence structure of the claim sizes process (X_n) and the ruin probability are. On the other hand, one needs to study how the interplay between the heavy tails and the dependence structure of the process affects the ruin probability. These are especially challenging problems because, when the tails are particularly heavy, the second moment of the claim sizes is infinite, hence it is impossible to use correlations to quantify the length or the strength of dependence. It is, of course, also clear that even if the second moment is finite, we are very far from the Gaussian case, and so correlation, even though being well defined, may not carry enough information.

We have chosen the class of stationary ergodic *symmetric α-stable* (SαS) processes with $\alpha \in (1, 2)$ to model the claim sizes. There are many reasons for that. First of all, stable processes are, arguably, the single most important class of heavy-tailed processes. Further, their structure is relatively well understood, and this allows one to focus on their dependence. Since stable processes do not have a finite second moment, we are forced to concentrate on what may be really important for dependence that far away from Gaussianity. Finally, there are good reasons to believe that, once we understood what happens when the claim sizes follow a stationary stable process, we will be able to treat more general classes of processes as well. Such results will be presented elsewhere, and there we will also remove the assumption of symmetry used in the present paper as a matter of (often only notational) convenience.

Let, therefore, $X_1, X_2, \dots$ be a stationary ergodic SαS process with $\alpha \in (1,2)$. This means, in particular, that each random variable (claim size) in this process has characteristic function

$$E\exp\{i\lambda X_j\} = \exp\{-\sigma^\alpha |\lambda|^\alpha\}, \quad \lambda \in \mathbb{R} \text{ for some } \sigma > 0. \tag{1.2}$$

Notice that X has infinite variance but a finite first moment. The statement that the whole process $X_1, X_2, \dots$ is SαS means that every finite linear combination of the coordinates of the process is a (one-dimensional) SαS random variable, that is, with a characteristic function of the form (1.2) for some $\sigma \geq 0$ that will depend on the coefficients of the linear combination. We refer the reader to [25] for more information on equivalent definitions of stability and other properties of stable random variables and processes.

The fact that the process (X_n) is SαS implies that it can be represented in the form

$$X_n = \int_E f_n(x)\, M(dx), \quad n = 1, 2, \dots, \tag{1.3}$$

where M is a SαS random measure on a measurable space $(E, \mathcal{E})$ with a σ-finite control measure m on $\mathcal{E}$, and $f_n \in L^\alpha(m, \mathcal{E})$ for all n; see [25], Section 3.3.

We consider a stationary SαS process. Integral representations of such processes can be chosen to be of a particularly descriptive form, due to [21]. Specifically, one can write

$$X_n = \int_E a_n(x) \left(\frac{dm \circ \phi_n}{dm}(x) \right)^{1/\alpha} f \circ \phi_n(x)\, M(dx), \quad n = 1, 2, \dots, \tag{1.4}$$

where ϕ_0 is the identity function on E, and for $n \geq 1$, $\phi_n = \phi_{n-1} \circ \phi$, where ϕ is a measurable non-singular map $E \to E$. Furthermore, (a_n) is a cocycle, taking values in $\{-1, 1\}$. That is, $a_0 \equiv 1$, and for $n \geq 1$, $a_{n+1}(x) = a_n(x)(a_1 \circ \phi_n)(x)$. Finally, f is a given function in $L^\alpha(m, \mathcal{E})$.

The importance of the representation (1.4) lies in the possibility that it opens for studying the properties of a stationary SαS process in terms of the properties of the flow (ϕ_n) and a single function f.

The ergodic decomposition of the flow (ϕ_n) immediately shows that one can decompose a stationary SαS process $\mathbf{X} = (X_1, X_2, \dots)$ as a sum of two independent stationary SαS processes,

$$\mathbf{X} = \mathbf{X}^{(1)} + \mathbf{X}^{(2)}, \tag{1.5}$$

where $\mathbf{X}^{(1)}$ is given by the representation (1.4) with a dissipative flow (ϕ_n) and $\mathbf{X}^{(2)}$ is given by the representation (1.4) with a conservative flow (ϕ_n); see [21] for details.

We are interested in studying ergodic stationary SαS processes. It turns out that any stationary SαS process with a dissipative flow is a so called *mixed moving average* and, hence ergodic [26], while it is fairly tricky (but possible) to construct examples of ergodic processes corresponding to conservative flows ([23]). In the present paper we consider ergodic stationary models both for claim sizes corresponding to dissipative flows and those corresponding to conservative flows.

Recall that a SαS random variable X with characteristic function given by (1.2) satisfies

$$P(X > x) \sim \tfrac{1}{2}\, C_\alpha \sigma^\alpha\, x^{-\alpha} \quad \text{as } x \to \infty \tag{1.6}$$

for some constant C_α depending only on α, see [25]. Therefore, if the claim sizes process (X_n) is an iid SαS sequence with common characteristic function given by (1.2), then the aforementioned result of [8] (cf. [7], Theorem 1.3.6) yields that the ruin probability $\psi(u)$ is asymptotically of the order

$$\psi(u) \sim \frac{C_\alpha \sigma^\alpha}{2(\alpha-1)\,\mu}\; u^{-(\alpha-1)}, \quad u \to \infty. \tag{1.7}$$

One can say that the order of magnitude of the ruin probability in (1.7) is a direct consequence of the heavy tails of SαS random variables. One of our main goals in this paper is to show that the dependence structure of ergodic stationary SαS processes can cause the asymptotic behavior of the ruin probability to be completely different from the classical result (1.7). Roughly speaking, one can summarize our findings as follows. In many cases the ruin probability $\psi(u)$ is of the same order of magnitude $u^{-(\alpha-1)}$ as in (1.7), but with a different, in general, multiplicative constant. We think of these classes of stationary SαS processes as *short-range dependent*. For other classes of stationary SαS processes even the order of magnitude of the ruin probability $\psi(u)$ changes, and we will see various examples of processes for which $\psi(u)$ is of the order $u^{-\gamma(\alpha-1)}L(u)$ for any $\gamma \in (0,1)$ and a slowly varying function L. We think of these SαS processes as *long-range dependent*. Note that in the absence of correlations the notion of the range of dependence is, by necessity, application specific and, hence, we gain here additional insight into the dependence structure of stationary SαS processes by studying the ruin probability.

As the reader will, undoubtedly, observe, in this paper we concentrate on what one can call *pure type* models. That is, we will always look at a process that has only one of the components in (1.5). While this, by itself, does not require justification, it is appropriate to add that, in the cases we are considering, the ruin probability is a regularly varying function of the level u, and a very simple and standard regular variation argument then allows one to compute the asymptotic behavior of the ruin probability when several independent components are present from the known behavior for pure type models.

Our paper is organized as follows. In Section 2 we present our main result (Theorem 2.5) which determines the asymptotic order of the ruin probability $\psi(u)$ for a rather general stationary ergodic SαS process (X_n). The main tool in this context is a series representation of a SαS process based on a particular kind of a Poisson random measure. We use large deviation ideas for such Poisson random measures. In Section 3 we consider various applications of Theorem 2.5 to different classes of SαS ergodic processes associated with conservative flow processes. We will show that a large variety of asymptotic orders for $\psi(u)$ is possible, depending on the strength of dependence of the step sizes of the random walk. In Section 4 we continue with applications of Theorem 2.5 to ergodic processes associated with a dissipative flow. Those include moving average processes and certain self-similar processes. Again, we can show that the order of $\psi(u)$ can vary significantly, depending on the kind of dependence of the step sizes. The results of this paper are a step towards a general theory of the ruin probability for ergodic SαS processes. Even though many details still have to be worked out in subsequent research, we believe that our results are quite representative and illustrate the kind of problems one has to face for any SαS process. Moreover, in [5] similar techniques as developed in the present paper have been used to derive the tail behavior of general subbadditive functionals acting on the paths of Lévy processes with heavy-tailed marginal distributions. The tail behavior of such subadditive functionals for stable and other infinitely divisible processes is the topic of future research.

2. How does ruin occur?

In this section we state and prove our general main result. It describes the most likely way in which ruin can occur when the claim sizes are distributed according to a stationary ergodic SαS process with a certain integral representation, which, for the moment, is allowed to have the general form of (1.3).

We introduce some notation first. Let

$$h_0(x) = 0 \quad \text{and} \quad h_n(x) = \sum_{k=1}^{n} f_k(x), \quad x \in E, \quad n \ge 1,$$

and define

$$m_n = C_\alpha^{1/\alpha} \left(\int_E |h_n(x)|^\alpha \, m(dx) \right)^{1/\alpha}, \tag{2.1}$$

where C_α is the constant appearing in (1.6). Observe that by ergodicity of the process we have

$$n^{-1} (X_1 + \cdots + X_n) \to 0 \qquad \text{a.s. as } n \to \infty. \tag{2.2}$$

Since m_n is just the scaling parameter (σ in (1.2)) of the sum $X_1 + \cdots + X_n$, we immediately conclude that

$$m_n = o(n) \qquad \text{as } n \to \infty. \tag{2.3}$$

Let η_0 be a probability measure on $\mathscr{E}$ equivalent to the control measure m in (1.3), and let

$$g = \frac{d\eta_0}{dm}.$$

A simple change of variable in the integral representation (1.3) (see [25], Section 3.5) shows that the process $X_1, X_2, \dots$ can alternatively be represented (at least, in law) in the form

$$X_n = \int_E g^{-1/\alpha}(x)\, f_n(x)\, M_0(dx), \quad n \geq 1, \tag{2.4}$$

where this time M_0 is a SαS random measure with control measure η_0.

The fact that the control measure η_0 of the random measure M_0 in (2.4) is a probability measure allows us to give yet another representation, again in law, of the process $X_1, X_2, \dots$ as a series

$$X_n = C_\alpha^{1/\alpha} \sum_{j=1}^{\infty} \varepsilon_j\, \Gamma_j^{-1/\alpha}\, g^{-1/\alpha}(V_j)\, f_n(V_j) \tag{2.5}$$

where $(\varepsilon_n)_{n\geq 1}$ is an iid sequence of Rademacher variables ($P(\varepsilon_n = -1) = P(\varepsilon_n = 1) = 1/2$), $(\Gamma_n)_{n\geq 1}$ are the points of a unit rate Poisson process on $(0, \infty)$, and $(V_n)_{n\geq 1}$ is an iid sequence of E-valued random variables with common distribution η_0. Moreover, the three sequences are mutually independent. See [25], Section 3.10.

The change of variable performed above resulted, effectively, in multiplying each function f_n by the same factor $g^{-1/\alpha}$, and the functions h_n now become

$$h_0^*(v) = 0 \quad \text{and} \quad h_n^*(v) := C_\alpha^{1/\alpha}\, g^{-1/\alpha}(v) \sum_{k=1}^{n} f_k(v), \quad v \in E, \quad n \geq 1.$$

We may, therefore, rewrite the ruin probability as follows:

$$\psi(u) = P\left(\sup_{n\geq 0}\left(\sum_{j=1}^{\infty} \varepsilon_j \Gamma_j^{-1/\alpha} h_n^*(V_j) - n\mu\right) > u\right). \tag{2.6}$$

In order to understand what is the most likely way for ruin to occur we look at the event on the right hand side of (2.6) from the point of view of heavy-tailed large deviations. Observe that the terms $\varepsilon_j \Gamma_j^{-1/\alpha} h_\cdot^*(V_j)$ in the sum above form the points of a Poisson random measure on $\mathbb{R}^\infty$. Now, the consequence (2.2) of the ergodicity of our process implies, in particular, that

$$n^{-1}\, h_n^*(V_j) \to 0 \quad \text{a.s. as } n \to \infty. \tag{2.7}$$

(see [20]), meaning that each of these Poisson points grows, as a function of time, slower than any linear function. It is, then, the factor $\Gamma_j^{-1/\alpha}$ and the sheer size of the jth of these functions that make the event on the right hand side of (2.6) occur.

The heavy-tailed large deviations intuition now tells us that it is most likely that this event happens because of a *single* unusually large (in overall size) of the Poisson points-functions, and so one expects that

$$\psi(u) \sim \psi_0(u) := \sum_{j=1}^{\infty} P\left(\sup_{n\geq 0}\left(\varepsilon_j \Gamma_j^{-1/\alpha} h_n^*(V_j) - n\mu\right) > u\right). \tag{2.8}$$

REMARK 2.1. In Theorem 2.5 below we show that the equivalence (2.8) indeed holds under very mild conditions. In fact, the mild conditions we are imposing make the ergodicity assumption unnecessary, even though our main interest is in the ergodic case.

REMARK 2.2. Related situations occur when one needs to study the tail behavior of functionals of stable (and, indeed, more general) processes under the assumption that the functional is dominated by an almost surely finite norm (or a semi-norm). The heavy-tailed large deviations work in that case too. See [22]. The difference between that situation and the present one is that in our case the single largest Poisson function is no longer necessarily the one corresponding to the largest one-dimensional scaling of $\Gamma_1^{-1/\alpha}$. That is, it is not necessarily the case that

$$\psi(u) \sim P\left(\sup_{n\geq 0}\left(\varepsilon_1 \Gamma_1^{-1/\alpha} h_n^*(V_1) - n\mu\right) > u\right). \tag{2.9}$$

In fact, (2.9) is false even in the case of SαS Lévy motion below. Rather, the functions $h_\cdot^*(V_j)$ can be very large on their own, and it is the interplay between those functions and the one-dimensional Poisson scales of the $\Gamma_j^{-1/\alpha}$s that determines how ruin occurs. However, in the case when a finite semi-norm dominates the functional of interest (e.g., when we are considering the supremum of a bounded process) all the other factors turn out to be small, and so it is only the scaling by $\Gamma_1^{-1/\alpha}$ that is likely to cause very high values.

Conditioning on the Γ_js on the right hand side of (2.8) and summing up, we obtain

$$\begin{aligned}
\psi_0(u) &= \int_0^\infty P\left(\sup_{n\geq 0}\ \left(\varepsilon_1\, h_n^*(V) - n\mu\, x^{1/\alpha}\right) > u x^{1/\alpha}\right)\, dx \\
&= \frac{1}{2}\int_E \int_0^\infty I\left\{h_n^*(v) > x^{1/\alpha}(u+n\mu) \quad \text{for some } n \geq 1\right\}\, dx\ \eta_0(dv) \\
&\quad + \frac{1}{2}\int_E \int_0^\infty I\left\{-h_n^*(v) > x^{1/\alpha}(u+n\mu) \quad \text{for some } n \geq 1\right\}\, dx\ \eta_0(dv) \\
&= \frac{C_\alpha}{2}\int_E \sup_{n\geq 1} \frac{\left(\sum_{k=1}^n f_k(v)\right)_+^\alpha}{(u+n\mu)^\alpha}\ m(dv) \\
&\quad + \frac{C_\alpha}{2}\int_E \sup_{n\geq 1} \frac{\left(-\sum_{k=1}^n f_k(v)\right)_+^\alpha}{(u+n\mu)^\alpha}\ m(dv).
\end{aligned} \tag{2.10}$$

REMARK 2.3. Certainly, the expression we obtained for $\psi_0(u)$ in (2.10) is more explicit than the original ruin probability $\psi(u)$. However, it is not very explicit and, in fact, $\psi_0(u)$ may be of various orders of magnitude. We will see a number of examples in the sequel.

It is illustrative to see how the large deviations equivalence (2.8) allows us to recover the classical [8] result (1.7) in the stable case.

EXAMPLE 2.4 (SαS Lévy motion with negative drift). Here $E = [0, \infty)$, $\mathscr{E}$ is the corresponding Borel σ-field and the control measure m is the appropriately scaled Lebesgue measure. That is, $m(dx) = \sigma^\alpha\, dx$, where σ is the scale parameter of the step size. Furthermore,

$$f_n(x) = I_{[n-1,n)}(x), \quad n \geq 1.$$

Therefore, as $u \to \infty$,

$$\begin{aligned} 2\,C_\alpha^{-1}\sigma^{-\alpha}\,\psi_0(u) &= \int_0^\infty \sup_{n\geq 1} \frac{I_{[0,n]}(v)}{(u+n\mu)^\alpha}\,dv \\ &= \sum_{n=1}^\infty (u+n\mu)^{-\alpha} \sim (\mu(\alpha-1))^{-1}\,u^{-(\alpha-1)}. \end{aligned}$$

The following theorem is our main general result for the ruin probability when the step sizes are distributed according to a symmetric stable process, and it justifies (2.8).

THEOREM 2.5. *Let (X_n) be a stationary ergodic* SαS process, $\alpha \in (1, 2)$, with integral representation (1.3). Assume that

$$m_n = O(n^\beta) \quad \text{as } n \to \infty \text{ for some } \beta \in (0, 1). \tag{2.11}$$

Then the relation $\psi(u) \sim \psi_0(u)$ holds as $u \to \infty$, where $\psi_0(u)$ is given in (2.10).

REMARK 2.6. Notice that assumption (2.11) is stronger than the automatic consequence (2.3) of ergodicity. There are examples of stationary SαS (X_n) such that $n^{-1}m_n \to 0$ at an arbitrarily slow rate. For example, it is clear that for a moving average process (which is always ergodic; see [14])

$$X_n = \int_{-\infty}^\infty f(x-n)\,M(dx), \quad n \geq 1,$$

with

$$f(x) = x^{-1/\alpha}(\log x)^{-p/\alpha} I_{(e,\infty)}(x)$$

and any $p > 1$, the assumption (2.11) does not hold. See Remark 3.7 for another, more interesting example. We believe that at least in the ergodic case, the assumption (2.11) can be relaxed and, perhaps, completely eliminated. However, our method of proof requires it.

PROOF OF THEOREM 2.5. We work with the process (X_n) given in the form of the series (2.5). Observe that the set $(\varepsilon_j \Gamma_j^{-1/\alpha} h^*_\cdot(V_j))_{j\ge 1}$ constitutes a Poisson random measure (PRM) N on $(\mathbb{R}^\infty, \mathscr{B}^\infty)$, with mean measure ν given by

$$\nu(A) = \int_0^\infty P\left(\varepsilon_1\, h^*_\cdot(V) \in A\, x^{1/\alpha}\right)\, dx\,, \quad A \in \mathscr{B}^\infty. \tag{2.12}$$

We refer the reader to [11] for the general theory of random measures.

Choose κ such that

$$\kappa > (\alpha+1)(1-\beta)^{-1}\,, \tag{2.13}$$

where $\beta \in (0,1)$ is the number for which (2.11) holds. Further, choose a number $K \ge 1$ such that

$$E\left|\sum_{j=K+1}^\infty \varepsilon_j \Gamma_j^{-1/\alpha}\right|^\kappa < \infty. \tag{2.14}$$

For $\epsilon \in (0, 1/K)$ we introduce the set

$$\mathbb{B}_\epsilon := \left\{\mathbf{a} = (a_n)_{n\ge 1} \in \mathbb{R}^\infty : \sup_{n\ge 1}\,(|a_n| - n\epsilon\mu) > 1\right\}. \tag{2.15}$$

The ergodic theorem implies that the stochastic process

$$\frac{X_1 + \cdots + X_n}{n}\,, \quad n \ge 1\,, \tag{2.16}$$

is a.s. bounded. Hence

$$\int_E \sup_{n\ge 1} \frac{|h_n(v)|^\alpha}{n^\alpha}\, m(dv) < \infty\,; \tag{2.17}$$

see Section 10.2 in [25]. In particular, the set $\mathbb{B}_\epsilon$ has finite ν-measure:

$$\nu\,(\mathbb{B}_\epsilon) = \int_0^\infty P\left(\sup_{n\ge 1}\,\left(|h^*_n(V)| - n\epsilon x^{1/\alpha}\mu\right) > x^{1/\alpha}\right)\, dx$$

$$= C_\alpha \int_E \sup_{n\ge 1}\, \frac{|h_n(v)|^\alpha}{(1+n\epsilon\mu)^\alpha}\, m(dv) < \infty.$$

Therefore, by (2.5),

$$\begin{aligned} X_1 + \cdots + X_n \;&=\; \sum_{j=1}^\infty \varepsilon_j \Gamma_j^{-1/\alpha} h^*_n(V_j) I\left\{\varepsilon_j \Gamma_j^{-1/\alpha} h^*_\cdot(V_j) \in \mathbb{B}_\epsilon\right\} \\ &\quad+\; \sum_{j=1}^\infty \varepsilon_j \Gamma_j^{-1/\alpha} h^*_n(V_j) I\left\{\varepsilon_j \Gamma_j^{-1/\alpha} h^*_\cdot(V_j) \in \mathbb{B}^c_\epsilon\right\} \\ &=: Y_n + Z_n. \end{aligned} \tag{2.18}$$

By the defining properties of a PRM, the sequences (Y_n) and (Z_n) are independent. Moreover, since the set $\mathbb{B}_\epsilon$ has finite ν-measure, the sequence (Y_n) can be represented in the form

$$Y_n = \sum_{j=1}^{N_\epsilon} A_{j,n}\,,$$

where N_ϵ is a Poisson random variable with mean $\nu(\mathbb{B}_\epsilon)$, independent of a sequence of iid random elements in $\mathbb{R}^\infty$, $(\mathbf{A}_j) = (A_{j,\cdot},\ j \geq 1)$, with common law given by

$$\nu(\cdot \cap \mathbb{B}_\epsilon)/\nu(\mathbb{B}_\epsilon). \tag{2.19}$$

We write

$$\widetilde{\beta} = \nu(\mathbb{B}_\epsilon) = EN_\epsilon \quad \text{and} \quad p(k) = P(N_\epsilon = k)\,, \quad k \geq 0.$$

Furthermore, one can represent the sequence (Z_n) as

$$Z_n = \sum_{j=1}^{\infty} B_{j,n},$$

where $(\mathbf{B}_j) = (B_{j,\cdot},\ j \geq 1)$, is an enumeration of the points of N restricted to $\mathbb{B}_\epsilon^c$.

We first study the probabilities

$$\psi_1(u) = P\left(\sup_{n\geq 1}(Y_n - n\mu) > u\right), \quad u > 0.$$

The following lemma shows that $\psi_1(u)$ is asymptotically equivalent to $\psi_0(u)$.

LEMMA 2.7. *As* $u \to \infty$, $\psi_0(u) \sim \psi_1(u)$.

PROOF. We make frequent use of the events

$$D_j(x,u) := \left\{\sup_{n\geq 1}(A_{j,n} - nx\mu) > ux\right\}.$$

Fix $\theta \in (0,1)$. The following bound follows from the easily verifiable inclusion of the events in the left and right hand sides for each fixed $N_\epsilon = k$:

$$\begin{aligned}\psi_1(u) \leq\ & P\left(\bigcup_{j\leq N_\epsilon} D_j(1-\theta,u)\right)\\ & + P\left(\bigcup_{j_1\neq j_2\leq N_\epsilon} D_{j_1}\left(N_\epsilon^{-1}\theta,u\right) \cap D_{j_2}\left(N_\epsilon^{-1}\theta,u\right)\right)\\ =:\ & \psi_{11}(u) + \psi_{12}(u).\end{aligned} \tag{2.20}$$

Then

$$\psi_{12}(u) \le \sum_{k=2}^{\infty} k^2 p_1^2(k)\; p(k)\,,$$

where $p_1(k) := P(D_1(k^{-1}\theta, u))$. Recall that the law of $\mathbf{A}_1$ is given by (2.19). Therefore

$$\begin{aligned} p_1(k) \le \frac{C_\alpha}{2\widetilde{\beta}} \int_E \Bigg[\sup_{n\ge 1} \; \frac{(\sum_{k=1}^n f_k(v))_+^\alpha}{(k^{-1}u\theta + k^{-1}n\theta\mu)^\alpha} \\ + \sup_{n\ge 1} \; \frac{(-\sum_{k=1}^n f_k(v))_+^\alpha}{(k^{-1}u\theta + k^{-1}n\theta\mu)^\alpha} \Bigg] \; m(dv) \\ = k^\alpha \theta^{-\alpha} \widetilde{\beta}^{-1} \psi_0(u). \end{aligned}$$

We conclude that

$$\psi_{12}(u) \le \psi_0^2(u) \widetilde{\beta}^{-2} \theta^{-2\alpha} \sum_{k=2}^{\infty} k^{2+2\alpha}\; p(k) =: c(\theta, \epsilon)\; \psi_0^2(u).$$

On the other hand,

$$\psi_{11}(u) \le \widetilde{\beta}\; P(D_1(1-\theta, u)) = (1-\theta)^{-\alpha} \psi_0(u).$$

Recalling (2.20) and the above estimates for $\psi_{11}(u)$ and $\psi_{12}(u)$, we conclude that for any $\theta \in (0,1)$,

$$\psi_1(u) \le (1-\theta)^{-\alpha}\psi_0(u) + c(\theta,\epsilon)\; \psi_0^2(u).$$

It follows immediately from (2.10) and (2.17) that $\psi_0(u) \to 0$ as $u \to \infty$. Hence, letting $\theta \to 0$, we obtain

$$\limsup_{u\to\infty} \psi_1(u)/\psi_0(u) \le 1. \tag{2.21}$$

Thus it remains to estimate $\psi_1(u)$ from below. For every fixed $\theta > 0$ we have

$$\psi_1(u) \ge \sum_{k=1}^{\infty} P\left(\bigcup_{j=1}^{k} C_{j,k} \right) \; p(k)\,,$$

where for $j = 1, \ldots, k$,

$$C_{j,k} := D_j(1+\theta, u) \cap \left\{ \sum_{i=1, i\ne j}^{k} A_{i,n} + n\theta\mu > -u\theta \quad \text{for all } n \ge 1 \right\}.$$

A Bonferroni inequality yields

$$\begin{aligned} \psi_1(u) &\ge \sum_{k=1}^{\infty} \left[kP(C_{1,k}) - \frac{k(k-1)}{2} P(C_{1,k} \cap C_{2,k}) \right] \; p(k) \\ &=: I_1(u) - I_2(u). \end{aligned} \tag{2.22}$$

For all u so large that $u(1+\theta) > 1$, the same argument as for (2.21) gives

$$P\left(D_1(1+\theta, u)\right) = \widetilde{\beta}^{-1} (1+\theta)^{-\alpha} \psi_0(u).$$

Therefore by the definition of Y_n and the symmetry of $\mathbf{A}_i$,

$$I_1(u) = (1+\theta)^{-\alpha}\,\psi_0(u)\,\sum_{k=1}^{\infty}[\widetilde{\beta}^{-1}\,k\,p(k)]\,P\left(\sum_{i=1}^{k-1} A_{i,n} - n\theta\mu < u\theta \quad \text{for all } n\right)$$

$$= (1+\theta)^{-\alpha}\,\psi_0(u)\,P(Y_n - n\theta\mu < u\theta \quad \text{for all } n).$$

Similar arguments as for (2.21) show that $P(Y_n - n\theta\mu < u\theta \quad \text{for all } n) \to 1$ as $u \to \infty$, and therefore

$$\liminf_{u\to\infty} \psi_0^{-1}(u) I_1(u) \geq (1+\theta)^{-\alpha}. \tag{2.23}$$

On the other hand, for every $k \geq 2$,

$$P(C_{1,k} \cap C_{2,k}) \leq \Big[P(D_1(1+\theta, u))\Big]^2 \leq (1+\theta)^{-2\alpha}\widetilde{\beta}^{-2}\psi_0^2(u).$$

Hence

$$I_2(u) \leq (1+\theta)^{-2\alpha}\widetilde{\beta}^{-2}\psi_0^2(u)\sum_{k=2}^{\infty}\frac{k(k-1)}{2}\,p(k). \tag{2.24}$$

Since $\psi_0(u) \to 0$ as $u \to \infty$ we can let $\theta \to 0$ in order to conclude from (2.22)–(2.24) that

$$\liminf_{u\to\infty} \psi_1(u)/\psi_0(u) \geq 1. \tag{2.25}$$

Finally, (2.21) and (2.25) establish the statement of the lemma. □

Denote now

$$\psi_2(u) := P\left(\sup_{n\geq 1}\,(Z_n - n\mu) > u\right).$$

It turns out that under our assumptions $\psi_2(u)$ is small compared to $\psi_0(u)$.

LEMMA 2.8. *Under the assumptions of Theorem* 2.5 *and with the choice of* $\epsilon \in (0, 1/K)$ *in* (2.15) *we have* $\psi_2(u) = o(\psi_0(u))$ *as* $u \to \infty$.

PROOF. We have

$$\psi_2(u) \leq \sum_{n=1}^{\infty} P\left(\sum_{j=1}^{\infty}\varepsilon_j\Gamma_j^{-1/\alpha}h_n^*(V_j)I\left\{\Gamma_j^{-1/\alpha}|h_k^*(V_j)| \leq k\epsilon\mu + 1 \quad \text{for all } k\geq 1\right\} - n\mu > u\right)$$

$$\leq 2\sum_{n=1}^{\infty} q(n),$$

where

$$q(n) := P\left(\sum_{j=1}^{\infty} \varepsilon_j \Gamma_j^{-1/\alpha} h_n^*(V_j) I\left\{\Gamma_j^{-1/\alpha} |h_n^*(V_j)| \le n\epsilon\mu + 1\right\} - n\mu > u\right).$$

The last inequality is a consequence of the contraction principle, applied to a sum of weighted Rademacher random variables, conditionally upon the sequences (Γ_j) and (V_j).

Observe that, for every n, the points $(\varepsilon_j \Gamma_j^{-1/\alpha} h_n^*(V_j))_{j\ge 1}$ constitute a symmetric PRM on $(\mathbb{R}, \mathscr{B})$ with mean measure of the set (x, ∞) equal to $x^{-\alpha} m_n^\alpha / 2$, $x > 0$, and the same PRM can be represented (in law) by the points

$$\varepsilon_j \Gamma_j^{-1/\alpha} m_n, \quad j \ge 1.$$

By the contraction principle and the Markov inequality, for every $u > K$ and $\kappa > 0$ we have

$$\begin{aligned} q(n) &= P\left(\sum_{j=1}^{\infty} \varepsilon_j \Gamma_j^{-1/\alpha} I\left\{\Gamma_j^{-1/\alpha} \le m_n^{-1}(n\epsilon\mu + 1)\right\} > m_n^{-1}(u + n\mu)\right) \\ &\le P\left(\sum_{j=K+1}^{\infty} \varepsilon_j \Gamma_j^{-1/\alpha} I\left\{\Gamma_j^{-1/\alpha} \le m_n^{-1}(n\epsilon\mu + 1)\right\}\right. \\ &\qquad\qquad \left. > m_n^{-1}(u - K + n\mu(1 - \epsilon K))\right) \\ &\le 2\, P\left(\sum_{j=K+1}^{\infty} \varepsilon_j \Gamma_j^{-1/\alpha} > m_n^{-1}(u - K + n\mu(1 - \epsilon K))\right) \\ &\le 2\, \frac{m_n^\kappa}{(u - K + n\mu(1 - \epsilon K))^\kappa} E\left|\sum_{j=K+1}^{\infty} \varepsilon_j \Gamma_j^{-1/\alpha}\right|^\kappa. \end{aligned}$$

Therefore, by (2.11), (2.14), the choice (2.13) of κ and Lemma 3.6 below, we have

$$\begin{aligned} \psi_2(u) &\le const \sum_{n=1}^{\infty} m_n^\kappa\, (u + n)^{-\kappa} \le const \sum_{n=1}^{\infty} n^{\beta\kappa}\, (u + n)^{-\kappa} \\ &\le const\; u^{-\kappa(1-\beta)+1} = o(u^{-\alpha}). \end{aligned}$$

On the other hand, cf. (2.10),

$$\begin{aligned} \psi_0(u) &\ge \frac{C_\alpha}{2} \int_E \sup_{n\ge 1} \frac{|\sum_{k=1}^n f_k(v)|^\alpha}{(u + n\mu)^\alpha}\, m(dv) \\ &\ge \frac{C_\alpha}{2}\, m_1\, (u + \mu)^{-\alpha}. \end{aligned}$$

This proves the lemma. □

We are now in a position to complete the proof of Theorem 2.5. For every $\theta \in (0, 1)$ we have

$$P\left(\sup_{n\geq 1}(Y_n - n(1+\theta)\mu) + \inf_{n\geq 1}(Z_n + n\theta\mu) > u\right)$$
$$\leq \psi(u) \leq P\left(\sup_{n\geq 1}(Y_n - n(1-\theta)\mu) + \sup_{n\geq 1}(Z_n - n\theta\mu) > u\right),$$

and so by Lemmas 2.7 and 2.8 and the fact that a random variable with a regularly varying probability tail keeps this tail unaffected if one adds to it another random variable with a probability tail of a lower order,

$$(1+\theta)^{-\alpha} \leq \liminf_{u\to\infty} \psi(u)/\psi_0(u) \leq \limsup_{u\to\infty} \psi(u)/\psi_0(u) \leq (1-\theta)^{-\alpha}.$$

Since $\theta \in (0, 1)$ is arbitrary, we conclude that the statement of the theorem holds. □

REMARK 2.9. It is clear from the proof of Theorem 2.5 that its conclusion also holds for non-stationary SαS processes $X_1, X_2, \ldots$ for which the sample mean process (2.16) is a.s. bounded, at least in the presence of the assumption (2.11).

REMARK 2.10. A minor modification in the last part of the proof of Theorem 2.5 shows that the lower bound

$$\liminf_{u\to\infty} \frac{\psi(u)}{\psi_0(u)} \geq 1 \tag{2.26}$$

holds even without the assumption (2.11). Indeed, fix $\epsilon > 0$ and $\theta > 0$ and let

$$N = \inf\{n \geq 1 : Y_n - n(1+\theta)\mu > u(1+\epsilon)\}.$$

Then

$$\psi(u) = P\left(\sup_{n\geq 1}(Y_n + Z_n - n\mu) > u\right) \geq P(N < \infty, Z_N + N\theta\mu > -\epsilon u)$$
$$\geq P\left(\sup_{n\geq 1}(Y_n - n(1+\theta)\mu) > u(1+\epsilon)\right) \inf_{n\geq 1} P(Z_n + n\theta\mu > -\epsilon u).$$

As in the proof of (2.23) and using Lévy's maximal inequality,

$$(1+\theta)^{\alpha} \liminf_{u\to\infty} \frac{\psi(u)}{\psi_0(u)}$$
$$\geq \liminf_{u\to\infty} \frac{\psi_0(u(1+\epsilon))}{\psi_0(u)} \inf_{n\geq 1}(1 - P(Z_n - n\theta\mu > \epsilon u))$$

$$\geq (1+\epsilon)^{-\alpha} \liminf_{u\to\infty} \inf_{n\geq 1} (1 - 2P(X_1 + \cdots + X_n - n\theta\mu > \epsilon u))$$

$$\geq (1+\epsilon)^{-\alpha} \liminf_{u\to\infty} \left(1 - 2P\left(\sup_{n\geq 1} (X_1 + \cdots + X_n - n\theta\mu) > \epsilon u\right)\right)$$

$$= (1+\epsilon)^{-\alpha}.$$

In the last step we used the ergodicity of the process. Letting both θ and ϵ go to 0 we obtain (2.26).

REMARK 2.11. The method of proof we use in Theorem 2.5 is the one where we split the Lévy measure of the process into parts concentrated "at the middle" and "at the wings" as in (2.18). One uses a similar approach in the situation described in Remark 2.2, and there the lighter-tailed of the two processes (that corresponding to the "middle part" of the Lévy measure) has, in fact, exponentially light tails (see [22]). This is not the case in our situation, as fairly easy counterexamples can show.

The following proposition is an immediate corollary of Theorem 2.5 and Remark 2.10.

PROPOSITION 2.12. *Let* (X_n) *be a stationary* SαS process given in the form (1.3) with $f_n \geq 0$ for all $n \geq 1$. Then

$$\liminf_{u\to\infty} u^{\alpha-1}\psi(u) \geq (1-\alpha^{-1})^{\alpha} \frac{C_\alpha}{2\mu(\alpha-1)} \int_E [f_1(v)]^\alpha \, m(dv). \tag{2.27}$$

PROOF. It is, clearly, enough to prove (2.27) with $\psi(u)$ replaced by $\psi_0(u)$. By (2.10) and stationarity, for any $a > 0$

$$\frac{2\,\psi_0(u)}{C_\alpha} = \int_E \sup_{n\geq 1} \frac{(\sum_{k=1}^n f_k(v))^\alpha}{(u+n\mu)^\alpha} \, m(dv)$$

$$\geq (1+a\mu)^{-\alpha} u^{-\alpha} \int_E \left(\sum_{k=1}^{[au]} f_k(v)\right)^\alpha m(dv)$$

$$\geq (1+a\mu)^{-\alpha} u^{-\alpha} \sum_{k=1}^{[au]} \int_E [f_k(v)]^\alpha \, m(dv)$$

$$\sim a(1+a\mu)^{-\alpha} u^{-(\alpha-1)} \int_E [f_1(v)]^\alpha \, m(dv)$$

as $u \to \infty$. Now select the optimal $a = 1/(\mu(\alpha-1))$. □

Under the assumptions of Proposition 2.12 the ruin probability $\psi(u)$ cannot decay faster than at the rate of $u^{-(\alpha-1)}$. Furthermore in all the many examples considered in this paper the rate of decay of the ruin probability is never faster than $u^{-(\alpha-1)}$ (but in many examples it is way slower than $u^{-(\alpha-1)}$!) We conjecture that for any non-trivial ergodic stationary SαS process the ruin probability cannot decay faster than $u^{-(\alpha-1)}$. In Section 4 we prove this for SαS mixed moving average processes. We should mention, however, that in certain cases of departure from the symmetry of the model one *can* get ruin probabilities that decay faster than $u^{-(\alpha-1)}$. See, for example, [15].

3. Ergodic processes associated with a conservative flow. In this section we study the asymptotic behavior of the ruin probability for step sizes forming a certain type of ergodic stationary SαS process associated with a conservative flow [i.e. a process of type $\mathbf{X}^{(2)}$ in the decomposition (1.5)]. The construction of processes of this type is due to [23]. In a certain sense, stationary ergodic SαS processes of this type have "the longest memory", and "the faster the flow returns to the starting point" the "longer is the memory" of the SαS process. In particular, we will see that the relatively fast "return time" of the flow can cause the ruin probability to decay very slowly.

We start by introducing the class of stationary ergodic SαS processes to be studied. Consider an irreducible null-recurrent Markov chain on $\mathbb{Z}$ with law $P_i(\cdot)$ on

$$E = \{\mathbf{x} = (x_0, x_1, x_2, \ldots) : x_i \in \mathbb{Z}\}$$

corresponding to the initial state $x_0 = i \in \mathbb{Z}$. Let $\pi = (\pi_i)_{i\in\mathbb{Z}}$ be the σ-finite invariant measure corresponding to the family (P_i) satisfying $\pi_0 = 1$.

We define a σ-finite measure on the cylindrical σ-field of E by

$$m(\cdot) = \sum_{i=-\infty}^{\infty} \pi_i \, P_i(\cdot).$$

That is, m is the measure generated on the path space by the Markov chain starting according to the (infinite) initial invariant measure π. Observe that the measure m is invariant under the shift $\theta : E \to E$:

$$\theta((x_0, x_1, x_2, \ldots)) = (x_1, x_2, \ldots), \quad \mathbf{x} = (x_0, x_1, x_2, \ldots) \in E.$$

We consider a SαS process defined by the stochastic integral representation (1.3), where M is a SαS random measure on E with control measure m. In this section we will use kernels f_n given by

$$f_n(\mathbf{x}) = I_{\{x_n=0\}}, \quad n \geq 0, \quad \mathbf{x} = (x_0, x_1, x_2, \ldots) \in E. \tag{3.1}$$

REMARK 3.1. The results below can be adapted in a straightforward way to a more general family of functions f_n, as, for example, in [19]. However, our main goal in this section is to study the connection between the first return time of the Markov chain and the memory properties of the stationary SαS process as reflected in the rate of decay of the ruin probability. This goal can be well achieved using a kernel as simple as in (3.1).

It follows from [23] that the process (X_n) with stochastic integral representation (1.3) is a stationary mixing process. In particular, it is ergodic. Note that the process would not be ergodic if the Markov chain were positive recurrent and, in particular, the invariant measure π and, hence, the control measure m of the SαS random measure M, were finite. See, for example, [10].

For a given $\mathbf{x} \in E$, let

$$\tau = \tau(\mathbf{x}) = \inf\{n \geq 1 : x_n = 0\}$$

be the first return time to 0. Since the Markov chain is null recurrent, we must have $E_0 \tau = \infty$. We will use a stronger assumption on the tail of the distribution of the first return time τ. Specifically, assume that there are $\gamma \in (0, 1]$ and a slowly varying function L such that

$$P_0(\tau \geq n) = n^{\gamma - 1} L(n). \tag{3.2}$$

We view the parameter γ in (3.2) (restricted to be non-negative by the null recurrence of the chain) as a measure of how fast the Markov chain returns to its initial state, hence of the strength of dependence in the SαS process. From this point of view, small values of γ (close to 0) correspond to frequent returns of the Markov chain and to longer memory of the SαS process. This interpretation is confirmed by the connection between the parameter γ and the rate of decay of the ruin probability in the theorem below.

Recall that the classical invariance principle (our favorite reference is [3], where it is spelled out in the Gaussian case) says that in the case $\gamma \in (0, 1)$

$$\big(a_n^{-1}(\tau_1 + \cdots + \tau_{[tn]})\big)_{t \geq 0} \Rightarrow (Z_{1-\gamma}(t))_{t \geq 0}, \quad n \to \infty, \tag{3.3}$$

in the Skorokhod space $\mathbb{D}[0, \infty)$ endowed with the J_1-metric and the corresponding Borel σ-field, where

$$a_k := \inf\{n \geq 1 : n^{1-\gamma} L^{-1}(n) \geq k\} \tag{3.4}$$

and $(Z_{1-\gamma})$ is a $(1-\gamma)$-stable subordinator, that is, a positive increasing strictly $(1-\gamma)$-stable Lévy motion with

$$E \exp\{i\lambda Z_{1-\gamma}(1)\} = \exp\left\{-C_{1-\gamma}^{-1} |\lambda|^{1-\gamma} \left(1 - i \tan \frac{\pi(1-\gamma)}{2}\right)\right\}, \quad \lambda \in \mathbb{R},$$

where the constant $C_{1-\gamma}$ is given in (1.6) with α replaced by $1 - \gamma$. See [25] for details.

We are now ready to state the main result of this section.

THEOREM 3.2. *Under assumption* (3.2) *the following relation holds*:

$$\psi(u) \sim A_{\alpha,\gamma} \mu^{\gamma(\alpha-1)-\alpha} u^{-\gamma(\alpha-1)} L^{-(\alpha-1)}(u), \quad u \to \infty, \tag{3.5}$$

where

$$A_{\alpha,\gamma} = \frac{C_\alpha A^*_{\alpha,\gamma} B(\gamma, \gamma(\alpha-1))}{2}.$$

C_α is the constant in (1.6), *B is the beta function*,

$$A^*_{\alpha,\gamma} := \begin{cases} E\left(\sup_{t\geq 1} \dfrac{t-1}{Z_{1-\gamma}(t)}\right)^{\alpha(1-\gamma)}, & \text{for } \gamma \in (0,1), \\ \Gamma(1+\alpha), & \text{for } \gamma = 1, \end{cases} \tag{3.6}$$

$(Z_{1-\gamma})$ is the $(1-\gamma)$-stable subordinator in (3.3) *and Γ is the gamma function. In particular, the constant $A^*_{\alpha,\gamma}$ is finite.*

PROOF. We will proceed through a sequence of intermediate results. As a first step, we establish the rate at which the scale m_n of the partial sums of the process [cf. (2.1)] grows.

LEMMA 3.3. *There exists a positive random variable η with all power moments finite such that*

$$m_n \sim \frac{C_\alpha^{1/\alpha}(E\eta^\alpha)^{1/\alpha}}{\gamma^{1/\alpha}} n^{(1-\gamma)+\gamma/\alpha} L^{-1+1/\alpha}(n), \quad n \to \infty. \tag{3.7}$$

In particular, (2.11) *holds for $\beta > 1 - \gamma(1 - 1/\alpha)$.*

PROOF. Observe that with the kernel f_n given by (3.1) we have

$$h_n(\mathbf{x}) = \sum_{j=1}^n f_j(\mathbf{x}) = \sum_{j=1}^n I_{\{x_j=0\}} =: N_n\,,$$

where N_n is the number of times the Markov chain visits the origin along a sample path $\mathbf{x}$ in the first n steps. We introduce a family of probability measures on E defined by

$$Q_n(\cdot) = m(\cdot \cap \{\tau \leq n\})/m(\tau \leq n)\,, \quad n \geq 1\,, \tag{3.8}$$

and a sequence of random variables defined by

$$\eta_n = N_n n^{\gamma-1} L(n)\,, \quad n \geq 1. \tag{3.9}$$

Then, by observing that

$$m(\tau \leq n) \sim \gamma^{-1} n^\gamma L(n)\,, \quad n \to \infty\,,$$

(see [19], Lemma 3.3) we conclude that

$$\begin{aligned} m_n^\alpha = C_\alpha \int_E N_n^\alpha \, dm &= m(\tau \leq n) \int_E N_n^\alpha \, dQ_n \\ &= m(\tau \leq n)\, n^{-\alpha(\gamma-1)} L^{-\alpha}(n) \int_E \eta_n^\alpha \, dQ_n \\ &\sim \gamma^{-1} n^{\gamma+\alpha(1-\gamma)} L^{1-\alpha}(n) \int_E \eta_n^\alpha \, dQ_n. \end{aligned}$$

It follows from Proposition 3.4 of [19] that the sequence of the distributions of η_n under Q_n converges weakly to the distribution of a positive random variable η, say, with all power moments finite, and all the corresponding moments converge as well. Hence $\lim_{n\to\infty}\int_E \eta_n^\alpha \, dQ_n =: E\eta^\alpha > 0$ exists, and so (3.7) follows. □

We immediately conclude from Theorem 2.5 that $\psi(u) \sim \psi_0(u)$ as $u \to \infty$, and it only remains to evaluate the asymptotic behavior of $\psi_0(u)$ as $u \to \infty$.

We continue with another auxiliary result.

LEMMA 3.4. *The following relation holds*:

$$g(u) := E_0\left(\sup_{n\ge 1} \frac{N_n^\alpha}{(u+n)^\alpha}\right) \sim A^*_{\alpha,\gamma}\, u^{-\gamma\alpha} L^{-\alpha}(u), \quad u \to \infty,$$

where E_0 *denotes expectation with respect to* P_0, *the constant* $A^*_{\alpha,\gamma}$ *is given in* (3.6) *and it is finite*.

PROOF. Consider the sequence of successive excursion times outside of zero

$$\tau_1 = \tau, \quad \tau_{n+1} = \inf\{k > \tau_n : x_k = 0\} - \tau_n, \quad n \ge 1.$$

It is clear that, under P_0, this is a sequence of iid random variables.

We start with the case $\gamma \in (0,1)$. The definitions of η_n and N_n [see (3.9)], the invariance principle (3.3) and self-similarity of the stable subordinator yield

$$\begin{aligned} P_0(\eta_n > x) &= P_0\left(n^{\gamma-1}L(n)N_n > x\right) = P_0\left(\tau_1 + \cdots + \tau_{[xn^{1-\gamma}L^{-1}(n)]} \le n\right) \\ &\sim P_0\left(\tau_1 + \cdots + \tau_{[xk]} \le a_k\right) \qquad (3.10) \\ &\to P\left(x^{1/(1-\gamma)} Z_{1-\gamma}(1) \le 1\right) = P\left(Z_{1-\gamma}^{\gamma-1}(1) > x\right), \quad x > 0, \end{aligned}$$

where $k = n^{1-\gamma}L^{-1}(n)$ and the norming sequence (a_n) is as in (3.4). We conclude that, under the probability measure P_0,

$$\eta_n = n^{\gamma-1}L(n)N_n \Rightarrow Z_{1-\gamma}^{\gamma-1}(1),$$

and an argument similar to that of Proposition 3.4 of [19] shows that all power moments converge as well. Hence

$$E_0 N_n^\alpha \sim n^{\alpha(1-\gamma)} L^{-\alpha}(n)\, E Z_{1-\gamma}^{-\alpha(1-\gamma)}(1). \qquad (3.11)$$

In the case $\gamma = 1$ the probability in (3.10) converges to $e^{-x} = P(Y > x)$ by Theorem 1 of [29]. Furthermore, a simple domination argument in

$$\begin{aligned} P_0(\eta_n > x) &= P_0\left(\tau_1 + \cdots + \tau_{[xL^{-1}(n)]} \le n\right) \\ &\le 2\left(P_0(\tau_1 \le n)\right)^{xL^{-1}(n)} \le 2e^{-x} \end{aligned}$$

for all n so large that $P(\tau_1 \le n) \ge 1/2$ shows that the moments converge as well. Then (3.11) turns into

$$E_0 N_n^\alpha \sim L^{-\alpha}(n)\, EY^\alpha = L^{-\alpha}(n)\Gamma(\alpha+1). \tag{3.12}$$

REMARK 3.5. Theorem 1 of [29] assumes that the distribution of the terms (τ_j) is continuous. However, the continuity assumption can be easily removed by applying the same result to the "smoothed" sequence $(\tau_j + U_j)$, where (U_j) is an iid sequence with common uniform distribution on $(0,1)$, independent of (τ_j).

For any $K > 0$ consider

$$g_K(u) := E_0\left(\sup_{1\le n\le uK} \frac{N_n^\alpha}{(u+n)^\alpha}\right) \quad \text{and} \quad g^K(u) := E_0\left(\sup_{n>uK} \frac{N_n^\alpha}{(u+n)^\alpha}\right).$$

We first bound $g^K(u)$. Choose an $\epsilon \in (0, \alpha\gamma)$ and observe that, by (3.11) for $\gamma \in (0,1)$ and (3.12) for $\gamma = 1$ and the properties of regularly varying functions, for all large u,

$$\begin{aligned} g^K(u) &\le \sum_{j=1}^\infty E_0 \sup_{uK2^{j-1}<n\le uK2^j} \frac{N_n^\alpha}{(u+n)^\alpha} \le u^{-\alpha} \sum_{j=1}^\infty E_0 \frac{N_{[uK2^j]}^\alpha}{(1+K2^{j-1})^\alpha} \\ &\le \text{const } u^{-\alpha} \sum_{j=1}^\infty \frac{(uK2^j)^{\alpha(1-\gamma)} L^{-\alpha}(uK2^j)}{(1+K2^{j-1})^\alpha} \\ &\le \text{const } u^{-\gamma\alpha} L^{-\alpha}(u)\, K^{-\gamma\alpha+\epsilon} \sum_{j=1}^\infty 2^{-(\gamma\alpha-\epsilon)j}. \end{aligned}$$

Therefore

$$\lim_{K\uparrow\infty} \limsup_{u\to\infty} u^{\gamma\alpha} L^\alpha(u)\, g^K(u) = 0.$$

Assume now $\gamma \in (0,1)$. To establish the statement of the lemma in this case it is enough to show that, for every $K > 0$,

$$\lim_{u\to\infty} u^{\gamma\alpha}\, L^\alpha(u)\, g_K(u) = E\left(\sup_{1\le t\le K+1} \frac{t-1}{Z_{1-\gamma}(t)}\right)^{\alpha(1-\gamma)}, \tag{3.13}$$

which we now proceed to do.

Observe that

$$\begin{aligned} &P_0\left(\sup_{1\le n\le uK} u^\gamma L(u)\, \frac{N_n}{u+n} \ge x\right) \\ &\quad = P_0\left(N_{[ut]} \ge xu^{1-\gamma} L^{-1}(u)(1+t) \quad \text{for some } t \le K\right) \\ &\quad = P_0\left(\tau_1 + \cdots + \tau_{[l(x,u)(1+t)]} \le ut \quad \text{for some } t \le K\right), \end{aligned}$$

where (a_n) is defined in (3.4) and, for every x,

$$l(x,u) := xu^{1-\gamma}L^{-1}(u) \to \infty \quad \text{as } u \to \infty.$$

Since $a_{[l(x,u)/x]}/u \to 1$ uniformly in x, a standard argument shows that the latter probability is asymptotically of the same order as

$$P_0\left(\tau_1 + \cdots + \tau_{[l(x,u)(1+t)]} \le a_{[l(x,u)/x]}t \quad \text{for some } t \le K\right).$$

Therefore,

$$\begin{aligned} &P_0\left(\sup_{1\le n\le uK} u^\gamma L(u)\,\frac{N_n}{u+n} \ge x\right) \\ (3.14)\qquad &\quad\sim P_0\left(\inf_{0\le t\le K}\left(a^{-1}_{l(x,u)}(\tau_1+\cdots+\tau_{[l(x,u)(1+t)]}) - \frac{a_{[l(x,u)/x]}}{a_{l(x,u)}}\,t\right) \le 0\right). \end{aligned}$$

An appeal to the invariance principle (3.3), the continuous mapping theorem and regular variation yield that the right hand side in (3.14) converges to

$$\begin{aligned} &P\left(\inf_{0\le t\le K}\left(Z_{1-\gamma}(1+t) - x^{-1/(1-\gamma)}\,t\right) \le 0\right) \\ &\quad = P\left(\inf_{0\le t\le K} t^{-1}Z_{1-\gamma}(1+t) \le x^{-1/(1-\gamma)}\right) \\ &\quad = P\left(\left(\sup_{0\le t\le K}\frac{t}{Z_{1-\gamma}(1+t)}\right)^{1-\gamma} \ge x\right), \end{aligned}$$

at least for those $x > 0$ that are continuity points of the distribution of $\sup_{0\le t\le K}(t/Z_{1-\gamma}(1+t))$. This shows that for every $K > 0$

$$\sup_{1\le n\le uK} u^\gamma L(u)\,\frac{N_n}{u+n} \Rightarrow \sup_{1\le t\le K+1}\left(\frac{t-1}{Z_{1-\gamma}(t)}\right)^{1-\gamma}. \qquad (3.15)$$

Moreover, by definition of N_n, for any fixed $K > 0$,

$$\sup_{1\le n\le uK} u^\gamma L(u)\,\frac{N_n}{u+n} \le \text{const } N_{[uK]}u^{\gamma-1}L(u),$$

and the random variables on the right hand side have all power moments finite and all moments, as above, converge. Hence (3.15) implies (3.13).

Finally, to establish that the constant $A^*_{\alpha,\gamma}$ is finite when $\gamma \in (0,1)$ we observe that for every $p > 0$ by self-similarity of the stable subordinator,

$$\begin{aligned} E\left(\sup_{t\ge1}\frac{t-1}{Z_{1-\gamma}(t)}\right)^p &\le \sum_{j=1}^\infty E\left(\sup_{2^{j-1}\le t\le 2^j}\frac{t-1}{Z_{1-\gamma}(t)}\right)^p \le \sum_{j=1}^\infty 2^{pj}E\frac{1}{Z^p_{1-\gamma}(2^{j-1})} \\ &\le 2^{p/(1-\gamma)}E\frac{1}{Z^p_{1-\gamma}(1)}\sum_{j=1}^\infty 2^{-pj\gamma/(1-\gamma)} < \infty. \end{aligned}$$

This concludes the proof of the lemma for $\gamma \in (0,1)$.

In the case $\gamma = 1$ the lemma is established once we show that, for every $K > 0$

$$\lim_{u\to\infty} u^\alpha\, L^\alpha(u)\, g_K(u) = EY^\alpha\,, \tag{3.16}$$

where Y is a standard exponential random variable. However,

$$\begin{aligned} p(x) &:= P_0\left(\sup_{1\le n\le uK} uL(u)\,\frac{N_n}{u+n} > x\right) \\ &\le\; P_0\left(L(u)N_{[uK]} > x\right) \\ &\le\; P_0\left(\tau_1+\cdots+\tau_{[x/L(u)]} \le u\,K\right) \to \mathrm{e}^{-x} \end{aligned} \tag{3.17}$$

by virtue of the mentioned result of [29]. Similarly, for small $0 < \epsilon < K$,

$$p(x) \ge P_0\left(L(\epsilon\, u)N_{[u\,\epsilon]} > (1+\epsilon)x\right) \to \mathrm{e}^{-(1+\epsilon)x}\,,$$

and since ϵ is arbitrary we conclude together with (3.17) that $p(x) \to \mathrm{e}^{-x}$. Furthermore, the same argument that leads to (3.12) also gives (3.16). This concludes the proof of the lemma. □

We now proceed with the evaluation of the asymptotic behavior of $\psi_0(u)$ as $u \to \infty$. Writing $\tau^*(\mathbf{x}) = \tau_1(\mathbf{x})I_{\{x_0\neq 0\}}$ and observing that $m(\tau^* = n) = P_0(\tau \ge n)$ (Lemma 3.3 in [19]), we have by (2.10) and the strong Markov property

$$\begin{aligned} 2\psi_0(u) &= C_\alpha \sum_{i=-\infty}^{\infty} \pi_i E_i\left(\sup_{n\ge 1}\frac{N_n^\alpha}{(u+n\mu)^\alpha}\right) \\ &= C_\alpha \sum_{k=0}^{\infty} m(\tau^*=k)\, E_0\left(\sup_{n\ge 1}\frac{N_n^\alpha}{(u+(n+k)\mu)^\alpha}\right) \\ &= C_\alpha\mu^{-\alpha}\sum_{k=0}^{\infty} P_0(\tau\ge k)\, g(k+u/\mu)\,, \end{aligned} \tag{3.18}$$

where $g(\cdot)$ is defined in the statement of Lemma 3.4. The right hand side of (3.18) may be viewed as a discrete analogue of the so-called Stieltjes transforms; see page 40 in [4]. The following lemma establishes the asymptotic behavior of such transforms.

LEMMA 3.6. *Let L_f and L_g be two non-negative slowly varying functions, and let $\rho_f > -1$ and $\rho_g > 1+\rho_f$ be two constants. Define regularly varying functions $f(x) = x^{\rho_f}L_f(x)$, $g(x) = x^{\rho_g}L_g(x)$. Then*

$$\sum_{k=1}^{\infty}\frac{f(k)}{g(u+k)} \sim u^{\rho_f-\rho_g+1}\frac{L_f(u)}{L_g(u)}\, B(\rho_f+1, \rho_g-\rho_f-1).$$

PROOF. Notice that we can assume without loss of generality that g is eventually monotone increasing. This is due to the fact that g is asymptotically equivalent to a monotone regularly varying function with the same index.

Using the monotonicity of g and Karamata's theorem, we obtain for every $K>0$,

$$I_1(u) := \sum_{k>uK} \frac{f(k)}{g(u+k)} \le \sum_{k>uK} \frac{f(k)}{g(k)} \sim \frac{f(uK)uK}{(-\rho_f+\rho_g-1)g(uK)}.$$

Analogously,

$$I_2(u) := \sum_{k<u/K} \frac{f(k)}{g(u+k)} \le \frac{1}{g(u)} \sum_{k<u/K} f(k) \sim \frac{f(u/K)u/K}{(\rho_f+1)g(u)}.$$

Therefore, for each $K>0$,

$$\lim_{K\uparrow\infty} \limsup_{u\to\infty} u^{-(\rho_f-\rho_g+1)} \frac{L_f(u)}{L_g(u)} I_j(u) = 0\,, \quad j=1,2. \tag{3.19}$$

Finally, for every $K>0$,

$$\sum_{u/K\le k\le uK} \frac{f(k)}{g(u+k)} \sim \left[\frac{L_f(u)}{L_g(u)} u^{\rho_f-\rho_g+1}\right] u^{-1} \sum_{u/K\le k\le uK} (k/u)^{\rho_f}(1+k/u)^{-\rho_g}$$

$$\sim \left[\frac{L_f(u)}{L_g(u)} u^{\rho_f-\rho_g+1}\right] \int_{1/K}^{K} y^{\rho_f}(1+y)^{-\rho_g}\, dy.$$

The statement of the lemma follows by letting $K\to\infty$, taking (3.19) into account and noticing that

$$\int_0^\infty y^{\rho_f}(1+y)^{-\rho_g}\, dy = B(\rho_f+1, \rho_g-\rho_f-1).$$

Now it is easy to complete the evaluation of the asymptotic behavior of $\psi_0(u)$ as $u\to\infty$ and, hence, to complete the proof of Theorem 3.2. By (3.2) and Lemma 3.4 we may apply Lemma 3.6 to (3.18) with $\rho_f=\gamma-1$ and $\rho_g=\gamma\alpha$. The statement (3.5) now follows, and so the proof of the theorem is complete. □

REMARK 3.7. Unfortunately, it is not clear from our approach how the interesting case of $\gamma=0$ can be treated. For example, Lemma 3.3 holds in this case, meaning that

$$nL^{-1+1/\alpha}(n) = o(m_n)$$

as $n\to\infty$. This implies in particular that assumption (2.11) fails. Nonetheless, we conjecture that in the case $\gamma=0$, the "borderline case" between positive and null recurrence, the ruin probability is asymptotically equivalent to a slowly varying function. This would be a case of very slowly disappearing risk indeed! For another example of a very slowly decaying ruin probability, see Remark 4.2.

4. Ergodic processes associated with a dissipative flow. In this section we switch to studying the asymptotic behavior of the ruin probability for step sizes forming a stationary SαS process associated with a dissipative flow [i.e. a process of the type $\mathbf{X}^{(1)}$ in the decomposition (1.5)]. These processes are automatically ergodic, and by Theorem 4.4 of [21] they have a mixed moving average representation

$$X_n = \int_W \int_{\mathbb{R}} f(v, x-n)\, M(dv, dx), \quad n = 1, 2, \dots, \tag{4.1}$$

where M is a SαS random measure on a product measurable space $(W \times \mathbb{R}, \mathscr{W} \times \mathscr{B})$ with the control measure $m = \nu \times \text{Leb}$, where ν is a σ-finite measure on $(W, \mathscr{W})$. Finally, $f \in L^\alpha(m, \mathscr{W} \times \mathscr{B})$. If the dissipative flow is, actually, ergodic then the stationary SαS process has a more familiar moving average representation

$$X_n = \int_{\mathbb{R}} f(x-n)\, M(dx), \quad n = 1, 2, \dots, x \in \mathbb{R}, \tag{4.2}$$

in which the space W in (4.1) becomes a singleton. Here M is a SαS random measure on $(\mathbb{R}, \mathscr{B})$ with Lebesgue control measure, and $f \in L^\alpha(\text{Leb})$. See Corollary 4.6 of [21].

Intuitively, the stationary SαS processes associated with dissipative flows have "shorter memory" than the stationary ergodic SαS processes associated with conservative flows, simply because "the flow does not come back". We will see in this section, however, that, at least as far as the ruin probability is concerned, sufficiently long dependence may be "caused by the kernel" f in (4.1) or even in (4.2). Put a bit differently, one of the conclusions of this section is that if the kernel f is "nice enough," then the ruin probability decreases at the fastest possible rate and, in this sense, the memory is short. This should be contrasted with the situation in Section 3, where even with the "nicest" possible kernel (the indicator function of a state) the long memory was caused by the conservative flow.

Let (X_n) be a mixed moving average process (4.1). For any $(v, x) \in W \times (0, 1)$ define

$$J_\pm(v, x) = \lim_{h \downarrow -\infty} \liminf_{m \to \infty} \sup_{h \le j \le m} \left(\sum_{k=j}^{m} f(v, x+k) \right)_\pm . \tag{4.3}$$

THEOREM 4.1. (a) *For any mixed moving average process* (4.1) *the following lower bound for the ruin probability holds*:

$$\liminf_{u \to \infty} u^{\alpha-1} \psi(u) \ge \frac{C_\alpha}{2(\alpha-1)\mu} I(f), \tag{4.4}$$

where

$$I(f) := \int_W \int_0^1 \left([J_+(v, x)]^\alpha + [J_-(v, x)]^\alpha \right) \nu(dv)\, dx. \tag{4.5}$$

(b) *Assume that for ν-almost every $v \in W$ there is a compact interval $[K_l(v), K_r(v)]$ such that $0 < K_r(v) - K_l(v) \le L$ for some finite constant L which does not depend on $v \in W$ and that $f(v, x) = 0$ for Leb-almost every $x \notin [K_l(v), K_r(v)]$. Then*

$$\lim_{u\to\infty} u^{\alpha-1}\psi(u) = \frac{C_\alpha}{2(\alpha-1)\mu}\, I(f) < \infty. \tag{4.6}$$

PROOF. By Remark 2.10, for part (a) of the theorem it is enough to prove the bound (4.4) with $\psi(u)$ replaced with $\psi_0(u)$. Defining

$$\psi_0^{(\pm)}(u) := \int_W \int_{\mathbb{R}} \sup_{n\ge 1} \frac{\left(\sum_{k=1}^n f(v, x-k)\right)_\pm^\alpha}{(u+n\mu)^\alpha}\, \nu(dv)\, dx,$$

we immediately see from (2.10) that

$$\psi_0(u) = \frac{C_\alpha}{2}\left[\psi_0^{(+)}(u) + \psi_0^{(-)}(u)\right]. \tag{4.7}$$

We will prove that

$$\liminf_{u\to\infty} u^{\alpha-1}\psi_0^{(+)}(u) \ge \frac{1}{(\alpha-1)\mu}\int_W \int_0^1 [J_+(v, x)]^\alpha\, \nu(dv)\, dx. \tag{4.8}$$

Since the second part of (4.4) is completely similar, (4.8) will be enough to prove part (a) of the theorem.

Assume first that $J_+(v, x) < \infty$ for $\nu\times$Leb-almost every $(v, x) \in W \times \mathbb{R}$. For any $(v, x) \in W \times \mathbb{R}$, $u > 0$, integers $h \le m$ and $\epsilon \in (0, 1)$ let $I_{\epsilon,h,m}(v, x)$ be an integer between h and m such that

$$\left(\sum_{k=I_{\epsilon,h,m}(v,x)}^{m} f(v, x+k)\right)_+ \ge (1-\epsilon)\sup_{h\le j\le m}\left(\sum_{k=j}^{m} f(v, x+k)\right)_+$$
$$=: (1-\epsilon) J_+^{(h,m)}(v, x).$$

Still keeping an integer h fixed, denote

$$J_+^{(h)}(v, x) = \liminf_{m\to\infty} J_+^{(h,m)}(v, x),$$

and let

$$N_{\epsilon,h}(v, x) = \inf\left\{i \ge h : J_+^{(h,m)}(v, x) \ge (1-\epsilon) J_+^{(h)}(v, x) \text{ for all } m \ge i\right\}.$$

Observe that

$$\psi_0^{(+)}(u) = \int_W \int_0^1 \sum_{i=-\infty}^{\infty} \sup_{n\ge 1} \frac{\left(\sum_{k=i-n}^{i-1} f(v, x+k)\right)_+^\alpha}{(u+n\mu)^\alpha}\, \nu(dv)\, dx.$$

The integrand can now be bounded from below as follows:

$$\geq \sum_{i=N_{\epsilon,h}(v,x)+1}^{\infty} \sup_{n\geq 1} \frac{\left(\sum_{k=i-n}^{i-1} f(v,x+k)\right)_+^{\alpha}}{(u+n\mu)^{\alpha}}$$

$$\geq \sum_{i=N_{\epsilon,h}(v,x)+1}^{\infty} \left[u+\mu(i-I_{\epsilon,h,i-1}(v,x))\right]^{-\alpha} \left[\sum_{k=I_{\epsilon,h,i-1}(v,x)}^{i-1} f(v,x+k)\right]_+^{\alpha}$$

$$\geq (1-\epsilon) \sum_{i=N_{\epsilon,h}(v,x)+1}^{\infty} \left[u+\mu(i-I_{\epsilon,h,i-1}(v,x))\right]^{-\alpha} \left[J_+^{(h,i-1)}(v,x)\right]^{\alpha}$$

$$\geq (1-\epsilon)^2 \left[J_+^{(h)}(v,x)\right]^{\alpha} \sum_{i=N_{\epsilon,h}(v,x)+1}^{\infty} \left[u+\mu(i-h)\right]^{-\alpha}$$

$$\sim (1-\epsilon)^2 \left[J_+^{(h)}(v,x)\right]^{\alpha} \left[(\alpha-1)\mu\right]^{-1} u^{1-\alpha}, \quad u\to\infty.$$

Therefore, by Fatou's lemma,

$$\liminf_{u\to\infty} u^{\alpha-1}\psi_0^{(+)}(u) \geq \frac{(1-\epsilon)^2}{(\alpha-1)\mu} \int_W \int_0^1 \left[J_+^{(h)}(v,x)\right]^{\alpha} \nu(dv)\, dx.$$

Letting $h\downarrow -\infty$ and $\epsilon\to 0$ we obtain (4.8), and so we have finished the proof of part (a) of the theorem.

If, on the other hand, $J_+(v,x)=\infty$ on a set of positive $\nu\times$Leb measure, then the same argument as above shows that

$$\lim_{u\to\infty} u^{\alpha-1}\psi_0^{(+)}(u) = \infty = \frac{1}{(\alpha-1)\mu} \int_W \int_0^1 \left[J_+(v,x)\right]^{\alpha} \nu(dv)\, dx.$$

For part (b), we may assume, without loss of generality, that $K_l(v) < K_r(v)$ are integers. Observe that for all $h\leq K_l(v)$ and $m>h$ we have $J_\pm^{(h,m)}(v,x) = J_\pm^{(K_l(v),m)}(v,x)$, implying that for all $h \leq K_l(v)$ we have $J_\pm^{(h)}(v,x) = J_\pm^{(K_l(v),K_r(v))}(v,x)$. Therefore also

$$J_\pm^{(h)}(v,x) = J_\pm^{(K_l(v),K_r(v))}(v,x) = \sup_{K_l(v)\leq j\leq K_r(v)} \left(\sum_{k=j}^{K_r(v)-1} f(v,x+k)\right)_\pm .$$

In particular, by Hölder's inequality,

$$I(f) \leq 2(L+1)^{\alpha-1} \int_W \int_{\mathbb{R}} |f(v,x)|^{\alpha}\, \nu(dv)\, dx < \infty,$$

and similarly,

$$C_\alpha^{-1}\, m_n^{\alpha} = \int_W \int_{\mathbb{R}} \left|\sum_{k=1}^{n} f(v,x-k)\right|^{\alpha} \nu(dv)\, dx$$

$$\leq (K_r(v)-K_l(v)+1)^{\alpha-1} \sum_{k=1}^{n} \int_W \int_{\mathbb{R}} |f(v,x-k)|^{\alpha}\, \nu(dv)\, dx$$

$$\leq n(L+1)^{\alpha-1} \int_W \int_{\mathbb{R}} |f(v,x)|^{\alpha}\, \nu(dv)\, dx.$$

Here we used the fact that for almost every $(v, x) \in W \times (0, 1)$ the sum under the integral on the left hand side has at most $L + 1$ nonzero terms. We conclude that assumption (2.11) holds with $\beta = 1/\alpha$. Therefore Theorem 2.5 is applicable and $\psi(u) \sim \psi_0(u)$.

Because of (4.7) and the already proved relation (4.8) it is enough to verify that

$$\limsup_{u\to\infty} u^{\alpha-1}\psi_0^{(+)}(u) \le \frac{1}{(\alpha-1)\mu} \int_W \int_0^1 [J_+(v, x)]^\alpha \, \nu(dv)\, dx. \tag{4.9}$$

Indeed, the corresponding statement for $\psi_0^{(-)}(u)$ will then follow by replacing f with $-f$ everywhere.

Notice that

$$\psi_0^{(+)}(u) =: \theta_1(u) + \theta_2(u), \tag{4.10}$$

where

$$\theta_1(u) = \int_W \int_0^1 \sum_{i=K_r(v)+1}^{\infty} \sup_{n\ge 1} \frac{\left(\sum_{k=i-n}^{i-1} f(v, x+k)\right)_+^\alpha}{(u+n\mu)^\alpha} \, \nu(dv)\, dx,$$

$$\theta_2(u) = \int_W \int_0^1 \sum_{i=K_l(v)+1}^{K_r(v)} \sup_{n\ge 1} \frac{\left(\sum_{k=i-n}^{i-1} f(v, x+k)\right)_+^\alpha}{(u+n\mu)^\alpha} \, \nu(dv)\, dx.$$

It is immediate that

$$\begin{aligned}
\theta_1(u) &\le \int_W \int_0^1 \sum_{i=K_r(v)}^{\infty} [u + \mu(i - K_r(v))]^{-\alpha} \\
&\qquad \times \sup_{n \ge i-K_r(v)} \left(\sum_{k=i-n}^{K_r(v)-1} f(v, x+k) \right)_+^\alpha \nu(dv)\, dx \\
&= \sum_{i=0}^{\infty} (u+i\mu)^{-\alpha} \int_W \int_0^1 \sup_{K_l(v)\le j\le K_r(v)} \left(\sum_{k=j}^{K_r(v)-1} f(v, x+k) \right)_+^\alpha \\
&\qquad \times \nu(dv)\, dx \\
&= \sum_{i=0}^{\infty} (u+i\mu)^{-\alpha} \int_W \int_0^1 [J_+(v, x)]^\alpha \, \nu(dv)\, dx.
\end{aligned} \tag{4.11}$$

On the other hand,

$$\begin{aligned}
\theta_2(u) &\le L u^{-\alpha} \int_W \int_0^1 \left(\sum_{k=K_l(v)}^{K_r(v)-1} f(v, x+k) \right)_+^\alpha \nu(dv)\, dx \\
&\le L^\alpha u^{-\alpha} \int_W \int_0^1 |f(v, x)|^\alpha \, \nu(dv)\, dx,
\end{aligned} \tag{4.12}$$

and so (4.9) follows from (4.10), (4.11) and (4.12). This completes the proof of the theorem. □

REMARK 4.2. An immediate conclusion from part (a) of Theorem 4.1 is that if $I(f) = \infty$ in (4.5) then the ruin probability $\psi(u)$ decays slower than $u^{-(\alpha-1)}$. An example is given by the moving average process of Remark 2.6. In fact, an easy manipulation with the function $\psi_0(u)$ in (2.10) and Remark 2.9 show that in this case

$$\liminf_{u\to\infty} (\log u)^{p-1} \psi(u) \geq \text{const}\, \mu^{-\alpha}$$

for some positive constant depending on α and p. Hence the ruin probability decays in this case very slowly indeed! Although not as dramatic as in the present example, we will see a whole range of possible rates of decay of the ruin probability while considering the increments of self-similar processes with stationary increments below.

REMARK 4.3. There is no doubt that the second part of Theorem 4.1 remains true if the assumption of the "uniformly compact" support of the kernel is replaced by an assumption of a suitably fast rate of decay of the kernel at infinity, but we are not pursuing this point here. As an example, consider the classical SαS Ornstein-Uhlenbeck process. This is a moving average process (4.2) with the kernel

$$f(x) = \mathrm{e}^{\rho x} I_{\{x\leq 0\}}, \quad n \geq 1$$

for some $\rho > 0$. It is very easy to check that assumption (2.11) holds with $\beta = \alpha^{-1}$, and it also easy to evaluate the asymptotic behavior of $\psi_0(u)$ directly from (2.10). However, this process is just an AR(1) linear process, and it follows from the general result of [15] that for this process

$$\psi(u) \sim \frac{C_\alpha}{2\alpha(\alpha-1)\rho} \frac{\mathrm{e}^{\alpha\rho}-1}{(\mathrm{e}^{\rho}-1)^\alpha} \frac{1}{\mu}\, u^{-(\alpha-1)} = \frac{C_\alpha}{2(\alpha-1)\mu}\, I(f), \quad u \to \infty,$$

and so (4.6) holds. In fact, (4.6) will also hold for any SαS linear process satisfying the assumptions of [15], and for any stationary SαS process that can be approximated appropriately well by such linear processes. The question of such approximations is not pursued in this paper either.

A more interesting question, which we cannot answer at this time is whether or not (4.6) *always holds* whenever $I(f) < \infty$.

In the remaining part of this section we concentrate on an interesting class of moving average SαS processes that arise naturally as the increments of self-similar SαS processes with stationary increments.

Recall that a process $(Y(t))_{t\geq 0}$ is said to be H-self-similar if

$$(Y(at),\ t \geq 0) \stackrel{d}{=} a^H (Y(t),\ t \geq 0)$$

(in terms of equality of finite-dimensional distributions) for any $a > 0$, and a process $(Y(t))_{t\geq 0}$ has stationary increments if

$$(Y(t+h) - Y(h),\ t \geq 0) \stackrel{d}{=} (Y(t) - Y(0),\ t \geq 0)$$

(in the same sense) for any $h > 0$. We will use the abbreviation H-sssi for an H-self-similar process with stationary increments.

Self-similar processes with their "fractal" nature have long been attractive for both probabilists and users of stochastic models. Self-similar processes with stationary increments have also been used to model the phenomenon of long range dependence. See, for example, [27] for an overview. Much work has been done in describing various classes of SαS H-sssi processes and studying their properties. We refer the reader to Chapter 7 of [25] for an extensive discussion. In particular, if $(Y(t))_{t\geq 0}$ is an SαS H-sssi process with $1 < \alpha < 2$, then we must have $0 < H \leq 1$, and the case $H = 1$ is possible only in degenerate situations ([30]).

A well-known class of H-sssi SαS processes is that of *linear fractional* SαS motions defined by

$$Y(t) = \int_{\mathbb{R}} g(t,x)\, M(dx)\,, \quad t \geq 0, \tag{4.13}$$

where M is a SαS random measure with Lebesgue control measure m on $\mathbb{R}$ and

$$\begin{aligned} g(t,x) = a &\left((t-x)_+^{H-1/\alpha} - (-x)_+^{H-1/\alpha}\right) \\ &+ b\left((t-x)_-^{H-1/\alpha} - (-x)_-^{H-1/\alpha}\right)\,, \quad t \geq 0,\ x \in \mathbb{R}, \end{aligned} \tag{4.14}$$

for some $H \in (0,1)$, $H \neq 1/\alpha$. Here a and b are two real constants, and we agree that $0^c = 0$ for all real c. The corresponding process for $H = 1/\alpha$ can be naturally defined in one of the following two ways: as the SαS Lévy motion of Example 2.4 corresponding to

$$g(t,x) = a I_{[0,t]}(x), \quad t \geq 0,\ x \in \mathbb{R} \tag{4.15}$$

for $a > 0$, or as *log-fractional* SαS motion with

$$g(t,x) = a(\ln|t-x| - \ln|x|)\,, \quad t \geq 0,\ x \in \mathbb{R} \tag{4.16}$$

also with $a > 0$. Interestingly enough, a general "unbalancing" of the positive and negative parts as in (4.14) is not productive in the case $H = 1/\alpha$: it does not lead to new processes when applied to the Lévy SαS motion, and it fails to define a self-similar process when applied to the log-fractional SαS motion.

It is elementary to check that the functions g defined above have, in all cases, the property that

$$g(ct, cx) = c^{H-1/\alpha} g(t,x) \quad \text{for all } c > 0,\ x \in \mathbb{R} \text{ and } t \geq 0. \tag{4.17}$$

The H-self-similarity property of the process $(Y(t))_{t\geq 0}$ follows immediately and the property of stationary increments is also clear. Linear fractional SαS motion was introduced by [28] and [13], while log-fractional SαS motion was introduced by [12]. See [25] for more details. In particular, different and non-proportional choices of a and b in (4.13) produce different and non-proportional linear fractional SαS motions.

The stochastic process (X_n) defined by

$$X_n = Y(n) - Y(n-1)\,, \quad n = 1, 2, \ldots \tag{4.18}$$

is stationary because the process $(Y(t))_{t\geq 0}$ has stationary increments. Moreover, it is a moving average process (4.2) with

$$f(x) = a\left((-x)_+^{H-1/\alpha} - (-x-1)_+^{H-1/\alpha}\right) + b\left((-x)_-^{H-1/\alpha} - (-x-1)_-^{H-1/\alpha}\right) \tag{4.19}$$

for the linear fractional SαS motion (4.14),

$$f(x) = aI_{[-1,0]}(x) \tag{4.20}$$

for the SαS Lévy motion (4.15) and

$$f(x) = a(\ln|x| - \ln|x+1|) \tag{4.21}$$

for the log-fractional SαS motion (4.16).

The following result describes the behavior of the ruin probability when the SαS process of the claim sizes is the increment process (4.18) of one of the H-sssi SαS processes (4.14)–(4.16).

PROPOSITION 4.4. *Let* (X_n) *be the stationary increment process* (4.18) *of the* H*-sssi process* $(Y(t))_{t\geq 0}$ *in* (4.13).

(a) *If* $1/\alpha < H < 1$ *then*

$$\psi(u) \sim \frac{C_\alpha K_g}{2\mu^{\alpha H}} u^{-\alpha(1-H)}, \quad u \to \infty, \tag{4.22}$$

where

$$K_g = \left[\int_{\mathbb{R}} \sup_{t\geq 0} \frac{(g(t,x))_+^\alpha}{(1+t)^\alpha}\, dx + \int_{\mathbb{R}} \sup_{t\geq 0} \frac{(g(t,x))_-^\alpha}{(1+t)^\alpha}\, dx\right] < \infty. \tag{4.23}$$

(b) *If* $H = 1/\alpha$ *then*

$$\psi(u) \sim \frac{C_\alpha a^\alpha}{2(\alpha-1)\,\mu}\; u^{-(\alpha-1)}, \quad u \to \infty \tag{4.24}$$

for SαS Lévy motion and

$$\psi(u) \sim \frac{C_\alpha a^\alpha}{2(\alpha-1)\mu}\, u^{-(\alpha-1)}\,(\ln u)^\alpha, \quad u \to \infty \tag{4.25}$$

for log-fractional SαS motion in (4.16).

(c) If $0 < H < 1/\alpha$ then

$$\psi(u) \sim \frac{C_\alpha}{2(\alpha-1)\mu}\, I(f)\, u^{-(\alpha-1)}, \quad u \to \infty, \tag{4.26}$$

where

$$I(f) = \frac{|a|^\alpha |b|^{-H(1-\alpha H)} + |b|^\alpha |a|^{-H(1-\alpha H)}}{\alpha H\,(|a|^{H-1/\alpha} + |b|^{H-1/\alpha})^{\alpha H}} \tag{4.27}$$

if $ab > 0$ and

$$I(f) = \frac{|a|^\alpha + |b|^\alpha}{\alpha H} \tag{4.28}$$

if $ab \leq 0$. In all cases $I(f) < \infty$.

PROOF. By definition of the process (X_n),

$$\sum_{k=1}^{n} f_k(\cdot) = \sum_{k=1}^{n} f(k-\cdot) = g(n,\cdot), \tag{4.29}$$

and so, by self-similarity of the process $(Y(t))_{t\ge 0}$,

$$m_n^\alpha = \int_{\mathbb{R}} |g(n,x)|^\alpha \, dx = \left(\int_{\mathbb{R}} |g(1,x)|^\alpha \, dx\right) n^{\alpha H}.$$

Thus, condition (2.11) is satisfied with $\beta = H \in (0,1)$ and Theorem 2.5 is applicable. Moreover, a change of variable argument in (2.10) yields that

$$\begin{aligned}\psi_0(u) = u^{-\alpha(1-H)} \frac{C_\alpha}{2\mu^{\alpha H}} \Bigg[&\int_{\mathbb{R}} \sup_{t=n\mu/u,\, n\ge 1} \frac{(g(t,x))_+^\alpha}{(1+t)^\alpha} \, dx \\ &+ \int_{\mathbb{R}} \sup_{t=n\mu/u,\, n\ge 1} \frac{(g(t,x))_-^\alpha}{(1+t)^\alpha} \, dx \Bigg].\end{aligned} \tag{4.30}$$

It is clear that for every $x \in \mathbb{R}$ the integrands on the right hand side in (4.30) converge, as $u \to \infty$, to the integrands of the corresponding integrals in (4.23) and, moreover, are bounded from above by the latter. Therefore, by the dominated convergence theorem the integrals on the right hand side in (4.30) will themselves converge to the corresponding integrals in (4.23) whenever the latter are finite.

It follows that in order to prove part (a) of the proposition it is enough to prove that $K_g < \infty$ if $1/\alpha < H < 1$, and we will show that

$$\int_{\mathbb{R}} \sup_{t\ge 0} \frac{|g(t,x)|^\alpha}{(1+t)^\alpha} \, dx < \infty. \tag{4.31}$$

Notice that the latter condition is necessary if the process $(Z(t)) = ((1+t)^{-1}Y(t))$ is almost surely bounded; see, for example, Theorem 10.2.3 in [25]. But boundedness follows by the following argument:

$$\begin{aligned} &P\left(\sup_{t\ge 0} |Z(t)| > u\right) \\ &\quad \le P\left(\left\{\sup_{0\le t\le 1} |Z(t)| > u\right\} \cup \bigcup_{n=0}^{\infty} \left\{\sup_{2^n\le t\le 2^{n+1}} |Z(t)| > u\right\}\right) \\ &\quad \le P\left(\sup_{0\le t\le 1} |Y(t)| > u\right) + \sum_{n=0}^{\infty} P\left(\sup_{1\le t\le 2} |Y(t)| > u\, 2^{n(1-H)}\right). \end{aligned} \tag{4.32}$$

In the last step we used the H-self-similarity of the process Y. Since Y is locally bounded when $1/\alpha < H < 1$ (see Theorem 12.4.1 in [25]), we know that

$$P\left(\sup_{1\le t\le 2} |Y(t)| > u\right) \sim c\, u^{-\alpha}, \quad u \to \infty,$$

for some constant $c > 0$. This shows that the probability on the left hand side of (4.32) goes to zero as $u \to \infty$, and so the process Z is globally bounded. Therefore the integral (4.31) is finite, so we have proved part (a) of the proposition.

For part (b) of the proposition, the statement (4.24) is, of course, simply the classical [8] result of Example 2.4. Now we consider the case of log-fractional SαS motion. Observe that the relation (4.30) still holds in this case, with $H = 1/\alpha$. We start with checking that

$$\int_{\mathbb{R}} \sup_{t \geq 0} \frac{(g(t, y))_+^\alpha}{(1+t)^\alpha}\, dy < \infty. \tag{4.33}$$

Once proved, this will show that the first term in (4.30) is of the order $u^{-(\alpha-1)}$ and, therefore, does not contribute to the asymptotic order of $\psi(u)$ in (4.25).

Since $(g(t, y))_+ \leq g(t, -y)$ for every $y > 0$ and $t > 0$, it suffices to show that

$$\int_{-\infty}^{0} \sup_{t \geq 0} \frac{(g(t, y))_+^\alpha}{(1+t)^\alpha}\, dy =: I_1 + I_2 < \infty. \tag{4.34}$$

Here I_1 and I_2 are the corresponding integrals over $[-1, 0]$ and $(-\infty, -1]$. We start by bounding I_2. Writing

$$r_i := \int_i^{i+1} \sup_{t \geq 0} \frac{(\ln(t+y) - \ln y)^\alpha}{(1+t)^\alpha}\, dy, \quad i \geq 1,$$

we see that $I_2 = \sum_{i=1}^\infty r_i$. Moreover, for all $i \geq 1$ and $y \in [i, i+1]$,

$$\sup_{t \geq 0} \frac{\ln(t+y) - \ln y}{1+t} \leq \sup_{t \geq 0} \frac{\ln(t+y) - \ln y}{t} = y^{-1} \sup_{u > 0} u^{-1} \ln(1+u) = c\, y^{-1},$$

where c is a finite positive constant, independent of i. Thus we have $r_i \leq c\, i^{-\alpha}$, and therefore $I_2 < \infty$. Next we turn to I_1. We have

$$\begin{aligned} I_1 &= \int_1^\infty z^{-2} \sup_{t \geq 0} \frac{(\ln(zt+1))^\alpha}{(1+t)^\alpha}\, dz \\ &\leq \int_1^\infty z^{-2} \sup_{t \geq z^{-1}} \frac{(\ln(2zt))^\alpha}{(1+t)^\alpha}\, dz + \int_1^\infty z^{-2} \sup_{t \leq z^{-1}} \frac{(\ln(zt+1))^\alpha}{(1+t)^\alpha}\, dz \\ &\leq \int_1^\infty z^{-2+\alpha} \sup_{v \geq 1} \frac{(\ln(2v))^\alpha}{(v+z)^\alpha}\, dz + \int_1^\infty z^{-2} \sup_{t \leq z^{-1}} \frac{(\ln 2)^\alpha}{(1+t)^\alpha}\, dz. \end{aligned}$$

The second integral is finite. The first integral is bounded from above by the infinite series:

$$\sum_{i=1}^\infty i^{-2+\alpha} \sup_{v \geq 1} \frac{(\ln(2v))^\alpha}{(v+i)^\alpha}.$$

Up to a multiplicative constant, each summand is bounded from above by $i^{-2}(\ln i)^\alpha$ and, therefore, the series is summable. This proves that (4.34) holds.

Thus it remains to consider the asymptotic order of the second term in (4.30) [or, equivalently, in (2.10)], which we denote by $I_3(u)$. Notice that $(g(n,x))_- = 0$ for $x < 0$, so that by (4.29),

$$I_3 = \frac{C_\alpha}{2} \int_0^\infty \sup_{n\ge 1} \frac{(\ln x - \ln|n-x|)_+^\alpha}{(u+n\mu)^\alpha}\,dx.$$

We have

$$\begin{aligned}
&(u+i\mu)^\alpha \int_i^{i+1} \sup_{n\ge 1} \frac{(\ln x - \ln|n-x|)_+^\alpha}{(u+n\mu)^\alpha}\,dx \\
&\qquad \ge \int_i^{i+1} (\ln x - \ln|i-x|)^\alpha\,dx \\
&\qquad \ge \left[\left(\int_i^{i+1}(\ln x)^\alpha\,dx\right)^{1/\alpha} - \left(\int_i^{i+1}(\ln|i-x|)^\alpha\,dx\right)^{1/\alpha}\right]^\alpha,
\end{aligned}$$

and, therefore,

$$I_3(u) \ge (1+o(1))\frac{C_\alpha}{2}\int_1^\infty \frac{(\ln x)^\alpha}{(u+x\mu)^\alpha}\,dx \sim \frac{1}{(\alpha-1)\mu}\,u^{-(\alpha-1)}\,(\ln u)^\alpha$$

as $u\to\infty$.

To obtain a corresponding upper bound, write

$$n(x) = n/x \quad \text{and } f(y) = (\ln|y-1|^{-1})_+^\alpha.$$

Then for $\epsilon\in(0,1)$,

$$\begin{aligned}
I_3(u) &\le \frac{C_\alpha}{2}\int_0^\infty \left(\sup_{n:\,n(x)\le 1-\epsilon} + \sup_{n:\,|n(x)-1|\le\epsilon} + \sup_{n:\,1+\epsilon\le n(x)\le 2}\right) \frac{f(n(x))}{(u+n\mu)^\alpha}\,dx \\
&=: I_{31}(u) + I_{32}(u) + I_{33}(u).
\end{aligned}$$

It immediate that for some constants $c_i = c_i(\epsilon) > 0$, $i = 1,2$,

$$I_{33}(u) \le c_1 \int_0^\infty (u+x)^{-\alpha}\,dx \le c_2\,u^{-(\alpha-1)}.$$

Moreover, by a Taylor expansion argument, for some $c_i = c_i(\epsilon) > 0$, $i = 3,4$,

$$I_{31}(u) \le c_3 \int_0^\infty \sup_{t\le 1-\epsilon} \frac{t^\alpha}{(u+t\mu\,x)^\alpha}\,dx \le c_3 \int_0^\infty \frac{1}{(u+\mu\,x)^\alpha}\,dx \le c_4\,u^{-(\alpha-1)}.$$

Finally, for $u\to\infty$,

$$\begin{aligned}
I_{32}(u) &= \frac{C_\alpha}{2}\sum_{i=0}^\infty \int_i^{i+1} \sup_{n:\,|n(x)-1|\le\epsilon} \frac{(\ln x - \ln|n-x|)_+^\alpha}{(u+n\mu)^\alpha}\,dx \\
&\le (1+o(1))\frac{C_\alpha}{2}\sum_{i=0}^\infty \frac{(\ln i)^\alpha}{(u+i(1-\epsilon)\mu)^\alpha} \\
&\sim \frac{C_\alpha}{2(\alpha-1)\mu(1-\epsilon)}u^{-(\alpha-1)}\,(\ln u)^\alpha.
\end{aligned}$$

In the last step we used Lemma 3.6. Since $\epsilon > 0$ can be taken as close to zero as we wish, this completes the proof of part (b) of the proposition.

For part (c) of the proposition, it is straightforward to check that $I(f)$ is given by the expressions in (4.27) which is finite. We will check (4.26) in the case $ab > 0$. The case $ab \le 0$ is entirely similar and the computations are even easier. By symmetry, it is enough to consider the case $a > 0$ and $b > 0$.

Since (4.30) is still applicable, we write

$$\begin{aligned}
\psi_0(u) = u^{-\alpha(1-H)} \frac{C_\alpha}{2\mu^{\alpha H}} \Bigg[\int_0^\infty \Bigg[&\sup_{t=n\mu/u,\, n\ge 1} \frac{(g(t,x))_+^\alpha}{(1+t)^\alpha} \\
&+ \sup\nolimits_{t=n\mu/u,\, n\ge 1} \frac{(g(t,x))_-^\alpha}{(1+t)^\alpha} \Bigg] dx \\
+ \int_{-\infty}^0 \Bigg[&\sup_{t=n\mu/u,\, n\ge 1} \frac{(g(t,x))_+^\alpha}{(1+t)^\alpha} \\
&+ \sup\nolimits_{t=n\mu/u,\, n\ge 1} \frac{(g(t,x))_-^\alpha}{(1+t)^\alpha} \Bigg] dx \Bigg] \\
=: u^{-\alpha(1-H)} \frac{C_\alpha}{2\mu^{\alpha H}} (I_1^+(u) + I_1^-(u) + I_2^+(u) + I_2^-(u)).&
\end{aligned} \tag{4.35}$$

Clearly, $I_2^-(u) = 0$, while

$$\begin{aligned}
a^{-\alpha} I_2^+(u) &= \int_0^\infty \sup_{t=n\mu/u,\, n\ge 1} \frac{\left(x^{H-1/\alpha} - (x+t)^{H-1/\alpha}\right)^\alpha}{(1+t)^\alpha} dx \\
&\le \int_0^\infty \sup_{t\ge 0} \frac{\left(x^{H-1/\alpha} - (x+t)^{H-1/\alpha}\right)^\alpha}{(1+t)^\alpha} dx \\
&\le \int_1^\infty \sup_{t\ge 0} \frac{\left(x^{H-1/\alpha} - (x+t)^{H-1/\alpha}\right)^\alpha}{t^\alpha} dx + \int_0^1 x^{\alpha H-1}\, dx \\
&\le \frac{1}{\alpha H} + \sup_{t\ge 0} \frac{\left(1 - (1+t)^{H-1/\alpha}\right)^\alpha}{t^\alpha} \int_1^\infty x^{-1-\alpha(1-H)}\, dx < \infty.
\end{aligned}$$

Since $0 < H < 1/\alpha$ we have $u^{-\alpha(1-H)} = o(u^{-(\alpha-1)})$, and so the last two terms $I_2^\pm(u)$ on the right hand side of (4.35) do not contribute to the asymptotic behavior of $\psi(u) \sim \psi_0(u)$ in (4.26).

Furthermore,

$$I_1^-(u) = \int_0^\infty \sup_{t=n\mu/u,\, t\ge x/\Delta} \frac{\left(bx^{H-1/\alpha} - a(t-x)^{H-1/\alpha}\right)^\alpha}{(1+t)^\alpha} dx$$

with

$$\Delta = \frac{1}{1+(b/a)^{1/(H-1/\alpha)}} \in (0,1). \tag{4.36}$$

Since

$$\int_0^\infty \sup_{t\ge x/\Delta} \frac{\big(bx^{H-1/\alpha}-a(t-x)^{H-1/\alpha}\big)^\alpha}{(1+t)^\alpha}\,dx$$

$$\le b^\alpha + \int_1^\infty \sup_{t\ge x/\Delta} \frac{\big(bx^{H-1/\alpha}-a(t-x)^{H-1/\alpha}\big)^\alpha}{t^\alpha}\,dx$$

$$= b^\alpha + \sup_{t\ge 1/\Delta} \frac{\big(b-a(t-1)^{H-1/\alpha}\big)^\alpha}{t^\alpha}\int_1^\infty x^{-1-\alpha(1-H)}\,dx < \infty,$$

we conclude that the corresponding term on the right hand side of (4.35) does not contribute to the asymptotic behavior of $\psi(u)\sim\psi_0(u)$ in (4.26) either.

It remains, therefore, to consider $I_1^+(u)$. Switching back to the language of (2.10) and taking into account Theorem 4.1, we need to prove that

$$\limsup_{u\to\infty} u^{\alpha-1}\int_0^\infty \sup_{n\ge1} \times \frac{\Big(a(n-x)_+^{H-1/\alpha}+b((n-x)_-^{H-1/\alpha}-x^{H-1/\alpha})\Big)_+^\alpha}{(u+n\mu)^\alpha}\,dx \le \frac{1}{(\alpha-1)\mu}\,I(f) \tag{4.37}$$

with $I(f)$ given by (4.27).

Choose $\epsilon\in(0,1)$. The integral on the left hand side of (4.37) is bounded from above by

$$\int_0^\infty \sup_{n\le(1-\epsilon)x}(\cdot)+\int_0^\infty \sup_{n>(1-\epsilon)x}(\cdot) := J_1(u)+J_2(u).$$

Observe that for some positive constants $c_i=c_i(\epsilon)$, $i=5,6$

$$J_1(u) = b^\alpha\int_0^\infty x^{\alpha H-1}\sup_{n\le(1-\epsilon)x}\frac{\big((1-n/x)^{H-1/\alpha}-1\big)^\alpha}{(u+n\mu)^\alpha}\,dx$$

$$\le c_5\int_0^\infty x^{\alpha H-1-\alpha}\sup_{n\le(1-\epsilon)x}\frac{n^\alpha}{(u+n\mu)^\alpha}\,dx$$

$$\le c_5\int_0^\infty x^{\alpha H-1}\frac{1}{(u+x\mu)^\alpha}\,dx$$

$$\sim c_6 u^{-\alpha(1-H)} \qquad \text{as } u\to\infty.$$

Since $H<1/\alpha$, we conclude that

$$\lim_{u\to\infty} u^{\alpha-1}J_1(u)=0. \tag{4.38}$$

Furthermore, for every noninteger $x>0$

$$\sup_{n>(1-\epsilon)x}\frac{\Big(a(n-x)_+^{H-1/\alpha}+b((n-x)_-^{H-1/\alpha}-x^{H-1/\alpha})\Big)_+^\alpha}{(u+n\mu)^\alpha}$$

$$= \max\left\{b^\alpha \sup_{(1-\epsilon)x<n<x} \frac{\left((x-n)^{H-1/\alpha} - x^{H-1/\alpha}\right)^\alpha}{(u+n\mu)^\alpha},\right.$$

$$\left. \sup_{n>x} \frac{\left(a(n-x)^{H-1/\alpha} - bx^{H-1/\alpha}\right)_+^\alpha}{(u+n\mu)^\alpha}\right\}$$

$$\le \max\left\{(1-\epsilon)^{-\alpha} b^\alpha \frac{(x-[x])^{\alpha H-1}}{(u+x\mu)^\alpha}, a^\alpha \frac{([x]+1-x)^{\alpha H-1}}{(u+x\mu)^\alpha}\right\}$$

$$\le (1-\epsilon)^{-\alpha}(u+x\mu)^{-\alpha} \max\left\{b^\alpha (x-[x])^{\alpha H-1}, a^\alpha([x]+1-x)^{\alpha H-1}\right\}$$

$$= \begin{cases} (1-\epsilon)^{-\alpha}(u+x\mu)^{-\alpha} b^\alpha (x-[x])^{\alpha H-1}, & \text{if } x-[x] \le \Delta, \\ (1-\epsilon)^{-\alpha}(u+x\mu)^{-\alpha} a^\alpha ([x]+1-x)^{\alpha H-1}, & \text{if } x-[x] > \Delta, \end{cases}$$

with Δ defined in (4.36). We conclude that

$$J_2(u) \le (1-\epsilon)^{-\alpha}\left(b^\alpha \sum_{i=0}^{\infty} \int_i^{i+\Delta} \frac{(x-i)^{\alpha H-1}}{(u+x\mu)^\alpha}\,dx + a^\alpha \sum_{i=0}^{\infty} \int_{i+\Delta}^{i+1} \frac{(i+1-x)^{\alpha H-1}}{(u+x\mu)^\alpha}\,dx\right)$$

$$\le (1-\epsilon)^{-\alpha}\left(b^\alpha \frac{\Delta^{\alpha H}}{\alpha H} \sum_{i=0}^{\infty}(u+i\mu)^{-\alpha} + a^\alpha \frac{(1-\Delta)^{\alpha H}}{\alpha H} \sum_{i=0}^{\infty}(u+i\mu)^{-\alpha}\right)$$

$$\sim (1-\epsilon)^{-\alpha}\left(b^\alpha \frac{\Delta^{\alpha H}}{\alpha H} + \frac{(1-\Delta)^{\alpha H}}{\alpha H}\right) \frac{1}{(\alpha-1)\mu} u^{-(\alpha-1)} \quad \text{as } u \to \infty.$$

Therefore,

$$\limsup_{u\to\infty} u^{\alpha-1} J_2(u) \le (1-\epsilon)^{-\alpha} \frac{1}{(\alpha-1)\mu} I(f), \tag{4.39}$$

and since we can take ϵ as small as we wish, (4.37) now follows from (4.38) and (4.39). This completes the proof of the proposition. □

REMARK 4.5. Proposition 4.4 provides additional substance to the common belief that H-sssi SαS processes with $1 < \alpha < 2$ are long-range dependent when $H > 1/\alpha$. Indeed, in that case the asymptotic behavior of the ruin probability for the linear fractional SαS motions is markedly different from the case $0 < H < 1/\alpha$. In fact, an argument similar to the one used in the proof of part (a) of Proposition 4.4 will work in far greater generality than just linear fractional SαS motions. All one needs is a scaling property of the kernel akin to that in (4.17) and an appropriate scaling property of the control measure of the SαS random measure in the integral representation of the process. See, for example, [24].

REMARK 4.6. Part (b) demonstrates that the asymptotic behavior of the ruin probability for H-sssi SαS processes is not determined by the values α and H! This interesting phenomenon deserves further study.

REMARK 4.7. We expect that an approach similar to the one used in this paper will allow us to treat the ruin probability in the case when the claim sizes are modeled by a rather general heavy tailed infinitely divisible stationary ergodic process. See, for example, [17] for some information. Our work on the ruin problem for such processes is now in progress.

Furthermore, the ruin probability is a special case of a whole class of exceedance probabilities for stochastic processes. It is natural to ask for the exceedance probability for threshold functions more general than linear ones. We expect that the methods of this paper allow one to derive the asymptotic order of such probabilities for a rather general class of threshold functions.

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