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DECOMPOSITION OF STATIONARY α-STABLE RANDOM FIELDS[1]

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This work is concerned with the structural analysis of stationary α-stable random fields. Three distinct classes of such random fields are characterized and it is shown that every stationary α-stable random field can be uniquely decomposed into the sum of three independent components belonging to these classes. Various examples of stationary α-stable random fields are discussed in this context.

1. Preliminaries and introduction. In this paper we study measurable stationary symmetric α-stable (SαS) random fields $\mathbf{X} = \{X_t\}_{t\in\mathbf{T}^d}$, where $\mathbf{T} = \mathbf{R}$ or $\mathbf{Z}$ and $d \geq 1$. Such processes can be represented (in distribution) by stochastic integrals

$$\mathbf{X} \overset{d}{=} \left\{ \int_S f_t(s) M(ds) \right\}_{t\in\mathbf{T}^d}, \tag{1.1}$$

where M is an independently scattered SαS random measure on some Borel space $(S, \mathscr{B}_S)$ with control measure μ, and $\{f_t\}_{t\in\mathbf{T}^d} \subset L^\alpha(S, \mu)$ is a collection of deterministic functions such that the map $(t, s) \to f_t(s)$ is jointly measurable (see, e.g., [7]). The condition for the stationarity of $\mathbf{X}$ is equivalent to

$$\left\| \sum a_k f_{t_k+t} \right\|_\alpha = \left\| \sum a_k f_{t_k} \right\|_\alpha, \tag{1.2}$$

for every $t, t_1, \dots, t_n \in \mathbf{T}^d$, $a_1, \dots, a_n \in \mathbf{R}$ and $n \geq 1$. We also consider complex-valued SαS random fields; the symmetry assumption means in this case the invariance under multiplication by complex numbers of modulus 1 (rotations). In the complex case, f_t are complex valued, M is invariant under rotations, and (1.2) must hold for all $a_1, \dots, a_n \in \mathbf{C}$. A family of functions $\{f_t\}_{t\in\mathbf{T}^d}$ satisfying (1.1) is called a representation of $\mathbf{X}$. Following verbatim the proof of Theorem 3.1 in [6] (given for $d = 1$), one shows that every stationary SαS random field ($\alpha < 2$) has a representation of the form

$$f_t(s) = c_t(s) \left\{ \frac{d(\mu \circ \phi_t)}{d\mu}(s) \right\}^{1/\alpha} f(\phi_t(s)), \qquad s \in S,\ t \in \mathbf{T}^d, \tag{1.3}$$

where $\phi_t \colon S \to S$ is a measurable nonsingular flow on (S, μ) [i.e., $\phi_{t_1+t_2}(s) = \phi_{t_1}(\phi_{t_2}(s))$, $\phi_0(s) = s$, and $\mu \sim \mu \circ \phi_t$, for every $t_1, t_2, t \in \mathbf{T}^d$, $s \in S$], and

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$f \in L^\alpha(S,\mu)$. Furthermore, $c_t\colon S \to \{\pm 1\}$ (in the complex case, $c_t\colon S \to \{|z| = 1\colon z \in \mathbf{C}\}$) is a measurable cocycle for $\{\phi_t\}_{t\in\mathbf{T}^d}$, that is, for every $t_1, t_2 \in \mathbf{T}^d$,

$$c_{t_1+t_2}(s) = c_{t_1}(s)c_{t_2}(\phi_{t_1}(s)), \quad \mu\text{-a.e.} \tag{1.4}$$

The flow $\{\phi_t\}_{t\in\mathbf{T}^d}$ is determined by the distribution of $\mathbf{X}$ up to the equivalence relation of flows and the cocycle $\{c_t\}_{t\in\mathbf{T}^d}$ up to the coboundary factor (see Theorem 3.6 in [6]).

It is easier to study solutions of (1.4) when this equation holds for all $s \in S$ as well. We say that a measurable cocycle is *strict* if (1.4) holds for all $(t_1, t_2, s) \in \mathbf{T}^d \times \mathbf{T}^d \times S$. It turns out that one can always choose a strict cocycle in (1.3). This follows from the following fact given (in a greater generality) in [10], Appendix B9. *For every measurable cocycle $\{c_t\}_{t\in\mathbf{T}^d}$ there exists a strict measurable cocycle $\{\tilde{c}_t\}_{t\in\mathbf{T}^d}$ on the same measure space such that, for each $t \in \mathbf{T}^d$, $\tilde{c}_t = c_t$ μ-a.e.*

Our structural analysis of stable non-Gaussian random fields is based on the representation (1.3). The idea is to decompose S into $\{\phi_t\}_{t\in\mathbf{T}^d}$-invariant parts and to characterize the flow and the cocycle acting on each part separately. This operation corresponds to a decomposition of $\mathbf{X}$ into the sum of simpler independent stationary SαS random fields. The relation between $\mathbf{X}$ and $\{c_t, \phi_t\}_{t\in\mathbf{T}^d}$ is not explicit, however. Therefore, we are seeking for criteria in terms of $\{f_t\}_{t\in\mathbf{T}^d}$ to characterize the equivalence classes of $\{c_t, \phi_t\}_{t\in\mathbf{T}^d}$ which, in turn, determine appropriate classes of stationary random fields.

The best known examples of stationary random fields include harmonizable and moving average random fields. Recall that a harmonizable random field is of the form

$$\mathbf{X} \stackrel{d}{=} \left\{ \int_{\widehat{\mathbf{T}}^d} e^{it\cdot s} M(ds) \right\}_{t\in\mathbf{T}^d}, \tag{1.5}$$

where $\widehat{\mathbf{T}} = [0, 2\pi)$ if $\mathbf{T} = \mathbf{Z}$, $\widehat{\mathbf{T}} = \mathbf{R}$ if $\mathbf{T} = \mathbf{R}$, and $\mu(\widehat{\mathbf{T}}^d) < \infty$; comparing this with (1.3) we have $\phi_t = \mathbf{id}$, $c_t(s) = e^{it\cdot s}$ and $f = 1$. A moving average is given by

$$\mathbf{X} \stackrel{d}{=} \left\{ \int_{\mathbf{T}^d} f(t+s) M(ds) \right\}_{t\in\mathbf{T}^d}, \tag{1.6}$$

where $f \in L^\alpha(\mathbf{T}^d, \mathrm{Leb})$; $\mu = \mathrm{Leb}$ (the Lebesgue measure when $\mathbf{T}^d = \mathbf{R}^d$ and the counting measure when $\mathbf{T}^d = \mathbf{Z}^d$); here $\phi_t(s) = t + s$ and $c_t = 1$. Taking superpositions of independent moving averages leads to a larger class of the so-called mixed moving averages (see [8]). A mixed moving average is represented by

$$\mathbf{X} \stackrel{d}{=} \left\{ \int_{W\times\mathbf{T}^d} g(w, t+s) M(dw, ds) \right\}_{t\in\mathbf{T}^d}, \tag{1.7}$$

where $g \in L^\alpha(W \times \mathbf{T}^d,\ \lambda \otimes \mathrm{Leb})$, λ is a σ-finite measure on a Borel space $(W, \mathscr{B}_W)$, and the control measure μ of M equals $\lambda \otimes \mathrm{Leb}$. In this case, $\phi_t(w, s) = (w, t + s)$ and $c_t = 1$. For other special cases of stationary SαS random processes and fields we refer the reader to the recent monographs by Janicki and Weron [2] and by Samorodnitsky and Taqqu [7].

This work extends results of [6] from stationary SαS processes to random fields, presents more direct approach to their structural analysis, and provides examples. The major obstacle in generalizing [6] to random fields was the unavailability of Krengel's theorem [4] classifying dissipative flows indexed by $\mathbf{R}^d$, $d \geq 2$. We resolve this difficulty and prove a characterization of mixed moving averages in Theorem 2.1 directly, without referring to dissipative flows or Hopf decomposition (see also Corollary 2.2). In Theorems 2.4 and 3.1 we characterize, respectively, harmonizable random fields and random fields which do not admit harmonizable or mixed moving average components. The latter class is not well understood at present. Therefore, in Section 3, we give some examples of processes from this class. We show examples of "cocycle processes" which are defined by (1.3) reduced to the cocycle term only

$$f_t = c_t, \qquad t \in \mathbf{T}^d, \tag{1.8}$$

$\left[\mu\right.$ is a ϕ_t-invariant finite measure and $f = 1$ in $\left.(1.3)\right]$. A cocycle (1.4) plays a similar role to the exponential function in (1.5); it changes the phase of the noise M (recall that the exponential function is a cocycle for the identity flow). Cocycle processes are doubly stationary by Theorem 7 in [1]. Our examples demonstrate that even real-valued cocycle processes, which are defined by ± 1-valued cocycles, form a large class.

The characterizations obtained in Theorems 2.1, 2.4 and 3.1 lead to a unique decomposition of a stationary SαS random field into three independent parts

$$\mathbf{X} \stackrel{d}{=} \mathbf{X}^1 + \mathbf{X}^2 + \mathbf{X}^3,$$

where $\mathbf{X}^1$ is a mixed moving average, $\mathbf{X}^2$ is harmonizable, and $\mathbf{X}^3$ is a stationary SαS process without mixed moving average or harmonizable components. Thus, time moving averages and harmonic frequencies constitute basic building blocks of a general SαS random field but they act independently of each other. This independence implies, in principle, a possibility of their identification from the statistics of a random field. This is Theorem 3.7, generalizing Theorem 6.1 in [6].

There is one technical condition that we will often assume in the proofs. Namely, we will assume that $\{f_t\}_{t\in\mathbf{T}^d}$, satisfying (1.1) and (1.2), also satisfies

$$\operatorname{supp}\left\{f_t\colon t \in \mathbf{T}^d\right\} = S. \tag{1.9}$$

This condition is unrestrictive because the truncation of S to $\operatorname{supp}\{f_t\colon t \in \mathbf{T}^d\}$ does not affect (1.1) or (1.2). Let F denote the closure in L^α of $\operatorname{lin}\{f_t\colon t \in \mathbf{T}^d\}$. A representation $\{f_t\}_{t\in\mathbf{T}^d}$ is said to be *minimal* if (1.9) holds and $\sigma\{g/h\colon g, h \in F\} = \mathscr{B}_S$ modulo μ. Every separable in probability SαS process has a minimal representation (see, e.g., [2]).

Many results of this paper can be generalized to processes indexed by groups. Since such generalizations raise some other interesting issues that can be considered in the group context, we have chosen to investigate them in a separate work.

2. Moving average and harmonizable SαS random fields.

THEOREM 2.1. *Let* $\mathbf{X} = \{X_t\}_{t \in \mathbf{T}^d}$ *be a stationary* SαS *random field with an arbitrary representation* (1.1). *Then* $\mathbf{X}$ *is a mixed moving average if and only if*

$$\int_{\mathbf{T}^d} |f_t(s)|^\alpha \, dt < \infty, \quad \mu\text{-a.e.} \tag{2.1}$$

PROOF. First we will prove the necessity of (2.1). Suppose that $\mathbf{X}$ has a representation (1.7); without loss of generality we may assume that $\int_{\mathbf{T}^d} |g(w,t)|^\alpha \, dt < \infty$ for all $w \in W$. By Theorem 3.1 in [5] there exist functions Φ: $S \to W \times \mathbf{T}^d$, $\Phi = (\Phi_1, \Phi_2)$, and h: $S \to \mathbf{R}$ ($\mathbf{C}$, resp.) such that

$$f_t(s) = h(s) g(\Phi_1(s), \Psi_2(s) + t), \qquad \text{Leb} \otimes \mu\text{-a.e.}$$

which yields (2.1).

The proof of the sufficiency goes through a series of steps modifying representation (1.1) until the desired form (1.7) is obtained.

STEP 1. Let $\{g_t: t \in \mathbf{T}^d\} \subset L^\alpha(W, \nu)$ be a minimal representation of $\mathbf{X}$ (see Section 1). Then

$$g^*(w) := \int_{\mathbf{T}^d} |g_t(w)|^\alpha \, dt < \infty, \quad \nu\text{-a.e.} \tag{2.2}$$

PROOF OF STEP 1. We may and do assume (1.9). By Remark 2.5 in [6] there exist measurable maps Ψ: $S \to W$ and h: $S \to \mathbf{C} \setminus \{0\}$ (into $\mathbf{R} \setminus \{0\}$, resp.) such that, for each $t \in \mathbf{T}^d$,

$$f_t(s) = h(s) g_t(\Psi(s)), \quad \mu\text{-a.e.} \tag{2.3}$$

and $\mu \circ \Psi^{-1} \sim \nu$. Let $W_\infty := \{w: \int_{\mathbf{T}^d} |g_t(w)|^\alpha \, dt = \infty\}$; from (2.1) and (2.3) we get $\mu(\Psi^{-1}(W_\infty)) = 0$, which gives $\nu(W_\infty) = 0$ and proves (2.2).

Using the same arguments as in [6], Theorem 3.1, we infer that there exists a measurable nonsingular flow ϕ_t: $S \to S$ on (W, ν) and a cocycle c_t: $S \to \{|z| = 1: z \in \mathbf{C}\}$ (or into $\{\pm 1\}$ if $\mathbf{X}$ is real valued) such that

$$g_t = c_t \left\{ \frac{d(\nu \circ \phi_t)}{d\nu} \right\}^{1/\alpha} g_0 \circ \phi_t, \qquad t \in \mathbf{T}^d. \tag{2.4}$$

STEP 2. Under the assumptions of Step 1 there exists a $\{\phi_t\}_{t \in \mathbf{T}^d}$-invariant measure λ on W which is equivalent to ν.

PROOF OF STEP 2. In view of (2.2), the measure λ given by

$$\lambda(dw) := g^*(w) \nu(dw)$$

is absolutely continuous with respect to ν. Let $W_0 := \{w: g^*(w) = 0\}$. We have

$$0 = \int_{W_0} g^* \, d\nu = \int_{\mathbf{T}^d} \int_{W_0} |g_t|^\alpha \, d\nu \, dt.$$

Hence there exists a dense set $D \subset \mathbf{T}^d$ such that for every $t \in D$,

$$\int_{W_0} |g_t|^\alpha \, d\nu = 0. \tag{2.5}$$

Consider the group of isometries V_t: $L^1(W, \nu) \to L^1(W, \nu)$ given by

$$V_t h = \frac{d(\nu \circ \phi_t)}{d\nu} h \circ \phi_t.$$

Notice that $|g_t|^\alpha = V_t(|g_0|^\alpha)$. The group $\{V_t\}_{t \in \mathbf{T}^d}$ is measurable, thus strongly continuous, and since (2.5) holds on a dense set D, it holds for all $t \in \mathbf{T}^d$. Consequently, W_0 is disjoint $(\operatorname{mod} \nu)$ with $\operatorname{supp}\{g_t\colon t \in \mathbf{T}^d\}$. From the minimality of $\{g_t\}_{t \in \mathbf{T}^d}$ we get $\nu(W_0) = 0$, showing that λ is equivalent to ν.

To prove that λ is invariant, choose $\tau \in \mathbf{T}^d$ and $A \in \mathscr{B}_W$. We have

$$\begin{aligned}
\lambda(\phi_\tau A) &= \int_W \mathbf{1}_A(\phi_{-\tau}(w)) g^*(w) \nu(dw) \\
&= \int_{\mathbf{T}^d} \int_W \mathbf{1}_A(\phi_{-\tau}(w)) \frac{d(\nu \circ \phi_t)}{d\nu}(w) |g_0(\phi_t(w))|^\alpha \nu(dw) \, dt \\
&= \int_{\mathbf{T}^d} \int_W \mathbf{1}_A(w) \frac{d(\nu \circ \phi_t)}{d\nu}(\phi_\tau(w)) |g_0(\phi_{t+\tau}(w))|^\alpha (\nu \circ \phi_\tau(dw)) \, dt \\
&= \int_W \mathbf{1}_A(w) \int_{\mathbf{T}^d} \frac{d(\nu \circ \phi_{t+\tau})}{d\nu}(w) |g_0(\phi_{t+\tau}(w))|^\alpha \, dt \, \nu(dw) = \lambda(A).
\end{aligned}$$

This completes the proof of Step 2.

STEP 3. Define

$$h_t(w) := c_t(w) h(\phi_t(w)),$$

where $h := (g^*)^{-1/\alpha} g_0$. Then $\{h_t\colon t \in \mathbf{T}^d\} \subset L^\alpha(W, \lambda)$ is a representation of $\mathbf{X}$ such that for λ-a.a. $w \in W$,

$$\int_{\mathbf{T}^d} |h(\phi_t(w))|^\alpha \, dt = 1. \tag{2.6}$$

PROOF. From the equality

$$1 = \frac{d(\lambda \circ \phi_t)}{d\lambda} = \frac{d(v \circ \phi_t)}{d\nu} \frac{g^* \circ \phi_t}{g^*}$$

we get

$$g_t = c_t \left\{ \frac{g^*}{g^* \circ \phi_t} \right\}^{1/\alpha} g_0 \circ \phi_t = (g^*)^{1/\alpha} h_t$$

or

$$h_t = (g^*)^{-1/\alpha} g_t,$$

proving that $\{h_t\}_{t \in \mathbf{T}^d} \subset L^\alpha(W, \lambda)$ is a representation of $\mathbf{X}$. Since the last equality holds $\operatorname{mod} \lambda$, for each $t \in \mathbf{T}^d$, by Fubini's theorem we get (2.6) λ-a.e., as needed.

Notice now that the set $\{w\colon \int_{\mathbf{T}^d} |h(\phi_t(w))|^\alpha\, dt = 1\}$ is $\{\phi_t\}$-invariant. Therefore, removing from W the complement of this set, which is of measure zero by (2.6), does not affect the representation $\{h_t\}$. Hence we may and do assume that (2.6) holds *for every* $w \in W$.

STEP 4. There exists a sequence of $\{\phi_t\}$-invariant real-valued Borel functions on W which separate the orbits of $\{\phi_t\}_{t\in\mathbf{T}^d}$.

PROOF OF STEP 4. We will now employ some topological arguments. By Theorem 8.7 of Varadarajan [9], W can be considered as a Borel subset of a compact metric space $\widetilde{W}$ on which the flow $\{\phi_t\}_{t\in\mathbf{T}^d}$ is defined, W is $\{\phi_t\}$-invariant, and the map $\mathbf{T}^d \times \widetilde{W} \ni (t, w) \to \phi_t(w) \in \widetilde{W}$ is jointly continuous.

Let $\{A_n\}$ be the sequence of finite unions of finite intersections of sets from a countable topological basis of W. Let

$$A_{nm} := A_n \cap \{w\colon |h(w)| > m^{-1}\}.$$

Since $\int_W |h|^\alpha\, d\lambda < \infty$, $\lambda(A_{nm}) < \infty$ for every $n, m \ge 1$. Define

$$u_{nm}(w) := \int_{\mathbf{T}^d} \mathbf{1}_{A_{nm}}(\phi_t(w))\, dt.$$

Notice that

$$u_{nm}(w) \le \int_{\mathbf{T}^d} m^\alpha |h(\phi_t(w))|^\alpha\, dt = m^\alpha < \infty,$$

for every $w \in W$, $n, m \ge 1$, and clearly u_{nm} is $\{\phi_t\}$-invariant. We will show that $\{u_{nm}\}_{n,m\ge 1}$ separate the orbits of $\{\phi_t\}_{t\in\mathbf{T}^d}$.

Suppose that w_1 and w_2 live on different orbits. We first claim that for some $n, m \ge 1$,

$$u_{nm}(w_1) > 0. \tag{2.7}$$

Indeed, from (2.6) (which now holds for all $w \in W$) we infer that there exist $m \ge 1$ and $t_0 \in \mathbf{T}^d$ such that for every open neighborhood G of t_0,

$$|\{t \in G\colon |h(\phi_t(w_1))| > m^{-1}\}| > 0$$

($|\cdot|$ stands for the Lebesgue measure). Furthermore, there exists an $n \ge 1$ such that $\phi_{t_0}(w_1) \in A_n$. By the continuity of ϕ, there is an open neighborhood G of t_0 such that $\phi_t(w_1) \in A_n$, provided $t \in G$. Hence

$$\begin{aligned} u_{nm}(w_1) &= |\{t\colon \phi_t(w_1) \in A_n,\ |h(\phi_t(w_1))| > m^{-1}\}| \\ &\ge |\{t \in G\colon |h(\phi_t(w_1))| > m^{-1}\}| > 0, \end{aligned}$$

proving (2.7). Now we will show that w_1 and w_2 can be separated.

Let n, m be as in (2.7). If $u_{nm}(w_2) \ne u_{nm}(w_1)$, then there is nothing to prove. Thus we assume

$$u_{nm}(w_2) = u_{nm}(w_1) = \delta > 0.$$

There exists a compact set $K \subset \mathbf{T}^d$ such that

$$\int_K \mathbf{1}_{A_{nm}}(\phi_t(w_i))\,dt > \delta/2, \qquad i = 1, 2. \tag{2.8}$$

Consider compact sets in W given by

$$\Gamma_i := \{\phi_t(w_i) \colon t \in K\}, \qquad i = 1, 2.$$

Since w_1 and w_2 live on different orbits, $\Gamma_1 \cap \Gamma_2 = \varnothing$. Hence, there is an $n_1 \geq 1$ such that

$$A_{n_1} \supset \Gamma_1 \quad \text{and} \quad A_{n_1} \cap \Gamma_2 = \varnothing.$$

By the definition of the sequence $\{A_k\}$, $A_n \cap A_{n_1} = A_{n'}$, for some $n' \geq 1$. Consider $u_{n'm}$. In view of (2.8) we get

$$\begin{aligned} u_{n'm}(w_1) &\geq \int_{\mathbf{T}^d} \mathbf{1}_{\Gamma_1 \cap A_{nm}}(\phi_t(w_1))\,dt \\ &\geq \int_K \mathbf{1}_{A_{nm}}(\phi_t(w_1))\,dt > \delta/2 \end{aligned}$$

and

$$u_{n'm}(w_2) = \int_{K^c} \mathbf{1}_{A_{n'm}}(\phi_t(w_2))\,dt < \delta/2.$$

Hence $u_{n'm}(w_2) < u_{n'm}(w_1)$, which ends the proof of Step 4.

By Step 4 and von Neumann's cross-section lemma (see [9], Corollary 8.2), there exists a Borel set $W_0 \subset W$ which intersects each orbit of $\{\phi_t\}_{t\in\mathbf{T}^d}$ at exactly one point. [To be exact, this step may require a reduction of W to some $\{\phi_t\}_{t\in\mathbf{T}^d}$-invariant subset, say, $\widetilde{W}$, such that $\lambda(W \setminus \widetilde{W}) = 0$.]

STEP 5. Let W_0 be as above. The map $\Phi \colon W_0 \times \mathbf{T}^d \to W$, given by $\Phi(w_0, t) = \phi_t(w_0)$, is a Borel isomorphism. The measure $\lambda \circ \Phi$, induced on $W_0 \times \mathbf{T}^d$ by the inverse map Φ^{-1} from W, is the product measure of certain measure λ_0 on W_0 and the Lebesgue measure on $\mathbf{T}^d$.

PROOF OF STEP 5. First we will show that Φ is one-to-one. Suppose that $\Phi(w_1, t_1) = \Phi(w_2, t_2)$. Then $w_1 = w_2 = w_0$ from the definition of W_0, thus $\phi_{t_0}(w_0) = w_0$, where $t_0 = t_1 - t_2$. Hence $h(\phi_{t+nt_0}(w_0)) = h(\phi_t(w_0))$, for every $n \in \mathbf{N}$ and $t \in \mathbf{T}^d$, which, in view of (2.6), is only possible when $t_0 = 0$. Hence Φ is one-to-one and clearly onto. Since Φ is Borel measurable, its inverse is measurable by Kuratowski's theorem.

Let $\Phi^{-1}(w) = (\pi(w), \tau(w))$ and consider $Q := \lambda \circ \Phi$. We have

$$Q(A \times B) = \lambda(\{w \colon \pi(w) \in A,\ \tau(w) \in B\}), \qquad A \in \mathscr{B}_{W_0},\ B \in \mathscr{B}_{\mathbf{T}^d}.$$

Since λ is $\{\phi_t\}$-invariant by Step 2, we get

$$\begin{aligned} Q(A \times (B + t)) &= \lambda(\{w \colon \pi(\phi_t(w)) \in A,\ \tau(\phi_t(w)) \in B + t\}) \\ &= \lambda(\{w \colon \pi(w) \in A,\ \tau(w) + t \in B + t\}) = Q(A \times B). \end{aligned}$$

Hence $Q(A\times\cdot)$ is proportional to the Lebesgue measure, and consequently, $Q(A\times B)=\lambda_0(A)|B|$, for some measure λ_0 on $(W_0,\mathscr{B}_{W_0})$.

The next step ends the proof of the theorem.

STEP 6. Let $\{c_t\}_{t\in\mathbf{T}^d}$ be a strict cocycle (see Section 1) and let

$$k(w_0,s):=c_s(w_0)h(\phi_s(w_0)),\qquad (w_0,s)\in W_0\times\mathbf{R}^d.$$

Then

$$k_t(w_0,s):=k(w_0,t+s),$$

is a representation of $\mathbf{X}$ in $L^\alpha(W_0\times\mathbf{T}^d,\lambda_0\otimes\mathrm{Leb})$.

PROOF OF STEP 6. For every $a_1,\dots,a_n\in\mathbf{R}$ ($\mathbf{C}$, resp.) and $t_1,\dots,t_n$ we have

$$\begin{aligned}\int_{W_0\times\mathbf{T}^d}&\Big|\sum a_jk_{t_j}(w_0,s)\Big|^\alpha\lambda_0(dw_0)\,ds\\&=\int_{W_0\times\mathbf{T}^d}\Big|\sum a_jc_{t_j}(\Phi(w_0,s))h(\phi_{t_j}(\Phi(w_0,s)))\Big|^\alpha\lambda_0(dw_0)\,ds\\&=\int_W\Big|\sum a_jh_{t_j}(w)\Big|^\alpha\lambda(dw)\end{aligned}$$

which ends the proof of Theorem 2.1. □

If $d=1$ and $\{f_t\}_{t\in\mathbf{T}}$ is given by (1.3) then (2.1), together with (1.9), characterize dissipative flows; in this case Theorem 2.1 is a consequence of Krengel's theorem [4] (see [6]). However, this line of reasoning does not extend to the multiparameter case. We propose the following extension of Krengel's theorem to flows indexed by $\mathbf{T}^d$.

Measurable nonsingular flows $\{\phi_t^{(1)}\}_{t\in\mathbf{T}^d}$ and $\{\phi_t^{(2)}\}_{t\in\mathbf{T}^d}$, defined on measure Borel spaces $(S_1,\mathscr{B}_{S_1},\mu_1)$ and $(S_2,\mathscr{B}_{S_2},\mu_2)$, resp., are said to be *equivalent* if there exists a measurable map Φ: $S_2\to S_1$ with the following properties:

1. There exist $N_i\subset S_i$ with $\mu_i(N_i)=0$ $(i=1,2)$ such that Φ is a Borel isomorphism between $S_2\backslash N_2$ and $S_1\backslash N_1$.
2. μ_1 and $\mu_2\circ\Phi^{-1}$ are mutually absolutely continuous.
3. $\phi_t^{(1)}\circ\Phi=\Phi\circ\phi_t^{(2)}$, μ_2-a.e. for each $t\in T$.

THEOREM 2.2. *Let $\{\phi_t\}_{t\in\mathbf{T}^d}$ be a measurable nonsingular flow on a σ-finite measure space $(S,\mathscr{B}_S,\mu)$. Then $\{\phi_t\}_{t\in\mathbf{T}^d}$ is equivalent to the flow,*

$$\psi_t(w,s):=(w,t+s),\qquad t\in\mathbf{T}^d$$

defined on $(W\times\mathbf{T}^d,\ \mathscr{B}_W\otimes\mathscr{B}_{\mathbf{T}^d},\ \lambda\otimes\mathrm{Leb})$, where $(W,\mathscr{B}_W,\lambda)$ is some σ-finite measure space, if and only if for some (equivalently, any) function $h\in L^1(S,\mathscr{B}_S,\mu)$ with $\operatorname{supp}\{h\circ\phi_t\colon t\in\mathbf{T}^d\}=S$ modulo μ,

$$\int_{\mathbf{T}^d}|h(\phi_t(s))|\frac{d(\mu\circ\phi_t)}{d\mu}(s)\,dt<\infty,\quad \mu\text{-a.e.}\tag{2.11}$$

For the proof, follow steps 2–5 of Theorem 2.1.

EXAMPLE 2.3. A (nonstationary) (α, H)-Takenaka random field $\mathbf{Y}$ is defined by

$$Y_t := M(V_t) = \int_{\mathbf{R}^d \times \mathbf{R}_+} \mathbf{1}_{V_t}(x, r) M(dx, dr), \qquad t \in \mathbf{R}^d,$$

where

$$V_t := \{(x, r)\colon \|x\| \le r\} \Delta \{(x, r)\colon \|x - t\| \le r\},$$

and M is a SαS random measure on $\mathbf{R}^d \times \mathbf{R}_+$ with control measure

$$\mu(dx, dr) = r^{\alpha H - d - 1}\, dx\, dr,$$

$H \in (0, 1/\alpha)$ (see [7], Chapter 8.4). Consider the increment field $\mathbf{X}$ of $\mathbf{Y}$ given by

$$X_t := Y_{t+e} - Y_t, \qquad t \in \mathbf{R}^d,$$

where $e \in \mathbf{R}^d$ is fixed. We will first compute a spectral representation of $\mathbf{X}$,

$$\begin{aligned} f_t(x, r) &= \mathbf{1}_{V_{t+e}}(x, r) - \mathbf{1}_{V_t}(x, r) \\ &= \mathbf{1}(\|x\| \le r) + \mathbf{1}(\|x - t - e\| \le r) - 2\mathbf{1}(\|x\| \le r,\ \|x - t - e\| \le r) \\ &\quad - [\mathbf{1}(\|x\| \le r) + \mathbf{1}(\|x - t\| \le r) - 2\mathbf{1}(\|x\| \le r,\ \|x - t\| \le r)] \\ &= \varepsilon(x, r)[\mathbf{1}(\|x - t - e\| \le r) - \mathbf{1}(\|x - t\| \le r)], \end{aligned}$$

where $\varepsilon(x, r) = \mathbf{1}(\|x\| > r) - \mathbf{1}(\|x\| \le r)$; $\varepsilon = \pm 1$. It is easy to check that $\{f_t\}_{t \in \mathbf{T}^d} \subset L^\alpha(\mathbf{R}^d \times \mathbf{R}_+, \mu)$ satisfies (1.2), so that $\mathbf{X}$ is a stationary random field.

Since, for each $(x, r) \in \mathbf{R}^d \times \mathbf{R}_+$,

$$\int_{\mathbf{R}^d} |f_t(x, r)|^\alpha\, dt = |B_r(x - e) \Delta B_r(x)| < \infty,$$

where $B_r(y)$ denotes the ball of radius r centered at y, $\mathbf{X}$ is a mixed moving average by Theorem 2.1. A mixed moving average representation of $\mathbf{X}$ is of the form

$$g_t(x, r) = \mathbf{1}(\|x + t + e\| \le r) - \mathbf{1}(\|x + t\| \le r).$$

THEOREM 2.4. *Let $\mathbf{X} = \{X_t\}_{t \in \mathbf{T}^d}$ be a stationary* SαS *random field with an arbitrary representation* (1.1). *Then $\mathbf{X}$ is a harmonizable process if and only if for* (Leb $\otimes$ Leb $\otimes$ μ)*-almost all $(t_1, t_2, s) \in \mathbf{T}^d \times \mathbf{T}^d \times S$,*

$$f_{t_1+t_2}(s) f_0(s) = f_{t_1}(s) f_{t_2}(s). \tag{2.12}$$

PROOF. The necessity of (2.12) follows by a similar argument to the proof of necessity in Theorem 2.1 and uses the fact that the representation of a harmonizable SαS random field $g_t(x) = e^{t \cdot x}$ satisfies (2.12) for every $x \in \mathbf{R}^d$.

To prove the sufficiency we first show, in the same way as Step 1 of Theorem 2.1, that (2.12) holds for a minimal representation. From this point on, the proof is identical to the proof of the case $d = 1$ given in [6], Theorem 5.7. □

EXAMPLE 2.5. We will describe the real part of a harmonizable random field $\mathbf{X}$ in terms of (1.3). Suppose that $\mathbf{X}$ is given by (1.5) with control measure μ of M. Let Z be a *real-valued* SαS random measure on $\widehat{T}^d \times [0, 2\pi)$ with control measure $\mu \otimes$ Leb. We claim that

$$\mathbf{X} \stackrel{d}{=} \left\{ k_\alpha^{-1} \int_{\widehat{T}^d \times [0, 2\pi)} e^{i(s+t\cdot w)} Z(dw, ds) \right\}_{t \in \mathbf{T}^d}, \tag{2.13}$$

where $k_\alpha = (\int_0^{2\pi} |\cos s|^\alpha \, ds)^{1/\alpha}$. Indeed, for any complex numbers $z_j = (x_j, y_j)$, $t_j \in \mathbf{T}^d$, $j = 1, \ldots, n$, we have

$$\begin{aligned}
&E \exp i\Re\left[\sum_j \bar{z}_j \int_{\widehat{T}^d \times [0, 2\pi)} e^{i(s+t_j\cdot w)} Z(dw, ds) \right] \\
&\quad = E \exp i \int_{\widehat{T}^d \times [0, 2\pi)} \left[\sum_j x_j \cos(s + t_j \cdot w) + y_j \sin(s + t_j \cdot w) \right] Z(dw, ds) \\
&\quad = \exp - \int_{\widehat{T}^d \times [0, 2\pi)} \left| \sum_j x_j \cos(s + t_j \cdot w) + y_j \sin(s + t_j \cdot w) \right|^\alpha \mu(dw)\, ds \\
&\quad = \exp - \int_{\widehat{T}^d} \int_{[0, 2\pi)} \left| \Re\Big(e^{is} \sum_j \bar{z}_j e^{it_j\cdot w} \Big) \right|^\alpha ds\, \mu(dw) \\
&\quad = \exp - k_\alpha^\alpha \int_{\widehat{T}^d} \left| \sum_j \bar{z}_j e^{it_j\cdot w} \right|^\alpha \mu(dw),
\end{aligned}$$

which proves (2.13). Hence the real part of $\mathbf{X}$ is given by

$$\Re\mathbf{X} \stackrel{d}{=} \left\{ k_\alpha^{-1} \int_{\widehat{T}^d \times [0, 2\pi)} \cos(s + t \cdot w) Z(dw, ds) \right\}_{t \in \mathbf{T}^d}. \tag{2.14}$$

This is a special case of (1.3). Indeed, let the flow $\{\phi_t\}_{t \in \mathbf{T}^d}$ be defined on $(\widehat{T}^d \times [0, 2\pi), \mu \otimes \text{Leb})$ by

$$\phi_t(w, s) := (w, s +_{2\pi} t \cdot w),$$

where "$+_{2\pi}$" denotes the addition on the one-dimensional torus $[0, 2\pi)$, $f(w, s) := k_\alpha^{-1} \cos s$, and $c_t = 1$. Notice that this flow is measure preserving. A possibility of representing the real part of a harmonizable process in the form (2.14) (with a different proof) was shown to us by Donatas Surgailis. We also record one fact that follows easily from (2.13): there are no real-valued harmonizable SαS random fields other than zero.

3. Decomposition. Consider the representation (1.1) and assume (1.9). Define

$$S_D := \{ s \in S \colon \int_{\mathbf{T}^d} |f_t(s)|^\alpha \, dt < \infty \} \tag{3.1}$$

and

$$S_H := \{s \in S\colon f_{t_1+t_2}(s) f_0(s) = f_{t_1}(s) f_{t_2}(s) \quad \text{for a.a. } (t_1, t_2) \in \mathbf{T}^{2d}\}. \tag{3.2}$$

If the representation satisfies some regularity conditions, such as the right continuity of the sections $t \to f_t(s)$, then

$$S_H = \{s\colon f_{t_1+t_2}(s) f_0(s) = f_{t_1}(s) f_{t_2}(s) \quad \text{for all } (t_1, t_2) \in \mathbf{T}^{2d}\}. \tag{3.3}$$

By the joint measurability of the representation $\{f_t\}$, the sets S_D and S_H are measurable. We claim that S_D and S_H are essentially disjoint, that is, $\mu(S_D \cap S_H) = 0$. Indeed, if $s \in S_H$, then

$$\begin{aligned}\left(\int_{\mathbf{T}^d} |f_t(s)|^\alpha \, dt\right)^2 &= \int_{\mathbf{T}^d}\int_{\mathbf{T}^d} |f_{t_1}(s)|^\alpha |f_{t_2}(s)|^\alpha \, dt_1 \, dt_2 \\ &= \int_{\mathbf{T}^d}\int_{\mathbf{T}^d} |f_{t_1+t_2}(s)|^\alpha |f_0(s)|^\alpha \, dt_1 \, dt_2 = \infty,\end{aligned}$$

unless $f_t(s) = 0$ for a.a. $t \in \mathbf{T}^d$. Consider $N := \{s \in S_H\colon f_t(s) = 0 \text{ for a.a. } t\}$. Since

$$\int_N \int_{\mathbf{T}^d} |f_t(s)|^\alpha \, dt \, \mu(ds) = 0,$$

$\int_N |f_t(s)|^\alpha \mu(ds) = 0$ for a.a. $t \in \mathbf{T}^d$. The latter equality holds for all t by the continuity of the map $\mathbf{T}^d \ni t \to f_t \in L^\alpha(S, \mu)$. Hence $\mu(N \cap \operatorname{supp}\{f_t\colon t \in \mathbf{T}^d\}) = 0$, giving $\mu(N) = 0$ by (1.9). This proves $\int_{\mathbf{T}^d} |f_t(s)|^\alpha \, dt = \infty$ μ-a.e. on S_H, and so S_D and S_H are essentially disjoint.

It is obvious that if $\{f_t\}$ is given by (1.3), then $\phi_t^{-1}(S_D) = S_D$, for every $t \in \mathbf{T}^d$, that is, S_D is $\{\phi_t\}_{t\in\mathbf{T}^d}$-invariant. However, it is not obvious that S_H is $\{\phi_t\}_{t\in\mathbf{T}^d}$-invariant. We will sketch the proof of the invariance of S_H under the some simplifying assumptions; a complete proof requires repeating tedious arguments of Lemma 5.6 in [6] given for $d = 1$. Let

$$u_t(s) := c_t(s)\left\{\frac{d(\mu \circ \phi_t)}{d\mu}\right\}^{1/\alpha}(s).$$

Then

$$u_{t_1+t_2}(s) = u_{t_1}(s) u_{t_2}(\phi_{t_1}(s)), \quad \mu\text{-a.e.}, \tag{3.4}$$

for every $t_1, t_2 \in \mathbf{T}^d$ and

$$f_t(s) = u_t(s) f_0(\phi_t(s)).$$

Assume that (3.4) holds *for all* $s \in S,\ t_1, t_2 \in \mathbf{T}^d$ and also assume (3.3). Let $s \in S_H$. We get, for every $t_1, t_2 \in \mathbf{T}^d$,

$$\begin{aligned}
f_{t_1+t_2}(\phi_t(s))f_0(\phi_t(s)) &= u_{t_1+t_2}(\phi_t(s))f_0(\phi_{t+t_1+t_2}(s))f_0(\phi_t(s)) \\
&= u_{t+t_1+t_2}(s)u_t(s)^{-1}f_0(\phi_{t+t_1+t_2}(s))f_0(\phi_t(s)) \\
&= f_{t+t_1+t_2}(s)u_t(s)^{-1}f_0(\phi_t(s)) \\
&= f_t(s)f_{t_1+t_2}(s)f_0(s)^{-1}u_t(s)^{-1}f_0(\phi_t(s)) \\
&= f_t(s)f_{t_1}(s)f_{t_2}(s)f_0(s)^{-2}u_t(s)^{-1}f_0(\phi_t(s)) \\
&= f_{t+t_1}(s)f_{t+t_2}(s)f_t(s)^{-1}u_t(s)^{-1}f_0(\phi_t(s)) \\
&= u_{t_1}(\phi_t(s))f_0(\phi_{t+t_1}(s))u_{t_2}(\phi_t(s))f_0(\phi_{t+t_2}(s)) \\
&= f_{t_1}(\phi_t(s))f_{t_2}(\phi_t(s)).
\end{aligned}$$

Thus $\phi_t(s) \in S_H$, proving the invariance of S_H.

We will say that a stationary SαS random field $\mathbf{X}$ admits a harmonizable (mixed moving average, resp.) component if

$$\mathbf{X} \overset{d}{=} \mathbf{X}^1 + \mathbf{X}^2, \tag{3.5}$$

where $\mathbf{X}^i = \{X_t^i\colon t \in \mathbf{T}^d\},\ i = 1, 2$, are mutually independent stationary SαS random fields $\mathbf{X}^1$ is harmonizable (mixed moving average, resp.).

THEOREM 3.1. *Let* $\mathbf{X} = \{X_t\}_{t\in\mathbf{T}^d}$ *be a stationary* SαS *random field with an arbitrary representation* (1.1). *Then* $\mathbf{X}$ *does not admit harmonizable* (*mixed moving average, resp.*) *components if and only if* $\mu(S_H) = 0$ [$\mu(S_D) = 0$, resp.].

PROOF. We will consider only the harmonizable component case; the proof for mixed moving average component is similar. Let $\{g_t\}_{t\in\mathbf{T}^d} \subset L^\alpha(W, \nu)$ be a minimal representation of $\mathbf{X}$ satisfying (2.3). Let W_H be defined by (3.2) for $\{g_t\}_{t\in\mathbf{T}^d}$; we infer from (2.3) that

$$\mu(S_H \Delta \Psi^{-1}(W_H)) = 0. \tag{3.6}$$

Suppose that $\mu(S_H) > 0$. Because $\mu \circ \Phi^{-1} \sim \nu$, we have $\nu(W_H) > 0$. Since $\{g_t\}_{t\in\mathbf{T}^d}$ has the form (2.4), we can apply the proof of Theorem 5.7 in [6] to our case proving that $\{g_t\}_{t\in\mathbf{T}^d}$ restricted to W_H is a representation of a harmonizable process, say $\mathbf{X}^1$. Since W_H is $\{\phi_t\}_{t\in\mathbf{T}}$-invariant, $\{g_t\}_{t\in\mathbf{T}^d}$ restricted to $W\backslash W_H$ is also a representation of a stationary process, say $\mathbf{X}^2$, which is independent from $\mathbf{X}^1$. Thus (3.5) holds.

Conversely, if $\mathbf{X}$ admits a harmonizable component, then it possesses a representation with $\mu(S_H) > 0$. By (3.6), $\mu(S_H) > 0$ for *any* representation satisfying (1.9). □

EXAMPLE 3.2. Let $\{r_k\}_{n\in\mathbf{Z}}$ be a bilateral sequence of independent Bernoulli random variables defined on a probability space (S, μ) such that

$$\mu\{r_k = 1\} = 1 - \mu\{r_k = -1\} = p.$$

Let

$$f_n := \begin{cases} \prod_{k=0}^{n-1} r_k, & \text{if } n \geq 1, \\ 1, & \text{if } n = 0, \\ \prod_{k=n}^{-1} r_k, & \text{if } n \leq -1. \end{cases}$$

Then $\{f_n\}_{n \in \mathbf{Z}}$ is a representation of a stationary SαS sequence $\mathbf{X} = \{X_n\}_{n \in \mathbf{Z}}$ which does not admit harmonizable or mixed moving average components [$\alpha \in (0, 2)$ and $p \in (0, 1)$ are arbitrary]. If $p = 1/2$, then it is easy to verify that $\mathbf{X}$ has a simpler representation,

$$\mathbf{X} \stackrel{d}{=} \left\{ \int_S r_n \, dM \colon n \in \mathbf{Z} \right\}.$$

Example 3.2 can be viewed as a special case of the following.

EXAMPLE 3.3. Let $S = \mathbf{R}^{\mathbf{Z}}$, and let μ be a finite measure on S which is invariant under the group of shifts $[\phi_n(s)]_k := s_{k+n}$, $s = (\dots, s_1, s_0, s_1, \dots) \in S$, $n, k \in \mathbf{Z}$. Put

$$\sigma_n(s) := \begin{cases} \sum_{k=0}^{n-1} s_k, & \text{if } n \geq 1, \\ 0, & \text{if } n = 0, \\ -\sum_{k=n}^{-1} s_k, & \text{if } n \leq -1, \end{cases}$$

and define

$$f_n(s) := e^{i\sigma_n(s)}, \qquad n \in \mathbf{Z}.$$

It is easy to verify that

$$f_{n+m}(s) = f_n(s) f_m(\phi_n(s)), \qquad n, m \in \mathbf{Z}, \; s \in S,$$

that is, $\{f_n\}_{n \in \mathbf{Z}}$ is a cocycle for the flow $\{\phi_n\}_{n \in \mathbf{Z}}$. Thus

$$X_n := \int_S e^{i\sigma_n(s)} \, dM, \qquad n \in \mathbf{Z}$$

is a cocycle SαS process described in (1.8). We have $S_D = \varnothing$ and

$$S_H = \{s \in S \colon s_k - s_0 \in 2\pi\mathbf{Z} \text{ for all } k \in \mathbf{Z}\}.$$

By Theorem 3.1, $\mathbf{X}$ does not have a mixed moving average component. Furthermore, $\mathbf{X}$ does not have a harmonizable component when $\mu(S_H) = 0$ [$\alpha \in (0, 2)$].

The next example is also interesting; it gives periodic real-valued stationary stable processes when $\theta = \pi$ and M is a real-valued SαS random measure.

EXAMPLE 3.4. Let $S = [0, 1)$, $\mu = \text{Leb}$ and

$$f_t(s) := e^{i\theta[t+s]}, \qquad t \in \mathbf{R},\ s \in S,$$

where $[x]$ denotes the largest integer not exceeding x; $\theta \in (0, 2\pi)$ is fixed. It is easy to verify that $\{f_t\}_{t\in\mathbf{R}}$ is a cocycle for the flow $\phi_t(s) := t+s-[t+s]$. Since this flow preserves μ, $\{f_t\}_{t\in\mathbf{R}}$ is a special case of (1.8). We will show that

$$X_t := \int_{[0,1)} e^{i\theta[t+s]} M(ds), \qquad t \in \mathbf{R},$$

does not admit harmonizable or mixed moving average components. Clearly, $S_D = \varnothing$; $S_H = \varnothing$ because, for every $s \in [0, 1)$ and (t_1, t_2) from the triangle $\{(t_1, t_2)\colon 0 \le t_1 < 1-s,\ 0 \le t_2 < 1-s,\ t_1 + t_2 \ge 1-s\}$, we have

$$f_{t_1+t_2}(s) f_0(s) = e^{i\theta} \neq f_{t_1}(s) f_{t_2}(s) = 1.$$

This example can be generalized to random fields as follows.

EXAMPLE 3.5. Let $S = [0, 1)^d$, $\mu = \text{Leb}$, and let

$$f_t(s) := e^{i\theta \sum_{k=1}^{d} [t_k + s_k]},$$

where $t = (t_1, \ldots, t_d) \in \mathbf{R}^d$ and $s = (s_1, \ldots, s_d) \in S$. Then the corresponding SαS random field does not admit moving average or harmonizable components.

The next example demonstrates the richness of the class of cocycle processes.

EXAMPLE 3.6. Let $S = \mathscr{N}(\mathbf{R})$ be the space of integer-valued Radon measures on $\mathbf{R}$ equipped with the topology of vague convergence of measures and let μ be the probability measure on S under which the identity map

$$(S, \mu) \ni s \to s \in \mathscr{N}(\mathbf{R})$$

is a Poisson point process with the Lebesgue intensity measure (see [3]). Define $\phi_t\colon S \to S$ by

$$[\phi_t(s)](A) := s(A + t), \qquad t \in \mathbf{R},\ A \in \mathscr{B}_{\mathbf{R}}.$$

The stationary of the Poisson point process implies that μ is invariant under ϕ_t. Fix $\theta \in (0, 2\pi)$ and define

$$f_t(s) := e^{i\theta\sigma_t(s)}, \qquad t \in \mathbf{R},$$

where

$$\sigma_t(s) := \begin{cases} s([0, t)), & \text{if } t > 0, \\ 0, & \text{if } t = 0, \\ -s([0, t)), & \text{if } t < 0. \end{cases}$$

Then

$$f_{t_1+t_2}(s) = f_{t_1}(s) f_{t_2}(\phi_{t_1}(s)) \quad \text{for all } t_1, t_2 \in \mathbf{R}, \ s \in S,$$

and again, $\{f_t\}_{t\in\mathbf{R}}$ is a special case of (1.8). It is obvious that $\mu(S_D) = 0$. Consider $s \in S_H$. By (3.2) we have, for almost all $t_1, t_2 > 0$,

$$s([0, t_1)) - s([t_2, t_1 + t_2)) \in 2\pi\theta^{-1}\mathbf{Z}, \tag{3.7}$$

which is only possible when $\pi\theta^{-1}$ is rational. The left continuity of $t \to s([0, t))$ yields (3.7) for all $t_1, t_2 > 0$. Take $t_1 = 1$ and $t_2 = n \in \mathbf{N}$ in (3.7). We get that $s([n, n+1)) - s([0, 1)) \in 2\pi\theta^{-1}\mathbf{Z}$, so that

$$s([2k, 2k+1)) - s([2k+1, 2k+2)) \in 2\pi\theta^{-1}\mathbf{Z}, \qquad k = 0, 1, \dots . \tag{3.8}$$

Since the sets $A_k = \{s\colon s([2k, 2k+1)) - s([2k+1, 2k+2)) \in 2\pi\theta^{-1}\mathbf{Z}\}$, considered as events for the probability space (S, μ), are independent and have the same probability less than 1 (the set $2\pi\theta^{-1}\mathbf{Z}$ does not contain 1, for example), (3.8) may hold only for s from a μ-null set. Hence $\mu(S_H) = 0$, proving that

$$X_t := \int_S e^{i\theta\sigma_t(s)} M(ds), \qquad t \in \mathbf{R},$$

is a stationary SαS process which does not have harmonizable or moving average components. This is a real-valued stationary process when $\theta = \pi$ and M is real-valued.

The majority of our examples are given for $d = 1$. We will now outline a procedure for generating stationary random fields from stationary SαS processes. Suppose we have stationary SαS processes ($d = 1$) with a representation of the form (1.3). Clearly,

$$U^t h := c_t \left\{ \frac{d(\mu \circ \phi_t)}{d\mu} \right\}^{1/\alpha} h \circ \phi_t, \qquad h \in L^\alpha(S, \mu), \ t \in \mathbf{T}, \tag{3.9}$$

is a one-parameter group of isometries on $L^\alpha(S, \mu)$. Suppose now that we have d such groups $\{U_1^t\}_{t\in\mathbf{T}}, \dots, \{U_d^t\}_{t\in\mathbf{T}}$, all acting on $L^\alpha(S, \mu)$ and commuting with each other (they can be identical, for example). Define

$$f_t := U_1^{t_1} \cdots U_d^{t_d} f, \qquad t = (t_1, \dots, t_d) \in \mathbf{T}^d, \tag{3.10}$$

where $f \in L^\alpha(S, \mu)$ is fixed. Then $\{f_t\}_{t\in\mathbf{T}^d} \subset L^\alpha(S, \mu)$ is a representation of a stationary SαS random field. Conversely, any representation of a stationary random field can be obtained in this way. Indeed, a representation (1.3) determines a d-parameter group $\{U^t\}_{t\in\mathbf{T}^d}$ by (3.9). Let

$$U_k^t := U^{te_k}, \qquad t \in \mathbf{T},$$

where e_k is the kth element of the standard basis in $\mathbf{R}^d$. Then $\{U_1^t\}_{t\in\mathbf{T}}, \dots,$ $\{U_d^t\}_{t\in\mathbf{T}}$ are commuting one-parameter groups of isometries on $L^\alpha(S, \mu)$ and (3.10) coincides with (1.3).

THEOREM 3.7. *Let* $\mathbf{X}$ *be a stationary* SαS *random field given by a stochastic integral*

$$X_t = \int_S f_t \, dM, \qquad t \in \mathbf{T}^d.$$

Then $\mathbf{X}$ *can be decomposed into the sum of three mutually independent stationary* SαS *random fields*

$$\mathbf{X} = \mathbf{X}^1 + \mathbf{X}^2 + \mathbf{X}^3, \tag{3.11}$$

such that $\mathbf{X}^1$ *is a mixed moving average,* $\mathbf{X}^2$ *is harmonizable, and* $\mathbf{X}^3$ *does not admit harmonizable or mixed moving average components. The decomposition* (3.11) *is unique in the sense that if* $\widetilde{\mathbf{X}} = \widetilde{\mathbf{X}}^1 + \widetilde{\mathbf{X}}^2 + \widetilde{\mathbf{X}}^3$ *is a version of* $\mathbf{X}$, *where* $\widetilde{\mathbf{X}}^1, \widetilde{\mathbf{X}}^2$ *and* $\widetilde{\mathbf{X}}^3$ *are mutually independent mixed moving average, harmonizable, and a stationary random field without harmonizable or moving average components, respectively, then* $\widetilde{\mathbf{X}}^1 =^d \mathbf{X}^1$, $\widetilde{\mathbf{X}}^2 =^d \mathbf{X}^2$, *and* $\widetilde{\mathbf{X}}^3 =^d \mathbf{X}^3$.

PROOF. Define

$$X_t^1 := \int_{S_D} f_t \, dM,$$

$$X_t^2 := \int_{S_H} f_t \, dM$$

and

$$X_t^3 := \int_{S_3} f_t \, dM,$$

where $S_3 = S \setminus (S_D \cup S_H)$. In order to apply Theorems 2.1, 2.4 and 3.1 we need to verify that $\mathbf{X}^i$ are stationary $i = 1, 2, 3$. We will prove this and, at the same time, the uniqueness in (3.11).

Let $\{g_t\}_{t \in \mathbf{T}^d} \subset L^\alpha(W, \mathscr{B}_W, \nu)$ be a minimal representation of $\mathbf{X}$ so that (2.3) and (2.4) hold. Using the same arguments as in the first part of the proof of Theorem 3.1 we get

$$\mu(S_D \Delta \Psi^{-1} W_D) = 0$$

and

$$\mu(S_H \Delta \Psi^{-1} W_H) = 0.$$

Since $\mu_h \circ \Psi^{-1} = \nu$, where $\mu_h(ds) = |h(s)|^\alpha \mu(ds)$, we obtain for every $a_1, \ldots, a_n \in \mathbf{R}$ ($\mathbf{C}$, resp.) and $t_1, \ldots, t_n \in \mathbf{T}^d$,

$$\begin{aligned}\int_{S_D} \Big| \sum a_k f_{t_k} \Big|^\alpha d\mu &= \int_{S_D} \Big| \sum a_k g_{t_k} \circ \Psi \Big|^\alpha |h|^\alpha \, d\mu \\ &= \int_{\Psi^{-1} W_D} \Big| \sum a_k g_{t_k} \circ \Psi \Big|^\alpha d\mu_h \\ &= \int_{W_D} \Big| \sum a_k g_{t_k} \Big|^\alpha d\nu.\end{aligned}$$

Hence $\{g_{t|W_D}\}$ is a representation of $\mathbf{X}^1$. Similarly, we show that $\{g_{t|W_H}\}$ is a representation of $\mathbf{X}^2$. This shows the uniqueness of the decomposition (3.11) and also the stationarity of $\mathbf{X}^i$, because W_D and W_H are $\{\phi_t\}$-invariant. □

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