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THE CRITICAL PARAMETER FOR THE HEAT EQUATION WITH A NOISE TERM TO BLOW UP IN FINITE TIME[1]

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Consider the stochastic partial differential equation

$$u_t = u_{xx} + u^\gamma \dot{W},$$

where $x \in \mathbf{I} \equiv [0, J]$, $\dot{W} = \dot{W}(t, x)$ is 2-parameter white noise, and we assume that the initial function $u(0, x)$ is nonnegative and not identically 0. We impose Dirichlet boundary conditions on u in the interval $\mathbf{I}$. We say that u blows up in finite time, with positive probability, if there is a random time $T < \infty$ such that

$$P\Big(\lim_{t\uparrow T} \sup_x u(t, x) = \infty\Big) > 0.$$

It was known that if $\gamma < 3/2$, then with probability 1, u does not blow up in finite time. It was also known that there is a positive probability of finite time blowup for γ sufficiently large.

We show that if $\gamma > 3/2$, then there is a positive probability that u blows up in finite time.

1. Introduction. We consider the heat equation with a nonlinear additive noise term,

$$\begin{aligned} u_t &= u_{xx} + g(u)\dot{W}, \qquad t > 0, x \in \mathbf{I} \equiv [0, J],\\ u(t, 0) &= u(t, J) = 0, \\ u(0, x) &= u_0(x). \end{aligned} \tag{1.1}$$

Here, $\dot{W} = \dot{W}(t, x)$ is 2-parameter white noise, and $g(u)$ is a locally Lipschitz function on $[0, \infty)$ satisfying $g(0) = 0$. We assume that $u_0(x)$ is a continuous nonnegative function on $\mathbf{I}$, vanishing at the endpoints, but not identically zero. In this paper, we will mainly consider the case

$$g(u) = u^\gamma, \qquad \gamma \geq 1.$$

Suppose that we are working on a probability space $(\Omega, \mathscr{F}, P)$ and fix a point $\omega \in \Omega$. If there exists a random time $T = T(\omega) < \infty$ such that

$$\lim_{t\uparrow T} \sup_{x\in\mathbf{I}} u(t, x) = \infty$$

then we say that u blows up in finite time (for the point ω).

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For deterministic partial differential equations, there is a large literature about blowup in finite time. See [1–4, 6], for example. Suppose that we are dealing with the equation

$$\frac{\partial w(t,x)}{\partial t} = \Delta w(t,x) + g(w(t,x)),$$
$$w(0,x) = w_0(x).$$

One basic idea is the following. Suppose that $g(w)$ increases faster than linearly. If a high peak forms in the solution $w(t,x)$, then the term $g(w(t,x))$ will win out over the term $\Delta w(t,x)$, and the growth of the peak will be governed by the ordinary differential equation

$$w'(t) = g(w(t)).$$

We can solve this equation explicitly, and its solutions often blow up in finite time.

On the other hand, for stochastic partial differential equations (SPDE) there are very few papers about finite time blowup. Apart from the heat equation, the author [8] studied the wave equation

$$\begin{aligned}
\frac{\partial^2 u(t,x)}{\partial t^2} &= \Delta u(t,x) + g(u(t,x))\,\dot{W}(t,x), \qquad t > 0, x \in R,\\
\frac{\partial u_0(x)}{\partial} &= h_1(x),\\
u_0(x) &= h_0(x),
\end{aligned}$$

where $g(u)$ was allowed to grow only barely faster than linearly. For $g(u) = |u|^\alpha$ with $\alpha > 1$, one would guess that solutions would blow up in finite time. But finite time blowup is not known for any value of α.

There is more precise information about the heat equation with noise. Suppose that u is a solution to (1.1). In [7] it was shown that if $\gamma < 3/2$, then, with probability 1, u does not blow up in finite time. Krylov [5] gave another proof of this fact for a more general class of equations. The papers [8, 9] are also relevant. We refer the reader to [11] for this and other questions about parabolic SPDE. Returning to the question of blowup, it was shown in [10] that there exists $\gamma_0 > 1$ such that if $\gamma > \gamma_0$, then with positive probability, u blows up in finite time. The argument in [10] was not sharp enough to give the best value of γ_0, and the question of whether one could take $\gamma_0 = 3/2$ was left open. The main theorem of this paper answers this question in the affirmative.

THEOREM 1. *Let $u(t,x)$ satisfy* (1.1)*, and suppose that $g(u) = u^\gamma$ with $\gamma > 3/2$. Then, with positive probability, u blows up in finite time.*

Of course, Theorem 1 does not tell us what happens at $\gamma = 3/2$. Surprisingly, the proof of Theorem 1 uses many of the same ideas as in [10], although in a sharper form.

Now we discuss the rigorous meaning of (1.1), following the formalism of Walsh [12], Chapter 3. Before giving details, we set up some notation. Let $G(t, x, y)$ be the fundamental solution of the heat equation on $\mathbf{I}$, with Dirichlet boundary conditions. If $G(t, x)$ is written as a function of two variables, we let $G(t, x)$ be the fundamental solution of the heat equation on $\mathbf{R}$. In other words,

$$G(t, x) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right).$$

It is well known that

$$G(t, x, y) \leq G(t, x - y).$$

We regard (1.1) as shorthand for the following integral equation:

$$u(t, x) = \int_{\mathbf{I}} G(t, x, y) u_0(y)\, dy + \int_0^t \int_{\mathbf{I}} G(t - s, x, y)\, g(u(s, y))\, W(dy\, ds), \tag{1.2}$$

where the final term in (1.2) is a white noise integral in the sense of [12], Chapter 2. Because $g(u)$ is locally Lipschitz, standard arguments show that (1.1) has a unique solution $u(t, x)$ valid up to the time σ_L at which $|u(t, x)|$ first reaches the level L for some $x \in \mathbf{I}$. Similar arguments are given in [12], Theorem 3.2 and Corollary 3.4, and Walsh's reasoning easily carries over to our case. Letting $L \to \infty$, we find that (1.1) has a unique solution for $t < \sigma$, where $\sigma = \lim_{L\to\infty} \sigma_L$. If $\sigma < \infty$, one has

$$\lim_{t\uparrow\sigma} \sup_{x\in\mathbf{I}} |u(t, x)| = \infty.$$

Our goal is to show that $\sigma = \infty$ with probability 1.

Last, we will always work with the σ-fields $\mathscr{F}_t = \mathscr{F}_t^W$ generated by the white noise up to time t. That is, $\mathscr{F}_t$ is the σ-field generated by the random variables $\int_0^t \int_{\mathbf{I}} \phi(s, x)\, W(dx\, ds)$, where ϕ varies over all continuous functions on $[0, t]\times\mathbf{I}$.

We now summarize the argument in [10], which is based on the analysis of the formation of high peaks. Such peaks will occur with positive probability. We wish to show that, with positive probability, such peaks grow until they blow up in finite time. If a high peak forms, we rescale the equation and divide the mass of the peak into a collection of peaks of smaller mass, and these peaks evolve almost independently. In this way we compare the evolution of u to a branching process. Large peaks are regarded as particles in this branching process. Offspring are peaks which are higher by some factor. We show that the expected number of offspring is greater than one when $\gamma > 3/2$, and thus the branching process survives with positive probability, corresponding to blowup in finite time.

Finally, we remark that we could replace our assumption that $g(u) = u^\gamma$ with the assumption that $g(u) > cu^\gamma$ for some $c > 0$ and that $g(0) = 0$. Then Theorem 1 would still hold, provided $\gamma > 3/2$.

2. Proof of Theorem 1. From now on, unless noted otherwise, we assume that $u(t, x)$ is a solution of (1.1) with

$$g(u) \geq u^{\gamma}.$$

We give a proof by contradiction. Assume that

$$P(\sigma < \infty) = 0. \tag{2.1}$$

For $x, y \geq 0$, define

$$b(x, y) \equiv \sqrt{(x+y)^{2\gamma} - y^{2\gamma}}. \tag{2.2}$$

If we are discussing different white noises, then $\mathscr{F}_t^W$ denotes the σ-field generated by $\dot{W}$.

Here is Lemma 2.4 of [10]. We have made a few changes in notation, to achieve consistency with the current article.

LEMMA 1. *Suppose that u solves* (1.1) *up to some $\mathscr{F}_t^W$ stopping time τ. Suppose that*

$$g(u) = b(u, \xi)$$

for some nonegative, adapted processes ξ. Let $L_0 > 0$. If we let

$$\tilde{v}(t, x) \equiv L_0^{-1} u\left(tL_0^{4(1-\gamma)}, xL_0^{2(1-\gamma)}\right), \qquad t \geq 0, x \in \mathbf{I}L_0^{2(\gamma-1)}$$

then there exists a nonnegative, $\mathscr{F}_t^{\widetilde{W}}$-adapted process $\tilde{\xi}$ such that $\tilde{v}(t, x)$ solves

$$\begin{aligned}
\frac{\partial \tilde{v}}{\partial t} &= \frac{\partial^2 \tilde{v}}{\partial x^2} + b(\tilde{v}, \tilde{\xi})\dot{\widetilde{W}}, \\
\tilde{v}(t, 0) &= \tilde{v}\big(t, JL_0^{2(\gamma-1)}\big) = 0, \\
\tilde{v}(0, \cdot) &= \tilde{v}_0, \qquad t \geq 0, 0 \leq x \leq JL_0^{2(\gamma-1)}
\end{aligned}$$

up to the $\mathscr{F}_t^{\widetilde{W}}$-stopping time $\tau L_0^{2(1-\gamma)}$, where $\tilde{v}_0(x) \equiv L_0^{-1} u_0\big(xL_0^{2(1-\gamma)}\big)$ for all $0 \leq x \leq JL_0^{2(\gamma-1)}$, for all $t \geq 0$ and $0 \leq x \leq JL_0^{2(\gamma-1)}$, and $\widetilde{W}$ is the white noise on $\mathscr{B}(\mathbf{R}_+ \times [0, JL_0^{2(\gamma-1)}])$ defined by

$$\widetilde{W}(A) \equiv L_0^{3(\gamma-1)} \int_{\mathbf{R}_+ \times \mathbf{I}} \chi_A\left(tL_0^{4(\gamma-1)}, xL_0^{2(\gamma-1)}\right) W(dt, dx)$$

for all A in $\mathscr{B}(\mathbf{R}_+ \times [0, JL_0^{2(\gamma-1)}])$ with finite Lebesgue measure.

In [10] we fixed L_0, which we called L, and took γ to be very large. In the current proof we wish to deal with all $\gamma > 3/2$, so we take L_0 as our large parameter. We will find that the probability of a peak getting up to level L_0 is about $p = 1/L_0$, from the gambler's ruin problem. Using Lemma 1, we will see that after rescaling, a peak of size L_0 gives rise to $N = L_0^{2(\gamma-1)}$ offspring. Thus, the expected number of offspring of our initial peak should be

$$pN = (1/L_0) \cdot L_0^{2(\gamma-1)} = L_0^{2\gamma-3}.$$

If $\gamma > 3/2$, then $2\gamma-3 > 0$ and $pN \to \infty$ as $L_0 \to \infty$. Thus, for large L_0, the branching process of high peaks has expected offspring size greater than 1. Therefore, this branching process survives with positive probability, and as we shall see, this fact implies that $u(t, x)$ blows up in finite time, with positive probability. Of course, the above heuristic calculation will suffer from the rough estimates we make during the course of the proof. Our hope is that taking pN large enough will compensate for all of our sloppiness.

Let $\varphi(t, x) = \varphi^{(T)}(t, x)$ be a solution of the backward heat equation

$$\varphi_t = -\varphi_{xx}, \qquad 0 \le t \le T, x \in \mathbf{R} \tag{2.3}$$

with "final condition,"

$$\varphi(T, x) = \frac{1}{\sqrt{4\pi T}} \exp\left(-\frac{x^2}{4T}\right).$$

Of course, $\varphi(T, x)$ is the heat kernel evaluated at time T, and therefore, for $0 \le t \le T$,

$$\varphi(t, x) = \frac{1}{\sqrt{4\pi(2T-t)}} \exp\left(-\frac{x^2}{4(2T-t)}\right).$$

For future use, we compute the $\mathbf{L}^1$ norm of $\varphi(t, x)^a$, for $a > 0$. We claim that there exists a constant $C = C(a) > 0$, not depending on T, such that for $0 \le t \le T$,

$$\begin{aligned}
\|\varphi(t, x)^a\|_1 &= \int_{\mathbf{I}} \varphi(t, x)^a \, dx \\
&= (4\pi(2T-t))^{-a/2} \int_{\mathbf{I}} \exp\left(-\frac{ax^2}{4(2T-t)}\right) dx \\
&= C'(a)(2T-t)^{(1-a)/2} \\
&\le C(a) T^{(1-a)/2}.
\end{aligned} \tag{2.4}$$

Recall that σ is the blowup time for u, and let

$$M(t) = \int_{\mathbf{I}} \varphi(t, x)\, u(t, x)\, dx, \qquad 0 \le t \le T \wedge \sigma$$

and note that $M(t)$ is a continuous local $\mathscr{F}_t$ martingale for $0 \le t \le T \wedge \sigma$. This assertion was proved in Lemma 2.3 of [10]. One can also check it heuristically, by formally differentiating $M(t)$ and applying (1.1) and (2.3). Lemma 2.3 of [10] also states that for $0 \le t \le T \wedge \sigma$, $M(t)$ has quadratic variation

$$\langle M \rangle_t = \int_0^t \int_{\mathbf{I}} g(u(s, x))^2 \, \varphi(s, x)^2 \, dx \, ds \ge \int_0^t \int_{\mathbf{I}} u(s, x)^{2\gamma} \, \varphi(s, x)^2 \, dx \, ds. \tag{2.5}$$

Of course, $M(t) = M^T(t)$ implicitly depends on T. We now prove the following lower bound on $\langle M \rangle_t$.

LEMMA 2. *There exists a constant $C_1 > 0$, not depending on T, such that if $0 \le t \le T \wedge \sigma$, then*

$$\langle M\rangle_t \ge C_1 T^{-1/2} \int_0^t M(s)^{2\gamma}\, ds.$$

PROOF. Let

$$a = \frac{2\gamma - 2}{2\gamma - 1}. \tag{2.6}$$

Note that

$$\frac{2-a}{2\gamma} + a = 1 \tag{2.7}$$

and

$$\frac{1-a}{2} \cdot (1 - 2\gamma) = -\frac{1}{2}. \tag{2.8}$$

Furthermore, for t fixed,

$$\frac{\varphi(t, x)^a}{\|\varphi(t, x)^a\|_1}$$

is a probability density over $x \in \mathbf{R}$. Using Jensen's inequality, (2.7), (2.4) and (2.8), we find that

$$\begin{aligned}
&\int_{\mathbf{I}} u(s, x)^{2\gamma} \varphi(s, x)^2\, dx \\
&\quad = \|\varphi(s, x)^a\|_1 \int_{\mathbf{I}} u(s, x)^{2\gamma} \varphi(s, x)^{2-a} \frac{\varphi(s, x)^a}{\|\varphi(s, x)^a\|_1}\, dx \\
&\quad \ge \|\varphi(s, x)^a\|_1 \left(\int_{\mathbf{I}} u(s, x) \varphi(s, x)^{(2-a)/(2\gamma)} \frac{\varphi(s, x)^a}{\|\varphi(s, x)^a\|_1}\, dx \right)^{2\gamma} \\
&\quad = \|\varphi(s, x)^a\|_1^{1-2\gamma} \left(\int_{\mathbf{I}} u(s, x) \varphi(s, x)\, dx \right)^{2\gamma} \\
&\quad \ge \left(C(a) T^{(1-a)/2} \right)^{1-2\gamma} M(s)^{2\gamma} \\
&\quad = C_1 T^{-1/2} M(s)^{2\gamma}.
\end{aligned} \tag{2.9}$$

where $C_1 = C(a)^{1-2\gamma}$, and a was defined in (2.6). After integrating (2.9) over $s \in [0, t]$ and putting this together with (2.5), we get Lemma 2. □

Using Lemma 2, it is possible to compare $M(t)$ to a time-changed Brownian motion. In the standard way, the new time scale is given by $\langle M\rangle_t$. Let

$$T(L) = 16 C_1^{-2} L^8$$

and consider the following gambler's ruin problem. Start with $M(0) = 2$. Let $\tau = \tau(L)$ be the first time t that $M(t) = 1$ or $M(t) = L$. Suppose for a moment that $M(t)$ were defined for $t \ge 0$ rather than for $0 \le t \le T$. Then, using the optional sampling theorem in the usual way, we could deduce that $EM(\tau) = 2$, and therefore [if $M(0) = 2$],

$$P(M(\tau) = L) = \frac{1}{L-1}. \tag{2.10}$$

In fact, we wish to show the following.

LEMMA 3. *Suppose* $\gamma > 1$. *If* $T = T(L) = 16C_1^{-2}L^8$, *then*

$$P(M(\tau \wedge T) = L) \geq \frac{1}{2(L-1)}.$$

PROOF. Recall that σ is the blowup time for $u(t, \cdot)$. First, note that $\tau < \sigma$. The definition of τ implies that for all $t \in [0, \tau]$, we have $M(t) \geq 1$. Therefore, by Lemma 2, if $t \in [0, \tau]$ then

$$\langle M \rangle_t \geq C_1 T^{-1/2} t.$$

Now $M(t)$ is a continuous supermartingale, so it follows that $M(t)$ is greater than or equal to a time-changed Brownian motion with time scale $\langle M \rangle_t$. In other words, for some Brownian motion $B(t)$, we have $M(t) \geq 2 + B(\langle M \rangle_t)$. Therefore, since

$$\langle M \rangle_T \geq CT^{-1/2}T = CT^{1/2},$$

we have

$$\begin{aligned}
P(T < \tau) &= P(T < \tau \leq \sigma,\ 1 < M(t) < L \text{ for } t \in [0, T]) \\
&\leq P(T < \tau,\ 1 < 2 + B(\langle M \rangle_t) < L \text{ for } 0 \leq t \leq T) \\
&= P(T < \tau,\ 1 < 2 + B(t) < L \text{ for } 0 \leq t \leq \langle M \rangle_T) \\
&\leq P(T < \tau,\ 1 < 2 + B(t) < L \text{ for } 0 \leq t \leq C_1 T^{1/2}) \\
&\leq P\Bigl(\sup_{t \in [0, C_1 T^{1/2}]} B(t) < L - 2\Bigr).
\end{aligned}$$

Using the reflection principle, we continue with

$$\begin{aligned}
P(T < \tau) &\leq 1 - P\Bigl(\sup_{t \in [0, C_1 T^{1/2}]} B(t) \geq L - 2\Bigr) \\
&= 1 - 2P\bigl(B(C_1 T^{1/2}) \geq L - 2\bigr) \\
&= P\bigl(|B(C_1 T^{1/2})| \leq L - 2\bigr) \\
&= \int_{-(L-2)}^{L-2} (2\pi C_1 T^{1/2})^{-1/2} \exp\left(-\frac{x^2}{2C_1 T^{1/2}}\right) dx \\
&\leq 2(L-2)(2\pi C_1 T^{1/2})^{-1/2} \\
&\leq C_1^{-1/2} L T^{-1/4}.
\end{aligned}$$

Therefore, if

$$T = T(L) = 16C_1^{-2}L^8$$

then

$$P(T < \tau) < \frac{1}{2(L-1)}.$$

Then, by (2.10),

$$\begin{aligned} P(M(\tau \wedge T) = L) &\geq P(M(\tau \wedge T) = L, T \geq \tau) \\ &= P(M(\tau) = L, T \geq \tau) \\ &= P(M(\tau) = L) - P(M(\tau) = L,\ T < \tau) \\ &\geq P(M(\tau) = L) - P(T < \tau) \\ &\geq \frac{1}{L-1} - \frac{1}{2(L-1)} \\ &= \frac{1}{2(L-1)}. \end{aligned}$$

This proves Lemma 3. □

Now we continue with the proof of Theorem 1. Recall that at the beginning of the current section, we defined $N = L_0^{2(\gamma-1)}$ and gave an informal definition of p. Let

$$p \equiv P(M(\tau \vee T) = L).$$

Lemma 3 implies that

$$p \geq \frac{1}{2(L-1)} \geq (2L)^{-1}.$$

Using Lemma 1, with $L_0 = L/2$, we deduce that

$$N \geq K^{-1}(L/2)^{2(\gamma-1)} - 1 \geq K^{-1}(L/4)^{2(1-\gamma)}$$

if L is large enough. Thus we find that

$$pN \geq K^{-1}(4L)^{-1}(L/4)^{2(\gamma-1)} = K^{-1}4^{1-2\gamma}L^{2(\gamma-3/2)} > 1 \tag{2.11}$$

if $\gamma > 3/2$ and L is large enough.

To complete the proof, we can apply the same argument as in [10], Sections 3 and 4. Since these arguments carry over, word for word, we will merely summarize the argument here and refer the reader to [10] for details.

First, we need to split up the solution u. For this, we quote Lemma 2.5 of [10]. Recall that we defined $b(x, y)$ in (2.2).

LEMMA 4. *For $t \geq 0, x \in \mathbf{I}, i = 1, 2, \ldots, N$ consider the N recursively defined equations*

$$\begin{aligned}\frac{\partial u^i}{\partial t} &= \frac{\partial^2 u^i}{\partial x^2} + b\left(u^i, \sum_{j=1}^{i-1} u^j\right)\dot{W}^i,\\ u^i(t,0) &= u^i(t,J) = 0, \\ u^i(0,\cdot) &= u_0^i,\end{aligned} \tag{2.12}$$

where $u^0 \equiv 0$ by definition. Here the $\{W^i\}$'s are independent white noises and the u_0^i are some collection of nonnegative initial functions. Let us then define the process

$$\tilde{u}(t,\cdot) \equiv \begin{cases} \sum_{i=1}^{N} u^i(t,\cdot), & \text{for } 0 \leq t < \min\{\sigma(u^i) : i = 1, 2, \ldots, N\}, \\ \infty, & \text{otherwise}\end{cases}$$

for all $t \geq 0$. Here, $\sigma(u^i)$ denotes the blowup time σ with respect to u^i. For $t \geq 0,\ x \in \mathbf{I}$, we have that $\tilde{u}$ is a solution of

$$\begin{aligned}\frac{\partial \tilde{u}}{\partial t} &= \frac{\partial^2 \tilde{u}}{\partial x^2} + \tilde{u}^{\gamma}\dot{\widetilde{W}},\\ \tilde{u}(t,0) &= \tilde{u}(t,J) = 0,\\ \tilde{u}(0,\cdot) &= \sum_{i=1}^{N} u_0^i.\end{aligned}$$

for some white noise $\widetilde{W}$ which is a linear combination of the $\{W^i\}$.

We use Lemma 4 to split up the solution u into the sum of solutions u^i. Later, we will further split up the u^i. Section 4 of [10] explains how to use Lemma 4 to split up u over and over again, at a sequence of stopping times. Each of these smaller solutions will have a larger noise term than in (1.1), so the corresponding total mass martingales $U^i(t) = \int_{\mathbf{I}} u^i(t,x)dx$ will have

$$\langle U^i \rangle_t \geq \int_0^t U^i(s)^{2\gamma} ds.$$

We need a way to split up u, given that a certain integral is sufficiently large. The following lemma is an easy modification of Proposition 3.2 of [10].

LEMMA 5. *Let*

$$\phi(t,x;y,J_1) \equiv G(2-t,x,y;J_1), \qquad 0 \leq t \leq 2,\ x \in [0,J_1],$$

where $G(2-t,x,y;J_1)$ is the Dirichlet heat kernel on the interval $[0,J_1]$ instead of $[0,J]$. Let $E_+^{\infty}(J_1)$ denote the class of nonnegative $\mathbf{C}^{\infty}$ functions on $[0,J_1]$. There exists a constant $K > 0$ such that the following holds.

Let $J > 4$ be fixed. Set $J_1 \equiv J2^{2(\gamma-1)}$. If $N > 0$ is and integer, and $f_0 \in E_+^\infty(J_1)$ satisfies

$$\int_0^{J_1} \phi(t, x2^{2(1-\gamma)}; z_0, J_1) f_0(x)\, dx > KN,$$

for some z_0 in $[1, J-1]$ and some $0 \le t \le 1$, then there are functions $\{f_i: i = 1, 2, \dots, N\} \subset E_+^\infty(J_1)$ such that

$$f_0 = \sum_{i=1}^N f_i$$

and for each $i = 1, 2, \dots, N$,

$$\int_0^{J_1} \phi(0, x; z_i, J_1) f_i(x)\, dx \ge 2. \tag{2.13}$$

for some z_i in $[1, J_1 - 1]$.

In [10], Lemma 5 was shown for $N = [2^{2\gamma-3}]$, but the proof given there also implies the above result.

Now we continue the main argument. We can assume without loss of generality that

$$\int_{\mathbf{I}} G(T, x, y) u(0, y)\, dy \ge 2.$$

Suppose that this condition fails, and that $u(t, x)$ has not blown up by time 1. Then, with positive probability,

$$\int_{\mathbf{I}} G(T, x, y) u(1, y)\, dy \ge 2.$$

Now wait until time T. By (2.10), we have that

$$P\left(\int_{\mathbf{I}} G(2T, x, y) u(0, y)\, dy \ge L \right) \ge \frac{1}{2(L-1)} = p. \tag{2.14}$$

Let

$$N = K^{-1} L^{2(\gamma-1)}. \tag{2.15}$$

Now perform the scaling as in Lemma 1, with $L_0 = L/2$. For the scaled function $\tilde{v}$, we see that

$$\int_{\mathbf{I}_1} G(2T, x, y; J_1) \tilde{v}(0, y)\, dy \ge L^{2(\gamma-1)} = KN.$$

where $G(2T, x, y; J_1)$ is the Dirichlet heat kernel on the rescaled interval $\mathbf{I}_1 = [0, J_1]$. Then, Lemma 5 shows that we can decompose

$$u(t, x) = \sum_{i=1}^N f_i(x)$$

such that for some set of points $\{z_i\}_{i-1}^N$,

$$\int_{\mathbf{I}_1} G(2T, x, y) f_i(y)\, dy \geq 2.$$

We use these f_i as initial conditions for new functions $u^i(t, x)$, which satisfy (2.12), and we call these $u^i(t, x)$ offspring of $u(t, x)$.

If

$$\int_{\mathbf{I}_1} G(2T, x, y) u(0, y)\, dy < L,$$

then we say that mass has died.

Repeating the argument, we find that there is mass alive at stage k if the branching process of the u's is alive at stage k. But this is a Galton–Watson process with expected number of offspring at least

$$pN = K^{-1} L^{2(\gamma-1)} \frac{1}{2(L-1)} \geq 2^{-1} K^{-1} L^{2(\gamma-3/2)}$$

by (2.14) and (2.15). Therefore, if $\gamma > 3/2$ and L is large enough, the expected number of offspring is at least

$$pN > 1$$

and there is a positive probability of survival. But survival means that there is mass present at each stage. This, in turn, means that $u(t, x)$ blows up in finite time. Indeed, it is easy to see that the times at which the stages begin have a finite accumulation point. Therefore, there is a positive probability of finite time blowup.

But this conclusion contradicts our assumption (2.1) that $P(\sigma < \infty) = 0$. Thus, Theorem 1 is proved. □

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