

UNIQUENESS FOR A CLASS OF ONE-DIMENSIONAL STOCHASTIC PDEs USING MOMENT DUALITY

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We establish a duality relation for the moments of bounded solutions to a class of one-dimensional parabolic stochastic partial differential equations. The equations are driven by multiplicative space-time white noise, with a non-Lipschitz multiplicative functional. The dual process is a system of branching Brownian particles. The same method can be applied to show uniqueness in law for a class of non-Lipschitz finite dimensional stochastic differential equations.

1. Introduction and statement of results. We consider solutions to the one-dimensional stochastic partial differential equation (SPDE)

$$(1.1) \quad \partial_t u = \frac{1}{2} \Delta u + b(u) + \sqrt{\sigma(u)} \dot{W}_{t,x}, \quad u_0 = \phi,$$

where $\dot{W}_{t,x}$ is space-time white noise. We study bounded solutions $u_t(x)$ that are continuous in t and x . When b and σ are analytic functions and the coefficients of their power series satisfy certain technical conditions, we establish a duality relation (Section 1.5) for the moments $E(u_t(x_1) \cdots u_t(x_N))$. This duality relation implies uniqueness in law for the solutions.

1.1. Motivation. Our interest in such equations is motivated by two key examples:

EXAMPLE 1 [$\mathbf{b}(\mathbf{u}) = \mathbf{b}\mathbf{u}$, $\sigma(\mathbf{u}) = \sigma\mathbf{u}^\gamma$]. When $b = 0$ and $\gamma = 1$, a solution of this SPDE is given by the density of one-dimensional super-Brownian motion [12, 6]. The solution can be obtained as a high density limit of a branching Brownian particle system [3]. The particles in the approximating system independently die at a constant rate and are replaced by a random number of offspring with mean one and variance σ . The noise arises from fluctuations in particle density due to the branching. When $b \neq 0$, a similar particle system approximation holds with the offspring mean suitably approaching one in the limit. One expects, for various $\sigma(u)$ and $b(u)$, a similar particle approximation holds for the solutions with the offspring mean and variance depending on local particle density. This has been shown in the case $b = 0$ and for $\gamma \in (1, 2)$ [7]. An exponential duality method can be used to show uniqueness in law for the above examples [9].

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EXAMPLE 2 [$\mathbf{b}(\mathbf{u}) = \mathbf{0}$, $\sigma(\mathbf{u}) = \mathbf{u}(\mathbf{1} - \mathbf{u})$]. This is the stepping stone model of populations genetics [13]. A formula for the moments $E(u_t(x_1) \cdots u_t(x_N))$ exists in terms of a dual system of coalescing Brownian motions. Since the solutions take values in the interval $[0, 1]$, these moments characterize the law.

Perkins [11] studies classes of interactive branching processes in \mathbb{R}^d and their high density limits. One expects, when $d = 1$, these processes to solve stochastic PDEs. The interactions have smoothness conditions that do not cover the zero-range interactions represented by equation (1.1).

An account of existence and properties of solutions to (1.1) can be found in [16, 10, 14]. Uniqueness for solution to (1.1) is important for justifying the equation as a viable model and is a useful step in showing the approximating particle system converges. When $\sqrt{\sigma(u)}$ is Lipschitz, pathwise uniqueness holds [16]. The above key examples motivate the importance of establishing uniqueness for the non-Lipschitz case. There is no known proof of pathwise uniqueness for this case. Uniqueness in law has been shown only in special cases (the above examples) using the method of duality. However, the proofs break down if small changes are made to the functions $b(u)$ and $\sigma(u)$.

1.2. *Objective.* The aim of this paper is to show that the duality method for finding moment formulae can be extended to a class of analytic b and σ that is closed under small analytic perturbations. We suppose that the functions b and σ have power series representations

$$b(u) = \sum_{k=0}^{\infty} b_k u^k, \quad \sigma(u) = \sum_{k=0}^{\infty} \sigma_k u^k.$$

The dual process will be a system of branching Brownian particles where the branching probabilities are given in terms of the coefficients of the power series for b and σ . These processes are rather complicated making the moment formulae hard to use directly, although for the stepping stone model they were exploited successfully by Mueller and Tribe [8] to investigate random interface solutions. Our main interest is that, in many cases, the moments $E(u_t(x_1) \cdots u_t(x_N))$ determine the law of u_t (as a random element of the space of continuous functions). It is then possible to recast the SPDE as a martingale problem and apply the well known result ([4], Theorem 4.4.2) that uniqueness of the one dimensional distributions implies the uniqueness for the martingale problem, which in turn can be shown to imply uniqueness in law for solutions. To check whether the moments do characterize the one-dimensional distributions we may try to check the Carleman condition. The following lemma shows that moments do suffice if the coefficients b and σ do not grow too rapidly.

LEMMA 1. *Suppose (u_t) is a solution to (1.1) with bounded initial condition. Suppose $|b(u)| \leq C(1+|u|^s)$, for some $s < 1$ and $|\sigma(u)| \leq C(1+|u|^r)$.*

1. *If $r < 1$, then the distribution of u_t is uniquely determined by its moments.*
2. *If u_t is bounded below (or above) and $r < 3/2$, then the distribution of u_t is uniquely determined by its moments.*

The proof of this lemma is based on estimates using the best constant in the Burkholder-Davis-Gundy inequality followed by a standard Gronwall argument as in Lemma 3.3 [16]. We do not include the proof as the main result of this paper only applies to bounded solutions to (1.1).

1.3. *Dual particle system.* Our dual process will be a finite system of one-dimensional Brownian particles with an interactive branching mechanism. For a detailed construction (which is not central to this paper) of the dual process we refer the reader to [1]. The particles are labeled by an index set I . The position of a particle with label $\gamma \in I$ at time t is denoted X_t^γ . I_t is the set of labels of particles that are alive at time t . At time zero there are N particles positioned at x_1, \dots, x_N . During their lifetime the particles perform independent Brownian motions. There are two independent types of births:

1. *Single-particle births.* Each particle independently dies at rate μdt and, on dying, is replaced by a random number Z_1 of offspring. The generating function for the offspring law is $E(s^{Z_1}) = \sum_{k \geq 0} q_k s^k$.
2. *Two-particle births.* Two particles X_t^β and X_t^γ simultaneously die (independent of other pairs) at the rate $\nu dL_t^{\beta,\gamma}$, where $L_t^{\beta,\gamma}$ denotes the local time of the process $X_t^\beta - X_t^\gamma$ at zero. At the moment of death the pair of particles is replaced by a random number Z_2 of offspring. The generating function for the offspring law is $E(s^{Z_2}) = \sum_{k \geq 0} p_k s^k$.

The offspring start at their place of birth and independently follow the above rules. The number of offspring is independent for each birth and independent of all other variables. For convenience we shall take a left continuous version of the process, in that at a death time τ the offspring emerge only for $t > \tau$. Finally we set

$$L_t = \frac{1}{2} \int_0^t \sum_{\beta \in I_s} \sum_{\gamma \in I_s \setminus \{\beta\}} dL_s^{\beta,\gamma},$$

the total local time accrued by all pairs of particles before time t .

The quadratic nature of the local time branching leaves the possibility that the dual system might obtain infinitely many particles in finite time. We impose a suitable non-explosion hypothesis [(H1) in Theorem 1] to prevent this.

1.4. *A formal duality relation.* The duality technique is described in Ethier and Kurtz [4], Section 4.9. We present the basic details of the required form keeping our SPDE in mind. Consider a function $H(u, X)$, where u is the solution to (1.1) and $X = x = (x_1, \dots, x_n)$ is the dual particle system. Suppose the following duality identity holds,

$$L^u H(u, x) = L^X H(u, x) + \alpha(x)H(u, x) \quad \text{for all } u, x,$$

where L^u (the generator of u) acts on the u variable and L^X (the generator of X) acts on the x variable. Then, under suitable integrability and regularity assumptions, one is able to show that

$$E(H(u_t, X_0)) = E(H(u_0, X_t) \exp\left(\int_0^t \alpha(X_s) ds\right)) \quad \text{for all } t \geq 0.$$

The function $H(u, x)$ is called the duality function. The duality function we use is based on the moment duality function $(f, \{x_1, \dots, x_n\}) \rightarrow \prod_{k=1}^n f(x_k)$.

For simplicity we consider the case $b(u) = 0$, in which case the dual process has no single particle births. Fix x_1, \dots, x_m . Writing $\stackrel{mg}{\equiv}$ for equality up to a martingale increment, we may formally apply Itô's formula to obtain

$$(1.2) \quad \begin{aligned} d \prod_i u_t(x_i) &\stackrel{mg}{\equiv} \sum_i \left(\prod_{j:j \neq i} u_t(x_j) \right) \frac{1}{2} \Delta u_t(x_i) dt \\ &\quad + \frac{1}{2} \sum_{i,j:j \neq i} \left(\prod_{l:l \neq i,j} u_t(x_l) \right) \delta_{\{x_i=x_j\}} \sum_{k=0}^{\infty} \sigma_k u_t^k(x_i) dt. \end{aligned}$$

We have written a delta function for the correlation of the space-time white noise. Also solutions are too rough for the derivative Δu_s to exist. We treat both these problems rigorously in the next section.

To match this generator (1.2) with that of the dual we shall include a crucial factor that counts the number of births having offspring numbers taking values in a set $S \subseteq \{0, 1, 2, \dots\}$ that will be chosen soon. Let K_t be the total number of two particle births by time t that involved offspring whose cardinality Z_2 lies in S . Then we have, for a fixed function $h(x)$,

$$(1.3) \quad \begin{aligned} &d(-1)^{K_t} \prod_{\beta \in I_t} h(X_t^\beta) \\ &\stackrel{mg}{\equiv} (-1)^{K_t} \sum_{\beta \in I_t} \left(\prod_{\gamma \in I_t \setminus \{\beta\}} h(X_t^\gamma) \right) \frac{1}{2} \Delta h(X_t^\beta) dt \\ &\quad + \frac{\nu}{2} (-1)^{K_t} \sum_{\beta, \gamma \in I_t; \beta \neq \gamma} \left(\prod_{\alpha \in I_t \setminus \{\beta, \gamma\}} h(X_t^\alpha) \right) \\ &\quad \quad \times \sum_{k=0}^{\infty} p_k \left((-1)^{I_{\{k \in S\}}} h^k(X_t^\beta) - h(X_t^\beta) h(X_t^\gamma) \right) dL_t^{\beta, \gamma} \\ &= (-1)^{K_t} \sum_{\beta \in I_t} \left(\prod_{\gamma \in I_t \setminus \{\beta\}} h(X_t^\gamma) \right) \frac{1}{2} \Delta h(X_t^\beta) dt \\ &\quad + \frac{\nu}{2} (-1)^{K_t} \sum_{\beta, \gamma \in I_t; \beta \neq \gamma} \left(\prod_{\alpha \in I_t \setminus \{\beta, \gamma\}} h(X_t^\alpha) \right) \\ &\quad \quad \times \sum_{k=0}^{\infty} p_k (-1)^{I_{\{k \in S\}}} h^k(X_t^\beta) dL_t^{\beta, \gamma} \\ &\quad - \nu (-1)^{K_t} \left(\prod_{\alpha \in I_t} h(X_t^\alpha) \right) dL_t. \end{aligned}$$

We follow the convention that $\prod_{\gamma \in I_t} \phi(X_t^\gamma) = 1$ whenever $I_t = \emptyset$. We can formally write the term $dL_t^{\beta, \gamma}$ as $2\delta_{X_t^\beta = X_t^\gamma} dt$. Then the expressions (1.2) and

(1.3) will satisfy the duality identity provided we set

$$\nu = \sum_k |\sigma_k|/2, \quad p_k = |\sigma_k|/2\nu, \quad S = \{k : \sigma_k < 0\}$$

and $\alpha(X_t)dt = (-1)^{K_t}\nu dL_t$. This suggests a duality relation

$$(1.4) \quad E\left(\prod_{i=1}^N u_t(x_i)\right) = E\left(\prod_{\gamma \in I_t} \phi(X_t^\gamma)(-1)^{K_t} \exp(\nu L_t)\right).$$

The idea of using the alternating term $(-1)^{K_t}$ is similar to that of annihilating duality used in interacting particle systems (see [5]). It does not seem to have been exploited for stochastic PDEs.

The integrability needed to establish (1.4) does not typically hold. Suppose for instance, the power series for $\sigma(u)$ converges for all u so that $\rho(u) = \sum_{k \geq 0} |\sigma_k|u^k$ is well defined. Replacing σ by ρ in (1.1) leads to an equation that has the same formal duality relation as (1.4) but without the alternating term $(-1)^{K_t}$. However the function $\sqrt{\rho(u)}$ has superlinear growth (except if $\sigma_k = 0$ for $k \geq 3$) and one would expect the moments of u to be infinite. This suggests that the exponential $\exp(\nu L_t)$ in (1.4) has infinite expectation and the integrability conditions needed to apply the duality theorem in Ethier and Kurtz may not hold. In cases where the moments are known to be finite, for example when σ has linear growth, it leaves the possibility that the right hand side of (1.4) might be interpreted as a limit of truncated expectations. This paper does not pursue this possibility but rather exploits a trick, described below, that allows a restricted class of functions σ to be treated.

The coefficients σ_2 and p_2 play a special role in the generator equations (1.2) and (1.3). The terms corresponding to σ_2 and p_2 are multiples of the duality function and can be included into the exponential factor $\alpha(X)$. In the dual process this has the effect of not allowing any births that have two offspring and changing the overall rate of births to $\nu' dL_t$, where $\nu' = \sum_{k \neq 2} |\sigma_k|/2$. Since two-particle births with two offsprings cannot be observed from viewing the particle positions alone, this leaves the dual process essentially unchanged. However if $\sigma_2 < 0$ the process K_t is changed as it no longer increases for births with two offspring. The exponential $\exp(\nu L_t)$ also changes to $\exp((\nu' + (\sigma_2/2))L_t)$. This leads to a slightly different duality relation where the exponential moment of L_t is reduced and the alternating term $(-1)^{K_t}$ is adjusted. We shall show that this small change solves the integrability problems for a class of functions σ , essentially where σ_2 is sufficiently negative. If $\sigma_2 > 0$ the formal duality relation is unchanged.

When $b(u)$ is non-zero we must include single particle branching where the probabilities of various size offspring are determined by the coefficients b_k . Again the term corresponding to the coefficient b_1 is included in the exponential factor $\alpha(X)$.

1.5. Main result. We need to restrict to bounded solutions, to help with integrability as we indicate later. Here is the situation we have in mind. Suppose: there is a finite interval $[a_1, a_2]$ with $\sigma(u) \geq 0$ for $u \in [a_1, a_2]$; $\sigma(a_1) = \sigma(a_2) = 0$; and $b(a_1) \geq 0$ and $b(a_2) \leq 0$. Then one can construct

solutions to (1.1) that remain bounded between a_1 and a_2 using the methods of Shiga [14] in his construction of non-negative solutions to non-Lipschitz SPDEs.

We define the parameters of the dual process X described in Section 1.3: the branching rates and probabilities as

$$\nu = \frac{1}{2} \sum_{k \neq 2} |\sigma_k|, \quad \mu = \sum_{k \neq 1} |b_k|,$$

$$p_k = \begin{cases} |\sigma_k|/2\nu, & \text{if } k \neq 2, \\ 0, & \text{if } k = 2, \end{cases} \quad q_k = \begin{cases} |b_k|/\mu, & \text{if } k \neq 1, \\ 0, & \text{if } k = 1; \end{cases}$$

the sets

$$S_1 = \{k \geq 0 : b_k < 0, k \neq 1\}, \quad S_2 = \{k \geq 0 : \sigma_k < 0, k \neq 2\};$$

and the counting process:

$$K_t^1 = \text{the number of single particle births with offspring size } Z_1 \in S_1 \text{ by time } t,$$

$$K_t^2 = \text{the number of two particle births with offspring size } Z_2 \in S_2 \text{ by time } t,$$

$$K_t = K_t^1 + K_t^2.$$

We write $|I_t|$ for the cardinality of the set I_t , the number of particles alive at time t . The main result below establishes a duality formula for bounded solutions under the conditions that σ_2 and b_1 are sufficiently negative. These can be expressed conveniently with the help of the following functions

$$(1.5) \quad \tilde{\sigma}(z) = \sum_{k \neq 2} |\sigma_k| z^{k-2}, \quad \tilde{b}(z) = \sum_{k \neq 1} |b_k| z^{k-1}.$$

THEOREM 1. *Suppose that u is a solution to (1.1) satisfying $|u_t(x)| \leq R_0$ for all t, x and the power series for σ and b are convergent in the interval $[-R_1, R_1]$ for some $R_1 > R_0$. Suppose that:*

- (H1) $R_1 > 1, \tilde{\sigma}'(1) \leq 0$ and $\tilde{b}'(1) \leq 0$.
- (H2) For some $R > R_0, \sigma_2 < -\tilde{\sigma}(R)$ and $b_1 < -\tilde{b}(R)$.

Then the following duality relation holds for any $T \geq 0$:

$$(1.6) \quad E \left(\prod_{i=1}^N u_T(x_i) \right) = E \left(\prod_{\gamma \in I_T} \phi(X_T^\gamma) (-1)^{K_T} \exp \left(\left(\nu + \frac{\sigma_2}{2} \right) L_T + (\mu + b_1) \int_0^T |I_s| ds \right) \right).$$

If $b = 0$, then the parts of the hypotheses that refer to b may be removed. (H2) alone implies uniqueness in law for the solutions to (1.1).

Hypothesis (H1) ensures that the branching rates ν and μ are finite and the dual process is non explosive. In fact, as we show in section 2.2, the equation can always be scaled so that this assumption holds and then a moment formula holds for the scaled equation. We conclude this subsection with two examples of σ satisfying the hypothesis of the theorem.

EXAMPLE 1. $b(u) = 0$, $\sigma(u) = u - u^2 - \varepsilon u^3$ for sufficiently small ε .

EXAMPLE 2. $b(u) = 0$, $\sigma(u) = \cos u$ (taking $R = 1.7$).

Various other examples of b and σ are discussed in Section 3.

1.6. *Layout of the paper.* The remaining sections of the paper are organized as follows. In Section 2 we establish an approximate duality relation, Proposition 1, for a modified version of the SPDE and the dual process. For these modified processes the integrability problems disappear but the duality formula no longer holds exactly, since there are error terms that arise from the modifications. We then establish the integrability, Lemma 2, needed to control these error terms and complete the proof of Theorem 1.

In Section 3 we give some examples, make several comments and describe some variants of our result. We comment on how the hypotheses changes if the SPDE is scaled or considered under a change of measure. The conditions for integrability needed to establish the duality formula are largely independent of the motion of the dual particles. Thereby we can consider generators other than the Laplacian in the SPDE. Systems of equations can also be treated by this method and we end by stating a variant for some finite dimensional stochastic ODEs that we do not believe are covered by current uniqueness results.

2. Proof of the duality relation.

2.1. *An approximate duality relation.* In this section we consider the dual particle system up to the time of the m th birth for each $m = 1, 2, \dots$. For details of a construction of the dual process see [1]. Define the following stopping times:

τ_m = the time of the m th birth;

$$\rho_l = \inf \left\{ t \geq 0 : L_t \geq l \text{ or } \int_0^t |I_s| ds \geq l \right\};$$

$$\tau_{l,m} = \rho_l \wedge \tau_m.$$

Similarly we consider the solution u of (1.1) up to a stopping time η_n , where

$$\eta_n = \inf \{ t \geq 0 : |u_t(x)| \geq n \text{ for some } x \in \mathbb{R} \}.$$

We shall assume that $P(\sup_n \eta_n = \infty) = 1$. This follows, for example, if we consider solutions that decay at infinity (Shiga [14] considers solutions in C_{rap} , the space of continuous functions decaying faster than any exponential at infinity).

The approximate duality relation, Proposition 1, is presented for all such solutions u , even though Theorem 1 considers only bounded solutions where the assumption is immediate. At present we are unable to establish a duality relationship for unbounded solutions. However we note that Proposition 1 clearly identifies the integrability problems that need to be solved to establish a duality relationship for unbounded solutions.

The basic idea is to fix n, m, l and to stop the stochastic PDE at the time η_n and the dual process at the time $\tau_{l,m}$. However it will be convenient to allow the PDE to follow the heat flow after time η_n but without forcing noise, and to allow the dual particles to perform Brownian motions after time $\tau_{l,m}$ but without any branching. So we define the modified process \tilde{u} by

$$\tilde{u}_t = \begin{cases} u_t, & \text{if } t \leq \eta_n, \\ P_{t-\eta_n} u_{\eta_n}, & \text{if } t > \eta_n, \end{cases}$$

where P_t is the Brownian semi-group. Note that \tilde{u} is still continuous and solves the SPDE

$$(2.7) \quad \partial_t \tilde{u} = \frac{1}{2} \Delta \tilde{u} + I_{\{t \leq \eta_n\}} b(\tilde{u}) + I_{\{t \leq \eta_n\}} \sqrt{\sigma(\tilde{u})} \dot{W}_{t,x}.$$

Choose $n \geq 1$ so that $\sup_x |\phi(x)| \leq n$. Then $|\tilde{u}_t(x)| \leq n$ for all t, x .

We also define a modified dual process \tilde{X} as follows. Let $\{W_t^\gamma : \gamma \in I\}$ be an independent collection of Brownian motions. Define

$$\begin{aligned} \tilde{X}_t^\gamma &= \begin{cases} X_t^\gamma, & \text{if } t < \tau_{l,m}, \\ X_{\tau_{l,m}}^\gamma + W_{t-\tau_{l,m}}^\gamma, & \text{otherwise;} \end{cases} \\ \tilde{I}_t &= \{ \gamma : \tilde{X}_t^\gamma \text{ that are alive at time } t \}; \\ \tilde{J}_t &= (\mu + b_1) \int_0^{t \wedge \tau_{l,m}} |\tilde{I}_s| ds + \left(\nu + \frac{\sigma_2}{2} \right) L_{t \wedge \tau_{l,m}}; \\ \tilde{K}_t &= K_{t \wedge \tau_{l,m}}; \\ \tilde{L}_t^{\beta,\gamma} &= \text{the local time of } \tilde{X}^\beta - \tilde{X}^\gamma \text{ at zero.} \end{aligned}$$

To clearly define the assumptions on σ and b , we consider two cases:

- (S1) If u is unbounded, then assume that the power series for σ and b is convergent everywhere.
- (S2) If u is bounded by R_0 , then assume that the power series for σ and b is convergent in an interval $[-R_1, R_1]$ for some $R_1 > \max\{R_0, 1\}$.

PROPOSITION 1. Assume (S1) or (S2). For all $T \geq 0$ we have

$$(2.8) \quad E \left(\prod_{i=1}^N \tilde{u}_T(x_i) \right) = E \left(\prod_{\gamma \in \tilde{I}_T} \phi(\tilde{X}_T^\gamma) (-1)^{\tilde{K}_T} \exp(\tilde{J}_T) \right) + \mathcal{E}_1 + \mathcal{E}_2$$

where the error term \mathcal{E}_1 is given by

$$\begin{aligned} & \frac{1}{4} E \int_0^T \left(I_{\{T-t \leq \eta_n\}} - I_{\{t \leq \tau_{l,m}\}} \right) (-1)^{\tilde{K}_t} \exp(\tilde{J}_t) \\ & \times \sum_{\beta, \gamma \in \tilde{I}_t; \beta \neq \gamma} \left(\prod_{\alpha \in \tilde{I}_t \setminus \{\beta, \gamma\}} \tilde{u}_{T-t}(\tilde{X}_t^\alpha) \right) \sigma(\tilde{u}_{T-t}(\tilde{X}_t^\gamma)) d\tilde{L}_t^{\beta,\gamma}, \end{aligned}$$

and the error term \mathcal{E}_2 is given by

$$E \int_0^T \left(I_{\{T-t \leq \eta_n\}} - I_{\{t \leq \tau_{l,m}\}} \right) (-1)^{\tilde{K}_t} \\ \times \exp(\tilde{J}_t) \sum_{\beta \in \tilde{I}_t} \left(\prod_{\gamma \in \tilde{I}_t \setminus \{\beta\}} \tilde{u}_{T-t}(\tilde{X}_t^\gamma) \right) b(\tilde{u}_{T-t}(\tilde{X}_t^\beta)) dt.$$

PROOF. For any function f , define $f^\varepsilon(x) = f * p_\varepsilon(x)$ which is the convolution of f with the function $p_\varepsilon(x) = (2\pi\varepsilon)^{-1/2} e^{-x^2/2\varepsilon}$. The process $\tilde{u}_s^\varepsilon(x)$ will then be a semimartingale and, as \tilde{u} solves (2.7), we know that for any $t \geq 0$,

$$\tilde{u}_t^\varepsilon(x) = \phi^\varepsilon(x) + \int_0^t \left(\frac{1}{2} \Delta \tilde{u}_s^\varepsilon(x) + I_{\{s \leq \eta_n\}} b(\tilde{u}_s) * p_\varepsilon(x) \right) ds \\ + \int_0^t \int I_{\{s \leq \eta_n\}} \sqrt{\sigma(\tilde{u}_s(y))} p_\varepsilon(x-y) dW_{y,s}.$$

Applying Itô's formula, and taking expectations, we have

$$E \prod_{i=1}^N \tilde{u}_t^\varepsilon(x_i) - E \prod_{i=1}^N \phi^\varepsilon(x_i) \\ = E \int_0^t \sum_{i=1}^N \left(\prod_{j=1, j \neq i}^N \tilde{u}_s^\varepsilon(x_j) \right) \left(\frac{1}{2} \Delta \tilde{u}_s^\varepsilon(x_i) + I_{\{s \leq \eta_n\}} b(\tilde{u}_s) * p_\varepsilon(x_i) \right) ds \\ + \frac{1}{2} E \int_0^t I_{\{s \leq \eta_n\}} \sum_{i,j=1: i \neq j}^N \left(\prod_{k=1: k \neq i,j}^N \tilde{u}_s^\varepsilon(x_k) \right) \\ \times \int p_\varepsilon(y-x_i) p_\varepsilon(y-x_j) \sigma(\tilde{u}_s(y)) dy ds.$$

(Since \tilde{u} is bounded by n , it is easy to see that the stochastic integral arising from Itô's formula is a true martingale.) For our purposes, we would like to replace x_i 's in the previous identity with a random, but independent, set of points $\{\tilde{X}_r^\beta : \beta \in \tilde{I}_r\}$ and insert the independent function $(-1)^{\tilde{K}_r} \exp(\tilde{J}_r)$ suitably. Namely, fixing $r \in [0, T]$,

$$E \prod_{\beta \in \tilde{I}_r} \tilde{u}_t^\varepsilon(\tilde{X}_r^\beta) (-1)^{\tilde{K}_r} \exp(\tilde{J}_r) - E \prod_{\beta \in \tilde{I}_r} \phi^\varepsilon(\tilde{X}_r^\beta) (-1)^{\tilde{K}_r} \exp(\tilde{J}_r) \\ = E \int_0^t (-1)^{\tilde{K}_r} \exp(\tilde{J}_r) \sum_{\beta \in \tilde{I}_r} \left(\prod_{\gamma \in \tilde{I}_r \setminus \{\beta\}} \tilde{u}_s^\varepsilon(\tilde{X}_r^\gamma) \right) \frac{1}{2} \Delta \tilde{u}_s^\varepsilon(\tilde{X}_r^\beta) ds \\ (2.9) \quad + E \int_0^{t \wedge \eta_n} (-1)^{\tilde{K}_r} \exp(\tilde{J}_r) \sum_{\beta \in \tilde{I}_r} \left(\prod_{\gamma \in \tilde{I}_r \setminus \{\beta\}} \tilde{u}_s^\varepsilon(\tilde{X}_r^\gamma) \right) b(\tilde{u}_s) * p_\varepsilon(\tilde{X}_r^\beta) ds \\ + \frac{1}{2} E \int_0^{t \wedge \eta_n} (-1)^{\tilde{K}_r} \exp(\tilde{J}_r) \sum_{\beta, \gamma \in \tilde{I}_r: \beta \neq \gamma} \left(\prod_{\alpha \in \tilde{I}_r \setminus \{\beta, \gamma\}} \tilde{u}_s^\varepsilon(\tilde{X}_r^\alpha) \right) \\ \times \int p_\varepsilon(y - \tilde{X}_r^\beta) p_\varepsilon(y - \tilde{X}_r^\gamma) \sigma(\tilde{u}_s(y)) dy ds.$$

As indicated earlier \tilde{u} is bounded by n and by definition of the stopping time $\tau_{l,m}$, \tilde{J}_r is bounded by $(\nu + (\sigma_2/2) + b_1 + \mu)l$. The integrand in the last term is bounded by

$$\begin{aligned}
 & C(l)(2\pi\varepsilon)^{-\frac{1}{2}} \sup_{|z|\leq n} |\sigma(z)| \int_0^t |\tilde{I}_r|^2 n^{|\tilde{I}_r|-2} dr \\
 (2.10) \quad & \leq C(\varepsilon, \sigma, l, n, T) \sup_{r\leq T} |\tilde{I}_r|^2 n^{|\tilde{I}_r|} \\
 & \leq C(\varepsilon, \sigma, l, n, T, \delta) \sup_{r\leq T} (n + \delta)^{|\tilde{I}_r|}
 \end{aligned}$$

for any $\delta > 0$. There are at most m births up to $\tau_{l,m}$. Hence, we may bound the maximum increase in $|\tilde{I}_s|$ by the sum of independent copies of the offspring random variables Z_1 and Z_2 . So the expected value of the dominating term (2.10) is less than or equal to

$$C(\varepsilon, \sigma, l, n, T, \delta, N) [E(n + \delta)^{Z_1}]^m [E(n + \delta)^{Z_2}]^m$$

Using (S1) or (S2), we can conclude that $E(n + \delta)^{Z_i} < \infty$. Hence, the last term is integrable. Analogously, the integrand in the other terms can be shown to be integrable. Identity (2.9) follows from Fubini’s theorem. In the calculations that follow, there will be repeated applications of Fubini’s theorem. All such calculations can be justified similarly with the help of the dominating term (2.10).

By considering the compensators for the jumps in $\{\tilde{X}_t^\gamma : \gamma \in I_t\}$ at the times of births we have, for $h \in C_b^2(\mathbb{R})$,

$$\begin{aligned}
 & E \prod_{\beta \in \tilde{I}_t} h(\tilde{X}_t^\beta) (-1)^{\tilde{K}_t} \exp(\tilde{J}_t) - E \prod_{i=1}^N h(x_i) \\
 & = E \int_0^t (-1)^{\tilde{K}_s} \exp(\tilde{J}_s) \sum_{\beta \in \tilde{I}_s} \left(\prod_{\gamma \in \tilde{I}_s \setminus \{\beta\}} h(\tilde{X}_s^\gamma) \right) \frac{1}{2} \Delta h(\tilde{X}_s^\beta) ds \\
 & \quad + E \int_0^{t \wedge \tau_{l,m}} (-1)^{\tilde{K}_s} \exp(\tilde{J}_s) \sum_{\beta \in \tilde{I}_s} \left(\prod_{\gamma \in \tilde{I}_s \setminus \{\beta\}} h(\tilde{X}_s^\gamma) \right) \\
 & \quad \quad \quad \times \left(\sum_{k=0}^\infty q_k \left(h^k(\tilde{X}_s^\beta) (-1)^{I(k \in S_1)} - h(\tilde{X}_s^\beta) \right) \right) \mu ds \\
 & \quad + E \int_0^{t \wedge \tau_{l,m}} (-1)^{\tilde{K}_s} \exp(\tilde{J}_s) \sum_{\beta, \gamma \in \tilde{I}_s : \beta \neq \gamma} \left(\prod_{\alpha \in \tilde{I}_s \setminus \{\gamma, \beta\}} h(\tilde{X}_s^\alpha) \right) \\
 & \quad \quad \quad \times \left(\sum_{k=0}^\infty p_k \left(h^k(\tilde{X}_s^\gamma) (-1)^{I(k \in S_2)} - h(\tilde{X}_s^\beta) h(\tilde{X}_s^\gamma) \right) \right) \frac{\nu}{2} d\tilde{L}_s^{\beta, \gamma}
 \end{aligned}$$

$$\begin{aligned}
 & + E \int_0^{t \wedge \tau_{l,m}} (-1)^{\tilde{K}_s} \exp(\tilde{J}_s) \left(\prod_{\beta \in \tilde{I}_s} h(\tilde{X}_s^\beta) \right) \left((\mu + b_1) |\tilde{I}_s| ds + \left(\nu + \frac{\sigma_2}{2} \right) d\tilde{L}_s \right) \\
 & = E \int_0^t (-1)^{\tilde{K}_s} \exp(\tilde{J}_s) \sum_{\beta \in \tilde{I}_s} \left(\prod_{\gamma \in \tilde{I}_s \setminus \{\beta\}} h(\tilde{X}_s^\gamma) \right) \frac{1}{2} \Delta h(\tilde{X}_s^\beta) ds \\
 & \quad + E \int_0^{t \wedge \tau_{l,m}} (-1)^{\tilde{K}_s} \exp(\tilde{J}_s) \sum_{\beta \in \tilde{I}_s} \left(\prod_{\gamma \in \tilde{I}_s \setminus \{\beta\}} h(\tilde{X}_s^\gamma) \right) b(h(\tilde{X}_s^\beta)) ds \\
 & \quad + \frac{1}{4} E \int_0^{t \wedge \tau_{l,m}} (-1)^{\tilde{K}_s} \exp(\tilde{J}_s) \sum_{\beta, \gamma \in \tilde{I}_s; \beta \neq \gamma} \left(\prod_{\alpha \in \tilde{I}_s \setminus \{\gamma, \beta\}} h(\tilde{X}_s^\alpha) \right) \sigma(h(\tilde{X}_s^\beta)) d\tilde{L}_s^{\beta, \gamma}.
 \end{aligned}$$

The last equality is a consequence of the definitions of μ, ν, q_k, p_k . The process \tilde{u}_r^ε is smooth and bounded. Hence the above identity holds with h replaced by the independent random function \tilde{u}_r^ε . Specifically,

$$\begin{aligned}
 & E \prod_{\beta \in \tilde{I}_t} \tilde{u}_r^\varepsilon(\tilde{X}_t^\beta) (-1)^{\tilde{K}_t} \exp(\tilde{J}_t) - E \prod_{i=1}^N \tilde{u}_r^\varepsilon(x_i) \\
 & = E \int_0^t (-1)^{\tilde{K}_s} \exp(\tilde{J}_s) \sum_{\beta \in \tilde{I}_s} \left(\prod_{\gamma \in \tilde{I}_s \setminus \{\beta\}} \tilde{u}_r^\varepsilon(\tilde{X}_s^\gamma) \right) \frac{1}{2} \Delta \tilde{u}_r^\varepsilon(\tilde{X}_s^\beta) ds \\
 (2.11) \quad & + E \int_0^{t \wedge \tau_{l,m}} (-1)^{\tilde{K}_s} \exp(\tilde{J}_s) \sum_{\beta \in \tilde{I}_s} \left(\prod_{\gamma \in \tilde{I}_s \setminus \{\beta\}} \tilde{u}_r^\varepsilon(\tilde{X}_s^\gamma) \right) b(\tilde{u}_r^\varepsilon(\tilde{X}_s^\beta)) ds \\
 & \quad + \frac{1}{4} E \int_0^{t \wedge \tau_{l,m}} (-1)^{\tilde{K}_s} \exp(\tilde{J}_s) \sum_{\beta, \gamma \in \tilde{I}_s; \beta \neq \gamma} \left(\prod_{\alpha \in \tilde{I}_s \setminus \{\gamma, \beta\}} \tilde{u}_r^\varepsilon(\tilde{X}_s^\alpha) \right) \sigma(\tilde{u}_r^\varepsilon(\tilde{X}_s^\beta)) d\tilde{L}_s^{\beta, \gamma}.
 \end{aligned}$$

We now follow the method of duality outlined in Theorem 4.4.11, [4]. Set

$$g(t, s, \varepsilon) = E \prod_{\beta \in \tilde{I}_s} \tilde{u}_t^\varepsilon(\tilde{X}_s^\beta) (-1)^{\tilde{K}_s} \exp(\tilde{J}_s).$$

For any $T \geq 0$ we have

$$\begin{aligned}
 & \int_0^T g(r, 0, \varepsilon) - g(0, r, \varepsilon) dr \\
 & = \int_0^T (g(T - r, r, \varepsilon) - g(0, r, \varepsilon)) dr - \int_0^T (g(r, T - r, \varepsilon) - g(r, 0, \varepsilon)) dr.
 \end{aligned}$$

We now use (2.9) and (2.12) to expand the integrands, obtaining

$$\begin{aligned}
 & \int_0^T g(r, 0, \varepsilon) - g(0, r, \varepsilon) dr \\
 (2.12) \quad &= E \int_0^T \int_0^{T-r} (-1)^{\tilde{K}_r} \exp(\tilde{J}_r) \sum_{\beta \in \tilde{I}_r} \left(\prod_{\gamma \in \tilde{I}_r \setminus \{\beta\}} \tilde{u}_s^\varepsilon(\tilde{X}_r^\gamma) \right) \frac{1}{2} \Delta \tilde{u}_s^\varepsilon(\tilde{X}_r^\beta) ds dr \\
 & \quad + E \int_0^T \int_0^{T-r} I_{\{s \leq \eta_n\}} (-1)^{\tilde{K}_r} \exp(\tilde{J}_r) \\
 (2.13) \quad & \quad \times \sum_{\beta \in \tilde{I}_r} \left(\prod_{\gamma \in \tilde{I}_r \setminus \{\beta\}} \tilde{u}_s^\varepsilon(\tilde{X}_r^\gamma) \right) b(\tilde{u}_s) * p_\varepsilon(\tilde{X}_r^\beta) ds dr \\
 & \quad + \frac{1}{2} E \int_0^T \int_0^{T-r} I_{\{s \leq \eta_n\}} (-1)^{\tilde{K}_r} \exp(\tilde{J}_r) \sum_{\beta, \gamma \in \tilde{I}_r: \beta \neq \gamma} \left(\prod_{\alpha \in \tilde{I}_r \setminus \{\beta, \gamma\}} \tilde{u}_s^\varepsilon(\tilde{X}_r^\alpha) \right) \\
 (2.14) \quad & \quad \times \int p_\varepsilon(y - \tilde{X}_r^\gamma) p_\varepsilon(y - \tilde{X}_r^\beta) \sigma(\tilde{u}_s(y)) dy ds dr \\
 & \quad - E \int_0^T \int_0^{T-r} (-1)^{\tilde{K}_s} \exp(\tilde{J}_s) \sum_{\beta \in \tilde{I}_s} \left(\prod_{\gamma \in \tilde{I}_s \setminus \{\beta\}} \tilde{u}_r^\varepsilon(\tilde{X}_s^\gamma) \right) \\
 (2.15) \quad & \quad \times \frac{1}{2} \Delta \tilde{u}_r^\varepsilon(\tilde{X}_s^\beta) ds dr \\
 & \quad - E \int_0^T \int_0^{T-r} I_{\{s \leq \tau_{l,m}\}} (-1)^{\tilde{K}_s} \exp(\tilde{J}_s) \\
 (2.16) \quad & \quad \times \sum_{\beta \in \tilde{I}_s} \left(\prod_{\gamma \in \tilde{I}_s \setminus \{\beta\}} \tilde{u}_r^\varepsilon(\tilde{X}_s^\gamma) \right) b(\tilde{u}_r^\varepsilon(\tilde{X}_s^\beta)) ds dr \\
 & \quad - \frac{1}{4} E \int_0^T \int_0^{T-r} I_{\{s \leq \tau_{l,m}\}} (-1)^{\tilde{K}_s} \exp(\tilde{J}_s) \\
 (2.17) \quad & \quad \times \sum_{\beta, \gamma \in \tilde{I}_s: \beta \neq \gamma} \left(\prod_{\alpha \in \tilde{I}_s \setminus \{\gamma, \beta\}} \tilde{u}_r^\varepsilon(\tilde{X}_s^\alpha) \right) \sigma(\tilde{u}_r^\varepsilon(\tilde{X}_s^\beta)) d\tilde{L}_s^{\beta, \gamma} dr.
 \end{aligned}$$

Using the domination given by (2.10) we may apply Fubini's theorem and terms (2.12) and (2.15) cancel. Now that $\Delta \tilde{u}^\varepsilon$ term has been canceled we let

$\varepsilon \rightarrow 0$ in the remaining terms to obtain

$$(2.18) \quad \int_0^T g(r, 0, 0) - g(0, r, 0)dr$$

$$= E \int_0^T \int_0^{T-r} I_{\{s \leq \eta_n\}} (-1)^{\tilde{K}_r} \exp(\tilde{J}_r)$$

$$(2.19) \quad \times \sum_{\beta \in \tilde{I}_r} \left(\prod_{\gamma \in \tilde{I}_r \setminus \{\beta\}} \tilde{u}_s(\tilde{X}_r^\gamma) \right) b(\tilde{u}_s(\tilde{X}_r^\beta)) ds dr$$

$$+ \frac{1}{4} E \int_0^T \int_0^{T-r} I_{\{s \leq \eta_n\}} (-1)^{\tilde{K}_r} \exp(\tilde{J}_r)$$

$$(2.20) \quad \times \sum_{\beta, \gamma \in \tilde{I}_r; \beta \neq \gamma} \left(\prod_{\alpha \in \tilde{I}_r \setminus \{\beta, \gamma\}} \tilde{u}_s(\tilde{X}_r^\alpha) \right) \sigma(\tilde{u}_s(\tilde{X}_r^\beta)) ds d\tilde{L}_r^{\gamma, \beta}$$

$$- E \int_0^T \int_0^{T-r} I_{\{s \leq \tau_{l,m}\}} (-1)^{\tilde{K}_s} \exp(\tilde{J}_s)$$

$$(2.21) \quad \times \sum_{\beta \in \tilde{I}_s} \left(\prod_{\gamma \in \tilde{I}_s \setminus \{\beta\}} \tilde{u}_r(\tilde{X}_s^\gamma) \right) b(\tilde{u}_r(\tilde{X}_s^\beta)) ds dr$$

$$- \frac{1}{4} E \int_0^T \int_0^{T-r} I_{\{s \leq \tau_{l,m}\}} (-1)^{\tilde{K}_s} \exp(\tilde{J}_s)$$

$$(2.22) \quad \times \sum_{\beta, \gamma \in \tilde{I}_s; \beta \neq \gamma} \left(\prod_{\alpha \in \tilde{I}_s \setminus \{\gamma, \beta\}} \tilde{u}_r(\tilde{X}_s^\alpha) \right) \sigma(\tilde{u}_r(\tilde{X}_s^\beta)) d\tilde{L}_s^{\beta, \gamma} dr.$$

Only the term (2.20) needs careful justification and we give this after completing the rest of the proof. The other terms (2.19), (2.21) and (2.22) follow immediately from dominated convergence using (2.10). Assuming we have justified (2.20), the final step is to show that the terms almost cancel, leaving behind the error terms mentioned in the statement of the proposition. Applying a change of variable and then Fubini's theorem, the previous terms reduce to

$$\int_0^T g(r, 0, 0) - g(0, r, 0)dr$$

$$= E \int_0^T \int_0^s (I_{\{s-t \leq \eta_n\}} - I_{\{t \leq \tau_{l,m}\}}) (-1)^{\tilde{K}_t} \exp(\tilde{J}_t)$$

$$\times \sum_{\beta \in \tilde{I}_t} \left(\prod_{\gamma \in \tilde{I}_t \setminus \{\beta\}} \tilde{u}_{s-t}(\tilde{X}_t^\gamma) \right) b(\tilde{u}_{s-t}(\tilde{X}_t^\beta)) dt ds$$

$$+ \frac{1}{4} E \int_0^T \int_0^s (I_{\{s-t \leq \eta_n\}} - I_{\{t \leq \tau_{l,m}\}}) (-1)^{\tilde{K}_t} \exp(\tilde{J}_t)$$

$$\times \sum_{\beta, \gamma \in \tilde{I}_t; \beta \neq \gamma} \left(\prod_{\alpha \in \tilde{I}_t \setminus \{\beta, \gamma\}} \tilde{u}_{s-t}(\tilde{X}_t^\alpha) \right) \sigma(\tilde{u}_{s-t}(\tilde{X}_t^\beta)) dL_t^{\gamma, \beta} ds$$

Using the continuity of \tilde{u}_t , the left continuity of the dual process and the domination given by (2.10) we can check that $r \rightarrow g(r, 0, 0)$ is continuous and $r \rightarrow g(0, r, 0)$ is left continuous. Then differentiating both sides of the above equation from the left at T , we have that $g(T, 0, 0) - g(0, T, 0) = \mathcal{E}_1 + \mathcal{E}_2$ and this completes the derivation of the approximate duality relation.

It remains only to justify the passage to the limit in the term (2.14). We need the following lemma, proved at the end of this section.

LEMMA 2. Consider two independent Brownian motions B_t^1, B_t^2 adapted to a filtration \mathcal{G}_t . Let $f : \mathbb{R} \rightarrow [0, \infty)$ be a bounded continuous function, $X_t = B_t^1 - B_t^2$, and Y_t be a (\mathcal{G}_t) predictable process satisfying $E(\int_0^t |Y_s| ds) < \infty$. Then, for $t \geq 0$,

$$(2.23) \quad 2 \int_0^t f(X_s) Y_s ds = \int \int_0^t f(z) Y_s dL_{s,z}^X dz \quad a.s.,$$

where $L_{s,z}^X$ is the local time of X at z until time s .

After the substitution $y \rightarrow y + \tilde{X}_r^\beta$ we may rewrite the term (2.14) in the form

$$\sum_{\beta, \gamma \in I: \beta \neq \gamma} \int \int_0^T p_\varepsilon(y + \tilde{X}_r^\beta - \tilde{X}_r^\gamma) p_\varepsilon(y) Y_r^{\beta, \gamma}(y) dr dy$$

where

$$Y_r^{\beta, \gamma}(y) = I_{\{\beta, \gamma \in \tilde{I}_r\}} (-1)^{\tilde{K}_r} \exp(\tilde{J}_r) \int_0^{T-r} I_{\{s \leq \eta_n\}} \left(\prod_{\alpha \in \tilde{I}_r \setminus \{\beta, \gamma\}} \tilde{u}_s^\varepsilon(\tilde{X}_r^\alpha) \right) \times \sigma(\tilde{u}_s(y + \tilde{X}_r^\beta)) ds.$$

Define a filtration by $\mathcal{G}_t = \sigma((\tilde{X}_q^\beta : \beta \in \tilde{I}_q), \tilde{K}_q, \tilde{J}_q : q \leq t) \vee \sigma(\tilde{u}_q : q \leq T)$. Then $Y_r^{\beta, \gamma}(y)$ is (\mathcal{G}_r) adapted and left continuous and (2.10) gives the integrability needed to apply Lemma 2. Note that the paths \tilde{X}_r^β and \tilde{X}_r^γ are defined only when $\beta, \gamma \in \tilde{I}_r$. However a careful construction of the dual can be made so that these paths are segments of entire Brownian paths defined for all $r \in [0, T]$ (see [1]). Moreover note that $Y_r^{\beta, \gamma}$ is zero whenever $\beta, \gamma \notin \tilde{I}_r$. Applying Lemma 2 for a fixed y we have

$$\int_0^T p_\varepsilon(y + \tilde{X}_r^\beta - \tilde{X}_r^\gamma) Y_r^{\beta, \gamma}(y) dr = \int \int_0^T p_\varepsilon(y + z) Y_r^{\beta, \gamma}(y) d\tilde{L}_{r,z}^{\beta, \gamma} dz$$

where $\tilde{L}_{r,z}^{\beta, \gamma}$ is the local time of the process $\tilde{X}_r^\beta - \tilde{X}_r^\gamma$ at the point z . Both sides of this last equality are continuous in y and so the equality holds for all y simultaneously, with probability one. Integrating over y we may rewrite the integrand in (2.14) as

$$(2.24) \quad \sum_{\beta, \gamma \in I: \beta \neq \gamma} \int \int \int_0^T p_\varepsilon(y + z) p_\varepsilon(y) Y_r^{\beta, \gamma}(y) d\tilde{L}_{r,z}^{\beta, \gamma} dz dy.$$

Note that the above sum is a finite sum with probability one. We can now let $\varepsilon \rightarrow 0$ to obtain

$$\sum_{\beta, \gamma \in I: \beta \neq \gamma} \int_0^T Y_r^{\beta, \gamma}(0) d\tilde{L}_r^{\beta, \gamma}.$$

To justify this we fix β, γ and argue pathwise. Note that $Y_r^{\beta, \gamma}(y)$ is bounded and is piecewise continuous in r . Since $y \rightarrow Y_r^{\beta, \gamma}(y)$ is continuous at zero uniformly in $r \leq T$ we may replace $Y_r^{\beta, \gamma}(y)$ by $Y_r^{\beta, \gamma}(0)$ at the cost of an error that is small in ε . Then replace the path $r \rightarrow Y_r^{\beta, \gamma}(0)$ by a Riemann sum approximation that is uniformly close. This replaces the $d\tilde{L}^{\beta, \gamma}$ integral by a sum where it is easy to take the limit $\varepsilon \rightarrow 0$ using the continuity of the local time.

It remains only to justify taking the limit $\varepsilon \rightarrow 0$ inside the expectation. Bounding \tilde{J}_t and \tilde{u}_t as before in (2.10) we can dominate the variable (2.24) by

$$\begin{aligned} C(T, \sigma, l) & \sum_{\beta, \gamma \in I: \beta \neq \gamma} \int \int \int_0^T p_\varepsilon(y+z) p_\varepsilon(y) n^{|\tilde{I}_r|} d\tilde{L}_{r,z}^{\beta, \gamma} dz dy \\ (2.25) \quad & = C(T, \sigma, l) \sum_{\beta, \gamma \in I: \beta \neq \gamma} \int \int_0^T p_{2\varepsilon}(z) n^{|\tilde{I}_r|} d\tilde{L}_{r,z}^{\beta, \gamma} dz \\ & \leq C(T, \sigma, l) \sum_{\beta, \gamma \in I: \beta \neq \gamma} \sup_{r \leq T} n^{|\tilde{I}_r|} \sup_z \tilde{L}_{T,z}^{\beta, \gamma}. \end{aligned}$$

The local time $\tilde{L}_{T,z}^{\beta, \gamma}$ can only be non-zero if both particles labeled β, γ came into existence before time T . We need here to be more explicit about the labeling scheme for the dual particles. Since there at most m births we may label particles by the pair $\beta = (j, k)$, for $j = 0, 1, \dots, m$ and $k \geq 1$, representing the k th offspring of the j th birth. When $j = 0$ then k runs between 1 and N labeling the original N particles. The j th birth might be a single or a two particle birth but we can construct the dual so that the probability the particle labeled (j, k) was ever born (when $j \geq 1$) is at most $p_k + q_k$.

We have shown in (2.10) that the variable $\sup_{r \leq T} (n + \delta)^{|\tilde{I}_r|}$ is integrable for small $\delta > 0$. Also, for fixed particles β, γ , the expectation $E(\sup_z (L_{T,z}^{\beta, \gamma})^p)$ is finite for any p . So applying Hölder's inequality we may bound (2.25) by

$$\begin{aligned} C(T, \sigma, l, R) & \sum_{\beta, \gamma \in I: \beta \neq \gamma} P(\text{particles } \beta, \gamma \text{ ever born})^{1/r} \quad \text{for some } r > 1 \\ & \leq C(T, \sigma, l, R, N) \\ & \quad \times \left(1 + \sum_{j, j'=1}^m \sum_{k' \leq k=1}^\infty P(\text{particles } (j, k) \text{ and } (j', k') \text{ ever born})^{1/r} \right) \\ & \leq C(T, \sigma, l, R, N, m) \left(1 + \sum_{k=1}^\infty k(p_k + q_k)^{1/r} \right) \\ & \leq C(T, \sigma, l, R, N, m, r) \left(1 + \left(\sum_{k=1}^\infty k^{2r}(p_k + q_k) \right)^{1/r} \right) \end{aligned}$$

which is finite using the convergence of $\tilde{b}(R)$ and $\tilde{\sigma}(R)$ for some $R > 1$. This domination allows us to take this limit inside the expectation and, after substituting in the expression for $Y_r(0)$, we obtain the term (2.20) in the limit. \square

PROOF OF LEMMA 2. As f is continuous, by Proposition 6.17. in [2], we have that for all $t \geq 0$

$$2 \int_0^t f(X_s) ds = \int f(y) L_y^t dy \quad \text{a.s.}$$

Now if Y_s was a simple predictable process, i.e. of the form, $\sum_{i=0}^n K_i 1_{(a_i, b_i]}(s)$, where K_i are bounded \mathcal{G}_{a_i} measurable and $0 = a_0 \leq b_0 \leq a_1 \leq b_1 \leq a_2 \leq \dots \leq b_n$, then

$$\begin{aligned} 2 \int_0^t f(X_s) Y_s ds &= \sum_{i=0}^n 2K_i \int_{a_i \wedge t}^{b_i \wedge t} f(X_s) ds \\ &= \sum_{i=0}^n K_i \int \int_{a_i \wedge t}^{b_i \wedge t} f(z) dL_{s,z}^X dz \\ &= \int \int_0^t f(z) Y_s dL_{z,s}^X dz \quad \text{a.s.} \end{aligned}$$

If Y_s is any bounded predictable process, then we approximate Y by a sequence of simple processes Y^n , that converge to Y pointwise. As (2.23) holds for Y^n , and f, Y are bounded, it holds for the limit Y via dominated convergence. For positive Y the result then holds via monotone convergence, approximating with $Y \wedge n$. Split a general Y into positive and negative parts and use the integrability to recombine the two parts. \square

2.2. *Proof of Theorem 1.* We first show that under assumption (H1) the dual process is non-explosive. The expected change in the number of particles at a two-particle birth (respectively one-particle birth) is $\sum_k p_k(k - 2) = \tilde{\sigma}'(1)/2\nu$ (respectively $\tilde{b}'(1)/\mu$). So, under (H1), the total number of particles $|I_t|$ is a supermartingale and hence converges. Since any death changes the number of particles, the deaths cease for large t . So the dual process is well defined for all t and $\tau_{l,m} \rightarrow \infty$ as $l, m \rightarrow \infty$.

The following lemma establishes the integrability that is needed to control the error terms \mathcal{E}_1 and \mathcal{E}_2 . Recall the definitions of \tilde{b} and $\tilde{\sigma}$ given in (1.5).

LEMMA 3. *Suppose that $\alpha, \beta, \gamma \in \mathbb{R}$ satisfy*

$$\alpha + \frac{1}{2} \tilde{\sigma}(e^\gamma) < \nu, \quad \beta + \tilde{b}(e^\gamma) < \mu.$$

Then for all $T \geq 0$,

$$E \left(\sup_{0 \leq t \leq T} \exp \left(\alpha L_t + \beta \int_0^t |I_s| ds + \gamma |I_t| \right) \right) < \infty.$$

PROOF. Considering the jumps of $|I_t|$ we see that the compensator of the pure jump process $\exp(\gamma|I_t|)$ is given by

$$\begin{aligned} & \sum_k \int_0^t \nu p_k \left(e^{\gamma(|I_s|+k-2)} - e^{\gamma|I_s|} \right) dL_s + \sum_k \int_0^t \mu q_k \left(e^{\gamma(|I_s|+k-1)} - e^{\gamma|I_s|} \right) |I_s| ds \\ &= \left(\frac{1}{2} \tilde{\sigma}(e^\gamma) - \nu \right) \int_0^t e^{\gamma|I_s|} dL_s + (\tilde{b}(e^\gamma) - \mu) \int_0^t e^{\gamma|I_s|} |I_s| ds \end{aligned}$$

provided that $\tilde{b}(e^\gamma)$ and $\tilde{\sigma}(e^\gamma)$ are finite. Set $Z_t = \exp(\alpha L_t + \beta \int_0^t |I_s| ds + \gamma |I_t|)$. By Itô's formula we have

$$(2.26) \quad dZ_t = \left(\alpha + \frac{1}{2} \tilde{\sigma}(e^\gamma) - \nu \right) Z_t dL_t + (\beta + \tilde{b}(e^\gamma) - \mu) Z_t |I_t| dt + dM_t$$

where M_t is a local martingale. Because of the strict inequalities in the assumptions of the lemma, we can find $\theta > 1$ so that $Z_t^\theta = \exp(\theta \alpha L_t + \theta \beta \int_0^t |I_s| ds + \theta \gamma |I_t|)$ is a non-negative supermartingale. This implies, by Doob's optional stopping argument, that $P(\sup_{t \geq 0} Z_t^\theta \geq c) \leq \exp(\theta \gamma N)/c$ (where N is the initial number of particles) and the conclusion of the lemma follows. \square

The above integrability lemma is the key to this method of establishing the duality relation. Any weakening of the hypotheses would lead directly to uniqueness for a larger class of equations. Increments in the local time dL_t are roughly the size of $|I_t|^2 dt$ so that if the coefficient of $Z_t dL_t$ in the decomposition (2.26) for Z_t is strictly negative this may allow the coefficient of $Z_t |I_t| dt$ in (2.26) to be arbitrary, which would lead to removing the hypothesis on the drift $b(u)$. In the finite dimensional setting in Section 3.4 we shall show that this is easy to implement. We have not been able to weaken the hypothesis on σ . In particular when considering unbounded SPDEs one needs to control expectations of terms of the form $\prod_{\alpha \in I_t} u_s(X_t^\alpha)$. These, after taking expectations conditional on the dual and using moment estimates for SPDEs, lead to terms of the form at least as bad as $\exp(c|I_t| \log |I_t|)$. The above lemma suggests that these will not be integrable and this is the reason that we treat only bounded SPDEs.

To complete the proof we must pass to the limit as $n, m, l \rightarrow \infty$ in the approximate duality given by Proposition 1. Our assumptions are that the solutions are bounded by R_0 so the stopping time η_n is infinite for $n > R_0$ and plays no role. Setting $\eta_n = \infty$ and using the bound $|u_t(z)| \leq R_0$ we can bound the error term \mathcal{E}_1 by

$$\begin{aligned} & \frac{1}{4} \sup_{|z| \leq R_0} |\sigma(z)| E \int_0^T I_{\{t \geq \tau_{l,m}\}} \exp(\tilde{J}_t) R_0^{|\tilde{I}_t| - 2} d\tilde{L}_t \\ &= C(\sigma) R_0^{-2} E \exp(\tilde{J}_{\tau_{l,m}}) R_0^{|\tilde{I}_{\tau_{l,m}}|} (\tilde{L}_T - \tilde{L}_{\tau_{l,m}})_+ \end{aligned}$$

Let (\mathcal{S}_t) be the natural filtration for the dual process. After time $\tau_{l,m}$ the modified dual particles have no more deaths. So the expected local time gained

by any pair of these particles after this time is at most $E(l_{\sqrt{2}T})$, where l_t is the local time of a standard Brownian motion at zero. So we estimate

$$E((\tilde{L}_T - \tilde{L}_{\tau_{l,m}})_+ | \mathcal{L}_{\tau_{l,m}}) \leq \frac{1}{2} |\tilde{I}_{\tau_{l,m}}| |\tilde{I}_{\tau_{l,m}} - 1| I_{\{\tau_{l,m} \leq T\}} E(l_{\sqrt{2}T}).$$

Using this we further bound the error term \mathcal{E}_1 by

$$\begin{aligned} & C(\sigma, T, R_0) EI_{\{\tau_{l,m} \leq T\}} \exp(\tilde{J}_{\tau_{l,m}}) |\tilde{I}_{\tau_{l,m}}|^2 R_0^{|\tilde{I}_{\tau_{l,m}}|} \\ & \leq C(\sigma, T, R_0, R) EI_{\{\tau_{l,m} \leq T\}} \exp(\tilde{J}_{\tau_{l,m}}) R^{|\tilde{I}_{\tau_{l,m}}|} \quad \text{for any } R > R_0 \\ & = C(\sigma, T, R_0, R) EI_{\{\tau_{l,m} \leq T\}} \sup_{t \leq T} \exp\left(\left(\mu + b_1\right) \int_0^t |I_s| ds \right. \\ & \quad \left. + \left(\nu + \frac{\sigma_2}{2}\right) L_t + \log(R) |I_t|\right). \end{aligned}$$

A similar but simpler estimate shows that the error term \mathcal{E}_2 is bounded by the same expression (with a different constant). The hypotheses on σ_2 and b_1 and Lemma 3 now ensure that both error terms vanish as $l, m \rightarrow \infty$. The same integrability gives the domination needed to pass to the limit on both sides of the approximate duality relation (2.8) and complete the proof of the true duality relation (1.6).

The duality relation implies uniqueness since we only consider bounded solutions. To show that assumption (H1) is not needed for uniqueness we make a series of reductions. We shall suppose that b and σ are non-zero functions. (Uniqueness when $\sigma = 0$ holds since b is Lipschitz. The case where $b = 0$ is covered by the argument below by ignoring all references to b .) We claim that without loss of generality we may assume that $b(0)$ and $\sigma(0)$ are non-zero. For if not then consider the equation for $\hat{u} = u + \varepsilon$. This solves the SPDE with new drift and diffusion functions $\hat{b}(\hat{u}) = b(\hat{u} - \varepsilon)$ and $\hat{\sigma}(\hat{u}) = \sigma(\hat{u} - \varepsilon)$. By the analyticity of b and σ we can find arbitrarily small ε so that $\hat{b}(0)$ and $\hat{\sigma}(0)$ are both non-zero. But the hypothesis (H2) involves strict inequalities and will still hold for small enough ε . Uniqueness for \hat{u} implies uniqueness for u .

Now we scale the solution by setting $v = Bu$. This solves the SPDE with new drift and diffusion functions $\hat{b}(v) = Bb(v/B)$ and $\hat{\sigma}(v) = B^2\sigma(v/B)$. Assumption (H1) for \hat{b} and $\hat{\sigma}$ becomes

$$\sum_{k=0}^{\infty} |\sigma_k| B^{2-k} (k-2) \leq 0, \quad \sum_{k=0}^{\infty} |b_k| B^{1-k} (k-1) \leq 0.$$

Using $b_0 \neq 0, \sigma_0 \neq 0$ and taking B large enough this assumption will hold. If u was bounded by R_0 then v is bounded by BR_0 and it is easy to check that hypothesis (H2) is unchanged by this scaling of the equation. Uniqueness for v implies uniqueness for u , completing the proof. \square

3. Remarks.

3.1. *Linear scaling.* Consider the scaled process defined by the linear change of variables $v_t(x) = Bu_{Ct}(Dx)$, where $B \neq 0$, $C, D > 0$. This satisfies the SPDE

$$\partial_t v = \frac{C}{2D^2} \Delta v + \hat{b}(v) + \sqrt{\hat{\sigma}(v)} \dot{W}_{t,x},$$

where $\hat{\sigma}(v) = (B^2 C/D) \sigma(v/B)$ and $\hat{b}(v) = CBb(v/B)$. If $C = D^2$ hypothesis (H2) is unchanged. If $C \neq D^2$ then a dual process with Brownian particles run at speed \sqrt{C}/D is needed and again it can be checked (see Section 3.2) that hypothesis (H2) is unchanged. So this linear scaling does not change our ability to establish uniqueness. For sufficiently large values of B we obtain a duality formula for v which can then be written as a duality formula for u . However these are different for each value of B since the branching probabilities for the dual depend on B .

The change $v_t(x) = u_t(x) - A$ produces an unintuitive effect on the hypotheses. The coefficients b and σ are now expanded around the base point A . This may lead to an improvement. For example, the equation with $b(u) = 0$ and $\sigma(u) = u - u^3$ has solutions bounded in the interval $[0, 1]$. The criteria fails since $\sigma_2 = 0$. However, considering the equation for $v = 1 - u$ the diffusion function becomes $\rho(v) = 2v - 3v^2 + v^3$ and it can be checked that hypothesis (H2) holds. Hence uniqueness in law holds for this equation.

3.2. *Spatial inhomogeneity.* The technique of this paper should apply to equations where the spatial motion process of the dual particles is different or where the drift of diffusion functions are spatially inhomogeneous. We do not seek generality but give some examples to indicate some of the changes that are necessary.

Suppose, for example, we consider the following spde $\partial_t u = Lu + \sqrt{\sigma(u)} \dot{W}_{t,x}$, where the dual particles will follow a diffusion with generator $Lu = (1/2)c(x)u''(x) + d(x)u'(x)$, where $c(x) > 0$ for all $x \in \mathbb{R}$. Then the two-particle branching rate should be $\nu c(X_t^\beta)^{-1} dL_t^{\beta,\gamma}$ and the process L_t in the duality relation should be changed to

$$L_t = \frac{1}{2} \int_0^t \sum_{\beta, \gamma \in I_s; \beta \neq \gamma} c(X_s^\beta)^{-1} dL_s^{\beta,\gamma}.$$

The definitions of ν, μ, p_k, q_k are unchanged. Some regularity conditions on c and d and growth conditions on c, d, c^{-1} will be needed. For example if c and d have bounded continuous derivatives and c^{-1} is bounded then the duality relation holds under the same hypotheses as before.

If an inhomogeneous diffusion $\sigma(x, u)$ has a power series expansion $\sum \sigma_k(x)u^k$ with continuous functions $\sigma_k(x)$ then the dual particles will have spatially dependent branching rates $\nu(x)$ and branching probabilities $p_k(x)$

given by the analogous formulae. The exponential of L_t in the duality formula must be replaced by

$$\exp\left(\frac{1}{2}\int_0^t \sum_{\beta,\gamma \in I_s; \beta \neq \gamma} \left(\nu(X_s^\beta) + \frac{\sigma_2(X_s^\beta)}{2}\right) dL_s^{\beta,\gamma}\right).$$

Suppose that the function $\nu(x)$ is bounded. If we replace the hypotheses with versions that hold uniformly in z [i.e., there exists $R > R_0$ so that $\sigma_2(z) < -\tilde{\sigma}(z, R)$ for all z] then the duality relation holds with the above changes.

The extension to colored noise, which would allow the equation to be treated in dimensions higher than one, does not follow. The generator for the SPDE would then include a term of the form

$$\sum_{i \neq j} \left(\prod_{k \neq i, j} u_t(x_k)\right) \sqrt{\sigma(u_t(x_i))\sigma(u_t(x_j))} f(x_i - x_j) dt$$

where f is the spatial correlation function of the colored noise. The square root singularities are evaluated at different points and there is no obvious dual particle system.

3.3. Change of measure. Girsanov’s theorem can sometimes be used to change the drift term $b(u)$ in the SPDE. However many drift terms cannot be treated by this method. Suppose that solutions are positive and $\sigma(u)$ vanishes at $u = 0$. If $b(0)$ is strictly positive then the law of solutions with the drift b should not be absolutely continuous with respect to that of solutions with no drift. When $b(0) = 0$ and the solutions are integrable in x then one can hope, with growth conditions on $b(u)$, to apply Girsanov’s theorem. There are also problems treating solutions non-integrable in space since one cannot expect absolute continuity. Tribe [15] gives an example (of an equation with non-integrable solutions) where, by changing the drift only over large intervals, uniqueness for the equation with drift can be deduced from uniqueness for the equation without drift.

We now give an example to show that use of Girsanov’s theorem can add a drift that helps to make hypothesis (H2) hold. To avoid the integrability problems in space we consider the SPDE acting on the one dimensional torus \mathbb{T} , where the dual particles perform Brownian motions on the torus. Consider the case $\sigma(u) = u(1 - u)$, where the drift satisfies $b(0) > 0$ and $b(1) < 0$ and where we consider solutions taking values in $[0, 1]$. As an example consider $b(u) = 1 - 2u^2$ for which the hypotheses of Theorem 1.1 fail. Under a change of measure, detailed below, this drift becomes $b(u) = 1 - 2u$ for which the hypotheses of Theorem 1.1 do hold. More generally if $b(u) = b^+(u) - b^-(u)$ where the functions b^+, b^- have power series expansions with coefficients given by $b_k^+ = b_k \vee 0$ and $b_k^- = (-b_k) \vee 0$. We shall check that a change of measure can be made to alter the drift to $b^+(u) - b^-(1)u$. Then the hypothesis (H2) for this new drift becomes $-b^-(1) < -\inf_{R>1} \sum_{k \neq 1} b_k^+ R^{k-1}$. The right hand side is at least $-b^+(1)$ and so the assumption that $b(1) < 0$ implies that the hypothesis (H2) holds. Theorem 1 then implies uniqueness. The change

of measure needed is the exponential martingale associated to the stochastic integral $\int_0^t \int h_s(x) dW_{s,x}$ where $h = (b^-(u) - b^-(1)u)/(u(1-u))^{1/2}$. Since $b(0) > 0$ we must have that $b^-(0) = 0$. Hence $b^-(u) - b^-(1)u$ vanishes at both $u = 0$ and $u = 1$. The integrand h is now bounded and the change of measure is valid.

3.4. *Stochastic ODEs.* Theorem 1 can be extended to systems (u_1, \dots, u_n) of stochastic PDEs provided the drift and noise functions are analytic functions of all the variables (u_1, \dots, u_n) . Rather than work with PDEs we illustrate the method for a multidimensional stochastic ODE. The method is similar to the one established for the SPDEs earlier. Hence to avoid being repetitive we shall leave out many of the details in the arguments that follow.

Consider the solution Y_t of the following finite dimensional stochastic differential equation:

$$(3.27) \quad \begin{aligned} dY_t^i &= B_i(Y_t)dt + \sum_{k=1}^r \Sigma_{ik}(Y_t^1, \dots, Y_t^d) dW_t^k, \\ Y_0^i &= y_i, \quad i = 1, \dots, d, \quad r \geq 1, \end{aligned}$$

where W_t^k are independent Brownian motions in \mathbb{R} . Let $A = \Sigma\Sigma^T$. Let $\mathbb{L}^d = \{\mathbb{N} \cup \{0\}\}^d$. For $\alpha \in \mathbb{L}^d$, we set $|\alpha| = \sum_{i=1}^d \alpha_i$. Let e_i denote the canonical coordinate vectors in \mathbb{R}^d . For each $y \in \mathbb{R}^d$ set $y^\alpha = \prod_{i=1}^d y_i^{\alpha_i}$. We suppose that the functions A_{ij} and B_i have power series representations

$$A_{ij}(y) = \sum_{\alpha} a_{\alpha}^{ij} y^{\alpha}, \quad B_i(y) = \sum_{\alpha} b_{\alpha}^i y^{\alpha},$$

where the summations are over all multi-indices $\alpha \in \mathbb{L}^d$. The dual process will be a multitype branching process, where the branching probabilities for each type i (resp. types i and j) are given in terms of the coefficients of the power series for B_i (resp. A_{ij}). The dual can be written as $\{Z_t^k : k = 1, \dots, d\}$, where Z_t^k is the number of particles of type k alive at time t , with initial condition $Z_0^k = N_k$. There are two types of births:

1. *Single-particle births.* Each particle of type i dies at a rate $\mu_i dt$ and, on dying, is replaced by a random vector of ξ^i of offspring. The generating function for the offspring law is $E(s^{\xi^i}) = \sum_{\alpha} q_{\alpha}^i s^{\alpha}$.
2. *Two-particle births.* Each pair of particles of type i and $j \leq i$ die at a rate $\nu_{ij} dt$ and, on dying, is replaced by a a random vector of ξ^{ij} of offspring. The generating function for the offspring law is $E(s^{\xi^{ij}}) = \sum_{\alpha} p_{\alpha}^{ij} s^{\alpha}$.

For each $i, j = 1, \dots, d$, we define: the branching rates to be

$$\nu_{ij} = \sum_{\alpha \neq e_i + e_j} |a_{\alpha}^{ij}|, \quad \mu_i = \sum_{\alpha \neq e_i} |b_{\alpha}^i|;$$

the branching probabilities to be

$$p_\alpha^{ij} = \begin{cases} |a_\alpha^{ij}|/\nu_{ij}, & \text{for } \alpha \neq e_i + e_j, \\ 0, & \text{for } \alpha = e_i + e_j, \end{cases} \quad q_\alpha^i = \begin{cases} |b_\alpha^i|/\mu_i, & \text{for } \alpha \neq e_i, \\ 0, & \text{for } \alpha = e_i; \end{cases}$$

the special sets

$$S_i^1 = \{\alpha \in \mathbb{L}^d : b_\alpha^i < 0, \alpha \neq e_i\}, \quad S_{ij}^2 = \{\alpha \in \mathbb{L}^d : a_\alpha^{ij} < 0, \alpha \neq e_i + e_j\};$$

and the factors $K_t^{1,i}$ the number of births (given by a particle of type i) with offspring size $\xi_i \in S_i^1$ by time t , $K_t^{2,ij}$ the number of births (given by particles of type i, j) with offspring size $\xi_{ij} \in S_{ij}^2$ by the time t , $K_t = \sum_{i=1}^d K_t^{1,i} + \sum_{j \leq i} K_t^{2,ij}$. For notational convenience, we introduce $\hat{a}_{ij} = a_{e_i+e_j}^{ij}$ and $\hat{b}_i = b_{e_i}^i$. Let

$$J_t = \int_0^t \sum_{i=1}^d (\hat{b}_i + \mu_i) Z_s^i + \sum_{j \leq i} (\hat{a}_{ij} + \nu_{ij}) \frac{Z_s^j (Z_s^i - \delta_{ij})}{1 + \delta_{ij}} ds.$$

An analogous approximate duality formula for the processes $\prod_{i=1}^d y_i^{Z_T^i} \exp(J_T) (-1)^{K_T}$ and $\prod_{i=1}^d (Y_T^i)^{N_i}$ can be obtained as in Proposition 1. The hypothesis of the integrability lemma analogous to Lemma 3 are expressed using the following functions of $x \in \mathbb{R}^d$:

$$\tilde{A}_{ij}(x) = \sum_{\alpha \neq e_i+e_j} |a_\alpha^{ij}| x^{\alpha-(e_i+e_j)}, \quad \tilde{B}_i(x) = \sum_{\alpha \neq e_i} |b_\alpha^i| x^{\alpha-e_i}.$$

LEMMA 4. For $i, j = 1, \dots, d$, if $\gamma \in \mathbb{R}^d$, $\alpha_{ij} \in \mathbb{R}$, satisfy $\tilde{B}_i(e^{\gamma_1}, \dots, e^{\gamma_d}) < \infty$,

$$\alpha_{ii} + \tilde{A}_{ii}(e^{\gamma_1}, \dots, e^{\gamma_d}) < \nu_{ii}$$

and

$$\alpha_{ij} + \tilde{A}_{ij}(e^{\gamma_1}, \dots, e^{\gamma_d}) \leq \nu_{ij} \quad \text{for } i \neq j$$

then

$$E \left(\sup_{0 \leq t \leq T} \exp \left(\int_0^t \left\{ \sum_{i=1}^d \beta_i Z_s^i + \sum_{j \leq i} \alpha_{ij} \frac{Z_s^j (Z_s^i - \delta_{ij})}{1 + \delta_{ij}} \right\} ds + \sum_{k=1}^d \gamma_k Z_t^k \right) \right) < \infty,$$

for all $\beta_i \in \mathbb{R}, i = 1, \dots, d$.

PROOF. Set

$$N_t = \exp \left(\int_0^t \left(\sum_{i=1}^d \beta_i Z_s^i + \sum_{j \leq i} \alpha_{ij} \frac{Z_s^j (Z_s^i - \delta_{ij})}{1 + \delta_{ij}} \right) ds + \sum_{k=1}^d \gamma_k Z_t^k \right).$$

As in Lemma 3, using Itô's formula we find that

$$dN_t = N_t \left[\sum_{i=1}^d (\beta_i + \tilde{B}_i(e^{\gamma_1}, \dots, e^{\gamma_d}) - \mu_i) Z_t^i + \sum_{i=1}^d \sum_{j < i} (\alpha_{ij} + \tilde{A}_{ij}(e^{\gamma_1}, \dots, e^{\gamma_d}) - \nu_{ij}) \frac{Z_t^j (Z_t^i - \delta_{ij})}{1 + \delta_{ij}} \right] dt + dM_t,$$

where M_t is a local martingale. If the hypothesis of the lemma are satisfied then the coefficients of $(Z^i)^2$ are strictly negative and the coefficients of $Z^i Z^j$ are less than or equal to zero. So there exists a $\theta > 1$ and $A < \infty$ so that $dN_t^\theta \leq AN_t^\theta dt + dM_t$. Now consider $\exp(-At)N_t$ and we can conclude that

$$d \exp(-At)N_t^\theta \leq \exp(-At)dM_t.$$

Now proceed as in Lemma 3 to complete the proof. \square

Using the above lemma the proof of the theorem stated below follows analogously. For $x, y \in \mathbb{R}^d$ we write $x < y$ if $x_i < y_i$ for all $i = 1, \dots, d$ and correspondingly for $x \leq y$. We also denote the vector $(1, \dots, 1)$ by the symbol $\mathbb{1}$.

THEOREM 2. *Suppose that the solution Y to (3.27) satisfies $-R_0 \leq Y_t \leq R_0$ for all t . Suppose the power series for A_{ij} and B_i are convergent in the set*

$$\{y \in \mathbb{R}^d : -R_1 \leq y \leq R_1\} \quad \text{for some } R_1 > R_0.$$

Suppose for all i, j that:

$$(H1)' \quad R_1 > \mathbb{1}, \quad \left. \frac{d}{da} \tilde{A}_{ij}(a \mathbb{1}) \right|_{a=1} \leq 0 \quad \text{and} \quad \left. \frac{d}{da} \tilde{B}_i(a \mathbb{1}) \right|_{a=1} \leq 0.$$

$$(H2)' \quad \text{For some } R > R_0 \text{ that } \hat{a}_{ii} < -\tilde{A}_{ii}(R) \text{ and } \hat{a}_{ij} \leq -\tilde{A}_{ij}(R) \text{ for } i \neq j.$$

Then the following duality relation holds between Y and Z for any $T \geq 0$:

$$E \prod_{i=1}^d (Y_T^i)^{N_i} = E \prod_{i=1}^d y_i^{Z_T^i} \exp(J_T)(-1)^{K_T}.$$

(H2)' alone implies uniqueness in law for solutions to (3.27).

We conclude the paper with a two-dimensional example. To the best of our knowledge uniqueness for this example is not covered in the literature. Consider $b(x, y) = (b_1(x, y), b_2(x, y))^T$ and

$$\sigma(x, y) = \begin{pmatrix} \sqrt{(1 + \varepsilon h_1(x, y))x(1-x)} & 0 \\ 0 & \sqrt{(1 + \varepsilon h_2(x, y))y(1-y)} \end{pmatrix},$$

where h_1, h_2, b_1, b_2 have power series expansions convergent on $\{(x, y) : |x| < 1 + 2\delta, |y| < 1 + 2\delta\}$ for some $\delta > 0$. Suppose that the drift vector $b(x, y)$ points into the unit square $[0, 1]^2$ at each point (x, y) in the boundary of the

unit square. Then one may construct solutions (X_t, Y_t) that lie inside the unit square for all time. One such drift is found by considering the linear case $b(x, y) = (Ay - Bx, Cx - Dy)^T$ where $B > A > 0$ and $D > C > 0$. This drift occurs when considering the superprocess constructed over a motion process that is a Markov chain on two states, when the coefficients A, B, C, D arise from the Q matrix of the underlying Markov chain.

For sufficiently small $\varepsilon > 0$, the hypothesis $(H2)'$ is satisfied and uniqueness in law holds.

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