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# BOUNDS FOR STABLE MEASURES OF CONVEX SHELLS AND STABLE APPROXIMATIONS

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The standard normal distribution $\Phi$ on $\mathbb{R}^d$ satisfies $\Phi\left((\partial C)^\varepsilon\right) \le c_d \varepsilon$, for all $\varepsilon > 0$ and for all convex subsets $C \subset \mathbb{R}^d$, with a constant $c_d$ which depends on the dimension $d$ only. Here $\partial C$ denotes the boundary of $C$, and $(\partial C)^\varepsilon$ stands for the $\varepsilon$-neighborhood of $\partial C$. Such bounds for the normal measure of convex shells are extensively used to estimate the accuracy of normal approximations.

We extend the inequality to all (nondegenerate) stable distributions on $\mathbb{R}^d$, with a constant which depends on the dimension, the characteristic exponent and the spectral measure of the distribution only. As a corollary we provide an explicit bound for the accuracy of stable approximations on the class of all convex subsets of $\mathbb{R}^d$.

**1. Introduction and formulation of results.** Let $\mathbb{R}^d$ denote the standard real Euclidean space with the norm defined by $|x|^2 = x_1^2 + \cdots + x_d^2$ and the corresponding inner product $\langle x, x\rangle = |x|^2$. Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed (i.i.d.) $\mathbb{R}^d$-valued random vectors with distribution $F$. Denote by $F_n$ the distribution of the sum

$$a_n^{-1} \sum_{i=1}^{n} X_i - b_n,$$

where $a_n > 0$ and $b_n \in \mathbb{R}^d$ are normalizing constants and centering vectors. It is well known that if $F_n$, as $n \to \infty$, converge weakly to a distribution, say $G$, it has to be a stable distribution with a characteristic exponent $0 < \alpha \le 2$. The case $\alpha = 2$ corresponds to a Gaussian law.

The characteristic function $\varphi(t) = \int_{\mathbb{R}^d} \exp\{i\langle t, x\rangle\} G(dx)$ of a stable law $G$ can be written as

$$\varphi(t) = \exp\left\{ i\langle t, a\rangle - \int_{S_{d-1}} \left|\langle t, y\rangle\right|^\alpha N(y, \alpha)\Gamma(dy)\right\} \tag{1.1}$$

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with

$$(1.2)\qquad \begin{aligned} N(y,\alpha) &\equiv N(t,y,\alpha) = 1 - i\,\mathrm{sign}\,(\langle t, y\rangle)\tan\frac{\pi\alpha}{2}, \qquad \alpha \neq 1,\\ N(y,\alpha) &= 1 + i\frac{2}{\pi}\,\mathrm{sign}\,(\langle t, y\rangle)\log|\langle t, y\rangle|, \qquad \alpha = 1, \end{aligned}$$

where $a \in \mathbb{R}^d$ and $\Gamma$ denotes a finite nonnegative $\sigma$-additive measure on the unit sphere $S_{d-1} = \{x \in \mathbb{R}^d\colon |x| = 1\}$. The measure $\Gamma$ is called the spectral measure of a stable distribution. The triple $(a, \alpha, \Gamma)$ completely characterizes the stable distribution. Since all our results are independent of shifts of distributions, without loss of generality throughout we assume that $a = 0$. We write $G_{\alpha,\Gamma}$, $\varphi_\alpha$, etc., in cases where we want to emphasize the dependence on the characteristic exponent $\alpha$ or on the spectral measure $\Gamma$. We denote the density of $G$ with respect to the Lebesgue measure on $\mathbb{R}^d$ as $g$ (if it exists). For more information about multivariate stable laws we refer to Samorodnitsky and Taqqu (1994).

A rather general formulation of the problem of convergence rates in the central limit theorem may be stated as follows [see, e.g., Bhattacharya and Rao (1976), Paulauskas (1975), Sazonov (1968)]. Let $\mathscr{F}$ be a class of measurable functions $f\colon \mathbb{R}^d \to \mathbb{R}$ such that the integral in (1.3) below exists. The goal is to estimate

$$(1.3)\qquad \Delta_n(\mathscr{F}) := \sup_{f\in\mathscr{F}} \left| \int_{\mathbb{R}^d} f(x)(F_n - G)(dx) \right|,$$

for example, as follows:

$$(1.4)\qquad \Delta_n(\mathscr{F}) \le c_d\,\zeta(\mathscr{F}, G)\nu(F, G)\delta_n$$

with some $\delta_n$ such that $\delta_n \to 0$, as $n \to \infty$. The constant $\zeta(\mathscr{F}, G)$ depends on $\mathscr{F}$ and $G$, and $\nu(F, G)$ usually is a moment or pseudo-moment related to the distributions $F$ and $G$.

Classes of indicator functions $\mathbf{I}(x; A)$ of subsets $A \subset \mathbb{R}^d$ are of special interest. We define $\mathbf{I}(x; A) = 1$ if $x \in A$, and $\mathbf{I}(x; A) = 0$ otherwise. A natural correspondence between classes $\mathscr{A}$ of Borel sets $A \in \mathscr{A}$ and classes $\mathscr{F}$ of indicator functions is given by $A \leftrightarrow \mathbf{I}(\cdot\,; A)$. Hence, for such classes we can rewrite (1.3) as

$$(1.5)\qquad \Delta_n(\mathscr{A}) := \sup_{A\in\mathscr{A}} \left|F_n(A) - G(A)\right|.$$

The constant $\zeta(\mathscr{F}, G) = \zeta(\mathscr{A}, G)$ in (1.4) usually depends on the quantities

$$\begin{aligned} \eta(\mathscr{A}, G, \varepsilon) &:= \sup_{A\in\mathscr{A}} G((\partial A)^\varepsilon), \qquad \varepsilon > 0,\\ \eta(\mathscr{A}, G) &:= \sup_{\varepsilon>0} \eta(\mathscr{A}, G, \varepsilon)/\varepsilon, \end{aligned}$$

$$\chi(g, w_1, \dots, w_s) := \int_{\mathbb{R}^d} \left| g^{(s)}(x) w_1, \dots, w_s \right| dx, \qquad w_1, \dots, w_s \in \mathbb{R}^d, \tag{1.6}$$

$$\chi_s(g) := \sup\{\chi(g, w_1, \dots, w_s) \colon |w_i| \le 1 \text{ for all } i\}, \tag{1.7}$$

where $g^{(s)}(x)$ denotes the Fréchet derivative. Using the directional derivatives,

$$d_w g(x) := \lim_{t \to 0} (g(x + tw) - g(x))/t,$$

we have

$$g^{(s)}(x) w_1 \cdots w_s = d_{w_1} \cdots d_{w_s} g(x).$$

The boundary of a set $A$ we denote as $\partial A$, and $(\partial A)^\varepsilon$ is the $\varepsilon$-neighborhood of $\partial A$.

In the case of the standard normal distribution $G = \Phi$, the quantities $\chi$ in (1.6) and (1.7) are obviously finite. However, one needs a special proof in order to show that $\eta(\mathscr{A}_c, \Phi) < \infty$ for the class $\mathscr{A}_c$ of convex subsets of $\mathbb{R}^d$ [see Bahr (1967), Bhattacharya and Rao (1976), Sazonov (1981)]. In the stable case $\alpha < 2$, the existence of $\chi_s(g)$ and $\eta(\mathscr{A}_c, G)$ was either imposed as a condition [see Paulauskas (1975), Bloznelis (1988)], or special cases were considered such that it was possible to show the existence of $\eta$ and $\chi$. A list of the special cases consists of (1) the class $\mathscr{A}_r$ of rectangles [Banys (1971)]; (2) spherically symmetric distributions [see Bloznelis (1989), Paulauskas (1975), Mikhailova (1983)]; (3) the two-dimensional case $d = 2$ [Paulauskas (1975)]; (4) stable random vectors with independent coordinates [Paulauskas (1975)]. The condition $\chi_s(g) < \infty$ is used to ensure the existence of some metrics related to stable distributions; see Chapter 14 in Rachev (1991).

The aim of the present paper is to show that all aforementioned quantities exist, for the class $\mathscr{A}_c$ of convex subsets and for any stable distribution which is nondegenerate in a subspace of $\mathbb{R}^d$. Furthermore, we obtain explicit bounds for these quantities.

A distribution $G$ we call nondegenerate if $G(L) = 0$, for any linear subspace $L \subset \mathbb{R}^d$ such that $\dim L < d$. A stable nondegenerate distribution is absolutely continuous and, hence has a density $g$. Write

$$\varkappa(\Gamma) := \inf_{|t|=1} \left( \int_{S_{d-1}} \left| \langle t, y \rangle \right|^\alpha \Gamma(dy) \right)^{1/\alpha}, \qquad \varkappa_0(\Gamma) := \Gamma(S_{d-1}). \tag{1.8}$$

Note that $\varkappa(\Gamma) > 0$ if and only if the stable distribution $G = G_\Gamma$ is nondegenerate, and $\varkappa_0(\Gamma) < \infty$ for any stable distribution. Write

$$K_\alpha(\Gamma) = \varkappa_0^d(\Gamma) \varkappa^{-d\alpha-1}(\Gamma), \qquad \alpha \ne 1, \tag{1.9}$$

$$K_\alpha(\Gamma) = \varkappa_0^d(\Gamma) \varkappa^{-d-1}(\Gamma) \Big(1 + \left| \log \varkappa(\Gamma) \right| \Big)^d, \qquad \alpha = 1. \tag{1.10}$$

If the distribution $G$ is symmetric, that is, the function $N(\cdot, \alpha)$ in the characteristic function (1.1) satisfies $N(\cdot, \alpha) \equiv 1$ or the measure $\Gamma$ is symmetric, then we define $K_\alpha(\Gamma)$ by (1.9), for all $0 < \alpha \le 2$.

THEOREM 1. *Let $G$ be a nondegenerate stable distribution. Then $\chi_s(g) < \infty$, for $s = 1, 2, \dots$ Moreover, we have*

$$\chi_1(g) \le c(\alpha, d) K_\alpha(\Gamma), \tag{1.11}$$

*where $K_\alpha(\Gamma)$ is defined in* (1.9)–(1.10)*, and*

$$\chi_s(g) \le (s^{1/\alpha}\chi_1(g))^s \quad \text{for } s = 2, 3, \dots. \tag{1.12}$$

*In particular, $g \in C^\infty(\mathbb{R}^d)$, for all $\alpha$, and $g$ is an analytic function, for $\alpha \ge 1$.*

Theorem 1 refines a result of Bogachev (1986) who showed that $\chi_1(g)$ exists. This fact easily implies (1.12) and the analyticity for $\alpha \ge 1$. Also it is necessary to note that in Bogachev 1986 is asserted only the existence of $\chi_1(g)$ while we provide a constructive proof and an explicit bound. To have a constructive proof is necessary in applications such as simulations of stable random vectors by LePage series in the multidimensional case [see Bentkus, Juozulynas and Paulauskas (1999b); in Ledoux and Paulauskas (1996), Bentkus, Götze and Paulauskas (1996) the case $d = 1$ is considered]. The constant $c(\alpha, d)$ in (1.11) allows the following explicit bound

$$\begin{aligned} &c(\alpha, d) \le 12(50/\alpha)^{2d}(2d)^{2d/\alpha}(1/\alpha)^{2/\alpha}(1 + |\tan(\pi\alpha/2)|)^d, \qquad \alpha \ne 1, \\ &c(1, d) \le 18(8d)^{3d}, \qquad \alpha = 1. \end{aligned} \tag{1.13}$$

If the distribution $G$ is symmetric, then

$$c(\alpha, d) \le 12(50/\alpha)^{2d}(2d)^{2d/\alpha}(1/\alpha)^{2/\alpha}. \tag{1.14}$$

Note that the bound (1.13), for $\alpha \ne 1$, is uniform in $\alpha$ from any compact subset of (0,1) $\cup$ (1,2], and it degenerates when $\alpha \downarrow 0$ or $\alpha \to 1$. It seems that the degeneration at $\alpha = 1$ is an artifact of our methods and is related to the parameterization (1.1) of stable laws, which is discontinuous as $\alpha \to 1$. In the symmetric case the bound (1.14) is satisfactory since it degenerates only as $\alpha \downarrow 0$. Of course, the bounds (1.13) and (1.14) are not optimal. Writing them down, we tried to reflect the uniformity in $\alpha$ and preferred simplicity of the form to accuracy.

Write

$$\zeta(\mathscr{A}, G) := \sup_{A \in \mathscr{A}} \int_{\partial A} g(x)\, ds,$$

where $ds$ denotes the surface area element on $\partial A$.

THEOREM 2. *Let $G$ be an arbitrary distribution on $\mathbb{R}^d$ such that its density $g$ exists and is a continuous function. Then*

$$\eta(\mathscr{A}_c, G) = 2\zeta(\mathscr{A}_c, G). \tag{1.15}$$

*Let $G$ be an arbitrary distribution on $\mathbb{R}^d$ such that $\chi_1(g)$ exists. Then*

$$\eta(\mathscr{A}_c, G) \le 4d^{3/2}\chi_1(g). \tag{1.16}$$

*In particular, any stable nondegenerate $G$ satisfies*

$$\eta(\mathscr{A}_c, G) \le 4d^{3/2}c(\alpha, d)K_\alpha(\Gamma)$$

*with $c(\alpha, d)$ as in* (1.13) *and* (1.14) *and $K_\alpha(\Gamma)$ as in* (1.9) *and* (1.10).

Using Theorem 2, the bounds for the accuracy of stable approximations in $\mathbb{R}^d$ obtained by Paulauskas (1975) and by Bloznelis (1988) extended to the whole class of nondegenerate stable distributions; see Theorem 3 below.

We hope that using (1.16) and applying the method used to proveTheorem 1, one can derive results similar to Theorem 2 for some classes of infinity divisible distributions. As an initial step in this direction we provide an extension to the case of mixtures of stable distributions with the varying $\alpha$; see Theorem 4 below.

In the special case of the standard normal distribution $G = \Phi$, simple calculations show that $\chi_1(g) \le (2/\pi)^{1/2}$ and the estimate (1.16) yields

$$\eta(\mathscr{A}_c, \Phi) \le (32/\pi)^{1/2}d^{3/2}, \tag{1.17}$$

which is worse than the best known estimate $\eta(\mathscr{A}_c, \Phi) \le 8d^{1/4}$ [see Ball (1993)]. A precise dependence of $\eta(\mathscr{A}_c, \Phi)$ on dimension is not known and would be of interest in the context of estimates of normal approximations [see Bentkus (1986)]. The bound (1.17) depends on $d^{3/2}$ which is worse than $d^{1/2}$ in Bhattacharya and Rao (1976), where a proof adapted to the structure of $\Phi$ is provided. Our proof applies to arbitrary $G$ such that $\chi_1(g) < \infty$ and seems to be simpler.

Recall that $X_1, X_2, \dots$ denote i.i.d. random vectors with common distribution $F$. Let henceforth $F_n$ denote the distribution of the sum $n^{-1/\alpha}\sum_{i=1}^n X_i$. Assume that

$$\int_{\mathbb{R}^d} \langle t, x\rangle (F - G)(dx) = 0 \quad \text{for all } t \in \mathbb{R}^d. \tag{1.18}$$

If $\alpha > 1$ then assume in addition that

$$\int_{\mathbb{R}^d} \langle t, x\rangle\langle s, x\rangle (F - G)(dx) = 0 \quad \text{for all } t, s \in \mathbb{R}^d. \tag{1.19}$$

Introduce the uniform distance

$$\rho = \rho(F, G) = \sup\left\{|F(A) - G(A)|\colon A \in \mathscr{A}_c\right\}$$

between distributions $F$ and $G$ on the class $\mathscr{A}_c$ of convex sets.

The class $H^r(\mathbb{R}^d)$, $r > 0$, consists of the functions $f\colon \mathbb{R}^d \to \mathbb{R}$ such that $f$ is $m$ times Fréchet differentiable and the derivative $f^{(m)}$ satisfies

$$\sup_{|w_1|=\cdots=|w_m|=1} \left|f^{(m)}(x)w_1\cdots w_m - f^{(m)}(y)w_1\cdots w_m\right| \le \left|x-y\right|^{\theta},$$

where a nonnegative integer $m$ and a positive $\theta$ satisfy $r = m+\theta$ and $0 < \theta \le 1$. Introduce the metric

$$\zeta_r = \zeta_r(F, G) = \sup\left\{\left|\int_{\mathbb{R}^d} f(x)(F-G)(dx)\right|\colon f \in H^r(\mathbb{R}^d)\right\}$$

and pseudo-moments

$$\nu_r = \nu_r(F, G) = \int_{\mathbb{R}^d} |x|^r \left|F-G\right|(dx),$$

where $|F-G|$ denotes the variation of the signed measure $F-G$. Note that $\zeta_1 \le \nu_1$. We have as well that $\zeta_{1+\alpha} \le \nu_{1+\alpha}$, for $0 < \alpha \le 1$, if (1.18) is fulfilled, and for $1 < \alpha \le 2$, if both (1.18) and (1.19) hold.

THEOREM 3. *Assume that $G$ is a stable nondegenerate distribution with the characteristic function given by* (1.1) *and* (1.2). *If $\alpha \ne 1$ then*

$$\Delta_n(\mathscr{A}_c) \le c_{\alpha,d}\, n^{-1/\alpha}\Big(\rho + K_\alpha(\Gamma)\zeta_1 + K_\alpha^{\alpha+1}(\Gamma)\zeta_{1+\alpha}\Big). \tag{1.20}$$

*For $\alpha = 1$ the bound* (1.20) *holds for strictly stable $G$.*

The quantities $\Delta_n(\mathscr{A}_c)$ and $K_\alpha(\Gamma)$ are defined by (1.5) and (1.9) and (1.10), respectively. We shall derive (1.20) combining a bound proved by Bloznelis (1988) and Theorems 1 and 2. Recall that $G$ is strictly stable if the distribution of the sum $n^{-1/\alpha}\sum_{i=1}^n Y_i$ equals $G$ when $Y_1, \dots, Y_n$ are i.i.d. with the distribution $G$. As it is well known, for $\alpha \ne 1$, the strict stability means that the shift $a$ from (1.1) and (1.2) equals zero. For $\alpha = 1$, it is equivalent to $a = 0$ and $\int_{S_{d-1}} \langle t, y\rangle \Gamma(dy) = 0$, for all $t \in \mathbb{R}^d$. The requirement of the strict stability in Theorem 3 seems to be superfluous and is inherited from the bound of Bloznelis. The constant in (1.20) satisfies

$$c_{\alpha,d} \le cc(\alpha)(20)^{1/\alpha} d^{3/2}(1 + c(\alpha, d))^{\alpha+1}$$

with $c(\alpha, d)$ defined by (1.13) and (1.14), where $c$ is an absolute constant and $c(\alpha) = 1$, for $\alpha \le 1$, $c(\alpha) = \alpha - 1$, for $\alpha > 1$. Once again, the bound for $c_{\alpha,d}$ degenerates as $\alpha \downarrow 0$ or $\alpha \to 1$.

We conclude the introduction with the aforementioned extension to mixtures of one-dimensional stable distributions with varying $\alpha$. Consider a measurable function $\alpha\colon S_{d-1} \to [0, 2]$. Let $G_m$ (respectively, $g_m$) denote a distribution (respectively, its density) which has the characteristic function defined by (1.1) and (1.2) with $\alpha$ and $N(y, \alpha)$ replaced by $\alpha(y)$ and $N(y, \alpha(y))$, respectively. Note that $G_m$ can be interpreted as a mixture of one-dimensional stable distributions, say $G_{\alpha(L)}$, with the varying characteristic exponent $\alpha = \alpha(L)$ such that $G_{\alpha(L)}$ degenerates in a one-dimensional subspace $L \subset \mathbb{R}^d$, and any stable distribution $G = G_\alpha$ as a similar mixture with the constant $\alpha$.

Let

$$\omega(\Gamma, \tau) = \inf_{|t|=1} \int_{S_{d-1}} \big|\langle t, y\rangle\big|^{\alpha(y)} \tau^{\alpha(y)} \Gamma(dy), \qquad \tau > 0. \tag{1.21}$$

If $\alpha$ is constant then $\omega(\Gamma, 1) = \varkappa^\alpha(\Gamma)$ [cf. (1.8)]. Let $\tau(\Gamma)$ denote a solution of the equation $\omega(\Gamma, \tau) = 1$. Define the following counterpart of $K_\alpha(\Gamma)$ [cf. (1.9)]:

$$K(\Gamma) = \varkappa_0^d(\Gamma) \max\{\tau^{2d+1}(\Gamma), \tau^{\delta d+1}(\Gamma)\}. \tag{1.22}$$

THEOREM 4. *Assume that* $\omega(\Gamma, 1) > 0$ *and that* $2 \geq \alpha(y) \geq \delta > 0$*, for some* $\delta > 0$*. Then there exists a unique solution, say* $\tau(\Gamma)$*, of the equation* $\omega(\Gamma, \tau) = 1$ *and*

$$\min\left\{\omega^{-1/\delta}, \omega^{-1/2}\right\} \leq \tau(\Gamma) \leq \max\left\{\omega^{-1/\delta}, \omega^{-1/2}\right\}, \qquad \omega := \omega(\Gamma, 1). \tag{1.23}$$

*Furthermore, assume that either* $G_m$ *is a mixture of symmetric distributions* [*i.e.,* $N(y, \alpha(y)) \equiv 1$], *or that* $|\alpha(y) - 1| > \delta$*, for all* $y \in S_{d-1}$*. Then there exists a constant* $c(\delta, d)$ *such that*

$$\chi_1(g_m) \leq c(\delta, d) K(\Gamma), \tag{1.24}$$

$$\eta(\mathscr{A}_c, G_m) \leq c(\delta, d) K(\Gamma). \tag{1.25}$$

**2. Proofs.** We start this section with an auxiliary lemma. Fourier transforms we denote as

$$\hat{f}(t) := \int_{\mathbb{R}^d} f(x) \exp\{i\langle x, t\rangle\}\, dx.$$

In particular, we have $\varphi = \hat{g}$ [see (1.1) and (1.2)].

LEMMA 5. *Let* $G$ *be a stable nondegenerate distribution on* $\mathbb{R}^d$*. Then its density* $g$ *is a function of the class* $C^\infty(\mathbb{R}^d)$*. Furthermore, the derivatives* $g^{(s)}(x) \times w_1 \cdots w_s$ *as functions of* $x$ *are square integrable and vanish as* $|x| \to \infty$*, for any* $w_1, \dots, w_s \in \mathbb{R}^d$.

PROOF. Without loss of generality we can assume that $|w_j| \leq 1$, for all $j$. Then

$$\big|\langle t, w_1\rangle \cdot \cdots \cdot \langle t, w_s\rangle \varphi(t)\big| \leq |t|^s \big|\varphi(t)\big| \leq |t|^s \exp\{-\varkappa^\alpha(\Gamma)|t|^\alpha\}. \tag{2.1}$$

The bound (2.1) implies that the functions $t \mapsto \langle t, w_1\rangle \cdot \cdots \cdot \langle t, w_s\rangle \varphi(t)\colon \mathbb{R}^d \to \mathbb{C}$ are in $L_2(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$. Hence, $g(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{g}(t) \exp\{-i\langle t, x\rangle\}\, dt$ almost everywhere. Differentiating under the sign of the integral, we see that $g \in C^\infty$.

The Parseval's equality shows that $g$ and its derivatives are square integrable. Finally, due to the Riemann–Lebesgue theorem, these functions vanish as $|x| \to \infty$. $\square$

Unfortunately, Lemma 5 does not imply directly the integrability of the derivatives of $g$. A proof of this integrability is rather involved; see the proof of Theorem 1 below.

PROOF OF THEOREM 1. The estimate (1.12) and the analyticity of $g$ is contained as Theorem 1 in Bogachev (1986). For the sake of completeness, we prove (1.12). Let $Y, Y_1, \dots, Y_s$ be i.i.d. random vectors such that $\mathscr{L}(Y) = G$. Since $G$ is $\alpha$-stable, we have $\mathscr{L}(s^{-1/\alpha}(Y_1 + \cdots + Y_s)) = G$, for any $s \in \mathbb{N}$. Hence, the function $g$ is an $s$-fold convolution, say $g_s^{*s}$, of the function

$$g_s(x) := s^{d/\alpha} g(s^{1/\alpha} x).$$

Using $\|f * g\|_1 \le \|f\|_1 \|g\|_1$, where $\|f\|_1$ denotes the $L_1(\mathbb{R}^d)$-norm of $f$, we have

$$\begin{aligned}\chi(g, w_1, \dots, w_s) &= \int_{\mathbb{R}^d} \left| d_{w_1} \cdots d_{w_s}(g_s^{*s}(x)) \right| dx = \int_{\mathbb{R}^d} \left| (d_{w_1} g_s * \cdots * d_{w_s} g_s)(x) \right| dx \\ &\le \prod_{j=1}^{s} \|d_{w_j} g_s\|_1 \le \left(s^{1/\alpha} \chi_1(g)\right)^s,\end{aligned}$$

whence (1.12) follows.

It remains to prove the bound (1.11) for $\chi_1(g)$. Let us describe the idea of the proof. It is based on integration by parts, repeating this integration $d$ times. In the case of the standard Gaussian distribution $G = \Phi$ one can proceed as follows. Write

$$\widehat{\mathbf{I}}(t; A) = (1 - \Delta)^d \, \widehat{\mathbf{I}}_*(t; A)$$

with

$$\mathbf{I}_*(x; A) := \mathbf{I}(x; A)/(1 + |x|^2)^d,$$

where $\widehat{\mathbf{I}}(t; A)$ denotes the Fourier transform of the function $x \mapsto \mathbf{I}(x; A)$ and $\Delta$ is the Laplace operator. The function $\mathbf{I}_*$ is integrable and $\left|\widehat{\mathbf{I}}_*(t)\right| \le c_d$ with some constant $c_d$ depending on the dimension. Integrating by parts we reduce [cf. (2.9)–(2.14)] the estimation of $\chi_1(g)$ to a proof that

$$\sup_{|w|=1} \int_{\mathbb{R}^d} \left| (1 - \Delta)^d \langle t, w \rangle \varphi(t) \right| dt \le c_d,$$

which clearly holds since $\varphi(t) = \exp\{-|t|^2/2\}$. In the non-Gaussian case $\alpha < 2$ such simple arguments are not applicable since the characteristic function $\varphi$ is not sufficiently smooth. For example, for $\alpha < 1$, it is differentiable at most once. This nondifferentiability enforce us to use a complicated construction of a sequence of (measurable) vector fields, say $b_1, \dots, b_d$. Each of the fields $b_j$ depends on $b_1, \dots, b_{j-1}$ and on many other variables related to the construction. Instead of $(1-\Delta)^d$ we take a differential operator of the form $P(\partial) := (1+id_{b_1})\cdots(1+id_{b_d})$, and instead of $\mathbf{I}_*$ we use a function of the type

$$\mathbf{I}(x;A)\prod_{i=1}^{d}(1+|\langle x, b_i\rangle|)^{-1}. \tag{2.2}$$

We choose the fields $b_j$ such that the function (2.2) is in $L_p(\mathbb{R}^d)$, for some $p > 1$. Furthermore, we have to construct the operator $P(\partial)$ such that the expression $P(\partial)\varphi(t)$ depends only on the first-order derivatives of $\log\varphi(t)$.

Let us return to the proof of (1.11). Consider the normalized measure $\sigma = \Gamma/\varkappa^\alpha(\Gamma)$ and define the function

$$\psi(t) = \exp\Big\{-\int_{S_{d-1}} H(\langle t, y\rangle)\sigma(dy)\Big\} \tag{2.3}$$

with

$$H(z) = |z|^\alpha\big(1 - i\ \mathrm{sign}(z)\tan(\pi\alpha/2)\big), \qquad \alpha \neq 1, \tag{2.4}$$

$$H(z) = |z|\Big(1 + (2i/\pi)\ \mathrm{sign}(z)\log\big|z/\varkappa(\Gamma)\big|\Big), \qquad \alpha = 1. \tag{2.5}$$

Notice that $\varkappa(\sigma) = 1$ and $\varkappa_0(\sigma) = \varkappa_0(\Gamma)/\varkappa^\alpha(\Gamma)$.

Introduce the class $D \subset C^\infty(\mathbb{R}^d)$ of the functions $v$ which satisfy

$$\big|v^{(s)}(t)w_1\cdots w_s\big| \le 1 + |t| \qquad \text{for all } s = 0, 1, \dots \text{ and } |w_i| \le 1.$$

Assuming that a Borel set $A \subset \mathbb{R}^d$ is bounded, define

$$J_0(A) := \sup_{v\in D}\left|\int_{\mathbb{R}^d}\bar{\widehat{\mathbf{I}}}(t;A)v(t)\psi(t)\,dt\right|. \tag{2.6}$$

By $\widehat{\mathbf{I}}(t;A)$ we denote the Fourier transform of the function $x \mapsto \mathbf{I}(x;A)$, and $\bar{z}$ is the complex conjugate of $z$. Let us show that (1.11) is implied by the following bounds

$$J_0(A) \le 2^{-1}(2\pi)^{-d}c(\alpha, d)\varkappa_0^d(\sigma), \qquad \alpha \neq 1, \tag{2.7}$$

$$J_0(A) \le 2^{-1}(2\pi)^{-d}c(1, d)\varkappa_0^d(\sigma)\Big(1 + \big|\log\varkappa(\Gamma)\big|\Big)^d, \qquad \alpha = 1, \tag{2.8}$$

respectively, where the constant $c(\alpha, d)$ satisfies (1.13) [we shall prove (2.7) and (2.8) below, for any bounded measurable subset $A \subset \mathbb{R}^d$]. The definition (1.7) yields

$$\chi_1(g) = \sup\{\chi_1(g, w)\colon w \in \mathbb{R}^d, |w| = 1\}. \tag{2.9}$$

It is clear that

$$\chi_1(g,w)=\int_{\mathbb{R}^d}|d_w g(x)|\,dx=\sup_A\int_A|d_w g(x)|\,dx \\ \le 2\sup_A\left|\int_A d_w g(x)\,dx\right|=2J_* \tag{2.10}$$

with

$$J_*=\sup_A\left|\int_{\mathbb{R}^d}\mathbf{I}(x;A)d_w g(x)\,dx\right|, \tag{2.11}$$

where $\sup_A$ is taken over all bounded Borel sets $A\subset\mathbb{R}^d$. Parseval's equality and the well-known properties of the Fourier transforms imply

$$J_*=(2\pi)^d\sup_A\left|\int_{\mathbb{R}^d}\bar{\widehat{\mathbf{I}}}(t;A)\langle t,w\rangle\varphi(t)\,dt\right|. \tag{2.12}$$

Changing variables $t=u/\varkappa(\Gamma)$ in (2.12), we obtain

$$J_*=(2\pi)^d\varkappa^{-1}(\Gamma)J_{**} \tag{2.13}$$

with

$$J_{**}=\sup_A\left|\int_{\mathbb{R}^d}\bar{\widehat{\mathbf{I}}}(t;A)\langle t,w\rangle\psi(t)\,dt\right|, \tag{2.14}$$

where $\psi$ is defined by (2.3). Obviously, the function $t\mapsto\langle t,w\rangle$ belongs to the class $D$ since $|w|=1$. Hence, $J_{**}\le\sup_A J_0(A)$. This inequality combined with (2.9)–(2.14) and $\varkappa_0(\sigma)=\varkappa_0(\Gamma)/\varkappa^\alpha(\Gamma)$ shows that instead of (1.11) it suffices to prove the estimates (2.7) and (2.8).

Let us prove (2.7) and (2.8). We shall integrate by parts $d$ times. Let us start with a description of the first integration by parts. Choose any vector $b_1\in R^d$ such that $|b_1|=1$. Splitting the set $A=B_1\cup B_2$, where

$$B_1=\{u\colon\langle b_1,u\rangle\ge 0\}\cap A\quad\text{ and }\quad B_2=\{u\colon\langle b_1,u\rangle<0\}\cap A,$$

we can write the following obvious identity

$$\mathbf{I}(x;A)=\mathbf{I}_1(x;B_1)(1+\langle b_1,x\rangle)+\mathbf{I}_1(x;B_2)(1-\langle b_1,x\rangle) \tag{2.15}$$

with

$$\mathbf{I}_1(x;C)=\frac{\mathbf{I}(x;C)}{1+|\langle b_1,x\rangle|},\qquad C=B_1,B_2.$$

Using the well-known properties of the Fourier transform, the identity (2.15) yields

$$\widehat{\mathbf{I}}(t;A)=(1-id_{b_1})\widehat{\mathbf{I}}_1(t;B_1)+(1+id_{b_1})\widehat{\mathbf{I}}_1(t;B_2). \tag{2.16}$$

Let $h(z) = H'(z)$ denote the derivative of the function $H(z)$ defined by (2.4) and (2.5), that is,

$$h(z) = \alpha|z|^{\alpha-1} \operatorname{sign}(z)(1 - i \operatorname{sign}(z) \tan(\pi\alpha/2)), \qquad \alpha \neq 1, \tag{2.17}$$

$$h(z) = \operatorname{sign}(z) + (2i/\pi) + (2i/\pi) \log |z/\varkappa(\Gamma)|, \qquad \alpha = 1. \tag{2.18}$$

Using (2.6) and (2.16), we obtain

$$J_0(A) \leq 2 \max_{B=B_1, B_2} \sup_{v \in D} (Q_1 + Q_2), \tag{2.19}$$

where

$$Q_1 = \Big| \int_{\mathbb{R}^d} \hat{\bar{\mathbf{I}}}_1(t; B) v(t) \psi(t)\, dt \Big|, \qquad Q_2 = \Big| \int_{\mathbb{R}^d} d_{b_1} \hat{\bar{\mathbf{I}}}_1(t; B) v(t) \psi(t)\, dt \Big|.$$

The case $d = 2$ is somewhat simpler compared to the general case $d > 2$ although it involves already some essential features of the general proof. In order to explain compact notation used in the general case, we provide a sketch of the proof for $d = 2$.

THE CASE $d = 2$. We shall show that $J_0(A) \leq M$, where $M$ is a generic constant depending on $d, \alpha$ and $\Gamma$. We shall integrate by parts twice.

Integrating by parts, using $|b_1| = 1$, $d_{b_1}(v\psi) = (d_{b_1} v)\psi + v d_{b_1} \psi$ and $d_{b_1} v \in D$ together with

$$d_{b_1} \psi(t) = \int_{S_1} \psi(t) h(\langle t, y_1 \rangle) \langle y_1, b_1 \rangle \sigma(dy_1),$$

we have

$$Q_2 \leq \int_{S_1} Q_3 \sigma(dy_1) + \sup_{v \in D} Q_1$$

with

$$Q_3 = \Big| \int_{\mathbb{R}^2} \hat{\bar{\mathbf{I}}}_1(t; B) v(t) \psi(t) h(\langle t, y_1 \rangle) \langle y_1, b_1 \rangle\, dt \Big|.$$

Hence

$$J_0(A) \leq 4 \sup_{*} Q_1 + 2 \int_{S_1} \sup_{*} Q_3 \sigma(dy_1), \tag{2.20}$$

where we write $\sup_{*} = \max_{B=B_1, B_2} \sup_{v \in D}$.

In the second integration by parts the choice of the second direction $b_2$ (such that $|b_2| = 1$) depends on the integral under the consideration. In the case of $Q_1$ we choose a unit vector $b_2$ to be orthogonal to $b_1$. Repeating the procedure which allowed us to derive (2.20) from (2.19), we obtain

$$Q_1 \leq 4 \sup_{*} R_1 + 2 \int_{S_1} \sup_{*} R_2 \sigma(dy_2) \tag{2.21}$$

with

$$R_1 = \Big| \int_{\mathbb{R}^2} \hat{\bar{\mathbf{I}}}_{\mathbf{2}}(t; B) v(t) \psi(t)\, dt \Big|, \qquad R_2 = \big| \int_{\mathbb{R}^2} \hat{\bar{\mathbf{I}}}_{\mathbf{2}}(t; B) v(t) \psi(t) h(\langle t, y_2\rangle) \langle y_2, b_2\rangle dt \big|$$

and

$$\mathbf{I}_2(x; B) = \mathbf{I}(x; B)\big(1 + |\langle x, b_1\rangle|\big)^{-1}\big(1 + |\langle x, b_2\rangle|\big)^{-1}. \tag{2.22}$$

In order to estimate $Q_3$ we choose a direction $b_2 = b_2(y_1) \in \mathbb{R}^2$ depending on $y_1$ such that

$$|b_2| = 1 \quad \text{and} \quad \langle y_1, b_2\rangle = 0. \tag{2.23}$$

Our choice of $b_2$ ensures that $d_{b_2} h(\langle t, y_1\rangle) \equiv 0$. Indeed, in the orthogonal basis $\{y_1, b_2\}$ of $\mathbb{R}^2$ we may write $t = t_1 y_1 + t_2 b_2$ with some $t_1, t_2 \in \mathbb{R}$, and $\langle t, y_1\rangle = t_1$ yields $d_{b_2} t_1 = (\partial/\partial t_2) t_1 \equiv 0$. Integrating by parts and repeating again the procedure which allowed us to derive (2.20) from (2.19), we obtain

$$Q_3 \le 4 \sup_* R_3 + 2 \int_{S_1} \sup_* R_4 \sigma(dy_2) \tag{2.24}$$

with $R_3$ defined as $R_2$ replacing $\langle t, y_2\rangle$ and $\langle y_2, b_2\rangle$ by $\langle t, y_1\rangle$ and $\langle y_1, b_1\rangle$, respectively, and

$$R_4 = \big| \int_{\mathbb{R}^2} \hat{\bar{\mathbf{I}}}_2(t; B) v(t) \psi(t) V W\, dt \big|,$$

where we write $V = h(\langle t, y_1\rangle) h(\langle t, y_2\rangle)$ and $W = \langle y_1, b_1\rangle \langle y_2, b_2\rangle$. The function $\hat{\bar{\mathbf{I}}}_2$ is given by (2.22) with $b_2$ from (2.23).

Collecting the bounds (2.20), (2.21) and (2.24), we obtain

$$\begin{aligned} J_0(A) \le 16 \sup_* R_1 + 8 \int_{S_1} \sup_* R_2 \sigma(dy_2) \\ + 8 \int_{S_1} \sup_* R_3 \sigma(dy_1) + 4 \int_{S_1} \int_{S_1} \sup_* R_4 \sigma(dy_1) \sigma(dy_2). \end{aligned}$$

To conclude the sketch of the proof, it suffices to verify that $R_i \le M$, for all $i$. Let us consider the most involved case of $R_4$ only. To simplify the considerations we shall assume as well that $\alpha \ne 1$. Using the Cauchy inequality and Parseval's equality $\|\hat{\bar{\mathbf{I}}}_2\|_2 = \|\mathbf{I}_2\|_2$, we have

$$R_4^2 \le W^2 \|\mathbf{I}_2\|_2^2 I \quad \text{with } I = \int_{\mathbb{R}^2} |v(t)|^2 |\psi(t)|^2 |V|^2\, dt. \tag{2.25}$$

In the case $\langle y_1, b_1\rangle = 0$ or $\langle y_2, b_2\rangle = 0$ we have that $R_4 = 0$. Hence, while estimating $R_4$, we may assume that both $\langle y_1, b_1\rangle$ and $\langle y_2, b_2\rangle$ are nonzero. By our choice, $\{y_1, b_2\}$ is an orthonormal basis of $\mathbb{R}^2$. Let $x_{(1)}, x_{(2)} \in \mathbb{R}$ be the coordinates of $x \in \mathbb{R}^2$ in this basis. Changing the variables

$u_1 = \langle x, b_1\rangle \equiv \langle b_1, y_1\rangle x_{(1)} + \langle b_1, b_2\rangle x_{(2)}$ and $u_2 = \langle x, b_2\rangle \equiv x_{(2)}$, we obtain

$$\begin{aligned} \|\mathbf{I}_2\|_2^2 &\le M \int_{\mathbb{R}^2} \frac{dx}{(1+\langle x, b_1\rangle^2)(1+\langle x, b_2\rangle^2)} \\ &= \frac{M}{|\langle y_1, b_1\rangle|}\Big(\int_{\mathbb{R}} \frac{ds}{1+s^2}\Big)^2 \le \frac{M}{|\langle y_1, b_1\rangle|}. \end{aligned} \tag{2.26}$$

To estimate the integral $I$ we use the basis $\{y_1, b_2\}$ again. Let $t_{(1)}, t_{(2)} \in \mathbb{R}$ be the coordinates of $t \in \mathbb{R}^2$. Introduce the variables

$$u_1 = \langle t, y_1\rangle \equiv t_{(1)}, \quad u_2 = \langle t, y_2\rangle \equiv \langle y_2, y_1\rangle t_{(1)} + \langle y_2, b_2\rangle t_{(2)}.$$

Notice that the vector $u = (u_1, u_2)$ satisfies $|u|^2 \le 2|t|^2$ since $y_1, y_2 \in S_1$. Using in addition the bounds $|v(t)|^2 \le M(1+|t|^2)$ and $|h(s)| \le M|s|^{\alpha-1}$, estimating $\psi(t) \le \exp\{-\varepsilon|t|^\alpha\}$ with some $\varepsilon = \varepsilon(d, \alpha, \Gamma) > 0$, we get

$$\begin{aligned} I &\le M \int_{\mathbb{R}^2} \exp\{-\varepsilon|u|^\alpha/4\} h^2(u_1) h^2(u_2)\, du / |\langle y_2, b_2\rangle| \\ &= \frac{M}{|(y_2, b_2)|}\Big(\int_{\mathbb{R}} \exp\{-\varepsilon s^\alpha/4\} h^2(s)\, ds\Big)^2 \le \frac{M}{|\langle y_2, b_2\rangle|}, \end{aligned} \tag{2.27}$$

provided that $\alpha > 1/2$. Combining (2.25)–(2.27), we obtain

$$R_4 \le M|W| / \sqrt{|\langle y_1, b_1\rangle\langle y_2, b_2\rangle|} = M\sqrt{|W|} \le M$$

since $|\langle y_i, b_i\rangle| \le 1$ and therefore $|W| \le 1$. The case $0 < \alpha \le 1/2$ may be considered similarly, just replacing Cauchy's inequality and Parseval's equality used in (2.25) by Hölder's and Hausdroff–Young inequalities, respectively; see (2.41)–(2.43).

THE CASE $d > 2$. Introducing the Dirac measure $\delta_b$ on $\mathbb{R}^d$ such that $\delta_b(C) = \mathbf{I}(b \in C)$, for $C \subset \mathbb{R}^d$, we can write

$$Q_1 = \int_{S_{d-1}} \Big| \int_{\mathbb{R}^d} \hat{\bar{\mathbf{I}}}_1(t; B) v(t) \psi(t) \langle y_1, b_1\rangle\, dt \Big| \delta_{b_1}(dy_1) \tag{2.28}$$

since $\langle b_1, b_1\rangle = 1$. Integrating by parts, using $|b_1| = 1$ and $d_{b_1} v \in D$ together with

$$d_{b_1}\psi(t) = \int_{S_{d-1}} \psi(t) h(\langle t, y_1\rangle)\langle y_1, b_1\rangle \sigma(dy_1),$$

we have

$$\begin{aligned} Q_2 \le \int_{S_{d-1}} &\Big| \int_{\mathbb{R}^d} \hat{\bar{\mathbf{I}}}_1(t; B) v(t) \psi(t) h(\langle t, y_1\rangle)\langle y_1, b_1\rangle\, dt \Big| \sigma(dy_1) \\ &+ \sup_{v \in D} Q_1. \end{aligned} \tag{2.29}$$

In order to rewrite the bounds (2.19), (2.28) and (2.29) in a more compact form and to proceed with integration by parts, let us introduce additional notation. We shall denote by $\Omega \subset \Omega_j$ a subset of the set $\Omega_j = \{1, 2, \dots, j\}$. Collecting (2.19), (2.28) and (2.29), we obtain

$$J_0(A) \le 6J_1(A), \tag{2.30}$$

where

$$J_1(A) = \max_{\Omega \subset \Omega_1} \sup_{v \in D} \sup_{B \subset A} \int_{S_{d-1}} \left| J_1^*(B) \right| \prod_{k \in \Omega} \sigma(dy_k) \prod_{l \in \Omega_1 \setminus \Omega} \delta_{b_l}(dy_l)$$

with

$$J_1^*(B) = \int_{\mathbb{R}^d} \hat{\bar{\mathbf{I}}}_1(t; B) v(t) \psi t \langle y_1, b_1 \rangle \prod_{k \in \Omega} h(\langle t, y_k \rangle)\, dt.$$

The inequality (2.30) provides the result of the first integration by parts. Let $2 \le j \le d$ and $y_1, \dots, y_d \in S_{d-1}$. In order to describe further integrations by parts, consider vector valued measurable functions

$$b_1, b_2 = b_2(y_1), \dots, b_d = b_d(y_1, \dots, y_{d-1}) \tag{2.31}$$

such that $b_j \in S_{d-1}$ and

$$\langle b_j, y_l \rangle = 0, \tag{2.32}$$

for all $1 \le j \le d$ and $1 \le l \le j - 1$. It is clear that such functions $b_j$ exist. Denote

$$\begin{aligned} J_j(A) = \max_{\Omega \in \Omega_j} \sup_{v \in D} \sup_{B \subset A} \int_{S_{d-1}} \cdots \int_{S_{d-1}} \\ \times \left| J_j^*(B) \right| \prod_{k \in \Omega} \sigma(dy_k) \prod_{l \in \Omega_j \setminus \Omega} \delta_{b_l}(dy_l), \end{aligned} \tag{2.33}$$

where $\Omega_j = \{1, \dots, j\}$, and

$$J_j^*(B) = \int_{\mathbb{R}^d} \hat{\bar{\mathbf{I}}}_j(t; B) v(t) \psi(t) \prod_{l=1}^{j} \langle y_l, b_l \rangle \prod_{k \in \Omega} h(\langle t, y_k \rangle)\, dt$$

with

$$\mathbf{I}_j(x; B) = \mathbf{I}(x; B) \prod_{i=1}^{j} (1 + |\langle x, b_i \rangle|)^{-1}.$$

The integrals $J_j(A)$ defined by (2.33) satisfy

$$J_{j-1}(A) \le 6J_j(A) \quad \text{for all } j = 1, \dots, d. \tag{2.34}$$

To see that (2.34) holds indeed, notice that our choice of $b_j$ as in (2.31) and (2.32) guarantees that

$$d_{b_j} \prod_{k\in\Omega} h\big(\langle t, y_k\rangle\big) = 0 \quad \text{for } \Omega \subset \Omega_{j-1}.$$

Hence, in order to prove (2.34) we can estimate $J_{j-1}(A)$ proceeding similarly as in (2.19), (2.28) and (2.29), which led to the bound (2.30) for $J_0(A)$.

In particular, the bound (2.34) yields

$$J_0(A) \le 6^d J_d(A), \tag{2.35}$$

where $J_d(A)$ is defined by (2.33) with $j = d$ and

$$J_d^*(B) \prod_{l=1}^{d} \langle y_l, b_l\rangle \int_{\mathbb{R}^d} \bar{\widehat{\mathbf{I}}}_d(t; B) v(t)\psi(t) \prod_{k\in\Omega} h\big(\langle t, y_k\rangle\big)\, dt. \tag{2.36}$$

We shall prove that

$$\big|J_d^*(B)\big| \le c_*(\alpha, d), \quad \alpha \ne 1 \tag{2.37}$$

and

$$\big|J_d^*(B)\big| \le c_*(1, d)\Big(1 + \big|\log \varkappa(\Gamma)\big|\Big)^d, \qquad \alpha = 1.$$

The constant $c_*(\alpha, d)$ is specified below, [see (2.52) and the text below], where estimates (1.13) and (1.14) for $c(\alpha, d)$ are proved. Using the definition (2.33) of $J_d(A)$ and integrating the bounds (2.37) with respect to the measure $\sigma$ on $S_{d-1}$, we obtain

$$J_d(A) \le c_*(\alpha, d) \max_{1\le i\le d} \varkappa_0^i(\sigma) \le c_*(\alpha, d)\varkappa_0^d(\sigma), \qquad \alpha \ne 1, \tag{2.38}$$

which combined with the inequality (2.35) proves (2.7). While proving (2.38) we used $1 = \varkappa(\sigma) \le \varkappa_0^\alpha(\sigma)$. Similarly, integrating the second inequality in (2.37), we derive (2.8).

To conclude the proof of the theorem we have to verify (2.37). Consider the matrix $\mathbb{E} = (\langle y_i, b_j\rangle)_{i,\, j=1,\dots,d}$. By our choice [see (2.31) and (2.32)] of the vectors $b_j$ all entries above the diagonal of the matrix $\mathbb{E}$ are equal to zero. Therefore,

$$\det \mathbb{E} = \prod_{l=1}^{d} \langle y_l, b_l\rangle \tag{2.39}$$

and it is clear that

$$|\det \mathbb{E}| \le 1 \tag{2.40}$$

since $|\langle y_l, b_l\rangle \le 1$, for $|b_l| = |y_l| = 1$.

If $\det \mathbb{E} = 0$ then $J_d^*(B) = 0$ [cf. (2.36) and (2.39)] and (2.37) is obviously fulfilled. Hence, without loss of generality we may assume in the proof of

(2.37) that $\det \mathbb{E} \neq 0$. Let $\|f\|_p$ stand for the $L_p(\mathbb{R}^d)$ norm of a function $f$. Using Hölder's inequality with $1/p + 1/q = 1$ such that $1 < p \leq 2$, $q \geq 2$, the relation (2.36) yields

$$\left|J_d^*(B)\right| \leq \left|\det \mathbb{E}\right| \|\widehat{\mathbf{I}}_d(\cdot; B)\|_q \|\vartheta\|_p, \tag{2.41}$$

where we denote for brevity

$$\vartheta(t) = v(t)\psi(t) \prod_{k\in\Omega} h(\langle t, y_k\rangle).$$

To estimate $\|\widehat{\mathbf{I}}_d(\cdot; B)\|_q$ we shall use the fact that the Fourier transform is a bounded operator from $L_p(\mathbb{R}^d)$ to $L_q(\mathbb{R}^d)$, $1 \leq p \leq 2$. The inequality of Hausdroff–Young says that $\|\hat{f}\|_q \leq \|f\|_p$ [see Chapter 5 in Stein and Weiss (1971)], whence

$$\|\widehat{\mathbf{I}}_d(\cdot; B)\|_q \leq \|\mathbf{I}_d(\cdot; B)\|_p.$$

Changing variables $t = u$ with $u = (u_1, \ldots, u_d)$ such that $u_i = \langle t, b_i\rangle$ and introducing the matrix

$$\mathbb{B} := (b_{i,j})_{i,j=1,\ldots,d}, \qquad b_i = (b_{i,1}, \ldots, b_{i,d}), \tag{2.42}$$

we obtain

$$\|\widehat{\mathbf{I}}_d(\cdot; B)\|_q \leq \|\mathbf{I}_d(\cdot; B)\|_p \leq |\det \mathbb{B}|^{-1/p} c_0 \tag{2.43}$$

with

$$c_0 = \left(\int_{\mathbb{R}} (1 + |u|)^{-p}\, du\right)^{d/p} = \big(2/(p-1)\big)^{d/p}. \tag{2.44}$$

Let us estimate $\|\vartheta\|_p$. The function $v$ belongs to the class $D$ and $\varkappa(\sigma) = 1$. Therefore $|\psi(t)| \leq \exp\{-|t|^\alpha\}$ and we get

$$|v(t)||\psi(t)| \leq (1 + |t|)|\psi(t)| \leq c_1 \exp\{-|t|^\alpha/2\}, \tag{2.45}$$

where $c_1 = 1 + \alpha^{-1/\alpha}$. The estimate (2.45) and the obvious inequality

$$d|t|^\alpha = \sum_{i=1}^d |t|^\alpha \geq \sum_{i=1}^d |\langle t, y_i\rangle|^\alpha \quad \text{for } |y_i| = 1, \tag{2.46}$$

yield

$$\|\vartheta\|_p^p \leq c_1^p \int_{\mathbb{R}^d} \prod_{j=1}^d \exp\left\{-\frac{p}{2d}|\langle t, y_j\rangle|^\alpha\right\} \prod_{k\in\Omega} |h(\langle t, y_k\rangle)|^p\, dt. \tag{2.47}$$

Changing in (2.47) variables $t = u$ with $u = (u_1, \ldots, u_d)$ such that $u_i = \langle t, y_i\rangle$ and introducing the matrix

$$\mathbb{A} := (y_{j,i})_{i,j=1,\ldots,d}, \qquad y_i = (y_{i,1}, \ldots, y_{i,d}), \tag{2.48}$$

we get (notice that $0 \le |\Omega| \le d$ and use $a^\beta b^\gamma \le a+b$, for nonnegative $\beta$ and $\gamma$ such that $\beta+\gamma=1$)

$$\|\vartheta\|_p \le \big|\det \mathbb{A}\big|^{-1/p} c_1 c_2^{|\Omega|/p} c_3^{(d-|\Omega|)/p} \le \big|\det \mathbb{A}\big|^{-1/p} c_1\big(c_2^{d/p}+c_3^{d/p}\big), \tag{2.49}$$

where

$$c_2=\int_{\mathbb{R}}\exp\left\{-\frac{p}{2d}|u|^\alpha\right\}|h(u)|^p\,du, \qquad c_3=\int_{\mathbb{R}}\exp\left\{-\frac{p}{2d}|u|^\alpha\right\}du. \tag{2.50}$$

Multiplying the matrices $\mathbb{A}$ and $\mathbb{B}$ we see that $\mathbb{A}\mathbb{B}=\mathbb{E}$. In particular, both changes (2.42) and (2.48) of variables are well defined since $\det\mathbb{E}=\det\mathbb{A}\times\det\mathbb{B}$ and we assume that $\det\mathbb{E}\ne 0$. The relation $\det\mathbb{A}\det\mathbb{B}=\det\mathbb{E}$ combined with the inequalities (2.41), (2.43), (2.49) yields

$$\big|J_d^*(B)\big| \le c_0c_1\big(c_2^{d/p}+c_3^{d/p}\big)\big|\det\mathbb{E}\big|^{1-1/p} \le c_0c_1\big(c_2^{d/p}+c_3^{d/p}\big), \tag{2.51}$$

since (2.40) implies $|\det\mathbb{E}|^{1-1/p}\le 1$, for $p\ge 1$. The inequality (2.37) follows from (2.51) with

$$\begin{aligned} c_*(\alpha,d)&=c_0c_1\big(c_2^{d/p}+c_3^{d/p}\big), \qquad \alpha\ne 1,\\ c_*(1,d)&=\sup_{\varkappa(\Gamma)}\big((1+\big|\log\varkappa(\Gamma)\big|)^{-d}c_0c_1\big(c_2^{d/p}+c_3^{d/p}\big)\big), \qquad \alpha=1, \end{aligned} \tag{2.52}$$

and with $c_*(\alpha,d)$ as in (2.52) in the symmetric case, for all $\alpha$. The bound (2.37) yields (2.7) and (2.8) with

$$c(\alpha,d)\le 2(12\pi)^d c_*(\alpha,d).$$

In order to prove bounds (1.13) and (1.14) for $c(\alpha,d)$ we have to estimate $c_*(\alpha,d)$. To bound $c_*(\alpha,d)$ it suffices to estimate constants $c_2$ and $c_3$ [they are defined in (2.50)] since $c_0$ is given by explicit formula (2.44) and $c_1=1+\alpha^{-1/\alpha}$. The estimation of $c_2$ and $c_3$ is in essence elementary, although somewhat cumbersome. Therefore we shall provide only a sketch of this estimation. Recall that $1<p\le 2$. The constant $c_3$ is obviously finite and can be simply estimated. Using (2.50) and the definition (2.17) and (2.18) of the function $h$, we have

$$\begin{aligned} c_2&=\int_{\mathbb{R}}\exp\left\{-\frac{p}{2d}|u|^\alpha\right\}|h(u)|^p\,du\\ &\le \alpha^p\big(1+|\tan(\pi\alpha/2)|\big)^p\int_{\mathbb{R}}\exp\left\{-\frac{p}{2d}|u|^\alpha\right\}|u|^{p(\alpha-1)}\,du, \end{aligned} \tag{2.53}$$

for $\alpha\ne 1$, and

$$\begin{aligned} c_2\le{}& 4^p\big(1+\big|\log\varkappa(\Gamma)\big|\big)^p\int_{\mathbb{R}}\exp\left\{-\frac{p|u|}{2d}\right\}du\\ &+2^p\int_{\mathbb{R}}\exp\left\{-\frac{p|u|}{2d}\right\}|\log|u||^p\,du, \end{aligned} \tag{2.54}$$

for $\alpha = 1$. The integrals in (2.53) and (2.54) exist if $p(\alpha - 1) > -1$. Therefore, we can choose

$$p = 2 \quad \text{for } \alpha \geq 1, \quad p = \frac{3}{2} \quad \text{for } \frac{1}{2} \leq \alpha < 1, \qquad (2.55)$$
$$p = \frac{1}{2}\left(1 + \frac{1}{1-\alpha}\right) \quad \text{for } 0 < \alpha < \frac{1}{2}.$$

Then, in particular, we change the variables $cu^{\alpha} = y$ and apply the following inequality $y^{\beta}\exp(-y) \leq c(\beta)\exp(-y/2)$. The symmetric case is less complicated since in this case $h(z) = \alpha|z|^{\alpha-1}$ which is simpler than $h$ defined by (2.17) and (2.18). □

PROOF OF THEOREM 2. We shall use a reduction to polyhedrons with finite number of faces as in Bhattacharya and Rao (1976). A convex set $P$ is called a polyhedron if there exist distinct unit vectors $u_1, \ldots, u_m \in S_{d-1}$ and $d_1, \ldots, d_m \in \mathbb{R}$ such that

$$P = \{x \in \mathbb{R}^d \colon \langle u_j, x\rangle \leq d_j, 1 \leq j \leq m\}. \qquad (2.56)$$

Let $\mathscr{P}$ denote the class of compact sets of the form (2.56) with nonempty interior. An inspection of the prof of Theorem 3.1 in Bhattacharya and Rao (1976) shows that

$$\frac{1}{2\varepsilon}\int_{(\partial A)^{\varepsilon}} g(x)\,dx \leq \zeta(\mathscr{P}, g), \qquad \varepsilon > 0, \qquad (2.57)$$

for $A \in \mathscr{A}_c$ and continuous $g$.

Let us prove (1.15). Taking in (2.57) the limit as $\varepsilon \to 0$ yields $\zeta(\mathscr{A}_c, g) \leq \zeta(\mathscr{P}, g)$, which together with the obvious reverse inequality $\zeta(\mathscr{A}_c, g) \geq \zeta(\mathscr{P}, g)$ implies the relation $\zeta(\mathscr{A}_c, g) = \zeta(\mathscr{P}, g)$. Dividing by $2\varepsilon$ the inequality

$$\int_{(\partial A)^{\varepsilon}} g(x)\,dx \leq \varepsilon\eta(\mathscr{A}_c, g) \quad \text{for } A \in \mathscr{A}_c,$$

and passing to the limit as $\varepsilon \to 0$ yields $\zeta(\mathscr{A}_c, g) \leq \eta(\mathscr{A}_c, g)/2$. The inequality (2.57) means that $\eta(\mathscr{A}_c, g) \leq 2\zeta(\mathscr{P}, g) = 2\zeta(\mathscr{A}_c, g)$, and (1.15) follows.

Let us prove (1.16). Due to (1.15) and $\zeta(\mathscr{A}_c, g) = \zeta(\mathscr{P}, g)$ it suffices to verify that

$$\int_{\partial P} g(x)\,ds \leq 2d^{3/2}\chi_1(g) \quad \text{for } P \in \mathscr{P}. \qquad (2.58)$$

Let $n(x)$ denote the unit outer normal vector of $\partial P$ at point $x \in \partial P$. The normal is defined for almost all $x \in \partial P$ with respect to the surface measure $ds$ on $\partial P$. Let $\mathscr{N}$ be the set of $x \in \partial P$ such that $n(x)$ is not defined. Introducing the standard orthonormal vectors $e_1, \ldots, e_d$ in $\mathbb{R}^d$ and writing $e_{-i} = -e_i$, $e_0 = 0$, it is clear that

$$(2.59)\qquad \partial P\Big\backslash \mathbb{N}\subset \bigcup_{i=-d}^{d} Q_i \quad \text{where } Q_i=\{x\in\partial P\colon \langle n(x),e_i\rangle\ge d^{-1/2}\},$$

since any vector $n\in S_{d-1}$ has at least one coordinate, say $n_j$, such that $|n_j|\ge d^{-1/2}$. Using the representation (2.59) and $\int_N g(x)\,ds=0$, we have

$$(2.60)\qquad \int_{\partial P} g(x)ds\le\sum_{i=-d}^{d}\int_{Q_i} g(x)\,ds.$$

The inequality $\langle n(x),e_i\rangle\ge d^{-1/2}$ yields

$$(2.61)\qquad \int_{Q_i} g(x)\,dx\le d^{1/2}\int_{Q_i}\langle n(x),e_i\rangle g(x)\,ds.$$

According to the construction (2.59) of the surface $Q_i$, it is the graph of a piecewise linear function, say $x_i=f(x_1,\dots,x_{i-1},x_{i+1},\dots,x_d)$. Introducing the subgraph $V_i=Q_i+(-\infty,0]e_i$ of the graph of $f$ and applying Stokes' theorem or just using Fubini's theorem and integrating with respect to the $i$th coordinate of $x$, we have

$$(2.62)\qquad \begin{aligned}\int_{Q_i}\langle n(x),e_i\rangle g(x)\,ds&=\int_{\partial V_i}\langle n(x),e_i\rangle g(x)\,ds=\int_{V_i} d_{e_i}g(x)\,dx\\&\le\int_{\mathbb{R}^d}|d_{e_i}g(x)|\,dx\le\chi_1(g).\end{aligned}$$

Combining (2.60)–(2.62), we obtain (2.58). □

PROOF OF THEOREM 3. Write $a=\eta(\mathscr{A}_c,G)$ and $B=\chi_1(g)$. Theorem 1 in Bloznelis (1988) says that

$$(2.63)\qquad \Delta_n(\mathscr{A}_c)\le cc(\alpha)(20)^{1/\alpha}n^{-1/\alpha}(\rho+a\zeta_1+R_\alpha\zeta_{1+\alpha}),$$

where $c$ is an absolute constant,

$$\begin{aligned}R_\alpha&=a+(a+1)(B+B^2),\qquad \alpha\le1,\\R_\alpha&=(a+(a+1)(B+B^2))B,\qquad 1<\alpha\le2,\end{aligned}$$

and $c(\alpha)=1$, for $\alpha\le1, c(\alpha)=\alpha-1$, for $\alpha>1$. Let a random variable $Y$ have the distribution $G$. Notice that $\Delta_n(\mathscr{A}_c)$ does not change if we replace $X_1,\dots,X_n$ by $\tau X_1,\dots,\tau X_n$ and the distribution $G$ by the distribution of $\tau Y$, respectively, for any fixed $\tau>0$. Similarly, the metric $\rho$ remains invariant under this scale transform. The quantities $a=\eta(\mathscr{A}_c,G)$, $B=\chi_1(g)$ and $\zeta_{1+\alpha}$ are transformed to $a/\tau$, $B/\tau$ and $\tau^{1+\alpha}\zeta_{1+\alpha}$, respectively. Hence, the bound (2.63) yields an estimate for $\Delta_n(\mathscr{A}_c)$ as (2.63) but with $R_\alpha$ replaced by $\tau^{\alpha+1}R_\alpha(\tau)$, where

$$R_\alpha(\tau) = a/\tau + (a/\tau + 1)(B/\tau + B^2/\tau^2), \qquad \alpha \le 1,$$
$$R_\alpha(\tau) = (a/\tau + (a/\tau + 1)(B/\tau + B^2/\tau^2))B/\tau, \qquad 1 < \alpha \le 2.$$

Choosing $\tau = B$, we see that (2.63) holds with $R_\alpha$ replaced by

$$B^{\alpha+1} R_\alpha(B) \le 3(a + B)B^\alpha.$$

By Theorems 1 and 2 we can estimate

$$a \le c d^{3/2} c(\alpha, d) K_\alpha(\Gamma), \quad B \le c(\alpha, d) K_\alpha(\Gamma)$$

and Theorem 3 follows. □

PROOF OF THEOREM 4. To prove (1.23) it suffices to use the definition (1.21) of $\omega(\Gamma, \tau)$. The bound (1.24) together with the estimate (1.16) yields (1.25). Therefore we have to prove (1.24) only. We may proceed similarly to the proof of Theorem 1 replacing everywhere $\alpha$ by $\alpha(y)$; for details see a paper of the authors (1999a).

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