The Annals of Probability
2000, Vol. 28, No. 2, 796–850

ON SUPPORT MEASURES IN MINKOWSKI SPACES AND CONTACT DISTRIBUTIONS IN STOCHASTIC GEOMETRY

BY DANIEL HUG AND GÜNTER LAST

Albert-Ludwigs-Universität and Universität Karlsruhe

This paper is concerned with contact distribution functions of a random closed set $\Xi = \bigcup_{n=1}^{\infty} \Xi_n$ in $\mathbb{R}^d$, where the Ξ_n are assumed to be random nonempty convex bodies. These distribution functions are defined here in terms of a distance function which is associated with a strictly convex gauge body (structuring element) that contains the origin in its interior. Support measures with respect to such distances will be introduced and extended to sets in the local convex ring. These measures will then be used in a systematic way to derive and describe some of the basic properties of contact distribution functions. Most of the results are obtained in a general nonstationary setting. Only the final section deals with the stationary case.

1. Introduction. We consider a random closed set Ξ in $\mathbb{R}^d$ (see [21]) of the form

$$\Xi = \bigcup_{n=1}^{\infty} \Xi_n,$$

defined on some probability space $(\Omega, \mathscr{F}, P)$. The *grains* Ξ_n are assumed to be random nonempty convex bodies (that is, nonempty compact convex sets) in $\mathbb{R}^d$ such that each bounded set is intersected by only a finite number of the grains. Such *grain models* can be used to describe a great variety of random patterns occurring in stochastic geometry, stereology and mathematical morphology (see, e.g., [21, 33, 34]). Our main interest in this paper is the investigation of the *contact distribution functions* of Ξ, which are defined as the distribution functions of the random variables

$$d_B(x) := \min\{r \geq 0\colon \Xi \cap (x + rB) \neq \varnothing\}, \qquad x \in \mathbb{R}^d,$$

where the *structuring element* (or *gauge body*) B is assumed to be a convex body which contains the origin in its interior. These functions summarize important information about Ξ and are a fundamental concept in stochastic geometry (see [34]). For a *stationary* (i.e., spatially homogeneous) grain model Ξ, the estimation of the contact distribution functions has been studied extensively (see [3, 4, 11, 10], and the survey in [1]). Such estimators provide a summary description of the random set Ξ and can be used to perform a first

Received June 1998; revised December 1999.
AMS 1991 *subject classifications*. Primary 60D05, 60G57, 52A21; secondary 60G55, 52A22, 52A20, 53C65, 46B20.
Key words and phrases. Stochastic geometry, Minkowski space, contact distribution function, germ-grain model, support (curvature) measure, marked point process, Palm probabilities, random measure.

model check. In [3], for instance, it has been proposed to use the *hazard rate* of the contact distribution function to judge whether the pattern is completely random (i.e., a Boolean model) or not.

Although stationarity is a common assumption in stochastic geometry, it is rather obvious that stationarity cannot be justified in certain applications. Hahn and Stoyan [9] (see also [8]), for instance, have been motivated by examples in materials science and biology to analyze surface processes with a *gradient*. These processes are not invariant under the full group of translations but only with respect to translations that are perpendicular to a specific direction. Further examples for the practical relevance of statistically inhomogeneous random media can be found in [25], a paper that uses methods from stochastic geometry to analyze a specific grain model with spherical grains. Our aim here is to provide an analysis of some of the basic properties of the contact distribution functions of the general nonstationary grain model introduced above. The absence of stationarity requires a careful analysis of the *local* behavior of Ξ and to reach that goal we will combine methods from convex and integral geometry with the theory of random measures and point processes (see [15]). Our main technical tool from convex geometry is the theory of *support* (*curvature*) *measures*. These support measures are associated with locally finite unions of convex bodies, and they are introduced as suitable nonnegative extensions of support measures of convex bodies in Minkowski spaces (finite-dimensional normed vector spaces). This theory is then applied to the support measures of the random set Ξ, and thus we arrive at random support measures. Random curvature measures (with respect to the Euclidean distance) are quite popular in stochastic geometry (see, e.g., [21, 2, 42, 43, 31, 34, 5, 36]) and their densities are important characteristics of stationary grain models.

Let

$$\bar{p}(x) := P(x \in \Xi), \qquad x \in \mathbb{R}^d,$$

denote the *volume density* of Ξ. From now on we adopt the general assumption that $B \subset \mathbb{R}^d$ is a convex body which contains 0 as an interior point. For $\bar{p}(x) < 1$ the contact distribution function of Ξ with respect to B is defined by

$$H_B(x, r) := P\big(d_B(x) \le r | x \notin \Xi\big), \qquad x \in \mathbb{R}^d,\ r \ge 0.$$

If $\bar{p}(x) = 1$, then we set $H_B(x, r) := 1$. Since we have not assumed that Ξ is stationary, all these quantities depend on $x \in \mathbb{R}^d$. Clearly, since $0 \in \operatorname{int} B$ and $\Xi \neq \varnothing$, the contact distribution functions are nondegenerate. If $B = B^d$ is the closed Euclidean unit ball in $\mathbb{R}^d$, then $d_{B^d}(x)$ is the Euclidean distance from x to Ξ and H_B is the *spherical contact distribution function of* Ξ. In the general case we define the *gauge function* $g(B, \cdot)$ of B by

$$g(B, x) := \min\big\{r \ge 0 \colon x \in rB\big\},$$

and then the distance function d_B can be represented as

$$d_B(x) = \min\big\{g(B, y - x) \colon y \in \Xi\big\}.$$

If B is additionally assumed to be centrally symmetric, then $g(B, \cdot)$ defines a norm with unit ball B. The space $\mathbb{R}^d$ equipped with such a norm is called a *Minkowski space* (see [35]). In the following, we use the term Minkowski space, although we do not adopt any symmetry assumptions. It should be emphasized that in such a space all measurements are based on the underlying gauge body alone.

Obviously, we have $0 < d_B(x) \leq r$ if and only if $x \in (\Xi + r\check{B})\backslash\Xi$ with $\check{B} := \{-x \colon x \in B\}$. If Ξ is stationary, this easily implies that $P(0 < d_B(x) \leq r)$ is the volume fraction of the "outer parallel" set $(\Xi + r\check{B})\backslash\Xi$, that is,

$$P\big(0 < d_B(x) \leq r\big) = \big(\mathscr{H}^d(A)\big)^{-1} E\big[\mathscr{H}^d(((\Xi + r\check{B})\backslash\Xi) \cap A)\big],$$

where A is any Borel set with positive and finite Lebesgue measure $\mathscr{H}^d(A)$. For a deterministic convex body $K \subset \mathbb{R}^d$, the volume $\mathscr{H}^d((K + r\check{B})\backslash K)$ can be computed with the aid of certain *mixed volumes* of K and $\check{B}$ (see, e.g., [28]), and a converse statement is also true. If, moreover, B is the Euclidean unit ball, then the classical *Steiner formula* allows us to express $\mathscr{H}^d((K + r\check{B})\backslash K)$ in terms of the intrinsic volumes $V_j(K)$, $j = 0, \ldots, d-1$. For a general element K of the *convex ring* and for B as described above, it is not clear at first glance whether a Steiner-type formula exists for the volume $\mathscr{H}^d((K + r\check{B})\backslash K)$. In the Euclidean context, however, it has recently been shown that even a local version of such a Steiner-type formula exists and that it involves nonnegative extensions $C_j^+(K, \cdot)$, $j = 0, \ldots, d-1$, of the *generalized* curvature (or support) measures on $\mathbb{R}^d \times \mathbb{R}^d$ (see [20]). For convex bodies K these measures, denoted by $C_j(K, \cdot)$, have been introduced in [26] (with a different notation and normalization) as a joint generalization of the *curvature* and the *surface area measures* of K. The treatment of more general structuring elements B suggests the introduction of support measures also in Minkowski spaces. Moreover, it is a fair conjecture that these measures should be useful in a nonstationary probabilistic setting as well. Some reasons for considering general structuring elements have been discussed in [10]. We merely wish to point out that the flexibility gained in being able to treat a great variety of test bodies makes it possible to obtain information about the geometric shape rather than just the size of a random pattern. Recent developments concerning integral geometry in Minkowski spaces (see [29, 32]) partly motivated the present approach.

We are now in a position to formulate the main aims of this paper. Generalizing a construction in [29], we first define support measures $C_j^B(K, \cdot)$, $j = 0, \ldots, d-1$, for convex bodies K with respect to a strictly convex gauge body B containing an open neighborhood of the origin. Secondly, we extend these measures to sets K in the *extended* (*local*) *convex ring*, that is, to typical realizations of the grain model Ξ. In the main part of the paper (Section 4), we will then exploit these support measures in a systematic way to study some of the basic properties of the contact distribution functions such as existence and form of densities. Because we do not assume stationarity, most of our results are new even in the Euclidean case. In the final section, we will show what some of our results look like for a stationary Ξ.

The detailed organization of the paper is as follows. In Section 2, we introduce the support measures $C_j^B(K, \cdot)$ and discuss some of their basic properties. As in the Euclidean case (see [27]) we define them using a local version of the Steiner formula in Minkowski spaces. A similar approach to so-called relative support measures has independently been developed by Kiderlen [16]. The corresponding results from [16] are contained in joint work of Kiderlen and Weil [17]. In Section 3, we discuss additive as well as nonnegative extensions $C_j^B(K, \cdot)$ and $C_j^{B,+}(K, \cdot)$, respectively, of these measures to sets K in the local convex ring. (In Sections 3–5 we will usually not indicate the dependence on B of these and other notions in order to simplify our notation.) The nonnegative measures $C_j^{B,+}(K, \cdot)$ will be obtained as restrictions of the additive extensions $C_j^B(K, \cdot)$ to the Minkowski *normal bundle* $N_B(K)$ of K with respect to B. In fact, we show (Theorem 3.4) that this particular nonnegative extension leads to the same result as another construction, using local parallel sets with multiplicities, which is due to Matheron [21] and Schneider [27] in a Euclidean space. The main result (Theorem 3.3) provides a Steiner-type formula in Minkowski spaces for sets from the local convex ring. An important prerequisite is Theorem 3.2 which shows that the *exoskeleton* of K (see [33, 6]) with respect to the gauge function $g(B, \cdot)$ has Lebesgue measure 0. In the special but important case $j = d - 1$, we show that $C_{d-1}^B(K, \cdot) = C_{d-1}^{B,+}(K, \cdot)$ (Theorem 3.9).

In Section 4, we turn our attention to the general grain model Ξ and consider the random support measures $C_j^{B,+}(\Xi, \cdot)$, $j = 0, \dots, d-1$. Our first result (Theorem 4.1) provides a fundamental relationship between the contact distribution function and the *intensity measure* $\Lambda_{d-1}^{B,+}(\cdot) := E[C_{d-1}^{B,+}(\Xi, \cdot)]$. The Euclidean special case of this result has been proposed in [8]. In fact, our result includes the more general functions

$$H_B(x, r, A) := P(d_B(x) \le r,\ u_B(x) \in A | x \notin \Xi), \tag{1.1}$$

where $A \subset \mathbb{R}^d$ is measurable and $u_B(x)$ is defined by the equality $d_B(x)u_B(x) = x - p_B(x)$, whenever there is a unique point $p_B(x) \in \Xi$ realizing the minimal distance of x from Ξ in the Minkowski space associated with B. Using these more general functions is essentially equivalent to considering the conditional distribution function of the Minkowskian contact vector $x - p_B(x)$ at x given that $x \notin \Xi$. In fact, once we know $H_B(x, \cdot, \cdot)$, we also know the conditional distribution of the random vector $(d_B(x), u_B(x))$ given that $x \notin \Xi$, and this again is equivalent to knowing the conditional distribution function of $x - p_B(x)$ given that $x \notin \Xi$. We remark that, for $\mathscr{H}^d$ almost all $x \in \mathbb{R}^d$, P-a.s. $p_B(x) - x$ is the unique vector which points from x to the unique intersection point of Ξ and $x + d_B(x)B$; moreover, the vector $u_B(x)$ has the same direction as $x - p_B(x)$, and it is normalized in such a way that its endpoint lies on the boundary of $\check{B}$. If $\Lambda_j^{B,+}(\cdot \times \mathbb{R}^d)$ is locally finite for $j = 0, \dots, d-1$ and $\Lambda_{d-1}^{B,+}(\cdot \times A)$ is absolutely continuous with density $\lambda_{d-1}^{B,+}(\cdot, A)$, then we prove that $(1-\bar{p}(x))H_B(x, \cdot, A)$ is "weakly" differentiable at $t = +0$ for $\mathscr{H}^d$-a.e. $x \in \mathbb{R}^d$ and the derivative is just $2\lambda_{d-1}^{B,+}(x, A)$. In the Euclidean case, for instance, $2\Lambda_{d-1}^{B,+}(C \times \mathbb{R}^d)$ is essentially the expected surface content of that part of the boundary of Ξ contained in the set C. Hence, the above result includes as a special case the pleasant fact that the *surface density* $2\lambda_{d-1}^{B,+}(x, \mathbb{R}^d)$ can be obtained as the (weak) limit of the difference quotient $t^{-1}(P(x \in \Xi + t\check{B}) - P(x \in \Xi))$ as $t \to +0$. Matheron (see page 50 of [21]) called such a result a probabilistic version of a well-known integral-geometric principle. In comparison to the Euclidean setting, the surface density $2\lambda_{d-1}^{B,+}(x, \mathbb{R}^d)$ involves an additional weighting function which takes into account the anisotropy of the gauge body B. We should emphasize that even the deterministic special case of Theorem 4.1 is new (compare Remark 4.8). A by-product is the formula

$$\mathscr{H}^d(K + \varepsilon B) = \mathscr{H}^d(K) + \varepsilon \int h_B(u) S_{d-1}(K, du) + o(\varepsilon)$$

as $\varepsilon \to +0$, where K is in the convex ring, h_B denotes the support function of B and $S_{d-1}(K, \cdot)$ is the additive (and nonnegative) extension of the Euclidean surface area measure of order $d-1$ to the convex ring (see Section 4.4 in [28]).

In the second and main part of Section 4 we proceed with a more detailed analysis using the *marked point process* $\Phi := \{(\xi_n, Z_n)\}$, where ξ_n is the "center" of Ξ_n and $Z_n = \Xi_n - \xi_n$ for all $n \in \mathbb{N}$. Under reasonable technical assumptions on Φ the function $(1 - \bar{p}(x))H_B(x, \cdot, A)$ turns out to be absolutely continuous for $\mathscr{H}^d$-a.e. $x \in \mathbb{R}^d$. The densities can be expressed in terms of the *Palm probabilities* $P_{(x,K)}$ of Φ satisfying the heuristic equation $P_{(x,K)} = P(\cdot|(x, K) \in \Phi)$ and the support measures $C_j^B(K, \cdot)$. These results are complemented by formulas for the intensity measures $\Lambda_j^{B,+}(\cdot) := E[C_j^{B,+}(\Xi, \cdot)]$, $j = 0, \dots, d-1$. Palm probabilities are a very important and powerful tool (see, e.g., [22, 15]) and can be used to describe and to analyze the dependency structure of a point process. In the fundamental special case of a Poisson process Φ the Palm probability $P_{(x,K)}$ of Φ arises by adding the point (x, K), that is, $P_{(x,K)}(\Phi \in \cdot) = P((\Phi \cup \{(x, K)\}) \in \cdot)$. This is Slivnyak's theorem. Assuming the intensity measure of Φ to be of the form $f(x, K)\mathscr{H}^d(dx)Q_0(dK)$, the density of the *direction dependent* contact distribution function, which we defined in (1.1), can then explicitly be expressed in terms of the integrals

$$\int \int \mathbf{1}\{b \in A\} f(x - z - tb, K) C_j^B(K, d(z, b)) Q_0(dK),$$

where $j = 0, \dots, d-1$. In more general cases one cannot expect such explicit results. However, important classes of point processes such as Gibbsian point processes, Cox processes and Poisson cluster processes can be characterized by their Palm probabilities. Therefore we will use these examples to illustrate our results.

The final Section 5 treats the stationary case and generalizes some results of [20, 3, 10]. Some further discussion of the stationary situation is contained in [19] and [14]. Section 5 does also contain an extensive discussion of the homogeneous Boolean model (see also [34, 37, 31]).

2. Some results from Minkowski geometry. Minkowski spaces provide a rich framework for geometric investigations. In Sections 2 and 3 we give an introduction to some notions and results which proved useful in Euclidean spaces, but which have not been considered before in the general setting of finite-dimensional normed linear spaces, where a priori no scalar product is available. Although there is a variety of results of a purely geometric nature, which could be investigated by the current approach, our main motivation for the present work is to treat applications in stochastic geometry concerning contact distribution functions.

We start by introducing a few facts from Minkowski geometry assuming, however, some familiarity with notation and basic results of the (Euclidean) geometry of convex bodies (see [28]). By a convex body we mean a nonempty compact convex set. Let $\mathscr{K}^d$ be the set of convex bodies in $\mathbb{R}^d$. In the following, the symbol B will always refer to a convex body belonging to the set $\mathscr{K}^d_{\mathrm{sc}}$ of convex bodies which are strictly convex and contain an open neighborhood of the origin. In Section 4, we will sometimes additionally assume that B is smooth. By this we mean that through each boundary point of B there passes precisely one support plane. We will *not* assume B to be centrally symmetric. Let us denote by $g(B, \cdot) = g_B(\cdot)$ the sublinear gauge function of B.

For a nonempty closed set $K \subset \mathbb{R}^d$, $K \neq \mathbb{R}^d$, we define the distance from $x \in \mathbb{R}^d$ to K with respect to B by

$$d_B(K, x) := \min\{g(B, y - x)\colon y \in K\}.$$

The distance function $d_B(K, \cdot)$ is convex and Lipschitz. It is easy to check that

$$\begin{aligned} d_B(K, x) &= \min\{r \geq 0\colon (x + rB) \cap K \neq \varnothing\} \\ &= \min\{r \geq 0\colon x \in K + r\check{B}\}. \end{aligned}$$

If K is a nonempty closed convex set and $t > 0$, then $x \in \partial(K + t\check{B})$ if and only if $d_B(K, x) = t$. It should be emphasized that all essential geometric notions introduced subsequently will be intrinsically defined, that is, they only depend upon Minkowskian quantities. Nonetheless it is convenient to introduce a (Euclidean) scalar product $\langle \cdot, \cdot \rangle$, which will be helpful in proofs and for reasons of comparison. By B^d and S^{d-1} we denote the corresponding Euclidean unit ball and the unit sphere centered at the origin, respectively. The *support function* of a convex body $L \in \mathscr{K}^d$ is defined by

$$h_L(u) := h(L, u) := \max\{\langle x, u\rangle\colon x \in L\}, \qquad u \in \mathbb{R}^d.$$

Of course, it would be more appropriate to define the support function h_L as a functional which is defined on the dual space of $\mathbb{R}^d$. But since the support function is merely used as an auxiliary tool, this definition, which resorts to Euclidean notions, seems to be legitimate. If L is strictly convex, then h_L is continuously differentiable on $\mathbb{R}^d\backslash\{0\}$, and $\{\nabla h_L(u)\}$ coincides with the support set $F(L,u)$ of L at $u \in \mathbb{R}^d\backslash\{0\}$; see [28] for explicit definitions. Further, if $L \in \mathscr{K}^d$ and $x \in \partial L$, then we define

$$N_{B^d}(L,x) := \{u \in S^{d-1}\colon \langle x,u\rangle = h(L,u)\}$$

and

$$N_{B^d}(L) := \{(x,u) \in \partial L \times \mathbb{R}^d\colon u \in N_{B^d}(L,x)\}.$$

It is well-known that the last definition is consistent with the one given below for general Minkowski spaces. A boundary point $x \in \partial L$ is said to be regular if the linear hull of $N_{B^d}(L,x)$ is one-dimensional, that is, if there exists precisely one hyperplane which separates L and x. The last formulation shows that this definition is independent of Euclidean notions. Finally, we write $\mathscr{H}^r$, $r \geq 0$, for the r-dimensional Hausdorff measure defined with respect to the auxiliary Euclidean metric. Observe, however, that up to a positive constant multiplier $\mathscr{H}^d$ is the unique translation invariant Haar measure on $\mathbb{R}^d$.

Henceforth, we will assume that $K \in \mathscr{K}^d$. Then for any point $x \in \mathbb{R}^d$ there is a unique $y \in K$ such that $d_B(K,x) = g(B, y-x)$. This easily follows from the strict convexity of B. We call $p_B(K,x) := y$ the Minkowski projection of x onto K with respect to B and define

$$u_B(K,x) := \frac{x - p_B(K,x)}{d_B(K,x)} \in \partial \check{B}$$

if $x \notin K$. The *Minkowski normal bundle* $N_B(K)$ of K with respect to B is defined by

$$N_B(K) := \{(p_B(K,x), u_B(K,x))\colon x \in \partial(K + t\check{B})\},$$

for any $t > 0$. That the last definition is independent of the particular choice of the distance parameter t can, for example, be seen from the following lemma, which again will be applied in Section 3.

LEMMA 2.1. *For any* $K \in \mathscr{K}^d$,

$$N_B(K) = \{(x, \nabla h_{\check{B}}(u))\colon (x,u) \in N_{B^d}(K)\}.$$

Let $t > 0$. *Then* $u_B(K,x) = \nabla h_{\check{B}}(u)$ *for any* $x \in \partial(K + t\check{B})$ *and any* $u \in \mathbb{R}^d$ *such that* $(x,u) \in N_{B^d}(K + t\check{B})$. *In particular, for any* $x \in \partial(K + t\check{B})$ *there is some* $u \in \mathbb{R}^d\backslash\{0\}$ *such that* $u_B(K,x) = \nabla h_{\check{B}}(u)$ *and* $(x,u) \in N_{B^d}(K + t\check{B})$.

PROOF. Let $t > 0$ be fixed, let $x \in \partial(K + t\check{B})$ and set $z := p_B(K,x)$. Then $z \in \partial K$, $(x + tB) \cap K = \{z\}$ and $d_B(K,x) = t$. Hence, there is some $u \in S^{d-1}$ such that the hyperplane $H = \{y \in \mathbb{R}^d\colon \langle y - z, u\rangle = 0\}$ separates K and $x + tB$. We can assume that $x + tB \subset \{y \in \mathbb{R}^d\colon \langle y - z, u\rangle \geq 0\}$ and $K \subset \{y \in \mathbb{R}^d\colon \langle y - z, u\rangle \leq 0\}$. The first inclusion implies that $\langle t^{-1}(x-z), u\rangle = h(\check{B}, u)$, and hence $u_B(K,x) = \nabla h_{\check{B}}(u)$. From the second inclusion we get $\langle z,u\rangle = h(K,u)$, and thus $(z,u) \in N_{B^d}(K)$.

Conversely, let $(x,u) \in N_{B^d}(K)$. First, we have $x + t\nabla h_{\check{B}}(u) \in \partial(K + t\check{B})$, since $\langle x + t\nabla h_{\check{B}}(u), u\rangle = h(K + t\check{B}, u)$. This yields that $d_B(K, x + t\nabla h_{\check{B}}(u)) = t$, and hence we obtain $p_B(K, x + t\nabla h_{\check{B}}(u)) = x$ and $u_B(K, x + t\nabla h_{\check{B}}(u)) = \nabla h_{\check{B}}(u)$. This shows that $(x, \nabla h_{\check{B}}(u)) \in N_B(K)$.

For the second statement, let $x \in \partial(K + t\check{B})$ and $t > 0$, and hence $t = d_B(K,x) > 0$. For any $u \in N_{B^d}(K + t\check{B}, x)$ we get $x \in F(K + t\check{B}, u) = F(K,u) + t\{\nabla h_{\check{B}}(u)\}$, and this implies $x - t\nabla h_{\check{B}}(u) \in F(K,u) \subset K$. In addition, it follows that

$$g\big(B, x - t\nabla h_{\check{B}}(u) - x\big) = g\big(B, -t\nabla h_{\check{B}}(u)\big) = tg\big(B, \nabla h_B(-u)\big) = t = d_B(K,x).$$

Thus $x - t\nabla h_{\check{B}}(u)$ satisfies the conditions which characterize $p_B(K,x)$. □

The next lemma implies that $N_B(K)$ is at least homeomorphic to $\partial(K + t\check{B})$, for any $t > 0$. The spaces $\mathscr{K}^d$ and $\mathscr{K}^d_{\mathrm{sc}}$ are endowed with the topology induced by the Hausdorff metric.

LEMMA 2.2. *The map* p: $\mathscr{K}^d_{\mathrm{sc}} \times \mathscr{K}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $(B, K, x) \mapsto p_B(K,x)$, *is continuous.*

PROOF. Let $B_i, B \in \mathscr{K}^d_{\mathrm{sc}}$, $K_i, K \in \mathscr{K}^d$ and x_i, $x \in \mathbb{R}^d$, for $i \in \mathbb{N}$, and assume that $B_i \to B$, $K_i \to K$ and $x_i \to x$ as $i \to \infty$. Let $I \subset \mathbb{N}$ be any infinite set. Then it is sufficient to show that $p_I = p_B(K,x)$, provided that $p_{B_i}(K_i, x_i) \to p_I$ as $i \to \infty$ and $i \in I$.

From Theorem 1.8.7 in [28] we get that $p_I \in K$, since $p_{B_i}(K_i, x_i) \in K_i$ for all $i \in I$. Let $y \in K$ be arbitrarily chosen. Then there are points $y_i \in K_i$, $i \in I$, such that $y_i \to y$ as $i \to \infty$ and $i \in I$. This follows again from Theorem 1.8.7 in [28]. Therefore, for all $i \in I$, $g(B_i, p_{B_i}(K_i, x_i) - x_i) \le g(B_i, y_i - x_i)$. Passing to the limit yields that $g(B, p_I - x) \le g(B, y - x)$. The last conclusion follows from the continuity of the map $\mathscr{K}^d_{\mathrm{sc}} \times \mathbb{R}^d \to [0,\infty)$, $(L,z) \mapsto g(L,z)$, that can easily be checked. □

The principal aim of this section is to provide a construction of *support* (*or generalized curvature*) *measures* for arbitrary convex bodies in a Minkowski space with a strictly convex gauge body $B \in \mathscr{K}^d_{\mathrm{sc}}$. Some of the arguments and underlying ideas have been inspired by the ones in [28], Sections 4.1 and 4.2 and [29]. The present setting, however, is more general.

Fix $K \in \mathscr{K}^d$ and $\rho > 0$ for the moment. By Lemma 2.2, the map

$$f^B_\rho\colon (K + \rho\check{B})\backslash K \to \mathbb{R}^d \times \mathbb{R}^d, \qquad x \mapsto \big(p_B(K,x), u_B(K,x)\big),$$

is continuous and hence measurable. Here and in the following, measurability always refers to the Borel σ-field $\mathscr{B}(T)$ of a topological space T. Thus, for any $D \in \mathscr{B}(\mathbb{R}^d \times \mathbb{R}^d)$, the set

$$M^B_\rho(K,D) := \big\{x \in \mathbb{R}^d\colon 0 < d_B(K,x) \le \rho,\ (p_B(K,x), u_B(K,x)) \in D\big\},$$

which is equal to $(f_\rho^B)^{-1}(D)$, is measurable. A measure $\mu_\rho^B(K,\cdot)$ is defined on the Borel subsets of $\mathbb{R}^d \times \mathbb{R}^d$ by setting

$$\mu_\rho^B(K,\cdot) := \mathscr{H}^d\big(M_\rho^B(K,\cdot)\big).$$

Note that $M_\rho^B(K, \mathbb{R}^d \times \mathbb{R}^d) = (K + \rho\check{B})\backslash K$, which implies that

$$\mu_\rho^B(K, \mathbb{R}^d \times \mathbb{R}^d) = \sum_{j=0}^{d-1} \rho^{d-j}\binom{d}{j} V\big(K[j], \check{B}[d-j]\big); \tag{2.1}$$

the mixed volumes $V(K[j], \check{B}[d-j])$ are, for example, introduced in Section 5.1 of [28].

Using Lemma 2.2, one can easily check that the map $\mu_\rho^B\colon \mathscr{K}^d \times \mathscr{B}(\mathbb{R}^d \times \mathbb{R}^d) \to \mathbb{R}$ enjoys the same properties as in the Euclidean case, that is, analogues of Theorems 4.1.1–4.1.3 in [28] remain true in the setting of Minkowski geometry (compare also [29]).

The measure $\mu_\rho^B(K,\cdot)$ is concentrated on $N_B(K)$. Again essentially the same argument as in the proof of Theorem 4.1.1 in [28] shows that the map $(B, K) \mapsto \mu_\rho^B(K,\cdot)$ from $\mathscr{K}^d_{\mathrm{sc}} \times \mathscr{K}^d$ into the space of finite Borel measures on $\mathbb{R}^d \times \mathbb{R}^d$ is weakly continuous.

In order to establish a *local Steiner formula* in Minkowski spaces, we first consider the case where $K = P$ is a polytope. Let us denote by $\mathscr{F}_j(P)$ the set of j-dimensional faces of P. Fix $j \in \{0, \dots, d-1\}$ and $F \in \mathscr{F}_j(P)$ for the moment. Then we set $N(P, F) := N_{B^d}(P, x_0)$ for an arbitrary $x_0 \in \operatorname{relint} F$. By Lemma 2.1,

$$\begin{aligned}&M_\rho^B\big(P, D \cap (\operatorname{relint} F \times \mathbb{R}^d)\big)\\ &\quad = \big\{a + t\nabla h_{\check{B}}(u)\colon t \in (0,\rho],\ u \in N(P,F),\ a \in \operatorname{relint} F,\ (a, \nabla h_{\check{B}}(u)) \in D\big\}.\end{aligned}$$

Let $F^\perp$ be the orthogonal complement of F, and let $\Pi(\cdot, F^\perp)\colon \mathbb{R}^d \to F^\perp$ denote the orthogonal projection onto $F^\perp$. Here orthogonality refers to our auxiliary scalar product. We define

$$W_\rho^F := \big\{\Pi\big(t\nabla h_{\check{B}}(u), F^\perp\big)\colon t \in (0,\rho],\ u \in N(P,F)\big\}$$

and

$$\begin{aligned}G_F\colon \big\{t\nabla h_{\check{B}}(u)\colon t \in (0,\rho],\ u \in N(P,F)\big\} &\to W_\rho^F,\\ t\nabla h_{\check{B}}(u) &\mapsto \Pi(t\nabla h_{\check{B}}(u), F^\perp).\end{aligned}$$

Since $\Pi\big(t\nabla h_{\check{B}}(u), F^\perp\big) \in t\partial\Pi(\check{B}, F^\perp)$, for $t \in (0,\rho]$ and $u \in N(P,F)$, and since B is strictly convex, it follows that G_F is injective. In fact, it is easy to see that G_F is a homeomorphism. Let $a_F \in \operatorname{relint} F$ be arbitrarily chosen, and set $a_F^\perp := \Pi(a_F, F^\perp)$. Set $G_F^-(\cdot) := g(\check{B}, G_F^{-1}(\cdot))^{-1} G_F^{-1}(\cdot)$ and note that this map is scaling invariant. Then

$$\Pi(\cdot, F^\perp)^{-1}\big(\{z + a_F^\perp\}\big) \cap M_\rho^B\big(P, D \cap (\operatorname{relint} F \times \mathbb{R}^d)\big)$$

is equal to $\{a + G_F^{-1}(z) \in \mathbb{R}^d\colon a \in \operatorname{relint} F,\ (a, G_F^-(z)) \in D\}$ if $z \in W_\rho^F$, and is equal to $\varnothing$ otherwise. An application of Fubini's theorem and the translation invariance of $\mathscr{H}^{d-j}$ hence yield that

$$\mu_\rho^B\big(P, D \cap (\operatorname{relint} F \times \mathbb{R}^d)\big) = \rho^{d-j} \int_{W_1^F} \int_F \mathbf{1}\big\{\big(a, G_F^-(z)\big) \in D\big\} \mathscr{H}^j(da) \mathscr{H}^{d-j}(dz).$$

For $j = 0, \dots, d-1$ and $D \in \mathscr{B}(\mathbb{R}^d \times \mathbb{R}^d)$, this suggests the definition

$$b_{d-j} C_j^B(P, D) := \sum_{F \in \mathscr{F}_j(P)} \int_{W_1^F} \int_F \mathbf{1}\big\{\big(a, G_F^-(z)\big) \in D\big\} \mathscr{H}^j(da) \mathscr{H}^{d-j}(dz),$$

where $b_i := \pi^{i/2}/\Gamma(i/2+1)$ is the volume of an i-dimensional Euclidean unit ball. Thus from

$$\mu_\rho^B(P, D) = \sum_{j=0}^{d-1} \sum_{F \in \mathscr{F}_j(P)} \mu_\rho^B\big(P, D \cap \big(\operatorname{relint} F \times \mathbb{R}^d\big)\big)$$

we finally obtain that

$$\mu_\rho^B(P, D) = \sum_{j=0}^{d-1} \rho^{d-j} b_{d-j} C_j^B(P, D).$$

Essentially in the same way as in Sections 4.1 and 4.2 of [28], the preceding considerations lead to the local Steiner formula (2.2) in a Minkowski space. In the special case $B = B^d$ the following theorem, except for the last statement, boils down to Theorem 4.2.1 from [28]. A function φ on $\mathscr{K}^d$ with values in some Abelian group is called *additive* if $\varphi(K_1 \cup K_2) + \varphi(K_1 \cap K_2) = \varphi(K_1) + \varphi(K_2)$, whenever $K_1, K_2, K_1 \cup K_2 \in \mathscr{K}^d$.

THEOREM 2.3. *For an arbitrary convex body $K \in \mathscr{K}^d$ and $j = 0, \dots, d-1$ there exist finite positive measures $C_j^B(K, \cdot)$ on $\mathscr{B}(\mathbb{R}^d \times \mathbb{R}^d)$ such that*

$$\mu_\rho^B(K, D) = \sum_{j=0}^{d-1} \rho^{d-j} b_{d-j} C_j^B(K, D) \tag{2.2}$$

holds for $\rho > 0$ and $D \in \mathscr{B}(\mathbb{R}^d \times \mathbb{R}^d)$. The mapping $K \mapsto C_j^B(K, \cdot)$ is additive and, for each $D \in \mathscr{B}(\mathbb{R}^d \times \mathbb{R}^d)$, the function $C_j^B(\cdot, D)$ is measurable. The measures $C_j^B(K, \cdot)$ are concentrated on $N_B(K)$. Moreover, the map $(B, K) \mapsto C_j^B(K, \cdot)$ from $\mathscr{K}_{sc}^d \times \mathscr{K}^d$ into the space of Borel measures on $\mathbb{R}^d \times \mathbb{R}^d$ is weakly continuous.

REMARK 2.4. The measures $C_j^B(K, \cdot)$ are called the support (or generalized curvature) measures of K (with respect to B). In the construction of the measures $b_{d-j} C_j^B(K, \cdot)$, Euclidean notions have been used. Nevertheless, these measures clearly are Minkowski quantities, since the measures $\mu_\rho^B(K, \cdot)$ are intrinsically defined and (2.2) holds for all $\rho > 0$. The normalization of the measures $C_j^B(K, \cdot)$ is chosen in such a way that for $B = B^d$ they do not depend on the dimension of the Euclidean space in which the convex body K is embedded and such that the full measures coincide with the intrinsic volumes of K.

We have already seen that the support measures in a Minkowski space possess similar properties as in a Euclidean space. There are some additional features such as the dependence on the gauge body. On the other hand, some properties cannot be preserved in general such as equivariance with respect to the full group of rigid motions. It is easy to see, however, that the support measures in an arbitrary Minkowski space are still equivariant under translations, that is, for all measurable $A, C \subset \mathbb{R}^d$ we have

$$C_j^B(K + x, A \times C) = C_j^B(K, (A - x) \times C), \qquad x \in \mathbb{R}^d. \tag{2.3}$$

Additional invariance properties may be satisfied for particular choices of B with distinguished symmetry properties. For such specific choices of gauge bodies it should be an interesting task to discover additional integral-geometric results which then could be applied to the investigation of random structures.

We finish this section with some further properties needed later in this paper. The map $(z, b, t) \mapsto z + tb$ from $N_B(K) \times (0, \infty)$ to $\mathbb{R}^d \setminus K$ is a homeomorphism with inverse $y \mapsto (p_B(K, y), u_B(K, y), d_B(K, y))$. Using standard arguments we can rewrite the Steiner formula (2.2) as

$$\int_{\mathbb{R}^d \setminus K} f(x) \mathscr{H}^d(dx) = \sum_{j=0}^{d-1} (d-j) b_{d-j} \int_0^\infty t^{d-j-1} \int f(z+tb) C_j^B(K, d(z,b))\, dt, \tag{2.4}$$

where $f: \mathbb{R}^d \to \mathbb{R}$ is measurable and nonnegative.

Next, for any $\rho > 0$, let us denote by p_ρ^B the map

$$p_\rho^B: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d, \qquad (z, b) \mapsto (z + \rho b, b).$$

By an obvious modification of the proof for Theorem 4.2.2 in [28], the following theorem can be established.

THEOREM 2.5. *Let $K \in \mathscr{K}^d$, $D \in \mathscr{B}(\mathbb{R}^d \times \mathbb{R}^d)$, $\rho > 0$, and let $m \in \{0, \dots, d-1\}$. Then*

$$b_{d-m} C_m^B\big(K + \rho \check{B}, p_\rho^B(D)\big) = \sum_{j=0}^{m} \rho^{m-j} \binom{d-j}{d-m} b_{d-j} C_j^B(K, D).$$

By combining (2.4) and Theorem 2.5 (with $m = d-1$), one can easily establish the following disintegration of Lebesgue measure. In a Euclidean space, different proofs have been given in [38], Lemmas 4.1 and 4.2 and [30], Hilfssatz 5.3.1.

COROLLARY 2.6. *Let $K \in \mathscr{K}^d$, and let $f: \mathbb{R}^d \to [0, \infty]$ be measurable. Then*

$$\int_{\mathbb{R}^d \setminus K} f(x) \mathscr{H}^d(dx) = 2 \int_0^\infty \int_{\partial(K + t\check{B})} f(y) C_{d-1}^B(K + t\check{B}, dy \times \mathbb{R}^d)\, dt.$$

The $(d-1)$st support measure $C^B_{d-1}(K,\cdot)$ admits an explicit representation in terms of the $(d-1)$st Euclidean support measure $C^s_{d-1}(K,\cdot)$, defined with respect to the Euclidean distance, and the support function of $\check{B}$. For its formulation we need the following notation. If x is a regular boundary point of a convex body K with $\dim K \neq d-1$ and $u_{B^d}(K,x)$ denotes the uniquely determined Euclidean exterior unit normal vector of K at x, then we set $u_B(K,x) := \nabla h_{\check{B}}(u_{B^d}(K,x))$. If x is a singular boundary point of K, then we give $u_B(K,x)$ some fixed value in $\partial\check{B}$.

PROPOSITION 2.7. *For any* $K \in \mathscr{K}^d$,

$$C^B_{d-1}(K,\cdot) = \int \mathbf{1}\{(x, \nabla h_{\check{B}}(u)) \in \cdot\} h_{\check{B}}(u) C^s_{d-1}(K, d(x,u));$$

moreover, if $\dim K \neq d-1$, *then*

$$2C^B_{d-1}(K, \cdot \times \mathbb{R}^d) = \int_{\partial K} \mathbf{1}\{x \in \cdot\} \langle u_B(K,x), u_{B^d}(K,x)\rangle \mathscr{H}^{d-1}(dx).$$

PROOF. Let P be a d-dimensional polytope. Choose any $F \in \mathscr{F}_{d-1}(P)$, and denote by u_F the uniquely determined Euclidean exterior unit normal vector of P at the facet F. Using the notation of the construction leading to Theorem 2.3, we get

$$W_1^F = \{t\langle u_F, \nabla h_{\check{B}}(u_F)\rangle u_F \colon t \in (0,1]\} = \{t u_F \colon t \in (0, h_{\check{B}}(u_F)]\}$$

and hence

$$\begin{aligned} 2C^B_{d-1}(P,D) &= \sum_{F \in \mathscr{F}_{d-1}(P)} \int_F \mathbf{1}\{(a, \nabla h_{\check{B}}(u_F)) \in D\} h_{\check{B}}(u_F) \mathscr{H}^{d-1}(da) \\ &= 2\int \mathbf{1}\{(x, \nabla h_{\check{B}}(u)) \in D\} h_{\check{B}}(u) C^s_{d-1}(P, d(x,u)). \end{aligned}$$

This proves the result for polytopes. The general case of the first statement can then be deduced by approximation, if the weak continuity of the support measures is exploited.

The second equation follows, for example, from Remark 1 in Schneider [27] and from the first assertion. Also note that $\mathscr{H}^{d-1}$-a.e. boundary point of K is regular so that, almost everywhere with respect to the boundary measure, $u_{B^d}(K,\cdot)$ is equal to the (Euclidean) exterior unit normal vector of K. $\square$

Our next result turns out to be particularly useful in the proof of Theorem 3.2 in Section 3. It is immediately implied by Corollary 2.6 and Proposition 2.7.

COROLLARY 2.8. *Let* $K \in \mathscr{K}^d$, *and let* $f \colon \mathbb{R}^d \to [0,\infty]$ *be measurable. Then*

$$\int_{\mathbb{R}^d \setminus K} f(x) \mathscr{H}^d(dx) = \int_0^\infty \int_{\partial(K + t\check{B})} f(y) h_{\check{B}}(u_{B^d}(K + t\check{B}, y)) \mathscr{H}^{d-1}(dy)\, dt.$$

Alternatively, Corollary 2.8 can be inferred by an application of Federer's coarea formula to the Lipschitz map $d_B(K,\cdot)$.

3. Support measures on the extended convex ring. In this section, we consider sets K in the extended convex ring, sets which can be represented as a union

$$K = \bigcup_{i \in \mathbb{N}} K_i \tag{3.1}$$

of convex sets $K_i \in \mathscr{K}^d$ which is *locally finite*, that is, such that each bounded set is intersected by only a finite number of the sets K_i. As in the Euclidean case (see [27]), one can define the *additive extension* of the support measures from the preceding section to sets from the convex ring, using the inclusion–exclusion principle and a general result on continuous valuations by Groemer [7]. The measures thus obtained are finite signed measures. A *nonnegative extension* of support measures has also been considered previously in the setting of Euclidean geometry. These two extensions have found various applications, for example, in stochastic geometry. One can construct both extensions by considering local parallel sets with multiplicities. Such explicit constructions have the advantage of leading to additional results which cannot be obtained from the valuation-theoretic approach alone.

Subsequently, we will first describe the additive extension of the support measures in a general Minkowski space. In order to state a Steiner-type formula for the volume of local outer parallel sets of sets from the extended convex ring, we then will consider a particular nonnegative extension of support measures to sets from the local convex ring. This extension is obtained by restricting the additive extension of the support measures to suitably defined subsets of $\mathbb{R}^d \times \partial \check{B}$. Then we prove that this particular nonnegative extension leads to the same result as another construction which is due to Matheron ([21], pages 119–122) and Schneider [27] in a Euclidean space. Apparently, this connection is new even in a Euclidean setting.

In a first step, we describe how Schneider's construction of the additive extension of the support measures in a Euclidean space has to be modified in Minkowski spaces. We assume that $K = \bigcup_{i=1}^{r} K_i$, where $K_i \in \mathscr{K}^d$. Let $\chi(\cdot)$ denote the Euler characteristic. Then, for $q, x \in \mathbb{R}^d$ we define the index of K at q with respect to x by

$$j_B(K,q,x) := \begin{cases} 1 - \lim\limits_{\delta \downarrow 0} \lim\limits_{\varepsilon \downarrow 0} \chi\big(K \cap (x + (g_B(q-x) - \varepsilon)B) \cap (q + \delta \check{B})\big), & q \in K, \\ 0, & q \notin K. \end{cases}$$

For convex K this definition yields that

$$j_B(K, q, x) = \begin{cases} 1, & \text{if } q = p_B(K, x), \\ 0, & \text{otherwise.} \end{cases}$$

The existence of the limit in the definition of $j_B(K, q, x)$ and the additivity of $j_B(\cdot, q, x)$ can be proved along similar lines as in the case of a Euclidean gauge body. For $\rho > 0$, $D \in \mathscr{B}(\mathbb{R}^d \times \mathbb{R}^d)$ and $x \in \mathbb{R}^d$, we set

$$c_\rho^B(K, D, x) := \sum_{*} j\big(K \cap (x + \rho B), q, x\big),$$

where the summation $\sum_*$ extends over all $q \in \mathbb{R}^d \backslash \{x\}$ with $(q, \overline{x-q}) \in D$ and $\overline{x-q} := g_B(g-x)^{-1}(x-q)$. We remark that this sum is finite, that $c_\rho^B(K, D, \cdot)$ is the characteristic function of $M_\rho^B(K, D)$ provided K is convex and that $c_\rho^B(\cdot, D, x)$ is additive. Finally, we define

$$\mu_\rho^B(K, D) := \int_{\mathbb{R}^d} c_\rho^B(K, D, x) \mathscr{H}^d(dx).$$

By repeating the argument in [28], pages 221 and 222, we find that

$$\mu_\rho^B(K, D) = \sum_{j=0}^{d-1} \rho^{d-j} b_{d-j} C_j^B(K, D),$$

where $C_j^B(K, \cdot)$ is a finite signed measure on the Borel sets of $\mathbb{R}^d \times \mathbb{R}^d$ for $j = 0, \dots, d-1$. Moreover, the mapping $C_j(\cdot, D)$ is additive on the convex ring for all measurable $D \subset \mathbb{R}^d \times \mathbb{R}^d$.

Two immediate consequences of the preceding construction should be mentioned. First, if $D \subset \mathbb{R}^d \times \mathbb{R}^d$ is measurable, then

$$C_j^B(K, D) = C_j^B\big(K, D \cap (\partial K \times \partial \check{B})\big).$$

Second, assume that K_1, K_2 lie in the convex ring, $A \subset \mathbb{R}^d$ is open, $K_1 \cap A = K_2 \cap A$ and $D \subset A \times \mathbb{R}^d$ is measurable. Then

$$C_j^B(K_1, D) = C_j^B(K_2, D).$$

We express this fact by saying that the support measures are locally defined. Therefore, if K lies in the extended convex ring and $D \subset \mathbb{R}^d \times \mathbb{R}^d$ is measurable and bounded in the first component (that is, $D \subset A \times \mathbb{R}^d$ for a bounded set $A \subset \mathbb{R}^d$), then $C_j^B(K, D)$ can unambiguously be defined by

$$C_j^B(K, D) := C_j^B(K \cap L, D),$$

where $L \in \mathscr{K}^d$ is an arbitrary convex body for which $D \subset \operatorname{int} L \times \mathbb{R}^d$.

In Section 2, we defined the Minkowski normal bundle of a convex set. Now we need a corresponding notion for sets from the extended convex ring. To prepare this definition we first define the set

$$\Pi_B(K, x) := \big\{y \in K: d_B(K, x) = g_B(y-x)\big\}, \qquad x \in \mathbb{R}^d.$$

Then the *exoskeleton* of K with respect to B is defined by

$$\operatorname{exo}_B(K) := \big\{x \in \mathbb{R}^d \backslash K: \operatorname{card} \Pi_B(K, x) \geq 2\big\}.$$

Below we will show that $\operatorname{exo}_B(K)$ is a measurable set of Lebesgue measure zero. For any $x \notin (K \cup \operatorname{exo}_B(K))$ we define $p_B(K, x)$ as the unique point $y \in \partial K$ which satisfies $d_B(K, x) = g_B(y - x)$, and then we define $u_B(K, x)$ as for convex K. For $x \in \operatorname{exo}_B(K) \backslash K$ we give $(p_B(K, x), u_B(K, x))$ some arbitrary but fixed value in $\mathbb{R}^d \times \partial \check{B}$.

Now we define the Minkowski normal bundle $N_B(K)$ of a set K from the extended convex ring with respect to the structuring element B by

$$N_B(K) := \{(p_B(K,x), u_B(K,x))\colon x \notin K \cup \mathrm{exo}_B(K)\},$$

and we set

$$\delta_B(K,z,b) := \inf\{r > 0\colon z + rb \in \mathrm{exo}_B(K)\}, \qquad (z,b) \in N_B(K),$$

where $\inf \varnothing := \infty$. For $(z,b) \notin N_B(K)$ we set $\delta_B(K,z,b) := 0$. Thus we have

$$N_B(K) = \{(z,b)\colon \delta_B(K,z,b) > 0\}.$$

Provided that K and B are clear from the context, we simply write $\delta(z,b)$ instead of $\delta_B(K,z,b)$. Similarly, we usually write $C_j(K,\cdot)$ instead of $C_j^B(K,\cdot)$ from now on.

LEMMA 3.1. *Let K_1, K_2 be in the convex ring, and assume that $D \subset N_B(K_1) \cap N_B(K_2)$ is measurable. Then $C_j(K_1,D) = C_j(K_2,D)$ for $j = 0,\dots,d-1$.*

PROOF. We first remark that $D = \bigcup_{n=1}^{\infty} D_n$, where

$$D_n := \{(z,b) \in D\colon \delta_B(K_1,z,b) > 1/n,\ \delta_B(K_2,z,b) > 1/n\}.$$

Certainly, it is sufficient to prove the assertion for D_n, $n \in \mathbb{N}$ instead of D. Let $x \in \mathbb{R}^d$ and $q \in \mathbb{R}^d \setminus \{x\}$ be given such that $(q, \overline{x-q}) \in D_n$. Then, for any $\rho \in (0, 1/n)$ we obtain that

$$j(K_i \cap (x+\rho B), q, x) = \begin{cases} 1, & \text{if } g_B(q-x) \le \rho, \\ 0, & \text{otherwise,} \end{cases}$$

holds for $i = 1,2$. In fact, $g_B(q-x) > \rho$ implies that $q \notin x + \rho B$ and hence, by the definition of j, $j(K_i \cap (x+\rho B), q, x) = 0$ for $i = 1,2$. Now we assume that $g_B(q-x) \le \rho$. Since $(q, \overline{x-q}) \in N_B(K_i)$, $\delta_B(K_i, q, \overline{x-q}) > 1/n$ and $g_B(q-x) \le \rho < 1/n$, we have

$$(x + g_B(q-x)B) \cap K_i = \{q\}, \qquad i = 1,2.$$

But then $K_i \cap (x + (g_B(q-x) - \varepsilon)B) = \varnothing$ is true for any $\varepsilon > 0$ and $i = 1,2$. This shows, again by the definition of j, that $j(K_i \cap (x+\rho B), q, x) = 1$ for $i = 1,2$. $\square$

The proof of the preceding lemma in particular shows, for any K in the convex ring, that the measures $C_j(K, \cdot \cap N_B(K))$ are nonnegative. Therefore, for any K in the extended convex ring, we can define nonnegative and locally finite measures

$$C_j^+(K,\cdot) := C_j(K, \cdot \cap N_B(K)), \qquad j = 0,\dots,d-1,$$

which are uniquely determined by their values on measurable subsets that are bounded in the first component. It is easy to check that Lemma 3.1 remains true for sets in the extended convex ring.

Our principle aim in this section is to establish an extension of Theorem 3.1 from [20] in the present more general framework. An important ingredient for our proof is the following theorem which generalizes the corresponding fact from Euclidean geometry (see [33]). The known proofs in the Euclidean setting (compare [33, 6]) do not seem to carry over to Minkowski spaces.

THEOREM 3.2. *Let K be an element of the extended convex ring. Then the exoskeleton* $\operatorname{exo}_B(K)$ *is Borel measurable and* $\mathscr{H}^d(\operatorname{exo}_B(K)) = 0$.

PROOF. It is easy to see that $E := \operatorname{exo}_B(K)$ is a countable union of closed sets, and hence E is Borel measurable. In fact, a more general assertion will be mentioned in the course of the proof for Lemma 3.12 below. Obviously, we have

$$E \subset \bigcup_{i\neq j} E_{ij},$$

where E_{ij} is the Borel set of all $x \in \mathbb{R}^d \backslash (K_i \cup K_j)$ such that $d_B(K_i, x) = d_B(K_j, x) > 0$ and $p_B(K_i, x) \neq p_B(K_j, x)$. We show that

$$\mathscr{H}^d(E_{12}) = 0. \tag{3.2}$$

Note that $E_{12} \subset \mathbb{R}^d \backslash K_1$. Hence, from Corollary 2.8 applied to K_1 and $f(x) := \mathbf{1}\{x \in E_{12}\}$ we see that it is sufficient to prove that

$$\int_0^\infty \int_{\partial(K_1+t\check{B})} \mathbf{1}\{y \in E_{12}\} \mathscr{H}^{d-1}(dy)\, dt = 0. \tag{3.3}$$

There is at most a single $t^* > 0$ such that

$$\operatorname{int}(K_1 + t^*\check{B}) \cap \operatorname{int}(K_2 + t^*\check{B}) = \varnothing \quad \text{and} \quad (K_1 + t^*\check{B}) \cap (K_2 + t^*\check{B}) \neq \varnothing.$$

Therefore (3.3) follows as soon as

$$\mathscr{H}^{d-1}(E_{12} \cap \partial(K_1 + t\check{B})) = 0 \tag{3.4}$$

has been established for an arbitrarily chosen $t \in (0, \infty)\backslash\{t^*\}$. To prove this, fix $t \in (0, \infty)\backslash\{t^*\}$. Let $x \in E_{12} \cap \partial(K_1 + t\check{B})$. Then we have $x \in \partial(K_1 + t\check{B}) \cap \partial(K_2 + t\check{B})$. For $i \in \{1, 2\}$ there is some $u_i \in N_{B^d}(K_i + t\check{B}, x)$ such that

$$\nabla h_{\check{B}}(u_i) = u_B(K_i, x) = \frac{x - p_B(K_i, x)}{t}.$$

This follows from Lemma 2.1. From $p_B(K_1, x) \neq p_B(K_2, x)$ we obtain that $\nabla h_{\check{B}}(u_1) \neq \nabla h_{\check{B}}(u_2)$, and thus $u_1 \neq u_2$. Then Theorem 2.2.1(b) in [28] implies that

$$\{u_1, u_2\} \subset N_{B^d}\big((K_1 + t\check{B}) \cap (K_2 + t\check{B}), x\big).$$

In particular, we have $u_1 \neq -u_2$, since $t \neq t^*$. But then x is a singular boundary point of $(K_1 + t\check{B}) \cap (K_2 + t\check{B})$. Thus we have shown that $E_{12} \cap \partial(K_1 + t\check{B})$ is contained in the set of singular boundary points of $(K_1 + t\check{B}) \cap (K_2 + t\check{B})$. Hence, (3.4) follows from Theorem 2.2.4 in [28]. □

The next theorem, which represents a Steiner-type formula, will repeatedly be used in the following sections. In particular, it provides a tool for calculating the volume of outer parallel sets for sets from the extended convex ring.

THEOREM 3.3. *For any K in the extended convex ring and for all measurable nonnegative functions $g\colon \mathbb{R}^d \times \mathbb{R}^d \times [0,\infty) \to [0,\infty]$,*

$$\int \mathbf{1}\{d_B(K,x) > 0\} g\big(p_B(K,x), u_B(K,x), d_B(K,x)\big) \mathscr{H}^d(dx)$$
$$= \sum_{j=0}^{d-1} (d-j) b_{d-j} \int\int \mathbf{1}\{\delta_B(K,z,b) > s\} g(z,b,s) s^{d-j-1} C_j^+(K, d(z,b))\, ds.$$

PROOF. Let K be given as in (3.1). Note that the convex bodies K_i in (3.1) are not uniquely determined by K. It is easy to see that $(z,b) \in N_B(K)$ implies that $(z,b) \in N_B(K_i)$ for some not uniquely determined $i \in \mathbb{N}$. Therefore we can inductively define a (not uniquely determined) decomposition of $N_B(K)$ by setting

$$N_B^i(K) := N_B(K) \cap N_B(K_i) \Big\backslash \bigcup_{j=1}^{i-1} N_B^j(K), \qquad i \in \mathbb{N}.$$

Since $N_B^i(K) \subset N_B(K) \cap N_B(K_i)$, $i \in \mathbb{N}$, Lemma 3.1 and the subsequent discussion imply that

$$\mathbf{1}\{(z,b) \in N_B^i(K)\} C_j(K_i, d(z,b)) = \mathbf{1}\{(z,b) \in N_B^i(K)\} C_j(K, d(z,b))$$

holds for $i \in \mathbb{N}$ and $j = 0, \ldots, d-1$. Summing over all $i \in \mathbb{N}$ shows that

$$C_j^+(K, \cdot) = \sum_{i=1}^{\infty} C_j\big(K_i, \cdot \cap N_B^i(K)\big), \qquad j = 0, \ldots, d-1. \tag{3.5}$$

In order to establish the Steiner-type formula, we set $K^* := \operatorname{exo}_B(K)$. We start by observing that, for every $i \in \mathbb{N}$,

$$x \notin K \cup K^* \quad \text{and} \quad (p_B(K,x), u_B(K,x)) \in N_B^i(K)$$

if and only if

$$\big(p_B(K_i,x), u_B(K_i,x)\big) \in N_B^i(K), \qquad d_B(K_i,x) > 0$$

and

$$\delta_B\big(K, p_B(K_i,x), u_B(K_i,x)\big) > d_B(K_i,x).$$

If either of these conditions is fulfilled, then

$$\big(p_B(K,x), u_B(K,x), d_B(K,x)\big) = \big(p_B(K_i,x), u_B(K_i,x), d_B(K_i,x)\big).$$

Hence, applying (2.4) to K_i and using Theorem 3.2, we find that

$$\begin{aligned}
&\int \mathbf{1}\{d_B(K,x) > 0\} g\big(p_B(K,x), u_B(K,x), d_B(K,x)\big)\mathscr{H}^d(dx) \\
&\quad = \int \mathbf{1}\{x \notin K \cup K^*\} g(p_B(K,x), u_B(K,x), d_B(K,x))\mathscr{H}^d(dx) \\
&\quad = \sum_{i=1}^{\infty} \int \mathbf{1}\{x \notin K \cup K^*\} \mathbf{1}\{(p_B(K,x), u_B(K,x)) \in N_B^i(K)\} \\
&\qquad\qquad \times g\big(p_B(K,x), u_B(K,x), d_B(K,x)\big)\mathscr{H}^d(dx) \\
&\quad = \sum_{i=1}^{\infty} \int \mathbf{1}\{d_B(K_i,x) > 0\} \mathbf{1}\{\delta_B(K, p_B(K_i,x), u_B(K_i,x)) > d_B(K_i,x)\} \\
&\qquad\qquad \times \mathbf{1}\{(p_B(K_i,x), u_B(K_i,x)) \in N_B^i(K)\} \\
&\qquad\qquad \times g\big(p_B(K_i,x), u_B(K_i,x), d_B(K_i,x)\big)\mathscr{H}^d(dx) \\
&\quad = \sum_{i=1}^{\infty} \sum_{j=0}^{d-1} (d-j) b_{d-j} \int\int \mathbf{1}\{\delta_B(K,z,b) > s\} \mathbf{1}\{(z,b) \in N_B^i(K)\} g(z,b,s) \\
&\qquad\qquad\qquad\qquad \times s^{d-j-1} C_j(K_i, d(z,b))\, ds.
\end{aligned}$$

By (3.5) the last sum boils down to the right-hand side of the asserted equality. This completes the proof of the theorem. □

By restricting the measures $C_j(K,\cdot)$, for sets K in the (local) convex ring, to the Minkowski normal bundle $N_B(K)$ we obtained nonnegative extensions $C_j^+(K,\cdot)$ of the Minkowski support measures introduced in Section 2. In Euclidean spaces, based on an idea of Matheron, nonnegative extensions $\overline{C}_j(K,\cdot)$ of support measures to the convex ring have been constructed by Schneider [27] in a different way. Subsequently, we describe how this construction can be carried out in Minkowski spaces and then we explain why the measures $C_j^+(K,\cdot)$ and $\overline{C}_j(K,\cdot)$ coincide.

For a set K in the convex ring and $x \in \mathbb{R}^d$, we let $\overline{\Pi}_B(K,x)$ be the set of all $q \in K$ for which there exists a neighborhood U of q such that $g_B(y-x) > g_B(q-x)$ for all $y \in U \cap K$ with $y \neq q$. Let $K = \bigcup_{i=1}^r K_i$ with $K_i \in \mathscr{K}^d$. Then $q \in \overline{\Pi}_B(K,x)$ if and only if $q = p_B(K_i,x)$ for all $i \in \{1,\dots,r\}$ with $q \in K_i$. For K in the convex ring, for a measurable set $D \subset \mathbb{R}^d \times \mathbb{R}^d$ and for any $\varepsilon > 0$, we set

$$\bar{c}_\varepsilon(K,D,x) := \operatorname{card}\big\{q \in \overline{\Pi}_B(K,x) \colon q \in (x+\varepsilon B)\setminus\{x\},\ (q, \overline{x-q}) \in D\big\}.$$

Imitating Schneider's [27] arguments, one can verify that

$$\int_{\mathbb{R}^d} \bar{c}_\varepsilon(K,D,x)\mathscr{H}^d(dx) = \sum_{j=0}^{d-1} b_{d-j}\varepsilon^{d-j}\overline{C}_j(K,D), \tag{3.6}$$

where the $\overline{C}_j(K,\cdot)$ are nonnegative measures on $\mathscr{B}(\mathbb{R}^d \times \mathbb{R}^d)$ that coincide, for convex bodies K, with the curvature measures from Section 2. These measures are locally defined and their definition can hence be extended to sets K in the extended convex ring.

Let K be given as at the beginning of this section. The following notation will be required. Let $S(\mathbb{N})$ be the set of all nonempty finite subsets of $\mathbb{N}$. For $v \in S(\mathbb{N})$ we write

$$K_v := \bigcap_{i\in v} K_i \quad \text{and} \quad K^{(v)} := \bigcup_{i\in \mathbb{N}\setminus v} K_i.$$

At first sight, the next theorem is surprising, since in the construction of $\overline{C}_j(K,\cdot)$ and $C_j^+(K,\cdot)$, respectively, different types of multiplicities are involved.

THEOREM 3.4. *For a set K in the extended convex ring and $j \in \{0,\dots, d-1\}$,*

$$C_j^+(K,\cdot) = \sum_{v\in S(\mathbb{N})} \int \mathbf{1}\{z \notin K^{(v)}\}\mathbf{1}\left\{(z,b) \in \cdot \cap \bigcap_{i\in v} N_B(K_i)\right\} C_j(K_v, d(z,b)).$$

Moreover, $C_j^+(K,\cdot) = \overline{C}_j(K,\cdot)$ holds for $j = 0,\dots,d-1$.

PROOF. From the definitions it is easy to check that we have the disjoint decomposition

$$N_B(K) = \bigcup_{v\in S(\mathbb{N})} \left[\big((\mathbb{R}^d\setminus K^{(v)}) \times \mathbb{R}^d\big) \cap \bigcap_{i\in v} N_B(K_i)\right]. \tag{3.7}$$

Let $D \subset \mathbb{R}^d \times \mathbb{R}^d$ be measurable. Since

$$D \cap \big((\mathbb{R}^d\setminus K^{(v)}) \times \mathbb{R}^d\big) \cap \bigcap_{i\in v} N_B(K_i) \subset N_B(K) \cap N_B(K_v),$$

for $v \in S(\mathbb{N})$, we obtain from Lemma 3.1 and the subsequent discussion that

$$\begin{aligned} C_j^+(K,D) &= C_j(K, D\cap N_B(K)) \\ &= \sum_{v\in S(\mathbb{N})} C_j\left(K, D\cap \big((\mathbb{R}^d\setminus K^{(v)}) \times \mathbb{R}^d\big) \cap \bigcap_{i\in v} N_B(K_i)\right) \\ &= \sum_{v\in S(\mathbb{N})} C_j\left(K_v, D\cap \big((\mathbb{R}^d\setminus K^{(v)}) \times \mathbb{R}^d\big) \cap \bigcap_{i\in v} N_B(K_i)\right). \end{aligned}$$

This establishes the first assertion.

The proof of the second assertion now immediately follows from a straightforward extension of (3.11) from [27] to Minkowski spaces, since the measures C_j^+ and $\overline{C}_j$ are locally defined. □

Let again $K = \bigcup_{i=1}^{\infty} K_i$ be in the extended convex ring. Then we denote by K^+ the set of all $z \in \partial K$ for which there is some $b \in \mathbb{R}^d$ such that $(z, b) \in N_B(K)$. Further, we simply write $K^{(n)}$ instead of $K^{(\{n\})}$ if $n \in \mathbb{N}$. A probabilistic application of the following corollary of Theorem 3.4 will be essential in the following section.

COROLLARY 3.5. *Let K be in the extended convex ring. Assume that*

$$K^+ = \bigcup_{n=1}^{\infty} \big(\partial K_n \backslash K^{(n)}\big).$$

Then

$$C_j^+(K, \cdot) = \sum_{i=1}^{\infty} C_j\Big(K_i, \cdot \cap \big((\mathbb{R}^d \backslash K^{(i)}) \times \mathbb{R}^d\big)\Big).$$

A pair $(z, b) \in \partial K \times \partial \check{B}$ is called a *support element* of K (with respect to B) if there is some $s > 0$ such that $(z + sb + sB)$ and $K \cap (z + s\check{B})$ can be separated by a hyperplane. The set of support elements of K (with respect to B) is denoted by $\Sigma_B(K)$.

LEMMA 3.6. *Let K be in the extended convex ring. Then $\Sigma_B(K) \subset N_B(K)$ with equality if B is smooth.*

PROOF. Let $(z, b) \in \Sigma_B(K)$, and choose $s > 0$ as in the definition of a support element. Then $z + tb + tB \subset z + sb + sB$ for any $t \in [0, s]$. If $t \in (0, s]$ is small enough, then $z + tb + tB \subset z + s\check{B}$ because $\operatorname{int} \check{B} \neq \varnothing$. This shows that $p_B(K, z + tb) = z$ and $u_B(K, z + tb) = b$, and hence $(z, b) \in N_B(K)$.

Now, assume that B is smooth and let $(z, b) \in N_B(K)$. Then there exists some $x \in \mathbb{R}^d \backslash (K \cup \operatorname{exo}_B(K))$ such that $(x + dB) \cap K = \{z\}$ with $d = d_B(K, x)$. Since K is a locally finite union of convex bodies K_i, $i \in \mathbb{N}$, there exists some $s \in (0, d]$ such that $(z + s\check{B}) \cap K_i = \varnothing$ if $z \notin K_i$. If $z \in K_i$, then $K_i \cap (x + dB) = \{z\}$. Therefore K_i and $(x + dB)$ can be separated by the unique support plane of $(x + dB)$ at z. But then $(z + sb + sB)$ and $K \cap (z + s\check{B})$ can be separated by this support plane, since $z + sb + sB \subset z + db + dB = x + dB$. □

The relationship between the Minkowski normal bundles of K with respect to different gauge bodies is depicted by the following lemma.

LEMMA 3.7. *Let K be in the extended convex ring. Then*

$$\big\{(x, \nabla h_{\check{B}}(u)) \colon (x, u) \in N_{B^d}(K)\big\} \subset N_B(K)$$

with equality if B is smooth.

PROOF. Let $(x,u)\in N_{B^d}(K)$. By Lemma 3.6, $(x,u)\in\Sigma_{B^d}(K)$. Hence there is some $s>0$ such that $(x+su+sB^d)$ and $K\cap(x+sB^d)$ can be separated by a hyperplane. Thus

$$K\cap(x+sB^d)\subset\{y\in\mathbb{R}^d:\ \langle y-x,u\rangle\le 0\}=:H^-.$$

If $\varepsilon>0$ is sufficiently small, then

$$\begin{aligned}x+\varepsilon\nabla h_{\check{B}}(u)+\varepsilon B&\subset(x+sB^d)\cap\big[(\mathbb{R}^d\backslash H^-)\cup\{x\}\big]\\&\subset(x+sB^d)\backslash\big[K\cap(x+sB^d)\big]\cup\{x\}\subset(\mathbb{R}^d\backslash K)\cup\{x\},\end{aligned}$$

and thus $(x+\varepsilon\nabla h_{\check{B}}(u)+\varepsilon B)\cap K=\{x\}$. This yields $x=p_B(K,x+\varepsilon\nabla h_{\check{B}}(u))$ and $\nabla h_{\check{B}}(u)=u_B(K,x+\varepsilon\nabla h_{\check{B}}(u))$, and hence $(x,\nabla h_{\check{B}}(u))\in N_B(K)$.

Conversely, let B be smooth and $(z,b)\in N_B(K)$. Again by Lemma 3.6 we have $(z,b)\in\Sigma_B(K)$. Thus $(z+sb+sB)$ and $K\cap(z+s\check{B})$ can be separated by a hyperplane H if $s>0$ is properly chosen. Since B is strictly convex, there exists a vector $u\in S^{d-1}$ such that $b=\nabla h_{\check{B}}(u)$. Since $z\in z+sb+sB$, $h(z+sb+sB,-u)=\langle z,-u\rangle$ and B is smooth, we see that $H=\{y\in\mathbb{R}^d:\ \langle y-z,u\rangle=0\}$ and

$$z+sb+sB\subset H^+:=\{y\in\mathbb{R}^d:\ \langle y-z,u\rangle\ge 0\}.$$

Hence

$$K\cap(z+s\check{B})\subset\{y\in\mathbb{R}^d:\ \langle y-z,u\rangle\le 0\}.$$

From $\varepsilon B^d\subset\check{B}$, for some $\varepsilon>0$, we get that $(z,u)\in\Sigma_{B^d}(K)=N_{B^d}(K)$. □

Simple examples show that the inclusions in the preceding two lemmas may be strict if B is not smooth. Nevertheless, the following theorem, which should be compared with Theorem 2.2 in [37], holds without any smoothness assumption on B. First, however, we need another preparatory lemma.

LEMMA 3.8. *Let $K=\bigcup_{i=1}^\infty K_i$, $K_i\in\mathscr{K}^d$, be in the extended convex ring. For such a representation of K, let T be the set of all $(q,u)\in\partial K\times S^{d-1}$ for which there is some $i\in\mathbb{N}$ such that $q\in\operatorname{relint}F(K_i,-u)$, $\dim F(K_i,-u)=d-1$ and $\dim K_i=d$. Then $C_j^s(K,\cdot\cap T)=0$ for $j=0,\dots,d-1$.*

PROOF. Let T_n^i, $i,n\in\mathbb{N}$, be the set of all $(q,u)\in T$ for which $q+tu+tB^d\subset K_i$ holds for $0<t<1/n$. An elementary geometric argument shows that

$$T=\bigcup_{i=1}^\infty\bigcup_{n=1}^\infty T_n^i.$$

It suffices to prove that $C_j^s(K,\cdot\cap T_n^i)=0$ for $j=0,\dots,d-1$ and $i,n\in\mathbb{N}$. Fix $i,n\in\mathbb{N}$. Take $\rho\in\mathbb{R}$ with $0<\rho<1/n$ and $q,x\in\mathbb{R}^d$, satisfying $x\ne q$ and $(q,\overline{x-q})\in T_n^i$. If $g_{B^d}(q-x)\le\rho$ then we may particularly choose $t=g_{B^d}(q-x)$ in the definition of T_n^i to obtain that $x+tB^d\subset K_i$. This shows that $j_{B^d}(K\cap(x+\rho B^d),q,x)=0$. Hence, $\mu_\rho^{B^d}(K,\cdot\cap T_n^i)=0$ for $0<\rho<1/n$, which in turn implies that $C_j^s(K,\cdot\cap T_n^i)=0$, $j=0,\dots,d-1$. □

THEOREM 3.9. *Let K be in the extended convex ring. Then $C_{d-1}(K,\cdot) = C^+_{d-1}(K,\cdot)$.*

PROOF. It suffices to assume that K lies in the convex ring. We consider a representation

$$K = \bigcup_{i=1}^{r} K_i, \qquad K_i \in \mathscr{K}^d.$$

Let T denote the set which is associated with this particular representation of K as in Lemma 3.8. We write $S(r)$ for the set of all nonempty subsets of $\{1,\dots,r\}$. For $v \in S(r)$, let $|v|$ be the number of elements of the set v and let $\operatorname{reg} K_v$ denote the set of regular boundary points of K_v (compare Section 2). Then we define

$$R := \bigcup_{v\in S(r)} \big(\partial K_v \backslash \operatorname{reg} K_v\big) \cap \partial K$$

and note that $\mathscr{H}^{d-1}(R) = 0$.

Let $v \in S(r)$ be arbitrarily chosen. We wish to show that if $x \in \partial K\backslash R$ and $(x,u) \in N_{B^d}(K_v)\backslash T$, then $(x,u) \in N_{B^d}(K)$. In order to verify this, we denote by I the set of all $i \in \{1,\dots,r\}$ such that $x \in K_i$ and we set $H := \{y \in \mathbb{R}^d\colon \langle y-x,u\rangle = 0\}$, $H^+ := \{y \in \mathbb{R}^d\colon \langle y-x,u\rangle \geq 0\}$ and $H^- := \{y \in \mathbb{R}^d\colon \langle y-x,u\rangle \leq 0\}$. For any $i \in I$ we have $K_i \subset H^+$ or $K_i \subset H^-$, since otherwise $x \notin \partial K$ or $x \in \partial(K_i \cap K_v)\backslash\operatorname{reg}(K_i\cap K_v)$, that is $x \in R$. Assume that there are $i, j \in I$ such that $x \in K_i \cap K_j$, $K_i \cap (H^-\backslash H) \neq \varnothing$ and $K_j \cap (H^+\backslash H) \neq \varnothing$. Then $x \in \operatorname{relint} K_{\{i,j\}} \subset H$ and $\dim K_{\{i,j\}} = d-1$, since $x \notin R$. But then $x \in \operatorname{int}(K_i \cup K_j) \subset \operatorname{int} K$, a contradiction. Therefore, $K_i \subset K^+$ holds for all $i \in I$ or $K_i \subset H^-$ holds for all $i \in I$. If the latter is true, then $(x,u) \in N_{B^d}(K)$. Now, assume that $K_i \subset H^+$ holds for all $i \in I$. Thus $K_v \subset H^+ \cap H^- = H$. From $x \in \operatorname{reg}(K_i \cap K_v)$, for all $i \in I$, we infer that $x \in \operatorname{relint}(K_i \cap H)$ and $\dim(K_i \cap H) = d-1$ for $i \in I$. This implies that $(x,u) \in \partial K \times S^{d-1}$, $x \in \operatorname{relint} F(K_i,-u)$ and $\dim F(K_i,-u) = d-1$. Since $(x,u) \notin T$, we get $\dim K_i \leq d-1$, and hence $K_i \subset H$, for all $i \in I$. This shows again that $(x,u) \in N_{B^d}(K)$.

Lemma 3.7 shows that $(x,u) \in N_{B^d}(K)$ implies $(x, \nabla h_{\check{B}}(u)) \in N_B(K)$. Using Proposition 2.7 and Lemma 3.8 we obtain for all measurable $D \subset \mathbb{R}^d \times \mathbb{R}^d$ that

$$\begin{aligned}
C_{d-1}(K,D) &= C_{d-1}(K, D\cap(\partial K\times\mathbb{R}^d))\\
&= \sum_{v\in S(r)} (-1)^{|v|-1} C_{d-1}\big(K_v, D\cap(\partial K\times\mathbb{R}^d)\big)\\
&= \sum_{v\in S(r)} (-1)^{|v|-1} \int \mathbf{1}\big\{(x,\nabla h_{\check{B}}(u)) \in D\big\}\mathbf{1}\{x\in\partial K\backslash R\} h_{\check{B}}(u) C^s_{d-1}(K_v, d(x,u))\\
&= \int \mathbf{1}\big\{(x,\nabla h_{\check{B}}(u)) \in D\big\}\mathbf{1}\{x\in\partial K\backslash R\} h_{\check{B}}(u)\mathbf{1}\{(x,u)\notin T\} C^s_{d-1}(K, d(x,u))
\end{aligned}$$

$$= \sum_{v\in S(r)} (-1)^{|v|-1} \int \mathbf{1}\{(x, \nabla h_{\check{B}}(u)) \in D\}\mathbf{1}\{x \in \partial K\setminus R\} h_{\check{B}}(u)$$
$$\times \mathbf{1}\{(x,u)\notin T\} C^s_{d-1}(K_v, d(x,u))$$
$$= \sum_{v\in S(r)} (-1)^{|v|-1} \int \mathbf{1}\{(x, \nabla h_{\check{B}}(u)) \in D\cap N_B(K)\}\mathbf{1}\{x \in \partial K\setminus R\} h_{\check{B}}(u)$$
$$\times \mathbf{1}\{(x,u)\notin T\} C^s_{d-1}(K_v, d(x,u))$$
$$= C_{d-1}(K, D\cap N_B(K)).$$

This yields the desired conclusion. □

A boundary point $x \in \partial K$ is called regular if there is a neighborhood U of x such that $K \cap U$ and $\{x\}$ can be separated by a uniquely determined hyperplane. Note that $x \in \partial K$ is regular if and only if the linear hull of all vectors $u \in \mathbb{R}^d$ such that $(x, u) \in N_{B^d}(K)$ is one-dimensional. If there exists precisely one such vector, then this vector is denoted by $u_{B^d}(K, x)$. By $\operatorname{reg} K$ we denote the set of regular boundary points of K. Note that our definition of a regular boundary point of a set from the extended convex ring coincides with the one proposed by Weil [37]. Finally, we set $(p_B(K, x), u_B(K, x)) := (x, \nabla h_{\check{B}}(u_{B^d}(K, x)))$ if $x \in \operatorname{reg} K$ and $u_{B^d}(K, x)$ is defined.

PROPOSITION 3.10. *Let K be in the extended convex ring. Then*

$$C^+_{d-1}(K,\cdot) = \int_{N_{B^d}(K)} \mathbf{1}\{(x, \nabla h_{\check{B}}(u)) \in \cdot\} h_{\check{B}}(u) C^s_{d-1}(K, d(x,u)),$$

where $C^s_{d-1}(K,\cdot)$ denotes the $(d-1)$st *support measure with respect to the Euclidean ball. Moreover, if $K = \operatorname{cl}\operatorname{int} K$, then*

$$2C^+_{d-1}(K,\cdot) = \int_{\operatorname{reg} K} \mathbf{1}\{(x, u_B(K,x)) \in \cdot\}\langle u_B(K,x), u_{B^d}(K,x)\rangle \mathscr{H}^{d-1}(dx).$$

PROOF. For the proof we can assume that K lies in the convex ring. By employing twice Theorem 3.9, the first assertion follows from an application of Proposition 2.7 to the sets K_v in the same way as in the proof of Theorem 3.9.

The second equation is implied by the first equation and by Theorem 2.2 in Weil [37]. □

In the remaining part of this section, we will establish some auxiliary results which are required to justify questions of measurability in the sequel. By $\mathscr{F}'(\mathbb{R}^d)$ we denote the class of nonempty closed subsets of $\mathbb{R}^d$ endowed with the usual Fell–Matheron "hit-or-miss" topology (compare [21]). Note that this topology is independent of any Euclidean metric.

LEMMA 3.11. *The map d_B: $\mathscr{F}'(\mathbb{R}^d) \times \mathbb{R}^d \to [0,\infty)$ is continuous.*

PROOF. Suppose that $(F_i, x_i) \to (F, x)$ in $\mathscr{F}'(\mathbb{R}^d)\times\mathbb{R}^d$ as $i\to\infty$. Set $d_i := d_B(F_i, x_i)$, for $i \in \mathbb{N}$, and $d := d_B(F, x)$. The sequence $(d_i)_{i\in\mathbb{N}}$ is bounded.

To see this note that there exists some $z \in (x + dB) \cap F$. Thus there are $z_i \in F_i$, for each $i \in \mathbb{N}$, with $z_i \to z$ as $i \to \infty$. But for $i \geq i_0$ we then have $z_i \in x+(d+1)B \subset x_i+(d+2)B$, and hence $d_i \leq d+2$. Therefore it is sufficient to show that $d_i \to d_0$ as $i \to \infty$ implies that $d = d_0$. For each $i \in \mathbb{N}$ there is some $z_i \in (x_i + d_i B) \cap F_i$. The sequence $(z_i)_{i \in \mathbb{N}}$ is bounded. Hence, for a subsequence we have $z_{i_j} \to z_0 \in \mathbb{R}^d$ as $j \to \infty$. This yields $z_0 \in F \cap (x + d_0 B)$, and thus $d \leq d_0$.

To obtain the reverse estimate, choose $z \in (x + dB) \cap F$. For each $i \in \mathbb{N}$ there is some $z_i \in F_i$ with $z_i \to z$ for $i \to \infty$. Let $\varepsilon > 0$ be arbitrarily chosen. Then, for $i \geq i_0$ we get $z_i \in (x_i + (d+\varepsilon)B) \cap F_i$; that is, $d_i \leq d + \varepsilon$, and hence $d_0 \leq d + \varepsilon$. Since $\varepsilon > 0$ was arbitrarily chosen, the result follows. □

The exoskeleton $\mathrm{exo}_B(F)$ of an arbitrary set $F \in \mathscr{F}'(\mathbb{R}^d)$ is defined in the same way as for sets from the extended convex ring.

LEMMA 3.12. *The set* $\mathscr{U} := \{(F, x) \in \mathscr{F}'(\mathbb{R}^d) \times \mathbb{R}^d\colon x \in \mathrm{exo}_B(F)\}$ *is measurable.*

PROOF. For $n \in \mathbb{N}$ and $F \in \mathscr{F}'(\mathbb{R}^d)$, we denote by $\mathrm{exo}_B^n(F)$ the set of all $x \in \mathbb{R}^d$ for which there exist $y_1, y_2 \in F$ such that $d_B(F, x) = g_B(y_1 - x) = g_B(y_2 - x)$ and $\|y_1 - y_2\| \geq 1/n$.

By definition we then have $\mathscr{U} = \bigcup_{n=1}^{\infty} \mathscr{U}_n$, where

$$\mathscr{U}_n := \{(F, x) \in \mathscr{F}'(\mathbb{R}^d) \times \mathbb{R}^d\colon x \in \mathrm{exo}_B^n(F)\}.$$

Using Lemma 3.11, one can easily check that $\mathscr{U}_n$ is a closed set. □

It is easy to see that $\mathscr{U}' := \{(F, x) \in \mathscr{F}'(\mathbb{R}^d) \times \mathbb{R}^d\colon x \in F\}$ is measurable. The proof of the next lemma follows again by an application of Lemma 3.11. Let $\mathscr{S}^d$ denote the extended convex ring in $\mathbb{R}^d$.

LEMMA 3.13. *The map* $p_B\colon (\mathscr{S}^d \times \mathbb{R}^d) \setminus (\mathscr{U} \cup \mathscr{U}') \to \mathbb{R}^d$ *is continuous.*

The preceding two lemmas immediately imply the next corollary.

COROLLARY 3.14. *The maps* $p_B, u_B\colon (\mathscr{S}^d \times \mathbb{R}^d) \setminus \mathscr{U}' \to \mathbb{R}^d$ *are measurable.*

LEMMA 3.15. *The map* $\delta_B\colon \mathscr{S}^d \times \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$, $(K, z, b) \mapsto \delta_B(K, z, b)$, *is measurable. Moreover, for any measurable set* $A \subset \mathbb{R}^d \times \mathbb{R}^d$, *the map* $\mathscr{S}^d \to [0, \infty]$, $K \mapsto C_j^+(K, A)$, *is measurable for every* $j \in \{0, \ldots, d-1\}$.

PROOF. For any $s \geq 0$ we obviously have

$$\begin{aligned}\big\{(K, z, b) &\in \mathscr{S}^d \times \mathbb{R}^d \times \mathbb{R}^d\colon \delta_B(K, z, b) > s\big\} \\ &= \bigcup_{n=1}^{\infty} \big\{(K, z, b) \in \mathscr{S}^d \times \mathbb{R}^d \times \partial \check{B}\colon K \\ &\qquad \cap (z + (s + n^{-1})\, b + (s + n^{-1})\, B) = \{z\}\big\},\end{aligned}$$

which yields the first assertion.

The first assertion implies in particular that the map $(K, z, b) \mapsto \mathbf{1}\{(z, b) \in N_B(K)\}$ is measurable. Hence, the second statement can be deduced from Lemma 2 in [39]. $\square$

4. Contact distributions in stochastic geometry. In the remainder of the paper we consider the grain model Ξ introduced in Section 1. It is convenient to use the abbreviation $(d(x), p(x), u(x)) := (d_B(\Xi, x), p_B(\Xi, x), u_B(\Xi, x))$, $x \in \mathbb{R}^d$. Recall that we always assume that $B \in \mathscr{K}^d$ and $o \in \operatorname{int} B$. For $x \in \mathbb{R}^d$, $r \geq 0$, and measurable $A \subset \mathbb{R}^d$ we recall the definitions

$$H_B(x, r, A) := P(d(x) \leq r, u(x) \in A | x \notin \Xi),$$

where $H_B(x, r, \cdot)$ equals some fixed probability measure on $\mathbb{R}^d$ if $\bar{p}(x) = 1$, and $H_B(x, r) := H_B(x, r, \mathbb{R}^d)$. Using the results of the previous section we will now analyze the contact distribution function $H_B(x, \cdot, A)$, which provides geometric information about the grain model. Our analysis will be based on the (nonnegative) random measures $C_j^+(\Xi, \cdot)$, $j = 0, \ldots, d-1$, on $\mathbb{R}^d \times \mathbb{R}^d$ having the *intensity measure*

$$\Lambda_j^+(\cdot) := E[C_j^+(\Xi, \cdot)].$$

Here and subsequently the superscript B is omitted.

It is appropriate to describe the aim of the present section. In Theorem 4.1 we will present a basic connection between the weak derivative of the contact distribution function and the intensity measure Λ_{d-1}^+ of the grain model Ξ. Later we will consider grain models Ξ which are defined via a random measure (marked point process) Φ on $\mathbb{R}^d \times \mathscr{K}^d$ with intensity measure α. Under some natural assumptions on α and the second factorial moment measure $\alpha^{(2)}$ of Φ, we prove (Theorem 4.16) that for $\mathscr{H}^d$-a.e. $x \in \mathbb{R}^d$ and all measurable $A \subset \mathbb{R}^d$ the function $(1 - \bar{p}(x))H_B(x, \cdot, A)$ is absolutely continuous and we exhibit its density function explicitly. A similar result (Theorem 4.17) is established for $\Lambda_j^+(dx \times A)$. Quite naturally, our results involve the Palm probabilities of Φ. Due to Slivnyak's theorem, the most explicit form of these theorems is obtained if Φ is an (inhomogeneous) Poisson process. The Poisson process is a very special example of a Gibbs point process, a Cox process, or a Poisson cluster process. We will discuss these substantially more general cases in the second part of the section. The main technical problem in each case is to treat the Palm probabilities and to verify that $\alpha^{(2)}$ is absolutely continuous with respect to a suitable measure.

Let us assume for the moment that the measures $\Lambda_j^+(\cdot \times \mathbb{R}^d)$ are locally finite. A sufficient condition will be provided in Proposition 4.10. Then, in particular, we can disintegrate Λ_{d-1}^+ according to

$$\Lambda_{d-1}^+(d(z, b)) = \mathscr{R}(z, db)\Lambda_{d-1}^+(dz \times \mathbb{R}^d), \tag{4.1}$$

where $\mathscr{R}$ is a stochastic kernel from $\mathbb{R}^d$ to $\mathbb{R}^d$. We might call $\mathscr{R}$ a position dependent *rose of directions* (see [34]) or *mean normal distribution* (see [36]) of Ξ.

THEOREM 4.1. *Assume that the measures* $\Lambda_j^+(\cdot \times \mathbb{R}^d)$, $j = 0, \ldots, d-1$, *are locally finite. Let* $A \subset \mathbb{R}^d$ *be a measurable set. Then*

$$\begin{aligned} &\lim_{t\to+0} \int g(x)t^{-1}(1-\bar{p}(x))H_B(x,t,A)\mathscr{H}^d(dx) \\ &\qquad = 2\int g(x)\Lambda_{d-1}^+(dx \times A) \end{aligned} \tag{4.2}$$

holds for any continuous function $g\colon \mathbb{R}^d \to \mathbb{R}$ *with compact support.*

REMARK 4.2. The assertion of the preceding theorem can be paraphrased by saying that the measure $t^{-1}(1-\bar{p}(x))H_B(x,t,A)\mathscr{H}^d(dx)$ converges vaguely to $2\Lambda_{d-1}^+(dx \times A)$ as $t \to +0$. The classical Portmanteau theorem then implies that the conclusion of the theorem still holds for any bounded function g with compact support for which the set of points of discontinuity of g has $\Lambda_{d-1}^+(dx \times A)$ measure zero.

PROOF OF THEOREM 4.1. Put $\delta(z,b) := \delta_B(\Xi, z, b)$. For $0 \le j \le d-1$,

$$\Psi_j^+(\cdot) := \int \mathbf{1}\{(z,b,\delta(z,b)) \in \cdot\}\, C_j^+(\Xi, d(z,b))$$

is a random measure on $\mathbb{R}^d \times \mathbb{R}^d \times [0,\infty]$. The intensity measure

$$\Lambda_j(\cdot) := E[\Psi_j^+(\cdot)]$$

of Ψ_j^+ satisfies $\Lambda_j(\cdot \times [0,\infty]) = \Lambda_j(\cdot \times (0,\infty]) = \Lambda_j^+(\cdot)$. Further, since $\Lambda_j^+(\cdot \times \mathbb{R}^d)$ is locally finite, we can make the disintegration

$$\Lambda_j(d(z,b,\rho)) = G_j^+(z, d(b,\rho))\Lambda_j^+(dz \times \mathbb{R}^d),$$

where G_j^+ is a stochastic kernel from $\mathbb{R}^d$ to $\mathbb{R}^d \times [0,\infty]$. (In fact, we will only need G_{d-1}^+.) Since $\delta(z,b) > 0$ for all $(z,b) \in N_B(\Xi)$, we can assume without loss of generality that

$$G_{d-1}^+(z, A \times (0,\infty]) = \mathscr{R}(z, A), \qquad z \in \mathbb{R}^d. \tag{4.3}$$

Applying Theorem 3.3 and writing $a_j := (d-j)b_{d-j}$, we obtain for $0 < t \le 1$ that

$$
\begin{aligned}
\int g(x)&(1-\bar p(x))H_B(x,t,A)\mathscr{H}^d(dx)\\
&= E\left[\int g(x)\mathbf{1}\{d(x)\in(0,t],u(x)\in A\}\,\mathscr{H}^d(dx)\right]\\
&= \sum_{j=0}^{d-1} a_j E\left[\iint g(z+sb)\mathbf{1}\{s\le t, b\in A, \delta(z,b)>s\}\right.\\
&\qquad\qquad\qquad\qquad \left.\times\, s^{d-j-1}C_j^+(\Xi, d(z,b))\,ds\right]\\
&= \sum_{j=0}^{d-2} a_j \iint g(z+sb)\mathbf{1}\{s\le t, b\in A\}\,\mathbf{1}\{\rho>s\}\,s^{d-j-1}\\
&\qquad\qquad \times \Lambda_j(d(z,b,\rho))\,ds\\
&\quad + a_{d-1}\iint (g(z+sb)-g(z))\mathbf{1}\{s\le t, b\in A\}\\
&\qquad\qquad \times \mathbf{1}\{\rho>s\}\,\Lambda_{d-1}(d(z,b,\rho))\,ds\\
&\quad + a_{d-1}\iint g(z)\mathbf{1}\{s\le t\}\,G_{d-1}^+(z, A\times(s,\infty])\Lambda_{d-1}^+(dz\times\mathbb{R}^d)\,ds\\
&=: R_1(t)+R_2(t)+R_3(t).
\end{aligned}
\tag{4.4}
$$

Write $\|g\| := \max\{|g(x)|: x\in\mathbb{R}^d\}$, let U be the (compact) support of g, and define $\overline{U} := U + \check{B}$. Then we get

$$
\begin{aligned}
|R_1(t)| &\le \sum_{j=0}^{d-2} \|g\| a_j \Lambda_j^+(\overline{U}\times\mathbb{R}^d)\frac{1}{d-j}t^{d-j}\\
&\le \left(\|g\|\sum_{j=0}^{d-2}\frac{a_j}{d-j}\Lambda_j^+(\overline{U}\times\mathbb{R}^d)\right)t^2.
\end{aligned}
\tag{4.5}
$$

The finite number in brackets is denoted by c_g, for short.

Now, let $\varepsilon > 0$ be an arbitrary positive number. Since g is uniformly continuous, there is some $t_\varepsilon \in (0,1]$ such that $|g(z+sb)-g(z)| < \varepsilon$ holds for all $s\in(0,t_\varepsilon]$ and $(z,b)\in\mathbb{R}^d\times\check{B}$. Hence we obtain for all $t\in(0,t_\varepsilon]$ that

$$
|R_2(t)| \le 2\varepsilon\int_0^t\int \mathbf{1}\left\{z\in\overline{U}\right\}\Lambda_{d-1}^+(dz\times\mathbb{R}^d)\,ds = 2\varepsilon\Lambda_{d-1}^+(\overline{U}\times\mathbb{R}^d)\,t.
$$

Since $\varepsilon > 0$ was arbitrary, we get

$$
\lim_{t\to+0} t^{-1}|R_2(t)| = 0.
\tag{4.6}
$$

Finally, using (4.3), we have

$$\int_0^t G_{d-1}^+(z, A \times (s, \infty])\, ds = R(z, A)\, t - \int_0^t G_{d-1}^+(z, A \times (0, s])\, ds$$

and

$$\left| \int_0^t G_{d-1}^+(z, A \times (0, s])\, ds \right| \le t G_{d-1}^+(z, A \times (0, t]). \tag{4.7}$$

Combining the relations (4.4)–(4.7), we obtain that

$$\begin{aligned} &\bigg| \int g(x) t^{-1} (1 - \bar{p}(x)) H_B(x, t, A) \mathscr{H}^d(dx) \\ &\qquad - 2 \int g(z) R(z, A) \Lambda_{d-1}^+(dz \times \mathbb{R}^d) \bigg| \\ &\quad \le c_g t + t^{-1} |R_2(t)| + 2 \int g(z) G_{d-1}^+(z, A \times (0, t]) \Lambda_{d-1}^+(dz \times \mathbb{R}^d) \\ &\quad \le c_g t + t^{-1} |R_2(t)| + 2 \|g\| \Lambda_{d-1}(U \times A \times (0, t]) \end{aligned} \tag{4.8}$$

and hence we see that the right-hand side of (4.8) converges to zero as $t \to +0$. In view of (4.1) this is precisely the desired conclusion. □

REMARK 4.3. Let the assumptions of Theorem 4.1 be satisfied, and assume that the measure $\Lambda_{d-1}^+(\cdot \times \mathbb{R}^d)$ is absolutely continuous with respect to $\mathscr{H}^d$ with density λ_{d-1}^+. Then Fatou's lemma implies that

$$\liminf_{t \to +0} \big[t^{-1} (1 - \bar{p}(x)) H_B(x, t, A) \big] \le 2 \lambda_{d-1}^+(x) \mathscr{R}(x, A)$$

holds for $\mathscr{H}^d$-a.e. $x \in \mathbb{R}^d$. Note that $\lambda_{d-1}^+(x, A) := \lambda_{d-1}^+(x) \mathscr{R}(x, A)$ is a density of the measure $\Lambda_{d-1}^+(\cdot \times A)$.

REMARK 4.4. If the assumptions of the preceding remark are fulfilled and if, in addition, the function

$$x \mapsto t^{-1} (1 - \bar{p}(x)) H_B(x, t, A)$$

can locally be dominated by a locally integrable function which is independent of t, then we also have

$$\limsup_{t \to +0} \big[t^{-1} (1 - \bar{p}(x)) H_B(x, t, A) \big] \ge 2 \lambda_{d-1}^+(x) \mathscr{R}(x, A)$$

for $\mathscr{H}^d$-a.e. $x \in \mathbb{R}^d$. This follows by another application of Fatou's lemma. Hence, in particular, if the contact distribution function is differentiable with respect to t at $t = +0$ for $\mathscr{H}^d$-a.e. $x \in \mathbb{R}^d$, then

$$\frac{\partial}{\partial t} \bigg|_{t=+0} (1 - \bar{p}(x)) H_B(x, t, A) = 2 \lambda_{d-1}^+(x) \mathscr{R}(x, A) \tag{4.9}$$

holds for $\mathscr{H}^d$-a.e. $x \in \mathbb{R}^d$.

REMARK 4.5. The preceding results are more explicit than might appear at first glance. In fact, it follows from Proposition 3.10 that

$$\Lambda_{d-1}^{+}(\cdot) = E\left[\int_{N_{B^d}(\Xi)} \mathbf{1}\{(x, \nabla h_{\check{B}}(u)) \in \cdot\}\, h_{\check{B}}(u) C_{d-1}^{s}(\Xi, d(x, u))\right].$$

Let $\Lambda_j^{s,+}$ be the intensity measure of $C_j^{+}(\Xi, \cdot)$ if B^d is the structuring element. Then, in particular, we have

$$\Lambda_{d-1}^{+}(\cdot) = \int \mathbf{1}\{(x, \nabla h_{\check{B}}(u)) \in \cdot\}\, h_{\check{B}}(u) \Lambda_{d-1}^{s,+}(d(x, u)).$$

Introducing the Euclidean rose of directions $\mathscr{R}^s$ as a stochastic kernel from $\mathbb{R}^d$ to $\mathbb{R}^d$ satisfying

$$\Lambda_{d-1}^{s,+}(d(x, u)) = \mathscr{R}^s(x, du) \Lambda_{d-1}^{s,+}(dx \times \mathbb{R}^d),$$

we get

$$\Lambda_{d-1}^{+}(dx \times \mathbb{R}^d) = \left[\int h_{\check{B}}(u) \mathscr{R}^s(x, du)\right] \Lambda_{d-1}^{s,+}(dx \times \mathbb{R}^d).$$

Hence we may choose $\mathscr{R}$ as

$$\mathscr{R}(x, \cdot) = \left[\int h_{\check{B}}(u) \mathscr{R}^s(x, du)\right]^{-1} \int \mathbf{1}\{\nabla h_{\check{B}}(u) \in \cdot\}\, h_{\check{B}}(u) \mathscr{R}^s(x, du).$$

COROLLARY 4.6. *Assume that the measures $\Lambda_j^{+}(\cdot \times \mathbb{R}^d)$, $j = 0, \dots, d-1$, are locally finite and that $\Lambda_{d-1}^{s,+}(\cdot \times \mathbb{R}^d)$ is absolutely continuous with respect to $\mathscr{H}^d$ with density $\lambda_{d-1}^{s,+}$. Let $A \subset \mathbb{R}^d$ be a measurable set. Then*

$$\begin{aligned} &t^{-1}(1 - \bar{p}(x)) H_B(x, t, A) \mathscr{H}^d(dx) \\ &\qquad \xrightarrow{v} 2\lambda_{d-1}^{s,+}(x) \int \mathbf{1}\{\nabla h_{\check{B}}(u) \in A\}\, h_{\check{B}}(u) \mathscr{R}^s(x, du) \mathscr{H}^d(dx) \end{aligned}$$

as $t \to +0$, where $\xrightarrow{v}$ denotes the vague convergence of measures.

REMARK 4.7. Assume that, for P-almost all $\omega \in \Omega$, the realization $\Xi(\omega)$ is the closure of its interior. Then

$$2\Lambda_{d-1}^{s,+}(\cdot \times \mathbb{R}^d) = E\left[\mathscr{H}^{d-1}(\partial \Xi \cap \cdot)\right]$$

is the mean surface measure of Ξ. By Theorem 2.2 in [37], more generally one has

$$2\Lambda_{d-1}^{s,+}(\cdot) = E\left[\mathscr{H}^{d-1}(\{x \in \operatorname{reg} \Xi\colon (x, u_{B^d}(\Xi, x)) \in \cdot\})\right]$$

and $\mathscr{H}^{d-1}(\partial \Xi \setminus \operatorname{reg} \Xi) = 0$.

REMARK 4.8. Finally, we obtain the following deterministic special cases of Theorem 4.1.

Let K be in the extended convex ring, let $A, C \subset \mathbb{R}^d$ be measurable and assume that C is bounded. Then

$$\begin{aligned}\mathscr{H}^d(\{x \in (K + \varepsilon\check{B})\backslash K\colon x \in C, u(x) \in A\})\\ = 2\varepsilon \int \mathbf{1}\{x \in C, \nabla h_{\check{B}}(u) \in A\}\, h_{\check{B}}(u) C^s_{d-1}(K, d(x, u)) + o(\varepsilon)\end{aligned}$$

as $t \to +0$, provided that

$$\int \mathbf{1}\{x \in \partial C\}\, \mathbf{1}\{\nabla h_{\check{B}}(u) \in A\}\, C^s_{d-1}(K, d(x, u)) = 0.$$

Now, let again K be in the extended convex ring and let $D \subset \mathbb{R}^d \times \mathbb{R}^d$ be measurable and bounded in the first component. Then

$$\begin{aligned}&\mathscr{H}^d\big(\{x \in (K + \varepsilon\check{B})\backslash K\colon (p(x), u(x)) \in D\}\big)\\ &\quad = 2\varepsilon \int \mathbf{1}\{(x, \nabla h_{\check{B}}(u)) \in D\}\, h_{\check{B}}(u) C^s_{d-1}(K, d(x, u)) + o(\varepsilon)\end{aligned} \tag{4.10}$$

as $\varepsilon \to +0$. To see this one merely has to repeat the proof of Theorem 4.1 with $g(\cdot)$ replaced by $\mathbf{1}\{(p(\cdot), u(\cdot)) \in D\}$. The argument then simplifies considerably and works without the additional assumption of continuity for g.

In the remainder of this paper it is often convenient to use the language of germ-grain models (see [34]). Let $\Phi = \{(\xi_n, Z_n)\colon n \in \mathbb{N}\}$ be a *point process* on $\mathbb{R}^d \times \mathscr{K}^d$ and set $\Xi_n := Z_n + \xi_n$ for $n \in \mathbb{N}$. If Φ satisfies the condition

$$\sum_{n=1}^{\infty} \mathbf{1}\{(Z_n + \xi_n) \cap C \neq \varnothing\} < \infty, \qquad P\text{-a.s.} \tag{4.11}$$

for all compact $C \subset \mathbb{R}^d$, then $\Xi := \bigcup_{n=1}^{\infty} \Xi_n$ is P-almost surely a closed set. Thus any such point process Φ defines a grain model Ξ which is derived from the point process $\{\Xi_n\colon n \in \mathbb{N}\}$ on $\mathscr{K}^d$. Conversely, any random closed set Ξ in the extended convex ring can be derived from a point process $\{\Xi_n\colon n \in \mathbb{N}\}$ on $\mathscr{K}^d$ such that the invariance properties of Ξ are preserved (see [40]) and from which we finally obtain a point process Φ on $\mathbb{R}^d \times \mathscr{K}^d$ (that is a germ-grain model) by setting $(\xi_n, Z_n) = (c(\Xi_n), \Xi_n - c(\Xi_n))$, where $c(\Xi_n)$ is the "center" of Ξ_n, that is, (for example) the midpoint of the smallest ball containing Ξ_n. Actually, it is not necessary to assume that $(\xi_n, Z_n) \neq (\xi_m, Z_m)$ for $n \neq m$. Therefore it is better to identify Φ with the random measure

$$\Phi \equiv \sum_{n=1}^{\infty} \delta_{(\xi_n, Z_n)},$$

where $\delta_{(x, K)}$ is the Dirac measure located at $(x, K) \in \mathbb{R}^d \times \mathscr{K}^d$. Note that we do not assume that the convex bodies in the second component have their centers at the origin. Here and in the following the summation index n formally ranges from 1 to ∞ even if the summation is merely from $n = 1$ to ν, where ν is a random variable with values in $\{0, 1, \ldots, \infty\}$.

Denote by $\mathbf{N}'$ the set of all $(\mathbb{Z}^+ \cup \{\infty\})$-valued measures φ on $\mathbb{R}^d \times \mathscr{K}^d$ such that $\varphi(\cdot \times \mathscr{K}^d)$ is locally finite and let $\mathscr{N}'$ be the σ-field generated by the vague topology on $\mathbf{N}'$ (see [15]). In the following, we always assume that Φ is given such that:

(A1) Φ is a random element of $(\mathbf{N}', \mathscr{N}')$.

(A2) For all compact $C \subset \mathbb{R}^d$ the condition

$$\int \mathbf{1}\{(K+x) \cap C \neq \varnothing\}\, \Phi(d(x,K)) < \infty \quad P\text{-a.s.}$$

is satisfied.

Let $\mathbf{N}'_s$ denote the set of all $\varphi \in \mathbf{N}'$ satisfying $\varphi(\{(x,K)\}) \leq 1$ for all (x,K). If $P(\Phi \in \mathbf{N}'_s) = 1$, then Φ is called *simple*. Although we will view Φ as a random measure, we will often write $\Phi = \{(\xi_n, Z_n)\colon n \in \mathbb{N}\}$ even if Φ is not simple. The intensity (or mean) measure α of Φ is defined as

$$\alpha(\cdot) := E\left[\sum_{n=1}^{\infty} \mathbf{1}\{(\xi_n, Z_n) \in \cdot\}\right].$$

We will often assume that the intensity measure α of Φ is σ-finite. This condition is, for example, satisfied if the intensity measure $\alpha(\cdot \times \mathscr{K}^d)$ of the point process $\Phi(\cdot \times \mathscr{K}^d) = \sum_{n=1}^{\infty} \delta_{\xi_n}$ is σ-finite. The *second factorial moment measure* $\alpha^{(2)}$ of Φ is defined by

$$\alpha^{(2)}(\cdot) := E\Big[\iint \mathbf{1}\{(x_1, K_1, x_2, K_2) \in \cdot\}\, (\Phi \backslash \delta_{(x_1, K_1)})(d(x_2, K_2))\Phi(d(x_1, K_1))\Big],$$

where $\Phi \backslash \delta_{(x,K)} := \Phi - \mathbf{1}\{\Phi(\{(x,K)\}) > 0\}\, \delta_{(x,K)}$. Recall that Ξ^+denotes the set of all boundary points $z \in \partial \Xi$ for which there is some $b \in \mathbb{R}^d$ with $(z,b) \in N_B(\Xi)$.

The following proposition will be essential for the calculations below. Here and subsequently we will assume that the structuring element B is smooth (i.e., has unique support planes). We will comment on this condition in Remark 4.18.

PROPOSITION 4.9. *Let B be smooth, let ν be a σ-finite measure on $\mathscr{K}^d \times \mathbb{R}^d \times \mathscr{K}^d$, and assume that $\alpha^{(2)}$ is absolutely continuous with respect to the product measure $\mathscr{H}^d \otimes \nu$. Then*

$$P\left(\Xi^+ = \bigcup_{n=1}^{\infty} (\partial \Xi_n \backslash \Xi^{(n)})\right) = 1, \tag{4.12}$$

where $\Xi^{(n)} := \bigcup_{i \neq n} \Xi_i$. In particular, for $j = 0, \ldots, d-1$ we have

$$C_j^+(\Xi, \cdot) = \sum_{n=1}^{\infty} C_j\big(\Xi_n, \cdot \cap \big((\mathbb{R}^d \backslash \Xi^{(n)}) \times \mathbb{R}^d\big)\big), \quad P\text{-a.s.} \tag{4.13}$$

PROOF. The Euclidean case of (4.12) has been proved in [13] (Theorem A.1). For the sake of completeness we outline the proof in the present more general setting. The inclusion

$$\Xi^+ \supset \bigcup_{n\in\mathbb{N}} (\partial\Xi_n \backslash \Xi^{(n)}) \tag{4.14}$$

is always true. Hence, if equality fails to hold in (4.14), then there is some $z \in \partial\Xi_n \cap \partial\Xi_m$, $m \neq n$, and some $b \in \mathbb{R}^d$ such that $(z, b) \in N_B(\Xi)$. The latter condition implies that there is some $\varepsilon > 0$ with $[(z+\varepsilon b)+\varepsilon B]\cap\Xi = \{z\}$. Since B is smooth, it follows that $z \in F(\Xi_n, u) \cap F(\Xi_m, u)$, where $-u \in S^{d-1}$ is the uniquely determined (Euclidean) exterior unit normal vector of $(z + \varepsilon b) + \varepsilon B$ at z and the support sets $F(\Xi_n, u)$ are defined as in [28] (see also Section 2). This shows that

$$\xi_n - \xi_m \in \Lambda(Z_m, Z_n) := \bigcup_{u\in S^{d-1}} [F(Z_m, u) + F(-Z_n, -u)].$$

It was proved in [13] (Theorem A.1) that $\mathscr{H}^d(\Lambda(Z_m, Z_n)) = 0$. Therefore, by essentially the same argument as in [13], we obtain $P(\Xi^+ \neq \bigcup_{n=1}^{\infty}(\partial\Xi_n\backslash \Xi^{(n)})) = 0$, which establishes the first assertion. The second assertion then is implied by Corollary 3.5. □

In the following, we will frequently assume that the intensity measure α of Φ can be represented in the form

$$\alpha(d(x, K)) = f(x, K)\mathscr{H}^d(dx)Q_0(dK), \tag{4.15}$$

where $f: \mathbb{R}^d \times \mathscr{K}^d \to [0, \infty)$ is a measurable function and Q_0 is a probability measure on $\mathscr{K}^d$. Sometimes we will have to assume that

$$\int \mathbf{1}\{(K + z) \cap C \neq \varnothing\}\, \alpha(d(z, K)) < \infty \tag{4.16}$$

for all compact $C \subset \mathbb{R}^d$.

PROPOSITION 4.10. *Assume that condition* (4.15) *is satisfied. Let $A \subset \mathbb{R}^d$ be measurable. Then, for $j = 0, \ldots, d-1$, $\Lambda_j^+(\cdot \times A)$ and*

$$E\left[\sum_{n=1}^{\infty} C_j(\Xi_n, \cdot \times A)\right] \tag{4.17}$$

are absolutely continuous. The density $\lambda_j(x, A)$ of the measure in (4.17) *fulfills*

$$\lambda_j(x, A) = \iint f(x - z, K)C_j(K, dz \times A)Q_0(dK).$$

If, in addition, (4.16) *is satisfied, then both measures are locally finite.*

PROOF. Fallert [5] has proved in the Euclidean case that the measure given in (4.17) is absolutely continuous and he has also determined the density. Moreover, he has proved that the additional assumption (4.16) implies that this measure is locally finite. The corresponding statements in the present more general setting of Minkowski geometry and for general sets A can be proved similarly. The only change which is required concerns the constant appearing in Lemma 2.1 of [5]. Using (3.5) and the notation of the proof for Theorem 3.3, we get

$$C_j^+(\Xi, \cdot) = \sum_{i\in\mathbb{N}} C_j(\Xi_i, \cdot \cap N_B^i(\Xi)) \le \sum_{i\in\mathbb{N}} C_j(\Xi_i, \cdot).$$

Hence, it is also true that $\Lambda_j^+(\cdot \times A)$ is absolutely continuous. Moreover, $\Lambda_j^+(\cdot \times A)$ is locally finite under the additional assumption (4.16). □

If Φ is a Poisson process, then we can compute Λ_j^+ quite easily as the next proposition shows. Clearly, a proof of Proposition 4.11 could also be obtained from the more general Theorem 4.17 below and by an application of Slivnyak's theorem.

PROPOSITION 4.11. *Let B be smooth. Assume that Φ is a Poisson process with an intensity measure α of the form* (4.15). *Let $j \in \{0, \dots, d-1\}$, and let $A \subset \mathbb{R}^d$ be measurable. Then $\Lambda_j^+(\cdot \times A)$ is absolutely continuous with density $\lambda_j^+(x, A) = (1-\bar{p}(x))\lambda_j(x, A)$, where $\lambda_j(\cdot, A)$ is the density of the measure in* (4.17).

PROOF. We use the equation

$$\begin{aligned} E\Big[\int \mathbf{1}\{(\Phi\backslash\delta_{(x,K)}, x, K) \in \cdot\}\,\Phi(d(x,K))\Big] \\ = E\Big[\int \mathbf{1}\{(\Phi, x, K) \in \cdot\}\,\alpha(d(x,K))\Big], \end{aligned} \tag{4.18}$$

which is characteristic for the Poisson process; see [23]. In particular, it follows that $\alpha^{(2)} = \alpha \otimes \alpha$ so that Proposition 4.9 is applicable. Representing Ξ as a measurable function $T(\Phi)$ such that $\Xi^{(n)} = T(\Phi\backslash\delta_{(\xi_n, Z_n)})$, $n \in \mathbb{N}$, we obtain from (4.13) that

$$\begin{aligned} \Lambda_j^+(\cdot) &= E\left[\sum_{n=1}^{\infty} \int \mathbf{1}\{(z,b) \in \cdot\}\,\mathbf{1}\left\{z \notin \Xi^{(n)}\right\} C_j(\Xi_n, d(z,b))\right] \\ &= E\Big[\iint \mathbf{1}\{(z,b) \in \cdot\}\,\mathbf{1}\{z \notin \Xi\}\, C_j(K+y, d(z,b))\alpha(d(y,K))\Big] \\ &= \iiint \mathbf{1}\{(z+y,b) \in \cdot\}\,(1-\bar{p}(z+y))C_j(K, d(z,b)) \\ &\qquad \times f(y,K)\mathscr{H}^d(dy)Q_0(dK) \\ &= \iiint \mathbf{1}\{(x,b) \in \cdot\}\,(1-\bar{p}(x))f(x-z,K)C_j(K,d(z,b))\mathscr{H}^d(dx)Q_0(dK). \end{aligned}$$

This proves the first assertion. The second assertion is then implied by Proposition 4.10 and the first assertion. □

REMARK 4.12. In a Euclidean setting and under the assumption that Φ is a Poisson process which satisfies conditions (4.15) and (4.16), Fallert has proved that the measures $E[C_j(\Xi, \cdot \times \mathbb{R}^d)]$, $j = 0, \dots, d-1$, are locally finite and absolutely continuous. He also determined the corresponding densities $D_j(\cdot)$ explicitly. In this situation and for $j = d-1$, one can easily check that $D_{d-1}(x) = \lambda_{d-1}^{+}(x)$ holds for $\mathscr{H}^d$-a.e. $x \in \mathbb{R}^d$. This is not surprising, since $C_{d-1}(\Xi, \cdot) = C_{d-1}^{+}(\Xi, \cdot)$ according to Theorem 3.9.

By saying that Φ is an *independently marked* point process we mean that $\Phi(\cdot \times \mathscr{K}^d)$ and (Z_n) are independent and that the Z_n, $n \in \mathbb{N}$, are independent with distribution Q_0. As usual we call Q_0 the distribution of a typical grain. If, for instance, $\Phi(\cdot \times \mathscr{K}^d)$ is a Poisson process with intensity measure α', then Φ is also Poisson with intensity measure $\alpha' \otimes Q_0$. In this case we can compute the contact distribution function $H_B(x, t)$ rather explicitly. In fact, we can even treat the following much more general situation.

PROPOSITION 4.13. *Assume that Φ is a Poisson process with an intensity measure α of the form* (4.15). *Then the contact distribution function $H_B(x, \cdot)$ is for all $x \in \mathbb{R}^d$ absolutely continuous and we have*

$$H_B(x, t) = 1 - \exp\Big[-\int_0^t \rho_B(x, s)\, ds\Big], \qquad t \geq 0,$$

where

$$\rho_B(x, t) := 2 \iint f(x - y, K) C_{d-1}(K + t\check{B}, dy \times \mathbb{R}^d) Q_0(dK).$$

PROOF. Following Heinrich [12] we first show that each bounded set is almost surely hit by only a finite number of the grains Ξ_n if and only if (4.16) is satisfied for each compact $C \subset \mathbb{R}^d$. We start with observing that $\beta := \alpha(\cdot \times \mathscr{K}^d)$ is locally finite, since Φ is a Poisson process which takes values in $\mathbf{N}'$. Moreover, (4.15) shows that β is diffuse, and hence $\Psi := \Phi(\cdot \times \mathscr{K}^d)$ and $\Phi = \{(\xi_n, Z_n): n \in \mathbb{N}\}$ are simple point processes. We write $\alpha(d(x, K)) = \gamma(x, dK)\beta(dx)$, where $\beta(dx) = \int f(x, K) Q_0(dK) \mathscr{H}^d(dx)$ and $\gamma(x, dK)$ is determined for β-a.e. $x \in \mathbb{R}^d$. By a fundamental property of marked Poisson processes (see Section 5.2 in [18]) we get that given Ψ the random variables Z_n are conditionally independent and $P(Z_n \in \cdot | \Psi) = \gamma(\xi_n, \cdot)$ P-a.s. Let $C \subset \mathbb{R}^d$ be compact. Then (4.11) is equivalent to

$$P\left(\sum_{n=1}^{\infty} \mathbf{1}\{(Z_n + \xi_n) \cap C \neq \varnothing\} < \infty | \Psi\right) = 1, \quad P\text{-a.s.}$$

By a Borel–Cantelli type argument we can conclude that this holds precisely if

$$\sum_{n=1}^{\infty} P((Z_n + \xi_n) \cap C \neq \varnothing | \Psi) < \infty, \quad P\text{-a.s.};$$

that is, if and only if

$$\iint \mathbf{1}\{(K+x)\cap C \neq \varnothing\}\,\gamma(x, dK)\Psi(dx) < \infty, \quad P\text{-a.s.}$$

Using the special form of the characteristic functional for the Poisson process Ψ, we see that the last condition yields that

$$\int \Big(1 - \exp\Big\{-\int \mathbf{1}\{(K+x)\cap C \neq \varnothing\}\,\gamma(x, dK)\Big\}\Big)\beta(dx) < \infty.$$

An application of the inequality $1 - e^{-x} \geq x/2$, $x \in [0, 1]$, then implies (4.16).

The reverse implication is obviously true.

Now take a compact $C \subset \mathbb{R}^d$. It is well known and easy to prove (see [34] for the stationary case) that

$$\begin{aligned}
-\ln P(\Xi \cap C = \varnothing) &= \iint \mathbf{1}\{(K+y)\cap C \neq \varnothing\}\, f(y, K) Q_0(dK)\mathscr{H}^d(dy) \\
&= \iint \mathbf{1}\Big\{y \in \check{K} + C\Big\}\, f(y, K) Q_0(dK)\mathscr{H}^d(dy).
\end{aligned}$$

Hence we obtain for all $x \in \mathbb{R}^d$ and any fixed $t \geq 0$ that

$$\begin{aligned}
-\ln P(d(x) > t) &= -\ln P(\Xi \cap (x + tB) = \varnothing) \\
&= \iint \mathbf{1}\Big\{y \in \check{K} + (x + tB)\Big\}\, f(y, K) Q_0(dK)\mathscr{H}^d(dy) \\
&= \iint \mathbf{1}\Big\{y \in (K - x) + t\check{B}\Big\}\, f(-y, K)\mathscr{H}^d(dy) Q_0(dK) \\
&= \iint f(-y, K)\mathbf{1}\{y \in K - x\}\,\mathscr{H}^d(dy) Q_0(dK) \\
&\qquad + 2\iiint \mathbf{1}\{s \leq t\}\, f(x - y, K) \\
&\qquad\qquad \times C_{d-1}(K + s\check{B}, dy \times \mathbb{R}^d)\, ds\, Q_0(dK),
\end{aligned}$$

where we have used Corollary 2.6. Relation (4.16) yields as a by-product that, for $t \geq 0$, $P(d(x) > t) > 0$ and, in particular, $P(d(x) > 0) = 1 - \bar{p}(x) > 0$. Letting $t = 0$, we see that the first term in the last sum equals $-\ln P(d(x) > 0) = \ln(1 - \bar{p}(x))$. Because

$$P(d(x) > t) = (1 - \bar{p}(x)) - (1 - \bar{p}(x)) H_B(x, t),$$

this completes the proof. □

REMARK 4.14. An alternative expression for ρ_B defined in Proposition 4.13 is

$$\begin{aligned}
\rho_B(x, t) = \sum_{j=0}^{d-1} (d - j) b_{d-j} t^{d-j-1} &\iint f(x - z - tb, K) \\
&\times C_j(K, d(z, b) Q_0(dK).
\end{aligned} \tag{4.19}$$

This can be deduced from Theorem 2.5.

REMARK 4.15. Let the assumptions of Proposition 4.13 be satisfied and assume moreover that $f(\cdot, K)$ is continuous for Q_0-a.e. K and that f is bounded. Then it follows that $\rho_B(x, \cdot)$ is continuous on $[0, \infty)$, provided we impose the integrability conditions

$$\int C_j(K, \mathbb{R}^d \times \mathbb{R}^d) Q_0(dK) < \infty, \qquad j = 0, \dots, d-1. \tag{4.20}$$

Alternatively, this condition can be expressed in terms of an integrability condition for Euclidean quermass-integrals (or intrinsic volumes). Hence $H_B(x, \cdot)$ is differentiable in this case and in particular we have

$$\frac{\partial}{\partial t}\bigg|_{t=+0} H_B(x, t) = 2 \iint f(x - y, K) C_{d-1}(K, dy \times \mathbb{R}^d) Q_0(dK).$$

If we merely assume that for Q_0-a.e. $K \in \mathcal{K}^d$ the set of points of discontinuity of $f(\cdot, K)$ has $\mathcal{H}^d$ measure zero, then the preceding equation still holds for $\mathcal{H}^d$-a.e. $x \in \mathbb{R}^d$. This is in accordance with Theorem 4.1 and Proposition 4.11.

In the general case it is difficult to treat the contact distribution function explicitly. However, under rather weak assumptions, we are still able to prove absolute continuity and to derive an expression for the density. The appropriate tool for formulating and proving the corresponding result are the *Palm probabilities* $\{P_{(x, K)}: (x, K) \in \mathbb{R}^d \times \mathcal{K}^d\}$ of Φ. Their definition requires that the intensity measure α of Φ is σ-finite, which is assumed from now on. Then $(x, K) \mapsto P_{(x, K)}(A)$ is for all $A \in \mathcal{F}$ a Radon–Nikodym derivative of the measure $E[\mathbf{1}_A \Phi(\cdot)]$ with respect to α. It is easy to see that this definition entails that

$$\iint H(\omega, x, K) \Phi(\omega, d(x, K)) P(d\omega) = \iint H(\omega, x, K) P_{(x, K)}(d\omega) \alpha(d(x, K)),$$

where $H: \Omega \times \mathbb{R}^d \times \mathcal{K}^d \to [0, \infty]$ is an arbitrary measurable function. Special cases of this equation will be used several times subsequently. As in Kallenberg ([15], page 84) we can assume without restricting generality that $(x, K) \mapsto P_{(x, K)}(\cdot)$ is a stochastic kernel, since all of our random elements take their values in Polish spaces. Moreover, by Lemma 10.2 in [15] we can also assume that $P_{(x, K)}(\Phi(\{(x, K)\}) \geq 1) = 1$ for all (x, K). If Φ is a simple point process, then $P_{(x, K)}(A)$ can be interpreted as the conditional probability of A given that $\Phi(\{(x, K)\}) = 1$. The distance $d(x)$ and other quantities which are used below depend on Φ. In order to make this dependence explicit we sometimes write, for example, $d(T(\Phi), x)$ or simply $d(\Phi, x)$.

THEOREM 4.16. *Let the assumptions of Proposition* 4.9 *be satisfied and assume also that* α *is of the form* (4.15). *Let* $A \subset \mathbb{R}^d$ *be measurable. Then*

$(1-\bar{p}(x))H_B(x,\cdot,A)$ *is for* $\mathscr{H}^d$-*a.e.* $x \in \mathbb{R}^d$ *absolutely continuous with density*

$$
\begin{aligned}
t \mapsto \sum_{j=0}^{d-1}(d-j)b_{d-j}t^{d-j-1} \iint P_{(x-z-tb,\,K)} \\
\times (d(\Phi\backslash\delta_{(x-z-tb,\,K)}, x) > t)\mathbf{1}\{b \in A\} \\
\times f(x-z-tb, K)C_j(K, d(z,b))Q_0(dK).
\end{aligned}
\tag{4.21}
$$

PROOF. For any measurable function $h : \mathbf{N}' \to \mathbb{R}$ (cf. the definitions before Proposition 4.11) we write $E^!_{(x,\,K)}[h(\Phi)] := E_{(x,\,K)}[h(\Phi\backslash\delta_{(x,\,K)})]$, where $E_{(x,\,K)}$ denotes expectation with respect to $P_{(x,\,K)}$. Then we have

$$
\begin{aligned}
&E\Big[\int h(\Phi\backslash\delta_{(x,\,K)}, x, K)\Phi(d(x,K))\Big] \\
&\quad = \iint E^!_{(x,\,K)}[h(\Phi, x, K)]f(x,K)\mathscr{H}^d(dx)Q_0(dK)
\end{aligned}
\tag{4.22}
$$

for all measurable functions h. Let g: $\mathbb{R}^d \to [0,\infty]$ be measurable. By Theorem 3.3, Proposition 4.9, using the same abbreviations as in the proof of Theorem 4.1, employing the map T defined in the proof of Proposition 4.11 and tacitly using Fubini's theorem (which is possible because of Lemma 3.11 and Corollary 3.14) we obtain

$$
\begin{aligned}
&\int g(x)(1-\bar{p}(x))H_B(x,t,A)\mathscr{H}^d(dx) \\
&\quad = \int E[g(x)\mathbf{1}\{x \notin \Xi, d(x) \le t, u(x) \in A\}]\mathscr{H}^d(dx) \\
&\quad = E\Big[\int g(x)\mathbf{1}\{x \notin \Xi, d(x) \le t, u(x) \in A\}\,\mathscr{H}^d(dx)\Big] \\
&\quad = \sum_{j=0}^{d-1} a_j E\Big[\iint g(z+sb)\mathbf{1}\{\delta(\Xi,z,b) > s\}\,\mathbf{1}\{s \le t, b \in A\} \\
&\qquad\qquad \times s^{d-j-1}C_j^+(\Xi, d(z,b))\,ds\Big] \\
&\quad = \sum_{j=0}^{d-1} a_j E\Big[\sum_{n=1}^{\infty}\iint g(z+sb)\mathbf{1}\{s \le t, b \in A\}\,s^{d-j-1}\mathbf{1}\Big\{z \notin \Xi^{(n)}\Big\} \\
&\qquad\qquad \times \mathbf{1}\{\delta(\Xi,z,b) > s\}\,C_j(\Xi_n, d(z,b))\,ds\Big] \\
&\quad = \sum_{j=0}^{d-1} a_j E\Big[\iiint g(z+sb)\mathbf{1}\{s \le t\}\,\mathbf{1}\{b \in A\}\,s^{d-j-1}\mathbf{1}\{z \notin T(\Phi\backslash\delta_{(y,\,K)})\} \\
&\qquad\qquad \times \mathbf{1}\{\delta(T(\Phi),z,b) > s\}\,C_j(K+y, d(z,b))\,ds\,\Phi(d(y,K))\Big].
\end{aligned}
$$

For $(z, b) \in N_B(K + y)$ and $\Phi(\{(y, K)\}) > 0$ we have

$$z \notin T(\Phi \backslash \delta_{(y,K)}) \quad \text{and} \quad \delta(T(\Phi), z, b) > s$$

if and only if

$$d(T(\Phi \backslash \delta_{(y,K)}), z + sb) > s.$$

In view of (4.22) the preceding chain of equalities can be continued with

$$\begin{aligned}
&= \sum_{j=0}^{d-1} a_j E\Big[\int \Big\{ \iint g(z+sb) \mathbf{1}\{s \le t\} \mathbf{1}\{b \in A\} s^{d-j-1} \\
&\qquad\qquad \times \mathbf{1}\big\{d(T(\Phi \backslash \delta_{(y,K)}), z+sb) > s\big\} C_j(K+y, d(z,b))\, ds \Big\} \\
&\qquad\qquad\qquad \times \Phi(d(y,K))\Big] \\
&= \sum_{j=0}^{d-1} a_j \iint E^!_{(y,K)}\Big[\iint g(z+sb)\mathbf{1}\{s \le t\}\mathbf{1}\{b \in A\} s^{d-j-1} \\
&\qquad\qquad \times \mathbf{1}\{d(T(\Phi), z+sb) > s\} C_j(K+y, d(z,b))\, ds\Big] \\
&\qquad \times f(y,K)\mathscr{H}^d(dy) Q_0(dK) \\
&= \iint E^!_{(y,K)}\Big[\int g(x)\mathbf{1}\{0 < d_B(K+y, x) \le t, u_B(K+y,x) \in A\} \\
&\qquad\qquad \times \mathbf{1}\{d(T(\Phi), x) > d_B(K+y,x)\}\mathscr{H}^d(dx)\Big] \\
&\qquad \times f(y,K)\mathscr{H}^d(dy)Q_0(dK) \\
&= \int g(x) \Big\{ \iint E^!_{(y,K)}[\mathbf{1}\{d(T(\Phi), x) > d_B(K+y, x)\}] \\
&\qquad \times \mathbf{1}\{0 < d_B(K+y,x) \le t, u_B(K+y,x) \in A\} \\
&\qquad\qquad \times f(y,K)\mathscr{H}^d(dy)Q_0(dK)\Big\}\mathscr{H}^d(dx).
\end{aligned}$$

Since $g\colon \mathbb{R}^d \to [0, \infty]$ was arbitrarily chosen, we obtain for $\mathscr{H}^d$-a.e. $x \in \mathbb{R}^d$ that

$$\begin{aligned}
(1 - \bar{p}(x))H_B(x, t, A) = \iint E^!_{(y,K)}&[\mathbf{1}\{d(T(\Phi), x) > d_B(K+y, x)\}] \\
&\times \mathbf{1}\{0 < d_B(K+y, x)\} \\
&\le t, u_B(K+y, x) \in A f(y,K)\mathscr{H}^d(dy)Q_0(dK).
\end{aligned}$$

Using for all $x, y \in \mathbb{R}^d$ and all convex K the easy to check relations

$$d_B(K+y, x) = d_B(K-x, -y) \quad \text{and} \quad u_B(K+y, x) = u_B(K-x, -y)$$

as well as the change of variables $y \mapsto -y$, we can continue with

$$= \iint E^!_{(y,K)}[\mathbf{1}\{d(T(\Phi), x) > d_B(K - x, y)\}]$$
$$\times \mathbf{1}\{0 < d_B(K - x, y) \le t, u_B(K - x, y) \in A\} f(-y, K)\mathscr{H}^d(dy) Q_0(dK)$$
$$= \sum_{j=0}^{d-1} a_j \iiint E^!_{(-z-sb,K)}[\mathbf{1}\{d(T(\Phi), x) > s\}]$$
$$\times \mathbf{1}\{s \le t\}\mathbf{1}\{b \in A\} s^{d-j-1} f(-z - sb, K) C_j(K - x, d(z, b))\, ds\, Q_0(dK)$$
$$= \sum_{j=0}^{d-1} a_j \iiint E^!_{(x-z-sb,K)}[\mathbf{1}\{d(T(\Phi), x) > s\}]$$
$$\times \mathbf{1}\{s \le t\}\mathbf{1}\{b \in A\} s^{d-j-1} f(x - z - sb, K) C_j(K, d(z, b))\, Q_0(dK)\, ds.$$

This proves the absolute continuity while the asserted form (4.21) of the density follows directly from the definition of the expectation $E^!_{(x-z-sb,K)}(\cdot)$. □

A similar argument will be used to determine explicit expressions for the densities of the intensity measures $\Lambda_j^+(\cdot \times A)$ for measurable sets $A \subset \mathbb{R}^d$.

THEOREM 4.17. *Let the assumptions of Proposition* 4.9 *be satisfied and assume also that* α *is of the form* (4.15). *Let* $A \subset \mathbb{R}^d$ *be measurable. Then* $\Lambda_j^+(\cdot \times A)$, $j = 0, \dots, d-1$, *is absolutely continuous with density*

$$\lambda_j^+(x, A) = \iint P_{(x-z,K)}(d(\Phi \backslash \delta_{(x-z,K)}, x) > 0)$$
$$\times f(x - z, K) C_j(K, dz \times A) Q_0(dK).$$

PROOF. Similarly as in the proof for Theorem 4.16 we obtain that

$$\Lambda_j^+(\cdot) = E\left[\sum_{n=1}^{\infty} \int \mathbf{1}\{(z, b) \in \cdot\}\mathbf{1}\left\{z \notin \Xi^{(n)}\right\} C_j(\Xi_n, d(z, b))\right]$$
$$= \iint E^!_{(x,K)}\left[\int \mathbf{1}\{(z, b) \in \cdot\}\mathbf{1}\{z \notin \Xi\} C_j(K + x, d(z, b))\right]$$
$$\times f(x, K)\mathscr{H}^d(dx) Q_0(dK)$$

which yields the result after the change of variables $x + z \mapsto y$. □

REMARK 4.18. In the preceding two theorems we have assumed that B is smooth. This assumption implies that only the first term of the expansion in Theorem 3.4 needs to be taken into account. The case of a general strictly convex body B can be treated by considering additional terms in this expansion. This requires the use of multivariate Palm probability measures as defined in [15].

REMARK 4.19. The previous result suggests taking another look at Theorem 4.1 and the subsequent discussion. Passing to the limit in (4.21) as $t \to +0$ in an informal way, yields indeed $2\lambda^+_{d-1}(x)\mathscr{R}(x, A)$. A formal justification of this convergence requires some additional assumptions such as absolute continuity of the Palm distributions and boundedness and continuity conditions on the densities. Rather than formulating a general theorem, we shall discuss this below by means of examples.

REMARK 4.20. In the proofs and statements of Theorems 4.16 and 4.17, effectively one merely uses the *reduced Palm distributions* $Q^!_{(x,K)}(\cdot) := P_{(x,K)}(\Phi\backslash\delta_{(x,K)} \in \cdot)$ and not the Palm probabilities themselves. They satisfy the equation

$$E\Big[\int \mathbf{1}\left\{(\Phi\backslash\delta_{(x,K)}, x, K) \in \cdot\right\}\Phi(d(x,K))\Big]$$
$$= \iint \mathbf{1}\{(\varphi, x, K) \in \cdot\}\, Q^!_{(x,K)}(d\varphi)\alpha(d(x,K)).$$

Although these distributions are only unique α-almost everywhere, one can use any version of them in Theorems 4.16 and 4.17.

In the remainder of this section we discuss some special cases of the preceding two results. First, we consider a *Gibbs process* Φ. Such a point process is a natural generalization of a Poisson process and can conveniently be defined by the equation

$$\begin{aligned} &E\Big[\int \mathbf{1}\left\{(\Phi\backslash\delta_{(x,K)}, x, K) \in \cdot\right\}\Phi(d(x,K))\Big] \\ &\quad = E\Big[\iint \mathbf{1}\{(\Phi, x, K) \in \cdot\}\,\lambda(\Phi, x, K)\mathscr{H}^d(dx)Q_0(dK)\Big], \end{aligned} \tag{4.23}$$

where λ is a nonnegative measurable function and Q_0 is a probability measure on $\mathscr{K}^d$. This is an integral definition of a *Gibbs process* with state space $\mathbb{R}^d \times \mathscr{K}^d$ and *local energy function* $-\ln\lambda$. We refer to Kallenberg [15] (see also [34]) for an extensive discussion of the point process approach to Gibbs processes. The intensity measure of Φ is given by $\alpha(d(x,K)) = f(x,K)\mathscr{H}^d(dx)Q_0(dK)$, where $f(x,K) := E[\lambda(\Phi, x, K)]$. We assume that these expectations are finite. Our first result on Gibbs processes generalizes Proposition 4.11.

PROPOSITION 4.21. *Assume that B is smooth and let Φ be a Gibbs process with local energy function $-\ln\lambda$ as described above. Then $\Lambda^+_j(\cdot \times A)$, $j = 0, \dots, d-1$, is for all measurable $A \subset \mathbb{R}^d$ absolutely continuous with density*

$$\lambda^+_j(x, A) = \int \overline{\lambda}(x, x-y, K)C_j(K, dy \times A)Q_o(dK),$$

where

$$\overline{\lambda}(x, y, K) := E[\mathbf{1}\{x \notin \Xi\}\,\lambda(\Phi, y, K)], \qquad x, y \in \mathbb{R}^d.$$

PROOF. Equation (4.23) easily implies that $\alpha^{(2)}$ is absolutely continuous with respect to $\mathscr{H}^d \otimes Q_0 \otimes \mathscr{H}^d \otimes Q_0$ with density

$$(x_1, K_1, x_2, K_2) \mapsto E[\lambda(\Phi + \delta_{(x_2, K_2)}, x_1, K_1)\lambda(\Phi, x_2, K_2)].$$

Therefore the assumptions of Proposition 4.9 are satisfied. Furthermore, note that as a consequence of (4.23) we have

$$P_{(x,K)}(\Phi \backslash \delta_{(x,K)} \in \cdot) = f(x, K)^{-1} E[\mathbf{1}\{\Phi \in \cdot\}\, \lambda(\Phi, x, K)] \tag{4.24}$$

for α-a.e. (x, K). In view of Theorem 4.17 and Remark 4.20 this yields the result. □

Having identified the density in Theorem 4.17 for Gibbs processes, we now show how Theorem 4.16 can be specified for such processes. Subsequently, we will only consider contact distribution functions, but intensity measures can be treated similarly.

PROPOSITION 4.22. *Let the assumptions of Proposition* 4.21 *be satisfied and let $A \subset \mathbb{R}^d$ be measurable. Then $(1 - \bar{p}(x))H_B(x, \cdot, A)$ is for $\mathscr{H}^d$-a.e. $x \in \mathbb{R}^d$ absolutely continuous with density*

$$\begin{aligned} t \mapsto \sum_{j=0}^{d-1} &(d-j) b_{d-j} t^{d-j-1} \\ &\times \iint E[\mathbf{1}\{d(x) > t\}\, \lambda(\Phi, x - z - tb, K)] \\ &\qquad \times \mathbf{1}\{b \in A\}\, C_j(K, d(z, b)) Q_0(dK). \end{aligned} \tag{4.25}$$

The proof follows from Theorem 4.16 in the same way as Proposition 4.21 was deduced from Theorem 4.17.

Now we generalize the result in Remark 4.15.

PROPOSITION 4.23. *Let the assumptions of Proposition* 4.21 *be satisfied and assume moreover that λ is bounded, that the set of points of discontinuity of $\lambda(\varphi, \cdot, K)$ has $\mathscr{H}^d$ measure zero for $(P(\Phi \in \cdot) \otimes Q_0)$-a.e. (φ, K), and that* (4.20) *is satisfied. Let $A \subset \mathbb{R}^d$ be measurable. Then $(1 - \bar{p}(x))H_B(x, t, A)$ is differentiable at $t = +0$ for $\mathscr{H}^d$-a.e. $x \in \mathbb{R}^d$ and the derivative satisfies*

$$\frac{\partial}{\partial t}\bigg|_{t=+0} (1 - \bar{p}(x))H_B(x, t, A) = 2 \iint \overline{\lambda}(x, x - z, K) C_{d-1}(K, dz \times A) Q_0(dK).$$

PROOF. By Fubini's theorem, for $\mathscr{H}^d$-a.e. $x \in \mathbb{R}^d$, and for $P(\Phi \in \cdot) \otimes C_{d-1}(K, \cdot) \otimes Q_0$-a.e. (φ, z, b, K), $x - z$ is not a point of discontinuity of $\lambda(\varphi, \cdot, K)$. Fix any such $x \in \mathbb{R}^d$. By Proposition 4.22 it suffices to show that then the sum in (4.25) tends to the right-hand side of the asserted equality as $t \to +0$. The boundedness of λ and the integrability assumption (4.20) imply that the jth integrand in the sum (4.25) is dominated by a function that is independent of t and integrable with respect to the measure $P(d\omega)C_j(K, d(z, b))Q_0(dK)$. By the dominated convergence theorem it suffices to show that for $C_{d-1}(K, \cdot)$-a.e. $(z, b) \in \mathbb{R}^d \times \mathbb{R}^d$, and for Q_0-a.e. $K \in \mathscr{K}^d$ we have

$$\mathbf{1}\{d(x) > t\}\,\lambda(\Phi, x - z - tb, K) \to \mathbf{1}\{x \notin \Xi\}\,\lambda(\Phi, x - z, K), \quad P\text{-a.s.}$$

as $t \to +0$. If $x \in \Xi$, then the above limit is indeed 0. If $x \notin \Xi$, then the above convergence is implied by the continuity property of λ. Hence the result is proved. □

Our next example concerns the class of *Cox processes*, that is, Poisson processes with a random intensity measure η (see, e.g., [15]). Formally, we introduce η as a random element of the measurable space $(\mathbf{M}', \mathscr{M}')$, where $\mathbf{M}'$ is the set of all measures μ on $\mathbb{R}^d \times \mathscr{K}^d$ such that $\mu(\cdot \times \mathscr{K}^d)$ is locally finite and $\mathscr{M}'$ is the σ-field generated by the vague topology. We let P_μ denote the distribution of a Poisson process with intensity measure $\mu \in \mathbf{M}'$. Then Φ is a Cox process directed by the random measure η if $P(\Phi \in \cdot \mid \eta) = P_\eta(\cdot)$ P-a.s. In this case η and Φ have the same intensity measure $\alpha(\cdot) = E[\eta(\cdot)]$. If the latter is σ-finite, then we can introduce the Palm distributions $V_{(x, K)}$, $(x, K) \in \mathbb{R}^d \times \mathscr{K}^d$, of η as a stochastic kernel from $\mathbb{R}^d \times \mathscr{K}^d$ to $\mathbf{M}'$ satisfying

$$E\Big[\int \mathbf{1}\{(\eta, x, K) \in \cdot\}\,\eta(d(x, K))\Big] = \iint \mathbf{1}\{(\mu, x, K) \in \cdot\}\,V_{(x, K)}(d\mu)\alpha(d(x, K)).$$

If η is deterministic, that is, $\eta \equiv \alpha$, then Φ is a Poisson process and $V_{(x, K)} = \delta_\alpha$ for α-a.e. (x, K). In the following theorem we choose a random measure η such that:

1. The intensity measure α of η is σ-finite.
2. The second moment measure $E[\eta \otimes \eta]$ is absolutely continuous with respect to the product measure $\mathscr{H}^d \otimes Q_0 \otimes \mathscr{H}^d \otimes Q_0$, where Q_0 is a probability measure on $\mathscr{K}^d$.

For example, we can choose a random measure $\eta(\omega, d(x, K)) = \zeta(\omega, x, K) \times \mathscr{H}^d(dx)Q_0(dK)$ with a nonnegative measurable function ζ such that $E[\zeta(\cdot, x, K)] < \infty$ for all (x, K) and $\eta(\omega) \in \mathbf{M}'$ for all $\omega \in \Omega$.

PROPOSITION 4.24. *Assume that B is smooth and let Φ be a Cox process directed by a random measure η as described above. Then the intensity measure α of Φ can be represented as in* (4.15). *Let $A \subset \mathbb{R}^d$ be measurable. Then $(1 - \bar{p}(x))H_B(x, \cdot, A)$ is for $\mathscr{H}^d$-a.e. $x \in \mathbb{R}^d$ absolutely continuous with density*

$$\begin{aligned} t \mapsto {} & \sum_{j=0}^{d-1} (d-j) b_{d-j} t^{d-j-1} \\ & \times \iiint \Big[\int \mathbf{1}\{d(\varphi, x) > t\}\,P_\mu(d\varphi)\Big] V_{(x-z-tb, K)}(d\mu) \\ & \times \mathbf{1}\{b \in A\}\, f(x - z - tb, K) C_j(K, d(z, b)) Q_0(dK), \end{aligned} \tag{4.26}$$

where the $V_{(x, K)}$, $(x, K) \in \mathbb{R}^d \times \mathscr{K}^d$, are a version of the Palm distributions of η.

PROOF. The second factorial moment measure of a Poisson process with intensity measure α' equals $\alpha' \otimes \alpha'$. Hence we have $\alpha^{(2)}(\cdot) = E[(\eta \otimes \eta)(\cdot)]$. According to the assumption (2), $\alpha^{(2)}$ is absolutely continuous with respect to $\mathscr{H}^d \otimes Q_0 \otimes \mathscr{H}^d \otimes Q_0$. Assumption 2 also implies that α satisfies condition (4.15). Thus we can apply Theorem 4.16. But (4.18) and the definition of a Cox process easily yield that

$$Q^!_{(x,K)}(\cdot) := \int P_\mu(\cdot) V_{(x,K)}(d\mu), \qquad (x, K) \in \mathbb{R}^d \times \mathscr{K}^d,$$

defines a version of the reduced Palm distributions of Φ (see Remark 4.20). Substituting this into (4.21) completes the proof of the result. □

EXAMPLE 4.25. For mixed Poisson processes, that is, if we choose the random measure $\eta(\omega, d(x, K)) = \bar{\zeta}(\omega)\bar{f}(x, K)\mathscr{H}^d(dx)Q_0(dK)$ with nonnegative measurable functions $\bar{\zeta}$ and $\bar{f}$, a probability measure Q_0 on $\mathscr{K}^d$, and under the assumption $E[\bar{\zeta}] < \infty$, the preceding proposition simplifies considerably. Indeed, the density is given by

$$\begin{aligned} t \mapsto & \sum_{j=0}^{d-1} (d-j) b_{d-j} t^{d-j-1} \\ & \times \iiint \Big[\int \mathbf{1}\{d(\varphi, x) > t\}\, P_{\lambda\overline{\alpha}}(d\varphi) \Big] \lambda P(\bar{\zeta} \in d\lambda) \\ & \qquad \times \mathbf{1}\{b \in A\}\, \bar{f}(x - z - tb, K) C_j(K, d(z, b)) Q_0(dK), \end{aligned} \tag{4.27}$$

where $\overline{\alpha}(d(x, K)) := \bar{f}(x, K)\mathscr{H}^d(dx)Q_0(dK)$. In particular, the integral in brackets can be further simplified with the help of Proposition 4.13.

The previous results contain an interesting generalization of Proposition 4.13.

PROPOSITION 4.26. *Assume that B is smooth. Let Φ be a Poisson process with intensity measure α of the form* (4.15), *and let $A \subset \mathbb{R}^d$ be measurable. Then $H_B(x, \cdot, A)$ is for $\mathscr{H}^d$-a.e. $x \in \mathbb{R}^d$ absolutely continuous and the density $t \mapsto h_B(x, t, A)$ satisfies*

$$\begin{aligned} h_B(x, t, A) = & \sum_{j=0}^{d-1} (d-j) b_{d-j} t^{d-j-1} (1 - H_B(x, t)) \\ & \times \iint \mathbf{1}\{b \in A\}\, f(x - z - tb, K) \\ & \qquad \times C_j(K, d(z, b)) Q_0(dK). \end{aligned} \tag{4.28}$$

PROOF. Taking into account that $P(d(x) > t) = (1 - \bar{p}(x))(1 - H_B(x, t)) > 0$, we can use (4.25) with $\lambda(\Phi, y, K) = f(y, K)$ to obtain the assertion. Of course, one could infer the result directly from Theorem 4.16 or from Proposition 4.24. □

In our final example we assume that the point process $\Psi := \sum_{n=1}^{\infty} \delta_{\xi_n}$ is a Poisson cluster process (see, e.g., [22, 15, 34]). To introduce such processes we let $\mathbf{N}$ denote the set of all locally finite $(\mathbb{Z}^+ \cup \{\infty\})$-valued measures ψ on $\mathbb{R}^d$ equipped with the Borel σ-field $\mathscr{N}$ generated by the vague topology. A Poisson cluster process (or infinitely divisible) point process Ψ is then completely characterized by its *KLM-measure* $\widetilde{P}$. This is a measure on $\mathbf{N}$ with $\widetilde{P}(\{0\}) = 0$ (0 is the zero measure) and $\int \mathbf{1}\{\psi(C) > 0\}\, \widetilde{P}(d\psi) < \infty$ for all compact $C \subset \mathbb{R}^d$. If such a measure is given and if Π is a Poisson process on $\mathbf{N}$ with intensity measure $\widetilde{P}$, then $\Psi = \int \psi \Pi(d\psi)$ is a Poisson cluster process with KLM-measure $\widetilde{P}$. Note that Ψ is a Poisson process if and only if $\widetilde{P}$ is concentrated on the Dirac measures, that is, $\widetilde{P}(\{\psi\colon \psi(\mathbb{R}^d) \neq 1\}) = 0$. If the intensity measure $\beta(\cdot) := E[\Psi(\cdot)] = \int \psi(\cdot)\widetilde{P}(d\psi)$ of Ψ is σ-finite, then we can define a stochastic kernel $x \mapsto \widetilde{P}_x(\cdot)$ from $\mathbb{R}^d$ to $\mathbf{N}$ by requiring that

$$\iint \mathbf{1}\{(\psi, x) \in \cdot\}\, \psi(dx)\widetilde{P}(d\psi) = \iint \mathbf{1}\{(\psi, x) \in \cdot\}\, \widetilde{P}_x(d\psi)\beta(dx).$$

The probability measures $\widetilde{P}_x$, $x \in \mathbb{R}^d$, are the *Palm distributions* of $\widetilde{P}$. Below we will assume that Φ is an independent marking of Ψ, where Q_0 is the distribution of the typical grain. For a given $\psi = \sum_n \delta_{x_n} \in \mathbf{N}$ we let $\Gamma(\psi, \cdot)$ denote the distribution of the point process $\sum_n \delta_{(x_n, Z_n)}$, where the Z_n are independent with distribution Q_0. The *reduced* Palm distributions of $\widetilde{P}$ are defined by $\widetilde{P}^!_x(\cdot) = \int \mathbf{1}\{\psi \backslash \delta_x \in \cdot\}\, \widetilde{P}_x(d\psi)$. Finally, we denote by $\tilde{\alpha}^!_x(\cdot) := \int \psi(\cdot)\widetilde{P}^!_x(d\psi)$ the intensity measure of $\widetilde{P}^!_x$.

In the following theorem we will have to impose the assumption $\bar{p}(x) < 1$. According to (2.6) in [12], for any compact set $C \subset \mathbb{R}^d$, we have the relation

$$\begin{aligned} &P(\Xi \cap C = \varnothing) = \\ &\qquad \exp\left\{-\int\left(1 - \prod_{x\in\mathbb{R}^d} P((Z_0 + x) \cap C = \varnothing)^{\psi(\{x\})}\right)\widetilde{P}(d\psi)\right\} \end{aligned} \tag{4.29}$$

with the convention $0^0 := 1$. This equation can be used for calculating the distribution functions $H_B(x, t)$. However, it does not seem to be possible to deduce $P(\Xi \cap C = \varnothing) > 0$, say for $C = \{x_0\}$ and $x_0 \in \mathbb{R}^d$, from (4.11) alone. A simple sufficient condition yielding $P(\Xi \cap C = \varnothing) > 0$ for a compact set $C \subset \mathbb{R}^d$ is

$$\int P((Z_0 + x) \cap C \neq \varnothing)\beta(dx) < \infty. \tag{4.30}$$

This can be inferred from an application of the estimate

$$1 - \prod_{x\in\mathbb{R}^d} P((Z_0 + x) \cap C = \varnothing)^{\psi(\{x\})} \le \int P((Z_0 + x) \cap C \neq \varnothing)\psi(dx)$$

to (4.29). Formally, (4.30) corresponds to the condition which one encounters for a Poisson process Ψ with intensity measure β. The stationary Euclidean special case of our next result is discussed in [19].

PROPOSITION 4.27. *Assume that B is smooth and let Φ be an independent marking of a Poisson cluster process Ψ with KLM-measure $\widetilde{P}$. Assume that the intensity measure β of Ψ is absolutely continuous with density f and that $\tilde{\alpha}^!_x$ is for β-a.e. $x \in \mathbb{R}^d$ absolutely continuous with respect to a σ-finite measure on $\mathbb{R}^d$. Let $A \subset \mathbb{R}^d$ be measurable. Then, for $\mathscr{H}^d$-a.e. $x \in \mathbb{R}^d$ with $\bar{p}(x) < 1$, $H_B(x, \cdot, A)$ is absolutely continuous and the density $t \mapsto h_B(x, t, A)$ satisfies*

$$
\begin{aligned}
h_B(x, t, A) = \sum_{j=0}^{d-1} (d-j) b_{d-j} t^{d-j-1} (1 - H_B(x, t)) \\
\times E\Big[\int \Big[\iint \mathbf{1}\{d(\varphi, x) > t\} \Gamma(\psi, d\varphi) \widetilde{P}^!_{x-z-tb}(d\psi) \Big] \\
\times f(x - z - tb) \mathbf{1}\{b \in A\} C_j(Z_0, d(z, b)) \Big],
\end{aligned}
\tag{4.31}
$$

where Z_0 is a random convex body with the distribution of the typical grain.

PROOF. The crucial property of a Poisson cluster process we shall use here is

$$
P_x(\Psi \backslash \delta_x \in \cdot) = \iint \mathbf{1}\{\psi_1 + \psi_2 \in \cdot\} \widetilde{P}^!_x(d\psi_1) P(\Psi \in d\psi_2), \tag{4.32}
$$

for β-a.e. $x \in \mathbb{R}^d$ (see Lemma 10.6 in [15]). Here P_x, $x \in \mathbb{R}^d$, are the Palm probabilities of Ψ which are defined similarly as $P_{(x,K)}$. (Note that β is σ-finite). In particular, we obtain for the second factorial moment measure $\beta^{(2)}$ of Ψ (defined similarly as $\alpha^{(2)}$) that for all measurable $A_1, A_2 \subset \mathbb{R}^d$ the relationship

$$
\beta^{(2)}(A_1 \times A_2) = (\beta \otimes \beta)(A_1 \times A_2) + \int \tilde{\alpha}^!_{x_1}(A_2) \mathbf{1}\{x_1 \in A_1\} \beta(dx_1)
$$

is satisfied. Since β and $\tilde{\alpha}^!_x$ are absolutely continuous and since it is easy to check from the definitions and from the assumption that Φ is an independent marking of Ψ that

$$
\alpha^{(2)}(d(x_1, K_1, x_2, K_2)) = \beta^{(2)}(d(x_1, x_2)) Q_0(dK_1) Q_0(dK_2),
$$

Proposition 4.9 applies. Because Φ is an independent marking of Ψ it also follows that $\alpha = \beta \otimes Q_0$ and furthermore it can easily be proved that

$$
P_{(y,K)}(\Phi \backslash \delta_{(y,K)} \in \cdot) = \iint \mathbf{1}\{\varphi \in \cdot\} \Gamma(\psi \backslash \delta_y, d\varphi) P_y(\Psi \in d\psi)
$$

for α-a.e. (y, K). The definition of Γ implies

$$
\int \mathbf{1}\{\varphi \in \cdot\} \Gamma(\psi_1 + \psi_2, d\varphi) = \iint \mathbf{1}\{\varphi_1 + \varphi_2 \in \cdot\} \Gamma(\psi_1, d\varphi_1) \Gamma(\psi_2, d\varphi_2).
$$

Using the latter two relationships and (4.32), we obtain for all $t \geq 0$, $x \in \mathbb{R}^d$ and α-a.e. (y, K) that

$$\begin{aligned}
&P_{(y,K)}(d(\Phi\backslash\delta_{(y,K)}, x) > t)\\
&\quad = \iiint \mathbf{1}\{d(\varphi, x) > t\}\,\Gamma(\psi_1 + \psi_2, d\varphi)P(\Psi \in d\psi_2)\widetilde{P}^!_y(d\psi_1)\\
&\quad = \iiiint \mathbf{1}\{d(\varphi_1, x) > t\}\,\mathbf{1}\{d(\varphi_2, x) > t\}\,\Gamma(\psi_1, d\varphi_1)\\
&\qquad\qquad \times \Gamma(\psi_2, d\varphi_2)P(\Psi \in d\psi_2)\widetilde{P}^!_y(d\psi_1)\\
&\quad = P(d(x) > t)\iint \mathbf{1}\{d(\varphi, x) > t\}\,\Gamma(\psi, d\varphi)\widetilde{P}^!_y(d\psi),
\end{aligned}$$

where we have also used that

$$\mathbf{1}\{d(\varphi_1 + \varphi_2, x) > t\} = \mathbf{1}\{d(\varphi_1, x) > t\}\,\mathbf{1}\{d(\varphi_2, x) > t\}.$$

Substituting this result into (4.21) we obtain the asserted formula. □

EXAMPLE 4.28. In this example we give a more specific discussion of the previous result. First, we introduce a Poisson cluster process in a more traditional way using a Poisson process Ψ_p of *parent* points. Each point $x \in \Psi_p$ generates another point process (cluster) with distribution $\kappa(x, \cdot)$. For given Ψ_p, these clusters are independent. The Poisson cluster process Ψ is then defined as the union of all clusters (see [22]). In fact, each infinitely divisible point process can be represented this way. Let us assume that the intensity measure of Ψ_p is absolutely continuous with density f_p. To ensure that Ψ is indeed well defined we have to assume that $x \mapsto \kappa(x, \cdot)$ is a stochastic kernel satisfying

$$\int \kappa(x, \{\psi\colon \psi(C) > 0\}) f_p(x)\mathscr{H}^d(dx) < \infty$$

for all compact $C \subset \mathbb{R}^d$. The KLM-measure $\widetilde{P}$ is then given by the formula

$$\widetilde{P}(\cdot) = \int \kappa(x, \cdot\backslash\{0\}) f_p(x)\mathscr{H}^d(dx).$$

We are now going to make some additional assumptions. First, we assume that the mean numbers of cluster points

$$\lambda_{\mathrm{cl}}(x, \cdot) := \int \psi(\cdot)\kappa(x, d\psi), \qquad x \in \mathbb{R}^d,$$

satisfy

$$\lambda_{\mathrm{cl}}(x, \cdot) = \int f_{\mathrm{cl}}(x, y)\mathbf{1}\{y \in \cdot\}\,\mathscr{H}^d(dy),$$

for some measurable $f_{\mathrm{cl}}\colon \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$. The intensity measure β of Ψ is then given by $\beta(dx) = f(x)\mathscr{H}^d(dx)$, where

$$f(x) := \int f_p(y) f_{\mathrm{cl}}(y, x)\mathscr{H}^d(dy)$$

is assumed to be finite for all $x \in \mathbb{R}^d$. Further, we assume that κ is of the form

$$\kappa(x, \cdot) = \int \mathbf{1}\{T_{-x}\psi \in \cdot\}\, g(x, \psi)\mu(d\psi), \qquad x \in \mathbb{R}^d,$$

where g: $\mathbb{R}^d \times \mathbf{N} \to [0, \infty)$ is measurable, μ is a probability measure on $\mathbf{N}$ and $T_x\psi$ is the measure $\psi(\cdot + x)$. The special case $g \equiv 1$ corresponds to *homogeneous clustering*. Let h: $\mathbb{R}^d \to [0, \infty)$ be measurable. A straightforward calculation yields that

$$\widetilde{P}_x(\cdot) = 1/f(x) \iint \mathbf{1}\left\{T_{y-x}\psi \in \cdot\right\} f_p(x-y)g(x-y, \psi)\psi(dy)\mu(d\psi), \qquad x \in \mathbb{R}^d,$$

is one possible choice of the Palm probabilities $\widetilde{P}_x$, $x \in \mathbb{R}^d$. In particular, it follows for β-a.e. $x \in \mathbb{R}^d$ that $\tilde{\alpha}^!_x$ is absolutely continuous with respect to the measure

$$\iiint \mathbf{1}\{z + x - y \in \cdot\}\, (\psi \backslash \delta_y)(dz)\psi(dy)\mu(d\psi),$$

which we assume to be σ-finite. If, for instance, the second reduced moment measure of μ is absolutely continuous with respect to $\mathscr{H}^d \otimes \mathscr{H}^d$, then $\tilde{\alpha}^!_x$ is even absolutely continuous for β-a.e. $x \in \mathbb{R}^d$. One possible choice of μ suitable for applications is the distribution of a Poisson process with σ-finite and absolutely continuous intensity measure. In view of Proposition 4.27 we finally compute, for $\mathscr{H}^d$-a.e. $x \in \mathbb{R}^d$ for which $\bar{p}(x) < 1$,

$$\begin{aligned}
f(x - z - tb) &\iint \mathbf{1}\{d(\varphi, x) > t\}\, \Gamma(\psi, d\varphi)\widetilde{P}^!_{x-z-tb}(d\psi) \\
&= \iiint \mathbf{1}\left\{d(T_{y-x+z+tb}\,\varphi, x) > t\right\} \Gamma(\psi \backslash \delta_y, d\varphi) f_p(x - z - tb - y) \\
&\qquad \times g(x - z - tb - y, \psi)\psi(dy)\mu(d\psi) \\
&= \iiint \mathbf{1}\{d(\varphi, y + z + tb) > t\}\, \Gamma(\psi \backslash \delta_y, d\varphi) f_p(x - z - tb - y) \\
&\qquad \times g(x - z - tb - y, \psi)\psi(dy)\mu(d\psi).
\end{aligned}$$

Assuming that f_p and g are bounded and continuous and that μ has finite intensity measure, we see that the last expression converges to

$$\iiint \mathbf{1}\{d(\varphi, y + z) > 0\}\, \Gamma(\psi \backslash \delta_y, d\varphi) f_p(x - z - y) g(x - z - y, \psi)\psi(dy)\mu(d\psi)$$

as $t \mapsto +0$. This has been announced in Remark 4.19.

5. Stationary grain models. In this section we assume that the grain model Ξ is *stationary*, that is, that the distribution of $\Xi + x$ is the same for all $x \in \mathbb{R}^d$. We find it convenient (see [24]) to express stationarity in terms of an abstract measurable flow θ: $\mathbb{R}^d \times \Omega \to \Omega$ for which the bijections θ_x: $\Omega \to \Omega$, $x \in \mathbb{R}^d$, defined by $\theta_x(\cdot) := \theta(x, \cdot)$, satisfy the flow property $\theta_x \circ \theta_y = \theta_{x+y}$ for all $x, y \in \mathbb{R}^d$. We then assume that the probability measure P is *invariant* under all θ_x and that

$$\Xi(\omega) - x = \Xi(\theta_x \omega), \qquad x \in \mathbb{R}^d.$$

The invariance property (2.3), which is also enjoyed by the additive extensions of the support measures, implies that the random measures $C_j^+(\Xi, \cdot)$, $j = 0, \dots, d-1$, inherit stationarity from Ξ; that is,

$$C_j^+(\Xi(\omega), (A + x) \times C) = C_j^+(\Xi(\theta_x \omega), A \times C), \tag{5.1}$$

for all measurable $A, C \subset \mathbb{R}^d$. In particular it follows that the intensity measures Λ_j^+ are of product form if $\Lambda_j^+(\cdot \times \mathbb{R}^d)$ is locally finite. Assuming that the *intensity*

$$\lambda_j^+ := E\big[C_j^+(\Xi, [0,1]^d \times \mathbb{R}^d)\big]$$

is finite, we have

$$\Lambda_j^+(d(x,b)) = \lambda_j^+ \mathscr{H}^d(dx) \mathscr{R}_j(db),$$

where $\mathscr{R}_j$ is a probability measure on $\mathbb{R}^d$. In fact, we find that

$$\mathscr{R}_j(\cdot) = (\lambda_j^+)^{-1} \Lambda_j^+([0,1]^d \times \cdot)$$

if $\lambda_j^+ > 0$. Note that $\mathscr{R} := \mathscr{R}_{d-1}$ is the rose of directions introduced in the previous section. Due to stationarity, the volume fraction $\bar{p} := P(x \in \Xi)$ and the contact distribution function

$$H_B(t, A) := P(d(x) \le t, u(x) \in A | x \notin \Xi)$$

are independent of x for all measurable $A \subset \mathbb{R}^d$. Therefore, in the present stationary situation Corollary 4.6 implies that

$$\frac{\partial}{\partial t}\bigg|_{t=+0} (1 - \bar{p}) H_B(t, A) = 2\lambda_{d-1}^{s,+} \int \mathbf{1}\{\nabla h_{\check{B}}(u) \in A\}\, h_{\check{B}}(u) R_{d-1}^s(du).$$

Our next aim is to introduce stochastic kernels κ_j, $j = 0, \dots, d-1$, from $\Omega \times \mathbb{R}^d$ to $\mathbb{R}^d$ satisfying

$$C_j^+(\Xi, d(z,b)) = \kappa_j(z, db) C_j^+(\Xi, dz \times \mathbb{R}^d), \quad P\text{-a.s.} \tag{5.2}$$

and being stationary in the sense that

$$\kappa_j(\omega, z, \cdot) = \kappa_j(\theta_z \omega, 0, \cdot), \qquad (\omega, z) \in \Omega \times \mathbb{R}^d. \tag{5.3}$$

To reach that goal we take a measurable set $D \subset \mathbb{R}^d$ with positive and finite volume. By (5.1) and the assumption of invariance,

$$\begin{aligned} Q_j(\cdot) := \frac{1}{\mathscr{H}^d(D)} \iint & \mathbf{1}\{z \in D\} \mathbf{1}\{(\theta_z \omega, b) \in \cdot\} \\ & \times C_j^+(\Xi(\omega), d(z,b)) P(d\omega) \end{aligned} \tag{5.4}$$

defines a measure on $\Omega \times \mathbb{R}^d$ which is independent of D. In the terminology of Mecke [23], $Q_j(\cdot \times A)$ is the *Palm measure* of the stationary random measure $C_j^+(\Xi, \cdot \times A)$, where $A \subset \mathbb{R}^d$ is measurable. If $\lambda_j^+ > 0$, then

$$P_j(\cdot) := (\lambda_j^+)^{-1} Q_j(\cdot \times \mathbb{R}^d)$$

is the *Palm probability* of $C_j^+(\Xi, \cdot \times \mathbb{R}^d)$. Just for completeness we define $P_j := P$ if $\lambda_j^+ = 0$. As in [34] we might interpret P_j as a conditional probability measure given that 0 is a *typical* point of $C_j^+(\Xi, \cdot \times \mathbb{R}^d)$. Equation (5.4) implies the *refined Campbell theorem*

$$\begin{aligned} &\iint \mathbf{1}\{(\omega, b, x) \in \cdot\}\, Q_j(d(\omega, b))\mathscr{H}^d(dx) \\ &\quad = \iint \mathbf{1}\{(\theta_z \omega, b, z) \in \cdot\}\, C_j^+(\Xi(\omega), d(z, b))P(d\omega). \end{aligned} \tag{5.5}$$

Next we introduce a stochastic kernel $\tilde{\kappa}_j$ from Ω to $\mathbb{R}^d$ by disintegrating Q_j according to

$$Q_j(d(\omega, b)) = \tilde{\kappa}_j(\omega, db) Q_j(d\omega \times \mathbb{R}^d).$$

The definition $\kappa_j(\omega, z, \cdot) := \tilde{\kappa}_j(\theta_z \omega, \cdot)$, $(\omega, z) \in \Omega \times \mathbb{R}^d$, yields a kernel with the property (5.3) while (5.2) follows from a straightforward application of the refined Campbell theorem (5.5).

The next result expresses expectations with respect to the stationary probability measure P in terms of the Palm probabilities P_j.

THEOREM 5.1. *For any measurable* $f\colon \Omega \to [0, \infty)$,

$$\begin{aligned} E[\mathbf{1}\{d(0) > 0\} f] = \sum_{j=0}^{d-1} (d-j) b_{d-j} \lambda_j^+ & \\ \times E_j\Big[\iint f \circ \theta_{sb}\, s^{d-j-1} \mathbf{1}\{\delta(0, b) > s\}\, \kappa_j(0, db)\, ds\Big], & \end{aligned}$$

where E_j denotes expectation with respect to the Palm probability P_j.

PROOF. Denote by W an arbitrary subset of $\mathbb{R}^d$ with $\mathscr{H}^d(W) = 1$. Stationarity gives

$$E[\mathbf{1}\{d(0) > 0\} f] = E\Big[\int \mathbf{1}\{x \in W\}\mathbf{1}\{d(x) > 0\} f \circ \theta_x \mathscr{H}^d(dx)\Big]. \tag{5.6}$$

Recall that $a_i = ib_i$. Writing $x = d(x)u(x) + p(x)$ and applying Theorem 3.3, we obtain that the right-hand side of equation (5.6) is equal to

$$\sum_{j=0}^{d-1} a_{d-j} E\Big[\iint \mathbf{1}\{z + sb \in W\} f \circ \theta_{sb} \circ \theta_z \mathbf{1}\{\delta(0, b) \circ \theta_z > s\} s^{d-j-1} C_j^+(\Xi, d(z, b))\, ds\Big],$$

where we have also used that $\delta(\omega, z, b) = \delta(\theta_z \omega, 0, b)$. Applying successively (5.2), (5.3) and the refined Campbell theorem for P_j we obtain that the last sum equals

$$\sum_{j=0}^{d-1} a_{d-j}\lambda_j^+ E_j\Big[\iiint \mathbf{1}\{z + sb \in W\} \times f \circ \theta_{sb}\, \mathbf{1}\{\delta(0, b) > s\}\, s^{d-j-1}\kappa_j(0, db)\mathscr{H}^d(dz)\, ds\Big],$$

which can be simplified to yield the right-hand side of the asserted equality. □

Theorem 5.1 allows several remarks and corollaries (cf. [20]). Using the probability measures

$$G_j(\cdot) := E_j\Big[\int \mathbf{1}\{(\delta(0, b), b) \in \cdot\}\, \kappa_j(0, db)\Big]$$

on $[0, \infty] \times \mathbb{R}^d$, we can generalize Theorem 2.1 in [20].

COROLLARY 5.2. *For any measurable set $A \subset \mathbb{R}^d$ and $r \geq 0$, we have*

$$(1 - \bar{p})H_B(r, A) = \sum_{j=0}^{d-1} (d - j)b_{d-j}\lambda_j^+ \int \mathbf{1}\{s \leq r\}\, s^{d-j-1} G_j((s, \infty] \times A)\, ds.$$

PROOF. Apply Theorem 5.1 with $f := \mathbf{1}\{(d(0), u(0)) \in (0, r] \times A\}$. □

The corollary says that $(1 - \bar{p})H_B(\cdot, A)$ is absolutely continuous with density

$$t \mapsto \sum_{j=0}^{d-1} (d - j)b_{d-j}\lambda_j^+ t^{d-j-1} G_j((t, \infty] \times A). \tag{5.7}$$

The fact that the contact distribution $H_B(\cdot, \mathbb{R}^d)$ is absolutely continuous has been proved in [10] using Federer's coarea theorem.

REMARK 5.3. If follows directly from the definitions that

$$E_j[\kappa_j(0, \cdot)] = G_j((0, \infty] \times \cdot) = \mathscr{R}_j(\cdot)$$

provided that $\lambda_j^+ > 0$, since $G_j(\{0\} \times \mathbb{R}^d) = 0$. Hence the value of the density (5.7) for $t = 0$ equals $2\lambda_{d-1}^+ \mathscr{R}(A)$, which is in accordance with Theorem 4.1.

In the remainder of this section we discuss our results in terms of the marked point process $\Phi = \sum_{n=1}^{\infty} \delta_{(\xi_n, Z_n)}$ and the point process $\Psi = \Phi(\cdot \times \mathscr{K}^d)$ introduced in the previous section. A result by Weil and Wieacker [40] justifies assuming that Φ is stationary, that is, we assume that

$$\sum_{n=1}^{\infty} \delta_{(\xi_n \circ \theta_x,\, Z_n \circ \theta_x)} = \sum_{n=1}^{\infty} \delta_{(\xi_n - x,\, Z_n)}$$

holds for all $x \in \mathbb{R}^d$. In other words, (5.1) is satisfied with $C_j^+(\Xi(\cdot), \cdot)$ replaced by Φ. We also assume that the intensity

$$\lambda_\Psi := E\big[\Psi([0,1]^d)\big]$$

is strictly positive and finite. The intensity measure α of Φ then satisfies

$$\alpha(d(x,K)) = \lambda_\Psi \mathscr{H}^d(dx) Q_0(dK),$$

where Q_0 is a probability measure on $\mathscr{K}^d$ which is called the distribution of the typical grain.

Just for simplicity we assume that Ψ is simple, that is, $\Psi(\{x\}) \le 1$ for all $x \in \mathbb{R}^d$. We can then define a $\mathscr{K}^d$-valued stochastic process $\{Z(x): x \in \mathbb{R}^d\}$ by letting $Z(x) := Z_n$ if $x = \xi_n$ for some $n \in \mathbb{N}$ and $Z(x) := K_0$ otherwise, where K_0 is some fixed convex body. This process is stationary in the sense that $Z(x) = Z(0) \circ \theta_x$, $x \in \mathbb{R}^d$. Denoting by P_Ψ the Palm probability of the point process Ψ, we have the following version of the refined Campbell theorem:

$$\begin{aligned}\lambda_\Psi \iint \mathbf{1}\{(\omega, x, Z(0)) \in \cdot\}\, P_\Psi(d\omega)\mathscr{H}^d(dx) \\ = \iint \mathbf{1}\{(\theta_x\omega, x, Z(x)) \in \cdot\}\, \Psi(\omega, dx) P(d\omega).\end{aligned} \tag{5.8}$$

Let $K \mapsto P_\Psi^K$ be a version of the conditional probability $P_\Psi(\cdot | Z(0) = K)$. By (5.8),

$$P_{(x,K)} = P_\Psi^K \circ \theta_x, \qquad (x,K) \in \mathbb{R}^d \times \mathscr{K}^d, \tag{5.9}$$

is one possible choice of the Palm probabilities introduced in the previous section, where we note that $Q_0 = P_\Psi(Z(0) \in \cdot)$. From Theorem 4.16 we easily obtain the following result.

PROPOSITION 5.4. *Let the assumptions of Proposition* 4.9 *be satisfied and let $A \subset \mathbb{R}^d$ be measurable. Then $(1-\bar{p})H_B(\cdot, A)$ is absolutely continuous with density*

$$\begin{aligned} t \mapsto \sum_{j=0}^{d-1} a_{d-j} t^{d-j-1} \lambda_\Psi \\ \times E_\Psi\Big[\int \mathbf{1}\big\{d(\Phi \setminus \delta_{(0, Z(0))}, z + tb) > t\big\} \mathbf{1}\{b \in A\}\, C_j(Z(0), d(z,b))\Big],\end{aligned} \tag{5.10}$$

where E_Ψ denotes expectation with respect to P_Ψ and $a_i = ib_i$.

REMARK 5.5. If the *second-order reduced moment measure*

$$\int (\psi \setminus \delta_0)(\cdot) P_\Psi\,(\Psi \in d\psi)$$

is absolutely continuous and Φ is an independent marking of Ψ, then the second factorial moment measure $\alpha^{(2)}$ of Φ satisfies the assumption of Proposition 4.9.

EXAMPLE 5.6. In Propositions 4.13 (see also Remark 4.14) and 4.26 we found explicit expressions for the direction dependent contact distribution of the inhomogeneous Boolean model. Under the additional assumption of stationarity, the function f which appears in these propositions is equal to the constant λ_Ψ. We write

$$\overline{V}_j := \int C_j(K, \mathbb{R}^d \times \mathbb{R}^d) Q_0(dK) = \frac{\binom{d}{j}}{b_{d-j}} \int V(K[j], \check{B}[d-j]) Q_0(dK)$$

for the mean of the total jth Minkowski support measure and

$$\overline{S}_j(\cdot) := \int C_j(K, \mathbb{R}^d \times \cdot) Q_0(dK)$$

for the mean jth Minkowski surface area measure. Then we obtain

$$H_B(t) := H_B(t, \mathbb{R}^d) = 1 - \exp\left[-\lambda_\Psi \sum_{i=0}^{d-1} b_{d-i} t^{d-i} \overline{V}_i\right]$$

and, if B is smooth,

$$h_B(t, A) := h_B(x, t, A) = \lambda_\Psi \sum_{j=0}^{d-1} (d-j) b_{d-j} t^{d-j-1} (1 - H_B(t)) \overline{S}_j(A),$$

independent of $x \in \mathbb{R}^d$. This implies, for any measurable function g: $[0, \infty) \times \mathbb{R}^d \to [0, \infty)$, that

$$\begin{aligned} &E\big[g(d(0), u(0)) | 0 \notin \Xi\big] \\ &\quad = \lambda_\Psi \sum_{j=0}^{d-1} (d-j) b_{d-j} \iint g(s, b) s^{d-j-1} \exp\left[-\lambda_\Psi \sum_{i=0}^{d-1} b_{d-i} s^{d-i} \overline{V}_i\right] ds\, \overline{S}_j(db). \end{aligned}$$

Of course, the latter equation can also be derived from Proposition 5.4 using Slivnyak's theorem $P_\Psi(\Psi \backslash \delta_0 \in \cdot) = P(\Psi \in \cdot)$ and taking into account that Φ also under P_Ψ is an independent marking of Ψ.

In the course of the proof of Proposition 4.13 we have especially shown that

$$\int \mathscr{H}^d(K + rB^d) Q_0(dK) < \infty, \qquad r > 0,$$

is a consequence of our general assumption (A2). Therefore we can assume that

$$\int V_j(K) Q_0(dK) < \infty, \qquad j \in \{0, \dots, d-1\},$$

where $V_j(\cdot)$, $j \in \{0, \dots, d\}$, are the classical (Euclidean) quermass-integrals.

Then

$$H_B(t) = 1 - \exp\left[-\lambda_\Psi \sum_{j=0}^{d-1} \binom{d}{j} t^{d-j} \int V(K[j], \check{B}[d-j]) Q_0(dK)\right] \tag{5.11}$$

holds true for an *arbitrary* convex body B containing the origin (see also [37]). For example, if B is a line segment, then we obtain the linear contact distribution. These extensions can be established by approximating B from outside by a decreasing sequence of strictly convex bodies. But, of course, this can also be verified more directly.

If, in addition, Q_0 is invariant with respect to rotations, then one can use (5.3.24) from [28] as well as

$$\rho_B(t) := \rho_B(x,t) = \lambda_\Psi \sum_{j=0}^{d-1} (d-j)\binom{d}{j} t^{d-j-1} \int V(K[j], \check{B}[d-j]) Q_0(dK)$$

(independent of $x \in \mathbb{R}^d$) to deduce the well-known relation

$$\rho_B(t) = \lambda_\Psi \sum_{k=1}^{d} k t^{k-1} \frac{b_k b_{d-k}}{b_d} \frac{V_k(B)}{\binom{d}{k}} \int V_{d-k}(K) Q_0(dK).$$

This completes the discussion of the homogeneous Boolean model.

Finally, we generalize Proposition 5.4. The assertion is similar to Theorem 5.1 and follows by combining the methods of the proofs of Theorem 5.1 and Theorem 4.16. The details are left to the reader.

THEOREM 5.7. *Let the assumptions of Proposition* 4.9 *be satisfied. For any measurable* $f\colon \Omega \to [0,\infty)$,

$$\begin{aligned} E[\mathbf{1}\{d(0) > 0\} f] &= \sum_{j=0}^{d-1} (d-j) b_{d-j} \lambda_\Psi \\ &\quad \times E_\Psi\Big[\iint f \circ \theta_{z+sb} s^{d-j-1} \mathbf{1}\{d(\Phi \setminus \delta_{(0,Z(0))}, z+sb) > s\} \\ &\qquad \times C_j(Z(0), d(z,b))\, ds\Big]. \end{aligned} \tag{5.12}$$

Acknowledgments. This work was initiated during a very stimulating conference on stochastic geometry in Oberwolfach. We thank Professor Rolf Schneider for helpful discussions on the subjects of this paper and in particular for suggesting the second assertion of Theorem 3.4. We also thank Professor Dietrich Stoyan for drawing our attention to the subject of Theorem 4.1.

REFERENCES

[1] BADDELEY, A. J. (1999). Spatial sampling and censoring. In *Stochastic Geometry: Likelihood and Computation* (O. Barndorff-Nielsen, W. Kendall and M. N. M. van Lieshout, eds.) 37–78. CRC Press, Boca Raton, FL.

[2] BADDELEY, A. J. and AVERBACK, P. (1983). Stereology of tubular structures. *J. Microscopy* **131** 323–340.

[3] BADDELEY, A. J. and GILL, R. D. (1994). The empty space hazard of a spatial pattern. Preprint 845, Dept. Mathematics, Univ. Utrecht.

[4] CHIU, S. N. and STOYAN, D. (1998). Estimators of distance distributions for spatial patterns. *Statist. Neerlandica* **52** 239–246.

[5] FALLERT, H. (1996). Quermaßdichten für Punktprozesse konvexer Körper and Boolesche Modelle. *Math. Nachr.* **181** 165–184.

[6] FREMLIN, D. H. (1997). Skeletons and central sets. *Proc. London Math. Soc.* (3) **74** 701–720.

[7] GROEMER, H. (1978). On the extension of additive functionals on classes of convex sets. *Pacific J. Math.* **75** 397–410.

[8] HAHN, U., MICHELETTI, A., POHLINK, R., STOYAN, D. and WENDROCK, H. (1999). Stereological analysis and modelling of gradient structures. *J. Microscopy* **195** 113–124.

[9] HAHN, U. and STOYAN, D. (1998). Unbiased stereological estimation of surface area density of gradient surface processes. *Adv. in Appl. Probab.* **30** 904–920.

[10] HANSEN, M. B., BADDELEY, A. J. and GILL, R. D. (1999). First contact distributions for spatial patterns: regularity and estimation. *Adv. in Appl. Probab.* **31** 15–33.

[11] HANSEN, M. B., GILL, R. D. and BADDELEY, A. J. (1998). Kaplan–Meier type estimators for linear contact distributions. *Scand. J. Statist.* **23** 129–155.

[12] HEINRICH, L. (1992). On existence and mixing properties of germ-grain models. *Statistics* **23** 271–286.

[13] HEINRICH, L. and MOLCHANOV, I. S. (1999). Central limit theorems for a class of random measures associated with germ-grain models. *Adv. in Appl. Probab.* **31** 283–314.

[14] HUG, D. (2000). Contact distributions of Boolean models. *Rend. Circ. Mat. Palerno* **2**.

[15] KALLENBERG, O. (1993). *Random Measures.* Academic Press, London.

[16] KIDERLEN, M. (1999). *Schnittmittelungen und äquivariante Endomorphismen konvexer Körper.* Ph.D. dissertation, Univ. Karlsruhe.

[17] KIDERLEN, M. and WEIL, W. (1999). Measure-valued valuations and mixed curvature measures of convex bodies. *Geom. Dedicata* **76** 291–329.

[18] KINGMAN, J. F. C. (1993). *Poisson Processes.* Oxford Univ. Press.

[19] LAST, G. and HOLTMANN, M. (1999). On the empty space function of some germ-grain models. *Pattern Recognition* **32** 1587–1600.

[20] LAST, G. and SCHASSBERGER, R. (1998). On the distribution of the spherical contact vector of stationary germ-grain models. *Adv. in Appl. Probab.* **30** 36–52.

[21] MATHERON, G. (1975). *Random Sets and Integral Geometry.* Wiley, New York.

[22] MATTHES, K., KERSTAN, J. and MECKE, J. (1978). *Infinitely Divisible Point Processes.* Wiley, Chichester.

[23] MECKE, J. (1967). Stationäre zufällige Maße auf lokalkompakten Abelschen Gruppen. *Z. Wahrsch. Verw. Gebiete* **9** 36–58.

[24] NEVEU, J. (1977). Processus ponctuels. *Ecole d'Eté de Probabilités de Saint-Flour VI. Lecture Notes in Math.* **598** 249–445. Springer, Berlin.

[25] QUINTANELLA, J. and TORQUATO, S. (1997). Microstructure functions for a model of statistically inhomogeneous random media. *Phys. Rev. E* **55** 1558–1565.

[26] SCHNEIDER, R. (1979). Bestimmung konvexer Körper durch Krümmungsmaße. *Comment. Math. Helv.* **54** 42–60.

[27] SCHNEIDER, R. (1980). Parallelmengen mit Vielfachheit und Steiner-Formeln. *Geom. Dedicata* **9** 111–127.

[28] SCHNEIDER, R. (1993). Convex bodies: the Brunn–Minkowski theory. *Encyclopedia of Mathematics and Its Applications* **44**. Cambridge Univ. Press.

[29] SCHNEIDER, R. (1994). An extension of the principal kinematic formula of integral geometry. *Rend. Circ. Mat. Palermo* (2) **35** 275–290.

[30] SCHNEIDER, R. and WEIL, W. (1992). *Integralgeometrie.* Teubner, Stuttgart.

[31] SCHNEIDER, R. and WEIL, W. (2000). *Stochastische Geometrie.* Teubner, Stuttgart.

[32] SCHNEIDER, R. and WIEACKER, J. A. (1997). Integral geometry in Minkowski spaces. *Adv. Math.* **129** 222–260.

[33] SERRA, J. (1988). *Image Analysis and Mathematical Morphology* **2**: *Theoretical Advances.* Academic Press, London.

[34] STOYAN, D., KENDALL, W. S. and MECKE, J. (1995). *Stochastic Geometry and Its Applications*, 2nd ed. Wiley, Chichester.

[35] THOMPSON, A. C. (1996). Minkowski Geometry. *Encyclopedia of Mathematics and Its Applications* **63**. Cambridge Univ. Press.

[36] WEIL, W. (1981). Zufällige Berührung konvexer Körper durch q-dimensionale Ebenen. *Results Math.* **4** 84–101.

[37] WEIL, W. (1997). The mean normal distribution of stationary random sets and particle processes. In *Advances in Theory and Applications of Random Sets*, (D. Jeulin, ed.) 21–33. World Scientific, Singapore.

[38] WEIL, W. (1997). Mean bodies associated with random closed sets. *Rend. Circ. Mat. Palermo (2) Suppl.* **50** 387–412.

[39] WEIL, W. and WIEACKER, J. A. (1984). Densities for stationary random sets and point processes. *Adv. in Appl. Probab.* **16** 324–346.

[40] WEIL, W. and WIEACKER, J. A. (1988). A representation theorem for random sets. *Probab. Math. Statist.* **9** 147–151.

[41] WEIL, W. and WIEACKER, J. A. (1993). Stochastic geometry. In *Handbook of Convex Geometry* (P. M. Gruber and J. M. Wills, eds.) 1391–1438. North-Holland, Amsterdam.

[42] ZÄHLE, M. (1986). Curvature measures and random sets. II. *Probab. Theory Related Fields* **71** 37–58.

[43] ZÄHLE, M. (1988). Random cell complexes and generalized sets. *Ann. Probab.* **16** 1742–1766.

MATHEMATISCHES INSTITUT
ALBERT-LUDWIGS-UNIVERSITÄT
ECKERSTR. 1
D-79104 FREIBURG I. BR.
GERMANY
E-MAIL: hug@sun1.mathematik.uni-freiburg.de

INSTITUT FÜR MATHEMATISCHE STOCHASTIK
UNIVERSITÄT KARLSRUHE (TH)
D-76128 KARLSRUHE
GERMANY
E-MAIL: g.last@math.uni-karlsruhe.de